Introduction to 1D FDTD

1 Maxwell's Equations in 1D

In one dimension, with the electric field E polarized in the x-direction and the wave traveling in the z-direction, Maxwell's equations reduce to two coupled partial differential equations:

$$\frac{\partial E}{\partial z} = -\mu_0 \frac{\partial H}{\partial t} \tag{1}$$

$$\frac{\partial H}{\partial z} = -\varepsilon_0 \frac{\partial E}{\partial t} \tag{2}$$

Physical interpretation:

- Equation (1): A spatial change in E creates a time change in H (Faraday's law)
- Equation (2): A spatial change in H creates a time change in E (Ampère's law)

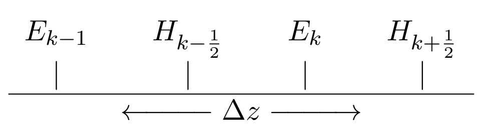
These fields "chase each other" through space, producing a wave traveling at speed:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \times 10^8 \text{ m/s}$$

2 The FDTD Idea: Discretizing Space and Time

FDTD (Finite-Difference Time-Domain) replaces continuous derivatives with finite differences. The key insight is the **Yee scheme**: stagger E and H in both space and time.

Staggered Grid Layout



Similarly in time: E is computed at integer time steps (n), and H at half-integer steps ($n + \frac{1}{2}$).

Why stagger? Centered differences are second-order accurate. If E and H are at the same points, you'd need one-sided differences (first-order, less accurate).

3 The Update Equations

Discretize equation (1) at position $k + \frac{1}{2}$ and time n:

$$\frac{E_{k+1}^{n} - E_{k}^{n}}{\Delta z} = -\mu_0 \frac{H_{k+\frac{1}{2}}^{n+\frac{1}{2}} - H_{k+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t}$$

Solve for the new H:

$$\boxed{H_{k+\frac{1}{2}}^{n+\frac{1}{2}} = H_{k+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{\Delta t}{\mu_0 \Delta z}\left(E_{k+1}^{n} - E_{k}^{n}\right)} \tag{3}$$

Similarly, discretize equation (2) at position k and time $n + \frac{1}{2}$:

$$\boxed{E_{k}^{n+1} = E_{k}^{n} - \frac{\Delta t}{\varepsilon_0 \Delta z}\left(H_{k+\frac{1}{2}}^{n+\frac{1}{2}} - H_{k-\frac{1}{2}}^{n+\frac{1}{2}}\right)} \tag{4}$$

The algorithm (leap-frog time stepping):

1. Update all H values using current E values (Eq. 3)
2. Update all E values using new H values (Eq. 4)
3. Repeat for each time step

4 Stability: The CFL Condition

The algorithm is only stable if information doesn't travel faster numerically than physically. This requires:

$$\boxed{\Delta t < \frac{\Delta z}{c}} \tag{5}$$

This is the **Courant-Friedrichs-Lewy (CFL) condition**.

Intuition: In one time step Δt, the wave travels distance $c \cdot \Delta t$. This must be less than one grid cell Δz, or the wave "skips" cells and the simulation blows up.

A safe choice is $\Delta t = \frac{\Delta z}{2c}$ (Courant number = 0.5).

5 Physical Constants

For reference:

$$\begin{aligned} \varepsilon_0 &= 8.854 \times 10^{-12} \text{ F/m} \quad \text{(permittivity of free space)} \\ \mu_0 &= 4\pi \times 10^{-7} \text{ H/m} \quad \text{(permeability of free space)} \\ c &= 2.998 \times 10^{8} \text{ m/s} \quad \text{(speed of light)} \end{aligned}$$

6 Next Steps

Once this vacuum code works:

1. Add a dielectric region: replace $\varepsilon_0 \to \varepsilon_0 \varepsilon_r$ in the E update
2. Add dispersion using the ADE (Auxiliary Differential Equation) method
3. Validate against analytical Fresnel coefficients