Analysis of Channel Impairments in Quadrature Demodulation Systems

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1 Introduction

Quadrature Amplitude Modulation (QAM) is foundational to modern communication systems, allowing two independent signals, $x_I(t)$ and $x_Q(t)$, to be sent on the same carrier frequency, f_c. This is achieved by using two carriers in quadrature (90° out of phase). This report analyzes the mathematical principles of quadrature modulation and the impact of several real-world channel impairments.

2 Quadrature Modulator Principle

The project first asks to prove the complex-number representation of the quadrature modulator output. The modulator diagram shows the output $x_{bp}(t)$ as:

$$x_{bp}(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

We must prove this is equivalent to $x_{bp}(t) = \mathrm{Re}\{[x_I(t) + jx_Q(t)]e^{j2\pi f_c t}\}$.

Proof

We start with the complex expression and use Euler's formula, $e^{j\theta} = \cos(\theta) + j\sin(\theta)$:

$$y(t) = \mathrm{Re}\{[x_I(t) + jx_Q(t)] \cdot [\cos(2\pi f_c t) + j\sin(2\pi f_c t)]\}$$

We expand the multiplication:

$$y(t) = \mathrm{Re}\{[x_I(t)\cos(2\pi f_c t) + jx_I(t)\sin(2\pi f_c t) + jx_Q(t)\cos(2\pi f_c t) + j^2 x_Q(t)\sin(2\pi f_c t)]\}$$

Using $j^2 = -1$ and grouping real and imaginary parts:

$$y(t) = \mathrm{Re}\{\underbrace{[x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)]}_{\text{Real Part}} + j[x_I(t)\sin(2\pi f_c t) + x_Q(t)\cos(2\pi f_c t)]\}$$

Taking the real part, $\mathrm{Re}\{\dots\}$, leaves:

$$y(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

This matches $x_{bp}(t)$, completing the proof.

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3 Analysis of Channel with Phase Error (Noise-Free)

This section analyzes the effect of a channel that introduces an attenuation α and a phase shift ϕ, but no noise. The received signal is:

$$y(t) = \mathrm{Re}\{\alpha x_I(t) e^{j\phi} e^{j2\pi f_c t}\}$$

Using the angle-addition identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$, we can write $y(t)$ in real terms:

$$y(t) = \mathrm{Re}\{\alpha x_I(t) e^{j(2\pi f_c t + \phi)}\} = \alpha x_I(t) \cos(2\pi f_c t + \phi)$$

This signal enters an ideal quadrature demodulator.

I-Channel (Top Path) Recovery

The signal $y(t)$ is multiplied by $\cos(2\pi f_c t)$:

$$v_1(t) = y(t) \cdot \cos(2\pi f_c t) = \alpha x_I(t) \cos(2\pi f_c t + \phi) \cos(2\pi f_c t)$$

Using the identity $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$:

$$v_1(t) = \frac{\alpha x_I(t)}{2}[\cos(\phi) + \cos(4\pi f_c t + \phi)]$$

The Low-Pass Filter (LPF) removes the high-frequency $2f_c$ term, leaving:

$$y_1(t) = \frac{\alpha x_I(t)}{2}\cos(\phi)$$

Q-Channel (Bottom Path) Recovery

The signal $y(t)$ is multiplied by $-\sin(2\pi f_c t)$:

$$v_2(t) = y(t) \cdot [-\sin(2\pi f_c t)] = -\alpha x_I(t) \cos(2\pi f_c t + \phi) \sin(2\pi f_c t)$$

Using the identity $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$:

$$v_2(t) = -\frac{\alpha x_I(t)}{2}[\sin(4\pi f_c t + \phi) - \sin(\phi)] = \frac{\alpha x_I(t)}{2}\sin(\phi) - \dots_{\text{High Freq}}$$

The LPF removes the high-frequency term, leaving:

$$y_2(t) = \frac{\alpha x_I(t)}{2}\sin(\phi)$$

This shows that a phase error ϕ causes the desired signal in the I-channel to be attenuated by $\cos(\phi)$, while "crosstalk" (an unwanted copy of the I-channel signal) leaks into the Q-channel, scaled by $\sin(\phi)$.

4 Analysis of Channel with Differential Delay

This section analyzes a different impairment model where the channel introduces a base delay t_a to both paths, but an additional *differential delay* t_d to the Q-path only. The received signal $y(t)$ is:

$$y(t) = \alpha x_I(t - t_a)\cos(2\pi f_c t) - \alpha x_Q(t - t_a - t_d)\sin(2\pi f_c t)$$

This signal enters an ideal demodulator.

I-Channel Recovery

$$v_1(t) = y(t) \cdot \cos(2\pi f_c t)$$
$$v_1(t) = \alpha x_I(t - t_a)\cos^2(2\pi f_c t) - \alpha x_Q(t - t_a - t_d)\sin(2\pi f_c t)\cos(2\pi f_c t)$$

After applying trigonometric identities and low-pass filtering, the high-frequency terms are removed:

$$y_1(t) = \frac{\alpha}{2} x_I(t - t_a)$$

Q-Channel Recovery

$$v_2(t) = y(t) \cdot [-\sin(2\pi f_c t)]$$
$$v_2(t) = -\alpha x_I(t - t_a)\cos(2\pi f_c t)\sin(2\pi f_c t) + \alpha x_Q(t - t_a - t_d)\sin^2(2\pi f_c t)$$

After filtering, the output is:

$$y_2(t) = \frac{\alpha}{2} x_Q(t - t_a - t_d)$$

This shows that the I-signal is recovered correctly (with delay t_a), but the Q-signal is recovered with an additional delay t_d, misaligning it with the I-signal.

5 BER Analysis with Attenuation and Phase Error

This section analyzes a Binary PAM system ($x_I = \pm A$, $x_Q = 0$) with attenuation α, phase error ϕ, and Complex White Gaussian Noise $n(t) \sim CN(0, 2\sigma^2)$.

5.1 Demodulator Model with Impairments

The received signal $r(t)$ includes the attenuated signal and noise:

$$r(t) = [\alpha x_I(t) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

The receiver's local oscillator has a phase error ϕ, so the I-path LO is $\cos(2\pi f_c t + \phi)$. The output of the I-path LPF is the decision variable Y_1:

$$Y_1 = \underbrace{\frac{\alpha x_I}{2}\cos(\phi)}_{\text{Signal Part}} + \underbrace{\frac{1}{2} n_I \cos(\phi) + \frac{1}{2} n_Q \sin(\phi)}_{\text{Noise Part } n'}$$

5.2 Noise and Error Probability Derivation

The noise $n(t) \sim CN(0, 2\sigma^2)$ means the total variance is $2\sigma^2$, so the independent components n_I and n_Q each have a variance of σ^2.

$$\text{Var}[n_I] = \sigma^2 \quad \text{and} \quad \text{Var}[n_Q] = \sigma^2$$

The variance of the noise at the decision point, $\sigma^2_{n'}$, is:

$$\sigma^2_{n'} = \text{Var}\left[\frac{1}{2} n_I \cos(\phi)\right] + \text{Var}\left[\frac{1}{2} n_Q \sin(\phi)\right]$$

$$\sigma_{n'}^2 = \frac{1}{4}\cos^2(\phi)\text{Var}[n_I] + \frac{1}{4}\sin^2(\phi)\text{Var}[n_Q]$$

$$\sigma_{n'}^2 = \frac{1}{4}\cos^2(\phi)(\sigma^2) + \frac{1}{4}\sin^2(\phi)(\sigma^2) = \frac{\sigma^2}{4}(\cos^2(\phi) + \sin^2(\phi)) = \frac{\sigma^2}{4}$$

The standard deviation of the noise is $\sigma_{n'} = \sqrt{\sigma^2/4} = \frac{\sigma}{2}$.

The general probability of error P_e (assuming $x_I = A$ was sent) is:

$$P_e = P(Y_1 < 0) = Q\left(\frac{\text{Signal Part}}{\text{Noise Std Dev}}\right) = Q\left(\frac{\frac{\alpha A}{2}\cos(\phi)}{\frac{\sigma}{2}}\right)$$

$$P_e = Q\left(\frac{\alpha A\cos(\phi)}{\sigma}\right)$$

5.3 Comparison of Ideal vs. Impaired BER

The project asks to find the error probability for two cases.

1. Probability of Error WITHOUT Channel Effect

This is the ideal case, where $\alpha = 1$ (no attenuation) and $\phi = 0$ (no phase error).

$$P_e(\text{ideal}) = Q\left(\frac{1\cdot A\cdot\cos(0)}{\sigma}\right) = Q\left(\frac{A}{\sigma}\right)$$

2. Probability of Error WITH Channel Effect

Here, we use the given parameters $\alpha = 0.2$ and $\phi = \pi/4$.

$$P_e(\text{channel}) = Q\left(\frac{0.2\cdot A\cdot\cos(\pi/4)}{\sigma}\right)$$

Since $\cos(\pi/4) = 1/\sqrt{2}$:

$$P_e(\text{channel}) = Q\left(\frac{0.2\cdot A\cdot(1/\sqrt{2})}{\sigma}\right) = Q\left(\frac{0.2A}{\sqrt{2}\sigma}\right) \approx Q\left(\frac{0.1414A}{\sigma}\right)$$

The channel impairments significantly reduce the argument of the Q-function, which will dramatically increase the bit error rate.

6 Analysis of the 3D Error Surface

The 3D surface plot visualizes the general P_e formula, $P_e = Q\left(\frac{\alpha A\cos(\phi)}{\sigma}\right)$, by plotting $\log_{10}(P_e)$ as a function of α and ϕ. The resulting plot confirms the theory:

- **Best Performance (Blue Valley):** The error is lowest (Z-axis is most negative) when α is close to 1 (low attenuation) and ϕ is 0 or 2π. Here, $\cos(\phi) = 1$, maximizing the signal component.
- **Catastrophic Error (Yellow Ridges):** The error is highest (Z-axis close to 0) in two scenarios:

1. **Phase Quadrature ($\phi = \pi/2, 3\pi/2$):** Here, $\cos(\phi) = 0$. The signal component becomes zero, and the receiver is just guessing based on noise. This leads to $P_e = Q(0) = 0.5$, so $\log_{10}(P_e) \approx -0.3$.
2. **Phase Inversion ($\phi = \pi$):** Here, $\cos(\phi) = -1$. The signal is inverted, causing the receiver to make the wrong decision on nearly every bit. This leads to $P_e \approx 1$, so $\log_{10}(P_e) = 0$.

- **Effect of Attenuation (α):** As α approaches 0, the argument of the Q-function goes to zero for all ϕ. This causes P_e to approach 0.5 (random guessing). The plot correctly shows the entire surface rising to the yellow region as $\alpha \to 0$.

7 Conclusion

This report successfully modeled several key principles and impairments in a quadrature communication system. The derivations showed how the complex-number notation represents the modulator, how phase error creates crosstalk, how differential delay de-synchronizes the I and Q channels, and how attenuation and phase error jointly impact the bit error rate. The final 3D plot provided a clear visualization of this relationship.