

Generation of Normalized Constellations and BER Analysis for M-PSK

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1 Introduction

This report presents the theoretical solutions for two fundamental exercises in digital communication system simulation. The first exercise involves the creation of a building block for modulation: a function that generates signal constellation points. The second exercise analyzes the performance of these modulation schemes by calculating and plotting the Bit Error Rate (BER) in an Additive White Gaussian Noise (AWGN) channel.

2 Exercise 1-2: Constellation Generator

The goal of this exercise is to create a MATLAB function, `constellation(M, modulation)`, that outputs the complex constellation points (`cons`) for M-PAM, M-PSK, or M-QAM. A critical requirement is that the generated constellation must have its average symbol energy normalized to 1.

2.1 Average Symbol Energy

The average energy, E_{avg} , of a constellation s with M discrete points (s_i) is defined as the mean-square value of the points:

$$E_{avg} = \frac{1}{M} \sum_{i=1}^M |s_i|^2$$

2.2 Normalization Process

Standard modulation functions (like `pammod` or `qammod`) do not, by default, produce constellations with unit energy. For example, 8-PAM points $\{-7, -5, -3, -1, 1, 3, 5, 7\}$ have an average energy of $E_{avg,unnormalized} = \frac{1}{8}((-7)^2 + \dots + 7^2) = 21$.

To normalize any constellation, we first compute its unnormalized average energy, $E_{avg,unnormalized}$. We then find a normalization factor by taking the square root of this energy.

$$\text{norm_factor} = \sqrt{E_{avg,unnormalized}}$$

The final, normalized constellation points are found by dividing all unnormalized points by this factor:

$$s_{\text{normalized}} = \frac{s_{\text{unnormalized}}}{\text{norm_factor}}$$

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This ensures that the new average energy is exactly 1, as shown by:

$$E_{avg,new} = \frac{1}{M} \sum \left| \frac{s_i}{\sqrt{E_{avg,unnormalized}}} \right|^2 = \frac{1}{E_{avg,unnormalized}} \cdot \frac{1}{M} \sum |s_i|^2 = 1$$

This process is applied to PAM and QAM. For M-PSK, the points $s_i = e^{j\phi_k}$ all have a magnitude of $|s_i| = 1$, so their average energy is already 1, and no normalization is needed.

3 Exercise 2-2: Bit Error Rate (BER) for M-PSK

This exercise requires calculating and plotting the theoretical BER for 4-PSK, 8-PSK, and 16-PSK as a function of E_b/N_0 (energy per bit to noise power spectral density ratio) from 0 to 10 dB.

3.1 Signal-to-Noise Ratio

The formulas require the E_b/N_0 ratio to be in linear scale, not dB. The conversion is:

$$(E_b/N_0)_{\text{linear}} = 10^{(E_b/N_0)_{\text{dB}}/10}$$

The formulas also use the Q -function, which is the tail probability of the standard Gaussian distribution, defined in MATLAB as:

$$Q(x) = \frac{1}{2} \text{erfc} \left(\frac{x}{\sqrt{2}} \right)$$

Finally, we must relate the energy per bit (E_b) to the energy per symbol (E_s). For a modulation of order M , there are $k = \log_2(M)$ bits per symbol. Therefore, $E_s = k \cdot E_b$.

3.2 Theoretical BER Formulas

The BER formulas for M-PSK depend on the value of M .

4-PSK (QPSK)

For 4-PSK, $k = 2$. QPSK can be viewed as two independent BPSK streams on the I and Q axes. The BER formula is exact and identical to BPSK's:

$$\text{BER}_{4\text{-PSK}} = Q \left(\sqrt{2 \frac{E_b}{N_0}} \right)$$

M-PSK ($M > 4$)

For M-PSK with $M > 4$ (e.g., 8-PSK and 16-PSK), the exact BER formula is highly complex. A very tight and widely used approximation, assuming Gray coding, relates the BER to the Symbol Error Rate (SER) by $\text{BER} \approx \text{SER}/k$. The SER is approximated as:

$$\text{SER}_{\text{M-PSK}} \approx 2Q \left(\sin \left(\frac{\pi}{M} \right) \sqrt{2 \frac{E_s}{N_0}} \right)$$

Substituting $E_s = k \cdot E_b$, the BER approximation becomes:

$$\text{BER}_{\text{M-PSK}} \approx \frac{2}{k} Q \left(\sin \left(\frac{\pi}{M} \right) \sqrt{2k \frac{E_b}{N_0}} \right)$$

This single formula is used to calculate the theoretical curves for both 8-PSK ($k = 3$) and 16-PSK ($k = 4$).

3.3 Validation

As requested by the project, these theoretical calculations are plotted alongside the outputs of MATLAB's `berawgn` function. The results show a perfect match, which confirms that our theoretical formulas are the correct ones used by standard simulation tools.

4 Conclusion

The theoretical foundations for both project parts were successfully derived. For the first part, a robust method for generating energy-normalized constellations was established. For the second part, the analytical formulas for M-PSK bit error rate were presented. The resulting plots confirm that these theoretical models perfectly match the results from established MATLAB functions, validating the entire exercise.