Detailed Analysis of Digital Signal Processing Problems

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This report details the mathematical solutions and computational approaches for a set of problems in digital signal processing. The topics covered include the fundamentals of frequency domain analysis, the application of the Discrete Fourier Transform (DFT) matrix as FIR filters, the spectral analysis of complex sinusoids, and the modeling of both ideal and non-ideal quadrature modulators. Each section provides the necessary theoretical derivations and a brief overview of the implementation in MATLAB.

Keywords. Digital Signal Processing, DFT, FFT, Quadrature Modulation, I/Q Imbalance, FIR Filter.

1 Introduction

This report presents the detailed solutions to four exercises focused on core concepts in digital signal processing (DSP). The analysis covers the relationship between time and frequency domains, the properties of the DFT, methods for spectral estimation, and the practical aspects of modulation and demodulation systems. For each problem, the underlying mathematical theory is derived in full, and the methodology for its computational verification using MATLAB is discussed.

2 Frequency Domain Analysis

2.1 Exercise 1: DFT Frequency Resolution

This problem explores the fundamental relationship between sampling frequency (f_s), the number of samples (N), and the frequency resolution (Δf) in the Discrete Fourier Transform. The core equation is:

$$\Delta f = \frac{f_s}{N}$$

a. Required Number of Samples (N)

Given a sampling frequency of $f_s = 44.1$ kHz and a desired frequency resolution of $\Delta f = 1$ Hz, the required number of samples N can be calculated by rearranging the formula:

$$N = \frac{f_s}{\Delta f} = \frac{44100 \text{ Hz}}{1 \text{ Hz}} = 44100 \text{ samples}$$

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b. Time Duration of the Sequence

The total time duration (T) of the signal segment is the number of samples divided by the sampling rate:

$$T = \frac{N}{f_s} = \frac{44100 \text{ samples}}{44100 \text{ samples/second}} = 1 \text{ second}$$

This result confirms that to resolve frequencies 1 Hz apart, a signal must be observed for at least 1 second.

2.2 Exercise 2: DFT Matrix Rows as FIR Filters

This exercise treats specific rows of the $N = 8$ DFT matrix as the coefficients of Finite Impulse Response (FIR) filters. The DFT of a signal $x[n]$ is calculated as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{where} \quad W_N = e^{-j2\pi/N}$$

Each row k of the DFT matrix contains the coefficients W_N^{kn} for $n = 0, \dots, N-1$. These coefficients can be interpreted as the impulse response of a band-pass filter centered at the frequency corresponding to bin k.

The MATLAB code for this problem first constructs the 8×8 DFT matrix using the `dftmtx(8)` command. It then extracts rows 1, 2, and 6 to serve as the impulse responses for three FIR filters. The frequency response of each filter is then calculated and plotted using the `freqz()` command. The plots confirm that each filter is a band-pass filter tuned to its corresponding frequency bin.

3 Spectrum Calculation and Analysis

3.1 Exercise 3: Spectrum of a Complex Sinusoid

This problem required computing the spectrum of a complex sinusoid, $x[n] = Ae^{j2\pi \frac{f_0}{f_s} n}$, using two methods and comparing the results in dBm.

Mathematical Derivation of Power

The key step is the conversion from the signal's properties to power in dBm.

1. **Physical Power**: The power of a complex exponential signal with amplitude A (in Volts) across a resistor R is given by $P = A^2/R$. With $A = 2\text{V}$ and $R = 50\,\Omega$, the power is:

$$P_{watts} = \frac{2^2}{50} = \frac{4}{50} = 0.08 \text{ W}$$

2. **Conversion to dBm**: The formula for conversion to dBm (decibels relative to 1 milliwatt) is:

$$P_{dBm} = 10\log_{10}\left(\frac{P_{watts}}{1 \text{ mW}}\right) = 10\log_{10}\left(\frac{0.08}{0.001}\right) = 10\log_{10}(80) \approx 19.03 \text{ dBm}$$

3. **FFT Scaling**: To obtain this physical power from the FFT output $X[k]$, the correct scaling must be applied. The power in Watts for any given bin k is:

$$P_{watts}[k] = \frac{|X[k]|^2}{N^2 R}$$

For the peak bin k_0, where $|X[k_0]| = A \cdot N$, this scaling correctly recovers the physical power: $P_{watts}[k_0] = \frac{(A \cdot N)^2}{N^2 R} = \frac{A^2}{R}$.

The MATLAB code implements this calculation for both an ideal spectrum (a single impulse) and the FFT-computed spectrum. The plots showed a near-perfect match with a peak power of 19.03 dBm, as the chosen frequencies fell exactly on DFT bins, avoiding spectral leakage.

4 Quadrature Modulation and Demodulation

4.1 Exercise 4: Ideal Quadrature Demodulation

This exercise required a step-by-step derivation of the output signals of an ideal quadrature demodulator.

Transmitted Signal

The transmitted signal, $s(t)$, is formed by modulating the in-phase signal $x_I(t)$ and the quadrature signal $x_Q(t)$:

$$s(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

I-Channel Demodulation

At the receiver, $s(t)$ is multiplied by the in-phase carrier, $\cos(2\pi f_c t)$. The signal before the low-pass filter (LPF), $v_1(t)$, is:

$$v_1(t) = s(t) \cdot \cos(2\pi f_c t) = [x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)] \cdot \cos(2\pi f_c t)$$

$$v_1(t) = x_I(t)\cos^2(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t)$$

Using the identities $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$ and $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$, we get:

$$v_1(t) = x_I(t)\left[\frac{1}{2} + \frac{1}{2}\cos(4\pi f_c t)\right] - x_Q(t)\left[\frac{1}{2}\sin(4\pi f_c t)\right]$$

$$v_1(t) = \underbrace{\frac{1}{2}x_I(t)}_{\text{Low Freq}} + \underbrace{\frac{1}{2}x_I(t)\cos(4\pi f_c t) - \frac{1}{2}x_Q(t)\sin(4\pi f_c t)}_{\text{High Freq}}$$

The LPF removes the high-frequency components at $2f_c$, leaving the final output:

$$y_1(t) = \frac{1}{2}x_I(t)$$

Q-Channel Demodulation

For the Q-channel, $s(t)$ is multiplied by $-\sin(2\pi f_c t)$. The signal before the LPF, $v_2(t)$, is:

$$v_2(t) = s(t) \cdot [-\sin(2\pi f_c t)] = -x_I(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + x_Q(t)\sin^2(2\pi f_c t)$$

Using the identity $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$, we get:

$$v_2(t) = -x_I(t)\left[\frac{1}{2}\sin(4\pi f_c t)\right] + x_Q(t)\left[\frac{1}{2} - \frac{1}{2}\cos(4\pi f_c t)\right]$$

$$v_2(t) = \underbrace{\frac{1}{2}x_Q(t)}_{\text{Low Freq}} - \underbrace{\frac{1}{2}x_I(t)\sin(4\pi f_c t) - \frac{1}{2}x_Q(t)\cos(4\pi f_c t)}_{\text{High Freq}}$$

After low-pass filtering, the output is:

$$y_2(t) = \frac{1}{2}x_Q(t)$$

5 Imperfections in Quadrature Converters

5.1 Exercise 5: I/Q Imbalance Analysis

This problem analyzes the effect of amplitude imbalance (α) and phase error (ϕ) in a quadrature demodulator. These imperfections create an unwanted "image" signal.

Mathematical Model of Imbalance

The imperfectly demodulated signal $y(t)$ can be modeled as a linear combination of the ideal signal $x(t)$ and its complex conjugate $x^*(t)$:

$$y(t) = K_1 x(t) + K_2 x^*(t)$$

where $K_1 x(t)$ represents the desired signal component and $K_2 x^*(t)$ represents the unwanted image component. The coefficients are given by:

$$K_1 = \frac{A}{2}[1 + (1+\alpha)e^{-j\phi}]$$

$$K_2 = \frac{A}{2}[1 - (1+\alpha)e^{j\phi}]$$

The problem requires an Image Rejection Ratio (IRR) of -60 dB, which corresponds to an amplitude ratio of $|K_2/K_1| \leq 10^{-60/20} = 0.001$.

Derivation of Error Constraints

For small imbalances ($\alpha \ll 1$) and phase errors ($\phi \ll 1$ radian), we can use the approximation $e^{j\phi} \approx 1 + j\phi$. The coefficients simplify to:

$$K_1 \approx \frac{A}{2}[1 + (1+\alpha)(1 - j\phi)] \approx \frac{A}{2}[2 + \alpha - j\phi] \approx A$$

$$K_2 \approx \frac{A}{2}[1 - (1+\alpha)(1 + j\phi)] \approx -\frac{A}{2}[\alpha + j\phi]$$

The ratio of their magnitudes becomes:

$$\left|\frac{K_2}{K_1}\right| \approx \frac{|-\frac{A}{2}(\alpha + j\phi)|}{|A|} = \frac{1}{2}|\alpha + j\phi| = \frac{1}{2}\sqrt{\alpha^2 + \phi^2}$$

Applying the -60 dB constraint yields the final requirement on the errors:

$$\frac{1}{2}\sqrt{\alpha^2 + \phi^2} \leq 0.001 \implies \sqrt{\alpha^2 + \phi^2} \leq 0.002$$

This defines a small circular region in the (α, ϕ) plane where the performance specification is met. The MATLAB code visualizes this by plotting the contour line corresponding to -60 dB, clearly delineating the acceptable region of operation.