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Continuous Multivariate Distributions

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Abstract: In this article, we present a concise review of developments on various continuous multivariate distributions. We first present some basic definitions and notations. Then, we present several important continuous multivariate distributions and list their significant properties and characteristics.

1 Definitions and Notations

We shall denote the k-dimensional continuous random vector by $\mathbf{X} = (X_1, \ldots, X_k)^T$, its **probability density function** by $p_{\mathbf{X}}(\mathbf{x})$, and its **cumulative distribution function** by $F_{\mathbf{X}}(\mathbf{x}) = Pr\{\bigcap_{i=1}^{k}(X_i \le x_i)\}$. We shall denote the **moment generating function** of $\mathbf{X}$ by $M_{\mathbf{X}}(\mathbf{t}) = E\{e^{\mathbf{t}^T\mathbf{X}}\}$, the cumulant **generating function** of $\mathbf{X}$ by $K_{\mathbf{X}}(\mathbf{t}) = \log M_{\mathbf{X}}(\mathbf{t})$, and the **characteristic function** of $\mathbf{X}$ by $\varphi_{\mathbf{X}}(\mathbf{t}) = E\{e^{i\mathbf{t}^T\mathbf{X}}\}$. Next, we shall denote the $\mathbf{r}$th mixed raw **moment** of $\mathbf{X}$ by $\mu'_{\mathbf{r}}(\mathbf{X}) = E\{\prod_{i=1}^{k} X_i^{r_i}\}$, which is the coefficient of $\prod_{i=1}^{k}(t_i^{r_i}/r_i!)$ in $M_{\mathbf{X}}(\mathbf{t})$, the $\mathbf{r}$th mixed **central moment** by $\mu_{\mathbf{r}}(\mathbf{X}) = E\{\prod_{i=1}^{k} [X_i - E(X_i)]^{r_i}\}$, and the $\mathbf{r}$th mixed **cumulant** by $\kappa_{\mathbf{r}}(\mathbf{X})$, which is the coefficient of $\prod_{i=1}^{k}(t_i^{r_i}/r_i!)$ in $K_{\mathbf{X}}(\mathbf{t})$. For simplicity in notation, we shall also use $E(\mathbf{X})$ to denote the mean vector of $\mathbf{X}$, $\mathrm{Var}(\mathbf{X})$ to denote the variance-covariance matrix of $\mathbf{X}$,

$$\mathrm{cov}(X_i, X_j) = E(X_{iX_j}) - E(X_i)E(X_j)$$

to denote the **covariance** of X_i and X_j, and

$$\mathrm{corr}(X_i, X_j) = \mathrm{cov}(X_i, X_j)/\{\mathrm{Var}(X_i)\mathrm{Var}(X_j)\}^{1/2}$$

to denote the **correlation** coefficient between X_i and X_j.

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2 Introduction

The past four decades have seen a phenomenal amount of activity on theory, methods, and applications of continuous multivariate distributions. Significant developments have been made with regard to nonnormal distributions, since much of the early work in the literature focused only on bivariate and multivariate normal distributions. Several interesting applications of continuous multivariate distributions have also been discussed in the statistical and applied literatures. The availability of powerful computers and sophisticated software packages have certainly facilitated the modeling of continuous multivariate distributions to data and in developing efficient inferential procedures for the parameters underlying these models. The book of Kotz, Balakrishnan, and Johnson [124] provides an encyclopedic treatment of developments on various continuous multivariate distributions and their properties, characteristics, and applications. In this article, we present a concise review of significant developments on continuous multivariate distributions.

3 Relationships Between Moments

Using binomial expansions, we can readily obtain the following relationships:

$$\begin{aligned}\mu_{\mathbf{r}}(\mathbf{X}) = &\sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_k=0}^{r_k} (-1)^{\ell_1+\cdots+\ell_k} \binom{r_1}{\ell_1} \cdots \\ &\times \binom{r_k}{\ell_k} \{E(X_1)\}^{\ell_1} \ldots \{E(X_k)\}^{\ell_k}\, \mu'_{\mathbf{r}-\ell}(\mathbf{X})\end{aligned} \tag{1}$$

and

$$\begin{aligned}\mu'_{\mathbf{r}}(\mathbf{X}) = &\sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_k=0}^{r_k} \binom{r_1}{\ell_1} \cdots \binom{r_k}{\ell_k} \\ &\times \{E(X_1)\}^{\ell_1} \ldots \{E(X_k)\}^{\ell_k}\, \mu_{\mathbf{r}-\ell}(\mathbf{X}).\end{aligned} \tag{2}$$

By denoting $\mu'_{r_1,\ldots,r_j,0,\ldots,0}$ by $\mu'_{r_1,\ldots,r_j}$ and $\kappa_{r_1,\ldots,r_j,0,\ldots,0}$ by $\kappa_{r_1,\ldots,r_j}$, Smith [188] established the following two relationships for computational convenience:

$$\begin{aligned}\mu'_{r_1,\ldots,r_{j+1}} = &\sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_j=0}^{r_j} \sum_{\ell_{j+1}=0}^{r_{j+1}-1} \binom{r_1}{\ell_1} \cdots \binom{r_j}{\ell_j} \\ &\times \binom{r_{j+1}-1}{\ell_{j+1}} \kappa_{r_1-\ell_1,\ldots,r_{j+1}-\ell_{j+1}} \\ &\times \mu'_{\ell_1,\ldots,\ell_{j+1}}\end{aligned} \tag{3}$$

and

$$\begin{aligned}\kappa_{r_1,\ldots,r_{j+1}} = &\sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_j=0}^{r_j} \sum_{\ell_{j+1}=0}^{r_{j+1}-1} \binom{r_1}{\ell_1} \cdots \binom{r_j}{\ell_j} \\ &\times \binom{r_{j+1}-1}{\ell_{j+1}},\, \mu'_{r_1-\ell_1,\ldots,r_{j+1}-\ell_{j+1}} \\ &\times \mu'^{*}_{\ell_1,\ldots,\ell_{j+1}},\end{aligned} \tag{4}$$

where $\mu'^{*}_{\boldsymbol{\ell}}$ denotes the $\boldsymbol{\ell}$th mixed raw moment of a distribution with cumulant generating function $-K_{\mathbf{X}}(\mathbf{t})$. Along similar lines, Balakrishnan, Johnson, and Kotz [32] established the following two relationships:

$$\begin{aligned}
\mu_{r_1,\ldots,r_{j+1}} &= \sum_{\ell_1=0}^{r_1} \cdots \sum_{\ell_j=0}^{r_j} \sum_{\ell_{j+1}=0}^{r_{j+1}-1} \binom{r_1}{\ell_1} \cdots \binom{r_j}{\ell_j} \\
&\quad \times \binom{r_{j+1}-1}{\ell_{j+1}} \{\kappa_{r_1-\ell_1,\ldots,r_{j+1}-\ell_{j+1}} \\
&\quad \times \mu_{\ell_1,\ldots,\ell_{j+1}} - E\{X_{j+1}\}\mu_{\ell_1,\ldots,\ell_j,\ell_{j+1}-1}\}
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\kappa_{r_1,\ldots,r_{j+1}} &= \sum_{\ell_1,\ell'_1=0}^{r_1} \cdots \sum_{\ell_j,\ell'_j=0}^{r_j} \sum_{\ell_{j+1},\ell'_{j+1}=0}^{r_{j+1}-1} \\
&\times \binom{r_1}{\ell_1, \ell'_1, r_1-\ell_1-\ell'_1} \cdots \\
&\times \binom{r_j}{\ell_j, \ell'_j, r_j-\ell_j-\ell'_j} \\
&\times \binom{r_{j+1}-1}{\ell_{j+1}, \ell'_{j+1}, r_{j+1}-1-\ell_{j+1}-\ell'_{j+1}} \\
&\times \mu_{r_1-\ell_1-\ell'_1,\ldots,r_{j+1}-\ell_{j+1}-\ell'_{j+1}} \\
&\times \mu'^{*}_{\ell_1,\ldots,\ell_{j+1}} \prod_{i=1}^{j+1} (\mu_i + \mu_i^*)^{\ell'_i} \\
&+ \mu_{j+1}\ I\{r_1 = \ldots = r_j = 0, r_{j+1} = 1\},
\end{aligned} \tag{6}$$

where $\mu_{r_1,\ldots,r_j}$ denotes $\mu_{r_1,\ldots,r_j,0,\ldots,0}$, $I\cdot$ denotes the indicator function, $\binom{n}{\ell,\ell',n-\ell-\ell'} = \frac{n!}{\ell!\ell'!(n-\ell-\ell')!}$, and $\sum_{\ell,\ell'=0}^{r}$ denotes the summation over all nonnegative integers ℓ and ℓ' such that $0 \le \ell + \ell' \le r$.

All these relationships can be used to obtain one set of moments (or cumulants) from another set.

4 Bivariate and Trivariate Normal Distributions

As mentioned before, early work on continuous multivariate distributions focused on bivariate and **multivariate normal distributions**; see, for example, Adrian [1], Bravais [45], Helmert [104], Galton [86,87], and Pearson [157,158]. Anderson [13] provided a broad account on the history of bivariate normal distribution.

4.1 Definitions

The pdf of the bivariate normal random vector $\mathbf{X} = (X_1, X_2)^T$ is

$$
\begin{aligned}
& p(x_1, x_2; \xi_1, \xi_2, \sigma_1, \sigma_2, \rho) \\
& \quad = \frac{1}{2\pi\sigma_1\ \sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\right. \\
& \qquad \times \left\{\left(\frac{x_1-\xi_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\xi_1}{\sigma_1}\right)\right. \\
& \qquad \left.\left.\times \left(\frac{x_2-\xi_2}{\sigma_2}\right) + \left(\frac{x_2-\xi_2}{\sigma_2}\right)^2\right\}\right], \\
& \qquad -\infty < x_1, x_2 < \infty.
\end{aligned} \tag{7}
$$

This bivariate normal distribution is also sometimes referred to as the *bivariate Gaussian, bivariate Laplace–Gauss*, or *Bravais distribution*.

In Equation 7, it can be shown that $E(X_j) = \xi_j$, $\mathrm{Var}(X_j) = \sigma_j^2$ (for $j = 1, 2$), and $\mathrm{Corr}(X_1, X_2) = \rho$. In the special case when $\xi_1 = \xi_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, Equation 7 reduces to

$$
\begin{aligned}
& p(x_1, x_2; \rho) \\
& \quad = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \times (x_1^2 - 2\rho x_1\ x_2 + x_2^2)\right\}, \\
& \quad -\infty < x_1, x_2 < \infty,
\end{aligned} \tag{8}
$$

which is termed the *standard bivariate normal* density function. When $\rho = 0$ and $\sigma_1 = \sigma_2$ in Equation 7, the density is called the **circular normal** density function, while the case $\rho = 0$ and $\sigma_1 \neq \sigma_2$ is called the *elliptical normal* density function.

The *standard trivariate normal* density function of $\mathbf{X} = (X_1, X_2, X_3)^T$ is

$$
\begin{aligned}
& p_{\mathbf{X}}(x_1, x_2, x_3) \\
& \quad = \frac{1}{(2\pi)^{3/2}\sqrt{\Delta}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\ x_{i x_j}\right\}, \\
& \quad -\infty < x_1, x_2, x_3 < \infty,
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
A_{11} &= (1-\rho_{23}^2)/\Delta,\ A_{22} = (1-\rho_{13}^2)/\Delta, \\
A_{33} &= (1-\rho_{12}^2)/\Delta, \\
A_{12} &= A_{21} = (\rho_{12}\ \rho_{23} - \rho_{12})/\Delta, \\
A_{13} &= A_{31} = (\rho_{12}\ \rho_{23} - \rho_{13})/\Delta, \\
A_{23} &= A_{32} = (\rho_{12}\ \rho_{13} - \rho_{23})/\Delta, \\
\Delta &= 1 - \rho_{23}^2 - \rho_{13}^2 - \rho_{12}^2 + 2\rho_{23}\ \rho_{13}\ \rho_{12},
\end{aligned} \tag{10}
$$

and ρ_23, ρ_{13}, ρ_{12} are the correlation coefficients between (X_2, X_3), (X_1, X_3) and (X_1, X_2) respectively. Once again, if all the correlations are zero and all the variances are equal, the distribution is called the *trivariate spherical normal* distribution, while the case when all the correlations are zero and all the variances are unequal is called the *ellipsoidal normal* distribution.

4.2 Moments and Properties

By noting that the standard bivariate normal pdf in Equation 8 can be written as

$$p(x_1, x_2; \rho) = \frac{1}{\sqrt{1-\rho^2}}\, \phi\left(\frac{x_2 - \rho x_1}{\sqrt{1-\rho^2}}\right) \phi(x_1), \tag{11}$$

where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the univariate standard normal density function, we readily have the conditional distribution of X_2, given $X_1 = x_1$, to be normal with mean ρx_1 and variance $1 - \rho^2$; similarly, the conditional distribution of X_1, given $X_2 = x_2$, is normal with mean ρx_2 and variance $1 - \rho^2$. In fact, Bildikar and Patil [39] have shown that among bivariate exponential-type distributions $\mathbf{X} = (X_1, X_2)^T$ has a bivariate normal distribution iff the regression of one variable on the other is linear and the marginal distribution of one variables is normal.

For the standard trivariate normal distribution in Equation 9, the regression of any variable on the other two is linear with constant variance. For example, the conditional distribution of X_3, given $X_1 = x_1$ and $X_2 = x_2$, is normal with mean $\rho_{13.2}x_1 + \rho_{23.1}x_2$ and variance $1 - R^2_{3.12}$, where, for example, $\rho_{13.2}$ is the partial correlation between X_1 and X_3, given X_2, and $R^2_{3.12}$ is the multiple correlation of X_3 on X_1 and X_2. Similarly, the joint distribution of $(X_1, X_2)^T$, given $X_3 = x_3$, is bivariate normal with means $\rho_{13}x_3$ and $\rho_{23}x_3$, variances $1 - \rho^2_{13}$ and $1 - \rho^2_{23}$, and correlation coefficient $\rho_{12.3}$.

For the bivariate normal distribution, zero correlation implies independence of X_1 and X_2, which is not true in general, of course. Further, from the standard bivariate normal pdf in Equation 8, it can be shown that the joint moment generating function is

$$\begin{aligned} M_{X_1,X_2}(t_1, t_2) &= E(\mathrm{e}^{t_1\ X_1 + t_2\ X_2}) \\ &= \exp\left\{-\frac{1}{2}(t_1^2 + 2\rho t_1\ t_2 + t_2^2)\right\} \end{aligned} \tag{12}$$

from which all the moments can be readily derived. A recurrence relation for product moments has also been given by Kendall and Stuart [121]. In this case, the **orthant probabilit**y $\Pr(X_1 > 0, X_2 > 0)$ was shown to be $\frac{1}{2\pi}\ \sin^{-1}\rho + \frac{1}{4}$ by Sheppard [177, 178]; also see Reference 119 and 120 for formulas for some incomplete moments in the case of bivariate as well as trivariate normal distributions.

4.3 Approximations of Integrals and Tables

With $F(x_1, x_2;\rho)$ denoting the joint cdf of the standard bivariate normal distribution, Sibuya [183] and Sungur [200] noted the property that

$$\frac{dF(x_1, x_2; \rho)}{d\rho} = p(x_1, x_2; \rho). \tag{13}$$

Instead of the cdf $F(x_1, x_2;\rho)$, often the quantity

$$\begin{aligned} L(h, k; \rho) &= \Pr(X_1 > h, X_2 > k) \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_h^\infty \int_k^\infty \\ &\quad \times \exp\left\{-\frac{1}{2(1-\rho^2)} \times (x_1^2 - 2\rho x_1\ x_2 + x_2^2)\right\} dx_2\ dx_1 \end{aligned} \tag{14}$$

is tabulated. As mentioned above, it is known that $L(0, 0; rho) = \frac{1}{2\pi}\ \sin^{-1}\rho + \frac{1}{4}$. The function $L(h, k;\ell)$ is related to joint cdf $F(h, k;\rho)$ as follows:

$$\begin{aligned} F(h, k; \rho) &= 1 - L(h, -\infty; \rho) - L(-\infty, k; \rho) \\ &\quad + L(h, k; \rho). \end{aligned}$$

An extensive set of tables of $L(h, k;\rho)$ was published by Karl Pearson [159] and The National Bureau of Standards [147]; tables for the special cases when $\rho = 1/\sqrt{2}$ and $\rho = 1/3$ were presented by Dunnett [72] and Dunnett and Lamm [74] respectively. For the purpose of reducing the amount of tabulation of $L(h, k;\rho)$ involving three arguments, Zelen and Severo [222] pointed out that $L(h, k;\rho)$ can be evaluated from a table with $k = 0$ by means of the formula

$$\begin{aligned} L(h, k; \rho) &= L(h, 0; \rho(h, k)) + L(k, 0; \rho(k, h)) \\ &\quad - \frac{1}{2}(1 - \delta_{hk}), \end{aligned} \tag{15}$$

where

$$\rho(h, k) = \frac{(\rho h - k) f(h)}{\sqrt{h^2 - 2\rho hk + k^2}},$$

$$f(h) = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0, \end{cases}$$

and

$$\delta_{hk} = \begin{cases} 0 & \text{if sign}(h)\text{sign}(k) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Another function that has been evaluated rather extensively in the tables of The National Bureau of Standards [147] is $V(h, \lambda h)$, where

$$V(h, k) = \int_0^h \phi(x_1) \int_0^{kx_1/h} \phi(x_2) dx_2\ dx_1. \tag{16}$$

This function is related to the function $L(h, k;\rho)$ in Equation 15 as follows:

$$\begin{aligned} L(h, k; \rho) &= V\left(h, \frac{k - \rho h}{\sqrt{1-\rho^2}}\right) \\ &\quad + V\left(k, \frac{k - \rho h}{\sqrt{1-\rho^2}}\right) + 1 \\ &\quad - \frac{1}{2}\{\Phi(h) + \Phi(k)\} - \frac{\cos^{-1}\rho}{2\pi}, \end{aligned} \tag{17}$$

where $\Phi(x)$ is the univariate standard normal cdf. Owen [153] discussed the evaluation of a closely related function

$$T(h,\lambda) = \frac{1}{2\pi}\int_h^\infty \phi(x_1)\int_{\lambda x_1}^\infty \phi(x_2)\,dx_2\,dx_1 = \frac{1}{2\pi}\left\{\tan^{-1}\lambda - \sum_{j=0}^{\infty} c_j\,\lambda^{2j+1}\right\}, \tag{18}$$

where

$$c_j = \frac{(-1)^j}{2j+1}\left\{1 - \mathrm{e}^{-h^2/2}\sum_{\ell=0}^{j}\frac{(h^2/2)^\ell}{\ell!}\right\}.$$

Elaborate tables of this function have been constructed by Owen [154], Owen and Wiesen [155], and Smirnov and Bol'shev [187].

Upon comparing different methods of computing the standard bivariate normal cdf, Amos [10] and Sowden and Ashford [194] concluded Equations 17 and 18 to be the best ones for use. While some approximations for the function $L(h, k;\rho)$ have been discussed by Drezner and Wesolowsky [70,71], Mee and Owen [140], Albers and Kallenberg [6], and Lin [129], Daley [63] and Young and Minder [221] proposed **numerical integration** methods.

4.4 Characterizations

Brucker [47] presented the conditions

$$X_1 \,|\, (X_2 = x_2) \stackrel{d}{=} N(a + bx_2, g) \quad \text{and}$$
$$X_2 \,|\, (X_1 = x_1) \stackrel{d}{=} N(c + dx_1, h)$$

for all $x_1, x_2 \in \mathbb{R}$, where $a, b, c, d, g(>0)$ and $h > 0$ are all real numbers, as sufficient conditions for the bivariate normality of $(X_1, X_2)^T$. Fraser and Streit [81] relaxed the first condition a little. Hamedani [101] presented 18 different characterizations of the **bivariate normal distribution**. Ahsanullah and Wesolowski [2] established a characterization of bivariate normal by normality of the distribution of $X_2|X_1$ (with linear conditional mean and nonrandom conditional variance) and the conditional mean of $X_1|X_2$ being linear. Kagan and Wesolowski [118] showed that if U and V are linear functions of a pair of independent random variables X_1 and X_2, then the conditional normality of $U|V$ implies the normality of both X_1 and X_2. The fact that $X_1|(X_2 = x_2)$ and $X_2|(X_1 = x_1)$ are both distributed as normal (for all x_1 and x_2) does not characterize a bivariate normal distribution has been illustrated with examples by Bhattacharyya [38], Stoyanov [198], and Hamedani [101]; for a detailed account of all bivariate distributions that arise under this conditional specification, one may refer to the book of Arnold, Castillo, and Sarabia [22].

4.5 Order Statistics

Let $\mathbf{X} = (X_1, X_2)^T$ have the bivariate normal pdf in Equation 7, and let $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$ be the order statistics. Then, Cain [48] derived the pdf and mgf of $X_{(1)}$ and showed, in particular, that

$$
\begin{aligned}
E(X_{(1)}) &= \xi_1\ \Phi\left(\frac{\xi_2-\xi_1}{\delta}\right)+\xi_2\ \Phi\left(\frac{\xi_1-\xi_2}{\delta}\right) \\
&\quad -\delta\phi\left(\frac{\xi_2-\xi_1}{\delta}\right),
\end{aligned}
\tag{19}
$$

where $\delta=\sqrt{\sigma_2^2-2\rho\sigma_1\ \sigma_2+\sigma_1^2}$. Cain and Pan [49] derived a recurrence relation for moments of $X_{(1)}$. For the standard bivariate normal case, Nagaraja [146] earlier discussed the distribution of $a_1X_{(1)}+a_2X_{(2)}$, where a_1 and a_2 are real constants.

Suppose now $(X_{1i}, X_{2i})^T$, $i=1,\ldots,n$, is a random sample from the bivariate normal pdf in Equation 7, and that the sample is ordered by the X_1-values. Then, the X_2-value associated with the rth order statistic of X_1 (denoted by $X_{1(r)}$) is called the "**concomitant** of the rth order statistic" and is denoted by $X_{2[r]}$. Then, from the underlying **linear regression** model, we can express

$$
\begin{aligned}
X_{2[r]} &= \xi_2+\rho\sigma_2\left(\frac{X_{1(r)}-\xi_1}{\sigma_1}\right)+\varepsilon_{[r]}, \\
&\qquad r=1,\ldots,n,
\end{aligned}
\tag{20}
$$

where $\epsilon_{[r]}$ denotes the ϵ_i that is associated with $X_{1(r)}$. Exploiting the independence of $X_{1(r)}$ and $\epsilon_{[r]}$, moments and properties of concomitants of order statistics can be studied; see, for example, References 214 and 64. Balakrishnan [29] and Song and Deddens [193] studied the concomitants of order statistics arising from ordering a linear combination $S_i=aX_{1i}+bX_{2i}$ (for $i=1,2,\ldots,n$), where a and b are nonzero constants.

4.6 Trivariate Normal Integral and Tables

Let us consider the standard trivariate normal pdf in Equation 9 with correlations ρ_{12}, ρ_{13}, and ρ_{23}. Let $F(h_1, h_2, h_3;\rho_{23}, \rho_{13}, \rho_{12})$ denote the joint cdf of $\mathbf{X}=(X_1, X_2, X_3)^T$, and

$$
\begin{aligned}
&L(h_1, h_2, h_3;\rho_{23}, \rho_{13}, \rho_{12}) \\
&\quad =\Pr(X_1>h_1, X_2>h_2, X_3>h_3).
\end{aligned}
\tag{21}
$$

It may then be observed that $F(0,0,0;\rho_{23},\rho_{13},\rho_{12})=L(0,0,0;\rho_{23},\rho_{13},\rho_{12})$ and that $F(0,0,0;\rho,\rho,\rho)=\frac{1}{2}-\frac{3}{4}\ \pi\cos^{-1}\rho$. This value as well as $F(h,h,h;\rho,\rho,\rho)$ has been tabulated by Ruben [171], Teichroew [208], Somerville [192], and Steck [196]. Steck has in fact expressed the trivariate cdf F in terms of the function

$$
\begin{aligned}
S(h,a,b) &= \frac{1}{4\pi}\ \tan^{-1}\left(\frac{b}{\sqrt{1+a^2+a^2\,b^2}}\right) \\
&\quad +\Pr(0<Z_1<Z_2+b\,Z_3, \\
&\qquad\quad 0<Z_2<h,\ Z_3>a\,Z_2),
\end{aligned}
\tag{22}
$$

where Z_1, Z_2, and Z_3 are independent standard normal variables, and provided extensive tables of this function.

Mukherjea and Stephens [144] and Sungur [200] have discussed various properties of the trivariate normal distribution. By specifying all the univariate conditional density functions (conditioned on one as well as both other variables) to be normal, Arnold, Castillo, and Sarabia [21] have derived a general trivariate family of distributions that includes the trivariate normal as a special case (when the coefficient of $x_1x_2x_3$ in the exponent is 0).

4.7 Truncated Forms

Consider the standard bivariate normal pdf in Equation 8 and assume that we select only values for which X_1 exceed h. Then, the distribution resulting from such a single **truncation** has pdf

$$\begin{aligned} p_h(x_1, x_2; \rho) &= \frac{1}{2\pi\sqrt{1-\rho^2}\{1-\Phi(h)\}} \\ &\quad \times \exp\left\{-\frac{1}{2(1-\rho^2)} \times (x_1^2 - 2\rho x_1\, x_2 + x_2^2)\right\}, \\ &\quad x_1 > h, -\infty < x_2 < \infty. \end{aligned} \tag{23}$$

Using now the fact that the conditional distribution of X_2, given $X_1 = x_1$, is normal with mean ρx_1 and variance $1 - \rho^2$, we readily get

$$\begin{aligned} E(X_2) &= E\{E(X_2|X_1)\} = \rho\, E(X_1), \\ \mathrm{Var}(X_2) &= E(X_2^2) - \{E(X_2)\}^2 \\ &= \rho^2\ \mathrm{Var}(X_1) + 1 - \rho^2, \\ \mathrm{cov}(X_1, X_2) &= \rho\ \ \mathrm{Var}(X_1) \end{aligned}$$

and

$$\mathrm{corr}(X_1, X_2) = \rho\left\{\rho^2 + \frac{1-\rho^2}{\mathrm{var}(X_1)}\right\}^{-1/2}.$$

Since $\mathrm{var}(X_1) \le 1$, we get $|\mathrm{corr}(X_1, X_2)| \le \rho$, meaning that the correlation in the truncated population is no more than in the original population, as observed by Aitkin [4]. Furthermore, while the regression of X_2 on X_1 is linear, the regression of X_1 on X_2 is nonlinear and is

$$\begin{aligned} &E(X_1|X_2 = x_2) \\ &= \rho x_2 + \frac{\phi\left(\frac{h-\rho x_2}{\sqrt{1-\rho^2}}\right)}{1 - \Phi\left(\frac{h-\rho x_2}{\sqrt{1-\rho^2}}\right)}\sqrt{1-\rho^2}. \end{aligned} \tag{24}$$

Chou and Owen [55] derived the joint mgf, the joint cumulant generating function, and explicit expressions for the cumulants in this case. More general truncation scenarios have been discussed by Shah and Parikh [176], Lipow and Eidemiller [131], and Regier and Hamadan [167].

Arnold et al. [17] considered sampling from the bivariate normal distribution in Equation 7 when X_2 is restricted to be in the interval $a < X_2 < b$ and that X_1-value is available only for untruncated X_2-values. When $\beta = \frac{b-\xi_2}{\sigma_2} \to \infty$, this case coincides with the case considered by Chou and Owen [55], while the case $\alpha = \frac{a-\xi_2}{\sigma_2} = 0$ and $\beta \to \infty$ gives rise to Azzalini's [24] **skew-normal distribution** for the marginal distribution of X_1. Arnold et al. [17] have discussed the estimation of parameters in this setting.

Since the conditional joint distribution of $(X_2, X_3)^T$, given X_1, in the trivariate normal case is bivariate normal, if truncation is applied on X_1 (selection only of $X_1 \ge h$), arguments similar to those in the bivariate case above will readily yield expressions for the moments. Tallis [206] discussed elliptical truncation of the form $a_1 < \frac{1}{1-\rho^2}(X_1^2 - 2\rho X_1\ X_2 + X_2^2) < a_2$ and discussed the choice of a_1 and a_2 for which the variance-covariance matrix of the **truncated distribution** is the same as that of the original distribution.

4.8 Related Distributions

Mixtures of bivariate normal distributions have been discussed by Akesson [5], Charlier and Wicksell [53], and Day [65].

By starting with the bivariate normal pdf in Equation 7 when $\xi_1 = \xi_2 = 0$ and taking absolute values of X_1 and X_2, we obtain the *bivariate half normal* pdf

$$\begin{aligned} f(x_1, x_2) &= \frac{2}{\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \\ &\quad \times \exp\left\{ -\frac{\left(\frac{x_1}{\sigma_1}\right)^2 + \left(\frac{x_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \right\} \\ &\quad \times \cosh\left(\frac{\rho x_1 x_2}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \right), \\ &\quad x_1, x_2 > 0. \end{aligned} \tag{25}$$

In this case, the marginal distributions of X_1 and X_2 are both half normal; in addition, the conditional distribution of X_2, given X_1, is folded normal corresponding to the distribution of the absolute value of a normal variable with mean $|\rho| X_1$ and variance $1 - \rho^2$.

Sarabia [173] discussed the *bivariate normal distribution with centered normal conditionals* with joint pdf

$$\begin{aligned} f(x_1, x_2) &= K(c) \frac{\sqrt{ab}}{2\pi} \\ &\quad \exp\{-\frac{1}{2}(a x_1^2 + b x_2^2 + abc x_1^2 \, x_2^2\}, \\ &\quad -\infty < x_1, x_2 < \infty, \end{aligned} \tag{26}$$

where $a, b > 0$, $c \geq 0$ and $K(c)$ is the normalizing constant. The marginal pdfs of X_1 and X_2 turn out to be

$$\begin{aligned} &K(c)\sqrt{\frac{a}{2\pi}} \frac{1}{\sqrt{1 + acx_1^2}} \; e^{-ax_1^2/2} \quad \text{and} \\ &K(c)\sqrt{\frac{b}{2\pi}} \frac{1}{\sqrt{1 + bcx_2^2}} \; e^{-bx_2^2/2}. \end{aligned} \tag{27}$$

Note that, except when $c = 0$, these densities are not normal densities.

The density function of the *bivariate skew-normal distribution*, discussed by Azzalini and Dalla Valle [26], is

$$f(x_1, x_2) = 2p(x_1, x_2; \omega)\Phi(\lambda_1 \, x_1 + \lambda_2 \, x_2), \tag{28}$$

where $p(x_1, X_2; \omega)$ is the standard bivariate normal density function in Equation 8 with correlation coefficient ω, $\Phi(\cdot)$ denotes the standard normal cdf, and

$$\lambda_1 = \frac{\delta_1 - \delta_2 \, \omega}{\sqrt{(1-\omega^2)(1 - \omega^2 - \delta_1^2 - \delta_2^2 + 2\delta_1 \, \delta_2 \, \omega)}}$$

and

$$\lambda_2 = \frac{\delta_2 - \delta_1\,\omega}{\sqrt{(1-\omega^2)(1-\omega^2-\delta_1^2-\delta_2^2+2\delta_1\,\delta_2\,\omega)}}.$$

It can be shown in this case that the joint moment generating function is

$$2\exp\{\frac{1}{2}(t_1^2+2\omega t_1\,t_2+t_2^2)\}\Phi(\delta_1\,t_1+\delta_2\,t_2), \tag{29}$$

the marginal distribution of $X_i (i = 1, 2)$ is

$$X_i \overset{d}{=} \delta_i\,|Z_0| + \sqrt{1-\delta_i^2}\,Z_i,\; i = 1, 2,$$

where Z_0, Z_1, and Z_2 are independent standard normal variables, and ω is such that

$$\delta_1\,\delta_2 - \sqrt{(1-\delta_1^2)(1-\delta_2^2)} < \omega <$$
$$\delta_1\,\delta_2 + \sqrt{(1-\delta_2^2)(1-\delta_2^2)}.$$

4.9 Inference

Numerous papers dealing with inferential procedures for the parameters of bivariate and trivariate normal distributions and their truncated forms based on different forms of data have been published. For a detailed account of all these developments, we refer the readers to Chapter 46 of Reference 124.

5 Multivariate Normal Distributions

The multivariate normal distribution is a natural multivariate generalization of the bivariate normal distribution in Equation 36. If $Z_1, \ldots, Z_k$ are independent standard normal variables, then the linear transformation $\mathbf{Z}^T + \xi^T = \mathbf{X}^T\mathbf{H}^T$ with $|\mathbf{H}| \neq 0$ leads to a multivariate normal distribution. It is also the limiting form of a **multinomial distribution**. The multivariate normal distribution is assumed to be the underlying model in analyzing multivariate data of different kinds and, consequently, many classical inferential procedures have been developed on the basis of the multivariate normal distribution.

5.1 Definitions

A random vector $\mathbf{X} = (X_1, \ldots, X_k)^T$ is said to have a multivariate normal distribution if its pdf is

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \frac{1}{(2\pi)^{k/2}|\mathbf{V}|^{1/2}} \\ &\quad \times \exp\left\{-\frac{1}{2}\,(\mathbf{x}-\xi)^T\,\mathbf{V}^{-1}(\mathbf{x}-\xi)\right\}, \\ &\quad \mathbf{x} \in \mathbb{R}^k; \end{aligned} \tag{30}$$

in this case, $E(\mathbf{X}) = \xi$ and $\text{var}(\mathbf{X}) = \mathbf{V}$, which is assumed to be a positive definite matrix. If $\mathbf{V}$ is a positive semidefinite matrix (i.e., $|\mathbf{V}| = 0$), then the distribution of $\mathbf{X}$ is said to be a *singular multivariate normal* distribution.

From Equation 30, it can be shown that the mgf of $\mathbf{X}$ is given by

$$M_{\mathbf{X}}(\mathbf{t}) = E(e^{\mathbf{t}^T\mathbf{X}}) = \exp\{\mathbf{t}^T\xi + \frac{1}{2}\,\mathbf{t}^T\mathbf{V}\mathbf{t}\} \tag{31}$$

from which the above expressions for the mean vector and the variance-covariance matrix can be readily deduced. Further, it can be seen that all cumulants and cross-cumulants of order higher than 2 are zero. Holmquist [105] has presented expressions in vectorial notation for raw and central moments of $\mathbf{X}$. The entropy of the multivariate normal pdf in Equation 30 is

$$E\left\{-\log p_{\mathbf{X}}(\mathbf{x})\right\} = \frac{k}{2}\log(2\pi) + \frac{1}{2}\log|\mathbf{V}| + \frac{k}{2} \tag{32}$$

which, as Rao [166] has shown, is the **maximum entropy** possible for any k-dimensional random vector with specified variance-covariance matrix $\mathbf{V}$.

Partitioning the matrix $\mathbf{A} = \mathbf{V}^{-1}$ at the sth row and column as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \tag{33}$$

it can be shown that $(X_{s+1}, \ldots, X_k)^T$ has a multivariate normal distribution with mean $(\xi_{s+1}, \ldots, \xi_k)^T$ and variance-covariance matrix $(\mathbf{A}_{22} - \mathbf{A}_{21}\,\mathbf{A}_{11}^{-1}\,\mathbf{A}_{12})^{-1}$; further, the conditional distribution of $\mathbf{X}_{(1)} = (X_1, \ldots, X_s)^T$, given $\mathbf{X}_{(2)} = (X_{s+1}, \ldots, X_k)^T = \mathbf{x}_{(2)}$, is multivariate normal with mean vector $\xi_{(1)}^T - (\mathbf{x}_{(2)} - \xi_{(2)})^T\,\mathbf{A}_{21}\,\mathbf{A}_{11}^{-1}$ and variance-covariance matrix $\mathbf{A}_{11}^{-1}$, which shows that the regression of $\mathbf{X}_{(1)}$ on $\mathbf{X}_{(2)}$ is linear and homoscedastic.

For the multivariate normal pdf in Equation 30, Šidák [184] established the inequality

$$\Pr\left\{\bigcap_{j=1}^{k}(|X_j - \xi_j| \le a_j)\right\} \ge \prod_{j=1}^{k}\Pr\{|X_j - \xi_j| \le a_j)\} \tag{34}$$

for any set of positive constants $a_1, \ldots, a_k$. Gupta [99] generalized this result to convex sets, while Tong [211] obtained inequalities between probabilities in the special case when all the correlations are equal and positive. Anderson [12] showed that, for every centrally symmetric convex set $C \subseteq \mathbb{R}_k$, the probability of C corresponding to the variance-covariance matrix $\mathbf{V}_1$ is at least as large as the probability of C corresponding to $\mathbf{V}_2$ when $\mathbf{V}_2 - \mathbf{V}_1$ is positive definite, meaning that the former probability is more concentrated about $\mathbf{0}$ than the latter probability. Gupta and Gupta [95] have established that the joint **hazard rate** function is increasing for multivariate normal distributions.

5.2 Order Statistics

Suppose $\mathbf{X}$ has the multivariate normal density in Equation 30. Then, Houdré [106] and Cirel'son, and Ibragimov and Sudakov [56] established some inequalities for variances of order statistics. Siegel [186] proved that

$$\begin{aligned}\operatorname{cov}\left(X_1, \min_{1\le i\le k} X_i\right)\\ = \sum_{j=1}^{k} \operatorname{cov}(X_1, X_j)\Pr\left\{X_j = \min_{1\le i\le k} X_i\right\},\end{aligned} \tag{35}$$

which has been extended by Rinott and Samuel-Cahn [168]. By considering the case when $(X_1, \ldots, X_k, Y)^T$ has a multivariate normal distribution, where $X_1, \ldots, K_k$ are exchangeable, Olkin and Viana [152] established that $\operatorname{cov}(\mathbf{X}_{()}, Y) = \operatorname{cov}(\mathbf{X}, Y)$, where $\mathbf{X}_{()}$ is the vector of order statistics corresponding to $\mathbf{X}$. They have also presented explicit expressions for the variance-covariance matrix of $\mathbf{X}_{()}$ in the case when $\mathbf{X}$ has a multivariate normal distribution with common mean ξ, common variance σ^2, and common correlation coefficient ρ; see also Reference 100 for some results in this case.

5.3 Evaluation of Integrals

For computational purposes, it is common to work with the standardized multivariate normal pdf

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{|\mathbf{R}|^{-1/2}}{(2\pi)^{k/2}} \exp\left(-\frac{1}{2}\,\mathbf{x}^T\,\mathbf{R}^{-1}\mathbf{x}\right), \mathbf{x} \in \mathbb{R}^k, \tag{36}$$

where $\mathbf{R}$ is the correlation matrix of $\mathbf{X}$, and the corresponding cdf

$$\begin{aligned}&F_{\mathbf{X}}(h_1, \ldots, h_k; \mathbf{R})\\ &= \Pr\left\{\bigcap_{j=1}^{k}(X_j \le h_j)\right\}\\ &= \frac{|\mathbf{R}|^{-1/2}}{(2\pi)^{k/2}} \int_{-\infty}^{h_k} \cdots \int_{-\infty}^{h_1}\\ &\quad\times \exp\left(-\frac{1}{2}\,\mathbf{x}^T\,\mathbf{R}^{-1}\mathbf{x}\right) dx_1 \ldots dx_k.\end{aligned} \tag{37}$$

Several intricate reduction formulas, approximations, and bounds have been discussed in the literature. As the dimension k increases, the approximations in general do not yield accurate results while the direct numerical integration becomes quite involved if not impossible. The MULNOR algorithm, due to Schervish [175], facilitates the computation of general multivariate normal integrals, but the computational time increases rapidly with k making it impractical for use in dimensions higher than 5 or 6. Compared to this, the MVNPRD algorithm of Dunnett [73] works well for high dimensions as well, but is applicable only in the special case when $\rho_{ij} = \rho_i\rho_j$ (for all $i \ne j$). This algorithm uses Simpson's rule for the required single integration and hence a specified accuracy can be achieved. Sun [199] has presented a Fortran program for computing the orthant probabilities $\Pr(X_1 > 0, \ldots, X_k > 0)$ for dimensions up to 9. Several computational formulas and approximations have been discussed for many special cases, with a number of them depending on the forms of the variance-covariance matrix $\mathbf{V}$ or the correlation matrix $\mathbf{R}$, as the one mentioned above of Dunnett [73]. A noteworthy algorithm is due to Lohr [132], that facilitates the computation of $\Pr(\mathbf{X} \in A)$, where $\mathbf{X}$ is a multivariate normal random vector with mean $\mathbf{0}$ and positive definite variance-covariance matrix $\mathbf{V}$ and A is a compact region that is star-shaped with respect to $\mathbf{0}$ (meaning that if $\mathbf{x} \in A$, then any point between $\mathbf{0}$ and $\mathbf{x}$ is also in A). Genz [90] has compared the performance of several methods of computing multivariate normal probabilities.

5.4 Characterizations

One of the early characterizations of the multivariate normal distribution is due to Fréchet [82], who proved that if $X_1, \ldots, X_k$ are random variables and the distribution of $\sum_{j=1}^{k} a_j X_j$ is normal for any set of real numbers $a_1, \ldots, a_k$ (not all zero), then the distribution of $(X_1, \ldots, X_k)^T$ is multivariate normal. Basu [36] showed that if $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are independent $k \times 1$ vectors and that there are two sets of constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that the vectors $\sum_{j=1}^{n} a_j \mathbf{X}_j$ and $\sum_{j=1}^{n} b_j \mathbf{X}_j$ are mutually independent, then the distribution of all $\mathbf{X}_j$'s for which $a_j b_j \neq 0$ must be multivariate normal. This generalization of **Darmois–Skitovitch theorem** has been further extended by Ghurye and Olkin [91]. By starting with a random vector $\mathbf{X}$, whose arbitrarily dependent components have finite second moments, Kagan [117] established that all uncorrelated pairs of linear combinations $\sum_{i=1}^{k} a_i X_i$ and $\sum_{i=1}^{k} b_i X_i$ are independent iff $\mathbf{X}$ is distributed as multivariate normal.

Arnold and Pourahmadi [23], Arnold, Castillo, and Sarabia [19,20], and Ahsanullah and Wesolowski [3] have all presented several characterizations of the multivariate normal distribution by means of conditional specifications of different forms. Stadje [195] generalized the well-known maximum likelihood characterization of the univariate normal distribution to the multivariate normal case. Specifically, it is shown that if $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is a random sample from a population with pdf $p(\mathbf{x})$ in $\mathbb{R}^k$ and $\bar{\mathbf{X}} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{X}_j$ is the **maximum likelihood estimator** of the translation parameter $\boldsymbol{\theta}$, then $p(\mathbf{x})$ is the multivariate normal distribution with mean vector $\boldsymbol{\theta}$ and a nonnegative definite variance-covariance matrix $\mathbf{V}$.

5.5 Truncated Forms

When truncation is of the form $X_j \geq h_j$, $j = 1, \ldots, k$, meaning that all values of X_j less than h_j are excluded, Birnbaum, Paulson, and Andrews [40] derived complicated explicit formulas for the moments. The elliptical truncation in which the values of $\mathbf{X}$ are restricted by $a \leq \mathbf{X}^T\mathbf{R}^{-1}\mathbf{X} \leq b$, discussed by Tallis [206], leads to simpler formulas because of the fact that $\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}$ is distributed as **chi-square** with k **degrees of freedom**. By assuming that $\mathbf{X}$ has a general truncated multivariate normal distribution with pdf

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{K(2\pi)^{k/2}|\mathbf{V}|^{1/2}} \\ &\quad \times \exp\left\{-\frac{1}{2} (\mathbf{x}-\xi)^T \mathbf{V}^{-1}(\mathbf{x}-\xi)\right\}, \\ &\quad \mathbf{x} \in R, \end{aligned} \tag{38}$$

where K is the normalizing constant and $R = \mathbf{x}{:}b_i \leq x_i \leq a_i$, $i = 1, \ldots, k$, Cartinhour [51,52] discussed the marginal distribution of X_i and displayed that it is not necessarily truncated normal. This has been generalized by Sungur and Kovacevic [201,202].

5.6 Related Distributions

Day [65] discussed methods of estimation for **mixtures of multivariate normal distributions**. While the method of moments is quite satisfactory for mixtures of univariate normal distributions with common variance, Day found that this is not the case for mixtures of bivariate and multivariate normal distributions with common variance-covariance matrix. In this case, Wolfe [218] has presented a computer program for the determination of the maximum likelihood estimates of the parameters.

Sarabia [173] discussed the *multivariate normal distribution with centered normal conditionals* that has pdf

$$\begin{aligned} p(\mathbf{x}) &= \frac{\beta_k(c)\sqrt{a_1 \ldots a_k}}{(2\pi)^{k/2}} \\ &\quad \times \exp\left\{-\frac{1}{2}\left(\sum_{i=1}^{k} a_{ix}^{2_i} + c\prod_{i=1}^{k} a_{ix}^{2_i}\right)\right\}, \end{aligned} \tag{39}$$

where $c \geq 0$, $a_i \geq 0$ $(i = 1, \ldots, k)$, and $\beta_k(c)$ is the normalizing constant. Note that when $c = 0$, Equation 39 reduces to the joint density of k independent univariate normal random variables.

By mixing the multivariate normal distribution $N_k(\mathbf{0}, \Theta\mathbf{V})$ by ascribing a **gamma distribution** $G(\alpha, \alpha)$ (shape parameter is α and scale is $1/\alpha$) to Θ, Barakat [35] derived the *multivariate K-distribution*. Similarly, by mixing the multivariate normal distribution $N_k(\xi + W\boldsymbol{\beta}\mathbf{V}, W\mathbf{V})$ by ascribing a generalized **inverse Gaussian distribution** to W, Blaesild and Jensen [41] derived the *generalized multivariate* **hyperbolic distribution**. Urzúa [212] defined a distribution with pdf

$$p(\mathbf{x}) = \theta(\mathbf{c})e^{-Q(\mathbf{x})}, \mathbf{x} \in \mathbb{R}^k, \tag{40}$$

where $\theta(\mathbf{c})$ is the normalizing constant and $Q(\mathbf{x})$ is a polynomial of degree ℓ in $x_1, \ldots, x_k$ given by $Q(\mathbf{x}) = \sum_{q=0}^{\ell} Q^{(q)}(\mathbf{x})$ with $Q^{(q)}(\mathbf{x}) = \sum c_{j_1 \ldots j_k}^{(q)} x_1^{j_1} \ldots x_k^{j_k}$ being a homogeneous polynomial of degree q, and termed it the *multivariate Q-exponential* distribution. If $Q(\mathbf{x})$ is of degree $\ell = 2$ relative to each of its components x_i's, then Equation 40 becomes the multivariate normal density function.

Ernst [77] discussed the *multivariate generalized Laplace* distribution with pdf

$$\begin{aligned} p(\mathbf{x}) &= \frac{\lambda\Gamma(k/2)}{2\pi^{k/2}\Gamma(k/\lambda)|\mathbf{V}|^{1/2}} \\ &\quad \times \exp\{-[(\mathbf{x}-\xi)^T\mathbf{V}^{-1}(\mathbf{x}-\xi)]^{\lambda/2}\}, \end{aligned} \tag{41}$$

which reduces to the multivariate normal distribution with $\lambda = 2$.

Azzalini and Dalla Valle [26] studied the *multivariate skew-normal* distribution with pdf

$$p(\mathbf{x}) = 2\varphi_k(\mathbf{x}, \Omega)\Phi(\alpha^T\mathbf{x}), \mathbf{x} \in \mathbb{R}^k, \tag{42}$$

where $\Phi(\cdot)$ denotes the univariate standard normal cdf and $\phi_k(\mathbf{x}, \boldsymbol{\Omega})$ denotes the pdf of the k-dimensional normal distribution with standardized marginals and correlation matrix $\boldsymbol{\Omega}$. An alternate derivation of this distribution has been given by Arnold et al. [17]. Azzalini and Capitanio [25] have discussed various properties and reparametrizations of this distribution.

If $\mathbf{X}$ has a multivariate normal distribution with mean vector ξ and variance-covariance matrix $\mathbf{V}$, then the distribution of $\mathbf{Y}$ such that $\log(\mathbf{Y}) = \mathbf{X}$ is called the *multivariate l***ognormal distribution**. The moments and properties of $\mathbf{X}$ can be utilized to study the characteristics of $\mathbf{Y}$. For example, the rth raw moment of $\mathbf{Y}$ can be readily expressed as

$$\begin{aligned} \mu_{\mathbf{r}}'(\mathbf{Y}) &= E(Y_1^{r_1} \ldots Y_k^{r_k}) = E(e^{r_1 X_1} \ldots e^{r_k X_k}) \\ &= E(e^{\mathbf{r}^T\mathbf{X}}) = \exp\{\mathbf{r}^T\xi + \frac{1}{2}\mathbf{r}^T\mathbf{V}\mathbf{r}\}. \end{aligned} \tag{43}$$

Let $Z_j = X_j + iY_j (j = 1, \ldots, k)$, where $\mathbf{X} = (X_1, \ldots, X_k)^T$ and $\mathbf{Y} = (Y_1, \ldots, Y_k)^T$, have a joint multivariate normal distribution. Then, the complex random vector $\mathbf{Z} = (Z_1, \ldots, Z_k)^T$ is said to have a *complex multivariate normal* distribution.

6 Multivariate Logistic Distributions

The univariate **logistic distribution** has been studied quite extensively in the literature. There is a book length account of all the developments on the logistic distribution by Balakrishnan [28]. However, relatively little has been done on multivariate logistic distributions as can be seen from Chapter 11 of this book written by B. C. Arnold.

6.1 Gumbel–Malik–Abraham Form

Gumbel [94] proposed bivariate logistic distribution with cdf

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) \\ &= \{1 + e^{-(x_1-\mu_1)/\sigma_1} + e^{-(x_2-\mu_2)/\sigma_2}\}^{-1}, \\ &\quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{44}$$

and the corresponding pdf

$$\begin{aligned} p_{X_1,X_2}(x_1, x_2) \\ &= \frac{2e^{-(x_1-\mu_1)/\sigma_1}\, e^{-(x_2-\mu_2)/\sigma_2}}{\sigma_1\, \sigma_2\, \{1 + e^{-(x_1-\mu_1)/\sigma_1} + e^{-(x_2-\mu_2)/\sigma_2}\}^3}, \\ &\quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned} \tag{45}$$

The standard forms are obtained by setting $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$.

By letting X_2 or x_1 tend to ∞ in Equation 44, we readily observe that both marginals are logistic with

$$F_{X_i}(x_i) = \{1 + e^{-(x_i-\mu_i)/\sigma_i}\}^{-1},\ x_i \in \mathbb{R}(i = 1, 2). \tag{46}$$

From Equations 45 and 46, we can obtain the conditional densities; for example, we obtain the conditional density of X_1, given $X_2 = X_2$, as

$$\begin{aligned} p(x_1|x_2) &= \frac{2e^{-(x_1-\mu_1)/\sigma_1}\,(1 + e^{-(x_2-\mu_2)/\sigma_2})^2}{\sigma_1\, \{1 + e^{-(x_1-\mu_1)/\sigma_1} + e^{-(x_2-\mu_2)/\sigma_2}\}^3}, \\ &\quad x_1 \in \mathbb{R}, \end{aligned} \tag{47}$$

which is not logistic. From Equation 47, we obtain the **regression** of X_1 on $X_2{=}x_2$ as

$$\begin{aligned} E(X_1|X_2 = x_2) &= \mu_1 + \sigma_1 - \sigma_1 \\ &\quad \times \log(1 + e^{-(x_2-\mu_2)/\sigma_2}), \end{aligned} \tag{48}$$

which is nonlinear.

Malik and Abraham [134] provided a direct generalization to the multivariate case as one with cdf

$$F_{\mathbf{X}}(\mathbf{x}) = \left\{1 + \sum_{i=1}^{k} e^{-(x_i-\mu_i)/\sigma_i}\right\}^{-1},\ \mathbf{x} \in \mathbb{R}^k. \tag{49}$$

Once again, all the marginals are logistic in this case. In the standard case, when $\mu_i = 0$ and $\sigma_i = 1$ for $(i = 1, \ldots, k)$, it can be shown that the mgf of $\mathbf{X}$ is

$$M_{\mathbf{X}}(\mathbf{t}) \; = E(e^{\mathbf{t}^T\mathbf{X}}) = \Gamma\left(1 + \sum_{i=1}^{k} t_i\right) \prod_{i=1}^{k} \Gamma(1 - t_i), \qquad |t_i| < 1 (i = 1, \ldots, k), \tag{50}$$

from which it can be shown that $\mathrm{Corr}(X_i, X_j) = 1/2$ for all $i \neq j$. This shows that this multivariate logistic model is too restrictive.

6.2 Frailty Models

For a specified distribution P on $(0,\infty)$, consider the standard multivariate logistic distribution with cdf

$$F_{\mathbf{X}}(\mathbf{x}) \; = \int_0^{\infty} \left\{\prod_{i=1}^{k} \Pr(U \le x_i)\right\}^{\theta} d\,P(\theta), \quad \mathbf{x} \in \mathbb{R}^k, \tag{51}$$

where the distribution of U is related to P in the form

$$\Pr(U \le x) = \exp\left\{-L_P^{-1}\left(\frac{1}{1 + e^{-x}}\right)\right\}$$

with $L_P(t) = \int_0^{\infty} e^{-\theta t} d\,P(\theta)$ denoting the Laplace transform of P. From Equation 51, we then have

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) \; &= \int_0^{\infty} \exp\left\{ -\theta \sum_{i=1}^{k} L_P^{-1}\left(\frac{1}{1 + e^{-x_i}}\right)\right\} d\,P(\theta) \\ &= L_P\left(\sum_{i=1}^{k} L_P^{-1}\left(\frac{1}{1 + e^{-x_i}}\right)\right), \end{aligned} \tag{52}$$

which is termed the *Archimedean distribution* by Genest and MacKay [89] and Nelsen [149]. Evidently, all its univariate marginals are logistic. If we choose, for example, $P(\theta)$ to be Gamma(α, 1), then Equation 52 results in a multivariate logistic distribution of the form

$$F_{\mathbf{X}}(\mathbf{x}) = \left\{1 + \sum_{i=1}^{k} \left(1 + e^{-x_i}\right)^{1/\alpha} - k\right\}^{-\alpha}, \quad \alpha > 0, \tag{53}$$

which includes the Malik–Abraham model as a particular case when $\alpha = 1$.

6.3 Farlie–Gumbel–Morgenstern Form

The standard form of Farlie–Gumbel–Morgenstern multivariate logistic distribution has a cdf

$$F_{\mathbf{X}}(\mathbf{x}) \; = \left(\prod_{i=1}^{k} \frac{1}{1 + e^{-x_i}}\right)\left(1 + \alpha \prod_{i=1}^{k} \frac{e^{-x_i}}{1 + e^{-x_i}}\right), \qquad -1 < \alpha < 1, \mathbf{x} \in \mathbb{R}^k. \tag{54}$$

In this case, $\mathrm{Corr}(X_i, X_j) = 3\alpha/\pi^2 < 0.304$ (for all $i \neq j$), which is once again too restrictive. Slightly more flexible models can be constructed from Equation 54 by changing the second term, but explicit study of their properties becomes difficult.

6.4 Mixture Form

A multivariate logistic distribution will all its marginals as standard logistic can be constructed by considering a scale-mixture of the form $X_i = UV_i$ $(i = 1, \ldots, k)$, where $V_i (i = 1, \ldots, k)$ are i.i.d. random variables, U is a nonnegative random variable independent of $\mathbf{V}$, and X_i's are univariate standard logistic random variables. The model will be completely specified, once the distribution of U or the common distribution of UV_i's is specified. For example, the distribution of U can be specified to be **uniform**, **power function**, etc. However, no matter what distribution of U is chosen, we will have $\mathrm{Corr}(X_i, X_j) = 0$ (for all $i \neq j$) since, due to the symmetry of the standard logistic distribution, the common distribution of V_i's is also symmetric about zero. Of course, more general models can be constructed by taking $(U_1, \ldots, U_k)^T$ instead of U and ascribing a multivariate distribution to $\mathbf{U}$.

6.5 Geometric Minima and Maxima Models

Consider a sequence of independent trials taking on values 0,1,...,k, with probabilities $p_0, p_1, \ldots, p_k$. Let $\mathbf{N} = (N_1, \ldots, N_k)^T$, where N_i denotes the number of times i appeared before 0 appeared for the first time. Then, note that $N_i + 1$ has a **Geometric**(p_i) distribution. Let $Y_i^{(j)}, j = 1, \ldots, k$ and $i = 1, 2, \ldots$, be k independent sequences of independent standard logistic random variables. Let $\mathbf{X} = (X_1, \ldots, X_k)^T$ be the random vector defined by $X_j = \min_{1 \le i \le N_j+1} Y_i^{(j)}, j = 1, \ldots, k$. Then, the marginal distributions of $\mathbf{X}$ are all logistic and the joint survival function can be shown to be

$$\begin{aligned} \Pr(\mathbf{X} \geq \mathbf{x}) &= p_0 \left(1 - \sum_{j=1}^{k} \frac{p_j}{1 + \mathrm{e}^{x_j}}\right)^{-1} \\ &\quad \times \prod_{j=1}^{k} (1 + \mathrm{e}^{x_j})^{-1}. \end{aligned} \tag{55}$$

Similarly, geometric maximization can be used to develop a multivariate logistic family.

6.6 Other Forms

Arnold, Castillo, and Sarabia [22] have discussed multivariate distributions obtained with logistic conditionals. Satterthwaite and Hutchinson [174] discussed a generalization of Gumbel's bivariate form by considering

$$F(x_1, x_2) = (1 + \mathrm{e}^{-x_1} + \mathrm{e}^{-x_2})^{-\gamma}, \gamma > 0, \tag{56}$$

which does possess a more flexible correlation (depending on γ) than Gumbel's bivariate logistic. The marginal distributions of this family are Type-I generalized logistic distributions discussed by Balakrishnan and Leung [33].

Cook and Johnson [60] and Symanowski and Koehler [203] have presented some other generalized bivariate logistic distributions. Volodin [213] discussed a spherically symmetric distribution with logistic marginals. Lindley and Singpurwalla [130], in the context of reliability, discussed a multivariate logistic distribution with joint survival function

$$\Pr(\mathbf{X} \geq \mathbf{x}) = \left(1 + \sum_{i=1}^{k} e^{x_i}\right)^{-1},$$

which can be derived from extreme value random variables.

7 Multivariate Pareto Distributions

Simplicity and tractability of univariate **Pareto distributions** resulted in a lot of work with regard to the theory and applications of multivariate Pareto distributions; see, for example, the book by Arnold [15] and Chapter 52 of Reference 124.

7.1 Multivariate Pareto of the First Kind

Mardia [135] proposed the *multivariate Pareto distribution of the first kind* with pdf

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= a(a+1)\ldots(a+k-1)\left(\prod_{i=1}^{k}\theta_i\right)^{-1} \\ &\times \left(\sum_{i=1}^{k}\frac{x_i}{\theta_i} - k + 1\right)^{-(a+k)}, \\ & x_i > \theta_i > 0,\ a > 0. \end{aligned} \tag{57}$$

Evidently, any subset of $\mathbf{X}$ has a density of the form in Equation 28 so that marginal distributions of any order are also multivariate Pareto of the first kind. Further, the conditional density of $(X_{j+1}, \ldots, X_k)^T$, given $(X_1, \ldots, X_j)^T$, also has the form in Equation 28 with a, k, and θ's changed. As shown by Arnold [15], the survival function of $\mathbf{X}$ is

$$\begin{aligned} \Pr(\mathbf{X} \geq \mathbf{x}) &= \left(\sum_{i=1}^{k}\frac{x_i}{\theta_i} - k + 1\right)^{-a} \\ &= \left(1 + \sum_{i=1}^{k}\frac{x_i - \theta_i}{\theta_i}\right)^{-a}, \\ & x_i > \theta_i > 0,\ a > 0. \end{aligned} \tag{58}$$

Jupp and Mardia [115], Arnold and Pourahmadi [23], Wesolowski and Ahsanullah [217], and Ruiz, Marin, and Zoroa [172] have all established different characterizations of the distribution.

7.2 Multivariate Pareto of the Second Kind

From Equation 29, Arnold [15] considered a natural generalization with survival function

$$\Pr(\mathbf{X} \geq \mathbf{x}) = \left(1 + \sum_{i=1}^{k} \frac{x_i - \mu_i}{\theta_i}\right)^{-a}, \qquad x_i \geq \mu_i,\ \theta_i > 0,\ a > 0, \tag{59}$$

which is the *multivariate Pareto distribution of the second kind*. It can be shown that

$$E(X_i) = \mu_i + \frac{\theta_i}{a-1},\ i = 1, \ldots, k,$$

$$E\{X_i | (X_j = x_j)\} = \mu_i + \frac{\theta_i}{a}\left(1 + \frac{x_j - \mu_j}{\theta_j}\right),$$

$$\operatorname{var}\{X_i | (X_j = x_j)\} = \theta_i^2 \frac{(a+1)}{a(a-1)}\left(1 + \frac{x_j - \mu_j}{\theta_j}\right)^2,$$

revealing that the regression is linear but heteroscedastic. The special case of this distribution, when $\boldsymbol{\mu} = \mathbf{0}$, has appeared in the works of Lindley and Singpurwalla [130] and Nayak [148]. In this case, when $\boldsymbol{\theta} = \mathbf{1}$, Arnold [15] has established some interesting properties for order statistics and the **spacings** $S_i = (k - i + 1)(X_{i:k} - X_{i-1:k})$ with $X_{0:k} \equiv 0$.

7.3 Multivariate Pareto of the Third Kind

Arnold [15] proposed a further generalization that is the *multivariate Pareto distribution of the third kind* with survival function

$$\Pr(\mathbf{X} > \mathbf{x}) = \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - \mu_i}{\theta_i}\right)^{1/\gamma_i}\right\}^{-1}, \qquad x_i > \mu_i,\ \theta_i > 0,\ \gamma_i > 0. \tag{60}$$

Evidently, marginal distributions of all order are also multivariate Pareto of the third kind, but the conditional distributions are not so. By starting with $\mathbf{Z}$ that has a standard multivariate Pareto distribution of the first kind (with $\theta_i = 1$ and $a = 1$), the distribution in Equation 31 can be derived as the joint distribution of $X_i = \mu_i + \theta_i Z_i^{\gamma_i}$ $(i = 1, 2, \ldots, k)$.

7.4 Multivariate Pareto of the Fourth Kind

A simple extension of Equation 31 results in the *multivariate Pareto distribution of the fourth kind* with survival function

$$\Pr(\mathbf{X} > \mathbf{x}) = \left\{1 + \sum_{i=1}^{k} \left(\frac{x_i - \mu_i}{\theta_i}\right)^{1/\gamma_i}\right\}^{-a}, \qquad x_i > \mu_i (i = 1, \ldots, k), \tag{61}$$

whose marginal distributions as well as conditional distributions belong to the same family of distributions. The special case of $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\theta} = \mathbf{1}$ is the multivariate Pareto distribution discussed by Takahasi [205]. This general family of multivariate Pareto distributions of the fourth kind possesses many interesting properties as shown by Yeh [220]. A more general family has also been proposed by Arnold [16].

7.5 Conditionally Specified Multivariate Pareto

By specifying the conditional density functions to be Pareto, Arnold, Castillo, and Sarabia [18] derived general multivariate families, which may be termed *conditionally specified multivariate Pareto* distributions. For example, one of these forms has pdf

$$p_{\mathbf{X}}(\mathbf{x}) = \left\{\prod_{i=1}^{k} x_i^{\delta_i - 1}\right\} \left\{\sum_{\mathbf{s} \in \xi_k} \lambda_{\mathbf{s}} \prod_{i=1}^{k} x_i^{s_i \ \delta_i}\right\}^{-(a+1)}, \qquad x_i > 0, \tag{62}$$

where ξ_k is the set of all vectors of 0's and 1's of dimension k. These authors have also discussed various properties of these general multivariate distributions.

7.6 Marshall–Olkin Form of Multivariate Pareto

The survival function of the *Marshall–Olkin form of multivariate Pareto* distribution is

$$\Pr(\mathbf{X} > \mathbf{x}) = \left\{\prod_{i=1}^{k} \left(\frac{x_i}{\theta}\right)^{-\lambda_i}\right\} \left\{\frac{\max(x_1, \ldots, x_k)}{\theta}\right\}^{-\lambda_0}, \qquad \lambda_0, \ldots, \lambda_k, \theta > 0; \tag{63}$$

this can be obtained by transforming the **Marshall–Olkin multivariate exponential distribution** (see section Multivariate Exponential Distributions). This distribution is clearly not absolutely continuous and X_i's become independent when $\lambda_0 = 0$. Hanagal [103] has discussed some inferential procedures for this distribution and the "dullness property", viz.

$$\Pr(\mathbf{X} > t\mathbf{s} | \mathbf{X} \geq \mathbf{t}) = \Pr(\mathbf{X} > \mathbf{s}) \text{for all } \mathbf{s} \geq \mathbf{1},$$

where $\mathbf{s} = (s_1, \ldots, s_k)^T$, $\mathbf{t} = (t, \ldots, t)^T$, and $\mathbf{1} = (1, \ldots, 1)^T$, for the case when $\theta = 1$.

7.7 Multivariate Semi-Pareto Distribution

Balakrishna and Jayakumar [27] have defined a *multivariate semi-Pareto* distribution as one with survival function

$$\Pr(\mathbf{X} > \mathbf{x}) = \{1 + \psi(x_1, \ldots, x_k)\}^{-1}, \tag{64}$$

where $\psi(x_1, \ldots, x_k)$ satisfies the functional equation

$$\begin{aligned} \psi(x_1, \ldots, x_k) &= \frac{1}{p}\,\psi(p^{1/\alpha_1} x_1, \ldots, p^{1/\alpha_k} x_k), \\ &\qquad 0 < p < 1, \alpha_i > 0, x_i > 0. \end{aligned} \tag{65}$$

The solution of this functional equation is

$$\psi(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i^{\alpha_i}\; h_i(x_i), \tag{66}$$

where $h_i(x_i)$ is a periodic function in $\log x_i$ with period $\frac{2\pi\alpha_i}{-\log p}$. When $h_i(x_i) \equiv 1 (i = 1, \ldots, k)$, the distribution becomes the multivariate Pareto distribution of the third kind.

8 Multivariate Extreme Value Distributions

Considerable amount of work has been done during the past 25 years on bivariate and multivariate extreme value distributions. A general theoretical work on the weak asymptotic convergence of multivariate extreme values was carried out by Deheuvels [67]. The books by Galambos [85] and Joe [112] discuss various forms of bivariate and multivariate extreme value distributions and their properties, while the latter also deals with inferential problems as well as applications to practical problems. Smith [189], Joe [111], and Kotz, Balakrishnan, and Johnson [Chapter 53, 124] have provided detailed reviews of various developments in this topic.

8.1 Models

The classical multivariate extreme value distributions arise as the asymptotic distribution of normalized componentwise maxima from several dependent populations. Suppose $M_{nj} = max(Y_{1j}, \ldots, Y_{nj})$, for $j = 1, \ldots, k$, where $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{ik})^T (i = 1, \ldots, n)$ are i.i.d. random vectors. If there exist normalizing vectors $\mathbf{a}_n = (a_{n1}, \ldots, a_{nk})^T$ and $\mathbf{b}_n = (b_{n1}, \ldots, b_{nk})^T$ (with each $b_{nj} > 0$) such that

$$\begin{aligned} \lim_{n\to\infty} \Pr\left(\frac{M_{nj} - a_{nj}}{b_{nj}} < x_j, j = 1, \ldots, k\right) \\ = G(x_1, \ldots, x_k), \end{aligned} \tag{67}$$

where G is a nondegenerate k-variate cdf, then G is said to be a *multivariate extreme value* distribution. Evidently, the univariate marginals of G must be **generalized extreme value distributions** of the form

$$F(x; \mu, \sigma, \xi) = \exp\left[-\left\{1 - \xi\left(\frac{x - \mu}{\sigma}\right)\right\}_+^{1/\xi}\right], \tag{68}$$

where $z_+ = \max(0, z)$. This includes all the three types of **extreme value distributions**, viz., Gumbel, Fréchet, and Weibull, corresponding to $\xi = 0$, $\xi > 0$, and $\xi < 0$, respectively; see, for example, Chapter 22 of Reference 114. As shown by de Haan and Resnick [66] and Pickands [160], the distribution G in Equation 38 depends on an arbitrary positive measure over (k-1)-dimensions.

A number of parametric models have been suggested by Coles and Tawn [57] of which the *logistic model* is one with cdf

$$G(x_1, \ldots, x_k) = \exp\left[-\left\{\sum_{j=1}^{k}\left(1 - \frac{\xi_j(x_j - \mu_j)}{\sigma_j}\right)^{1/(\alpha\xi_j)}\right\}^{\alpha}\right], \tag{69}$$

where $0 \leq \alpha \leq 1$ measures the dependence, with $\alpha = 1$ corresponding to independence and $\alpha = 0$ complete dependence. Setting $Y_i = (1 - \frac{\xi_i(X_i - \mu_i)}{\sigma_i})^{1/\xi_i}$ (for $i = 1, \ldots, k$) and $Z = (\sum_{i=1}^{k} Y_i^{1/\alpha})^{\alpha}$, a transformation discussed by Lee [127], and then taking $T_i = (Y_i/Z)^{1/\alpha}$, Shi [179] showed that $(T_1, \ldots, T_{k-1})^T$ and Z are independent, with the former having a multivariate beta $(1, \ldots, 1)$ distribution and the latter having a mixed gamma distribution.

Tawn [207] has dealt with models of the form

$$\bar{G}(y_1, \ldots, y_k) = \exp\{-t\,B(w_1, \ldots, w_{k-1})\}, \quad y_i \geq 0, \tag{70}$$

where $w_i = y_i/t$, $t = \sum_{i=1}^{k} y_i$, and

$$B(w_1, \ldots, w_{k-1}) = \int_{S_k} (\max_{1 \leq i \leq k} w_{iq_i})\,d\,H(q_1, \ldots, q_{k-1}) \tag{71}$$

with H being an arbitrary positive finite measure over the unit simplex

$$S_k = \{q \in \mathbb{R}^{k-1} : q_1 + \cdots + q_{k-1} \leq 1, q_i \geq 0, \; (i = 1, \ldots, k-1)\} \tag{72}$$

satisfying the condition

$$1 = \int_{S_k} q_i \; d\,H(q_1, \ldots, q_{k-1}) \qquad (i = 1, 2, \ldots, k).$$

B is the so-called "dependence function". For the logistic model given in Equation 40, for example, the dependence function is simply

$$B(w_1, \ldots, w_{k-1}) = \left(\sum_{i=1}^{k} w_i^r\right)^{1/r}, \; r \geq 1.$$

In general, B is a convex function satisfying $\max(w_1, \ldots, w_k) \leq B \leq 1$.

Joe [110] has also discussed *asymmetric logistic* and *negative asymmetric logistic* multivariate extreme value distributions. Gomes and Alperin [92] have defined a *multivariate generalized extreme value* distribution by using von Mises–Jenkinson distribution.

8.2 Some Properties and Characteristics

With the classical definition of the multivariate extreme value distribution presented earlier, Marshall and Olkin [138] showed that the convergence in distribution in Equation 38 is equivalent to the condition

$$\lim_{n\to\infty} n\{1 - F(\mathbf{a}_n + \mathbf{b}_n\, \mathbf{x})\} = -\log H(\mathbf{x}) \tag{73}$$

for all $\mathbf{x}$ such that $0 < H(\mathbf{x}) < 1$. Takahashi [204] established a general result that implies that if H is a multivariate extreme value distribution, then so is H^T for any $t > 0$.

Tiago de Oliveira [209] noted that the components of a multivariate extreme value random vector with cdf H in Equation 43 are positively correlated, meaning that $H(x_1, \ldots, x_k) \geq H_1(x_1) \cdots H_k(x_k)$. Marshall and Olkin [138] established that multivariate extreme value random vectors $\mathbf{X}$ are associated, meaning that $\text{cov}(\theta(\mathbf{X}), \psi(\mathbf{X})) \geq 0$ for any pair θ and ψ of nondecreasing functions on $\mathbb{R}^k$. Galambos [84] presented some useful bounds for $H(x_1, \ldots, x_k)$.

8.3 Inference

Shi [179,180] discussed the likelihood estimation and the **Fisher information** matrix for the multivariate extreme value distribution with generalized extreme value marginals and the logistic dependence function. The moment estimation has been discussed by Shi [181], while Shi, Smith, and Coles [182] suggested an alternate simpler procedure to the MLE. Tawn [207], Coles and Tawn [57,58], and Joe [111] have all applied multivariate extreme value analysis to different environmental data. Nadarajah, Anderson, and Tawn [145] discussed inferential methods when there are order restrictions among components and applied them to analyze rainfall data.

9 Dirichlet, Inverted Dirichlet, and Liouville Distributions

9.1 Dirichlet Distribution

The *standard Dirichlet* distribution, based on a multiple integral evaluated by Dirichlet [68], with parameters $(\theta_1, \ldots, \theta_k, \theta_0)$ has its pdf as

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \frac{\Gamma(\sum_{j=0}^{k} \theta_j)}{\prod_{j=0}^{k} \Gamma(\theta_j)} \prod_{j=1}^{k} x_j^{\theta_j - 1} \left(1 - \sum_{j=1}^{k} x_j\right)^{\theta_0 = 1}, \\ & 0 \leq x_j, \sum_{j=1}^{k} x_j \leq 1, \end{aligned} \tag{74}$$

It can be shown that this arises as the joint distribution of $X_j = Y_j / \sum_{i=0}^{k} Y_i$ (for $j = 1, \ldots, k$), where $Y_0, Y_1, \ldots, Y_k$ are independent χ^2 random variables with $2\theta_0, 2\theta_1, \ldots, 2\theta_k$ degrees of freedom respectively. It is evident that the marginal distribution of X_j is $\text{Beta}(\theta_j, \sum_{i=0}^{k} \theta_i - \theta_j)$ and, hence, Equation 44 can be regarded as a *multivariate beta* distribution.

From Equation 44, the **r**th raw moment of **X** can be easily shown to be

$$\mu'_{\mathbf{r}}(\mathbf{X}) = E\left(\prod_{i=1}^{k} X_i^{r_i}\right) = \prod_{j=0}^{k} \theta_j^{[r_j]} / \Theta^{[\sum_{j=0}^{k} r_j]}, \tag{75}$$

where $\Theta = \sum_{j=0}^{k} \theta_j$ and $\theta_j^{[a]} = \theta_j(\theta_j+1)\ldots(\theta_j+a-1)$; from Equation 45, it can be shown, in particular, that

$$\begin{aligned} E(X_i) &= \frac{\theta_i}{\Theta}, \ \mathrm{Var}(X_i) = \frac{\theta_i(\Theta-\theta_i)}{\Theta^2(\Theta+1)}, \\ \mathrm{corr}(X_i, X_j) &= -\sqrt{\frac{\theta_i \theta_j}{(\Theta-\theta_i)(\Theta-\theta_j)}}, \end{aligned} \tag{76}$$

which reveals that all pairwise correlations are negative, just as in the case of multinomial distributions; consequently, the Dirichlet distribution is commonly used to approximate the multinomial distribution. From Equation 44, it can also be shown that the marginal distribution of $(X_1, \ldots, X_s)^T$ is Dirichlet with parameters $(\theta_1, \ldots, \theta_s, \Theta - \sum_{i=1}^{s} \theta_i)$, while the conditional joint distribution of $X_j/(1-\sum_{i=1}^{s} X_i)$, $j = s+1, \ldots, k$, given $(X_1, \ldots, X_s)$, is also Dirichlet with parameters $(\theta_{s+1}, \ldots, \theta_k, \theta_0)$.

Connor and Mosimann [59] discussed a *generalized Dirichlet* distribution with pdf

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \prod_{j=1}^{k} \left\{ \frac{1}{B(a_j, b_j)} x_j^{a_j-1} \right. \\ &\quad \left. \times \left(1 - \sum_{i=1}^{j-1} x_i\right)^{b_{j-1}-(a_j+b_j)} \right\}, \\ &\quad \left(1 - \sum_{j=1}^{k} x_j\right)^{b_k - 1}, \\ &\quad 0 \le x_j, \sum_{j=1}^{k} x_j \le 1, \end{aligned} \tag{77}$$

which reduces to the Dirichlet distribution when $b_{j-1} = a_j + b_j (j = 1, 2, \ldots, k)$. Note that in this case the marginal distributions are not beta. Ma [133] discussed a *multivariate rescaled Dirichlet* distribution with joint survival function

$$\begin{aligned} S(\mathbf{x}) &= \left(1 - \sum_{i=1}^{k} \theta_{i x_i}\right)^{a}, \\ &\quad 0 \le \sum_{i=1}^{k} \theta_{i x_i} \le 1, a, \theta_i > 0, \end{aligned} \tag{78}$$

which possesses a strong property involving residual life distribution.

9.2 Inverted Dirichlet Distribution

The *standard inverted Dirichlet* distribution, as given in Reference 210, has its pdf as

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma(\Theta)}{\prod_{j=0}^{k} \Gamma(\theta_j)} \frac{\prod_{j=1}^{k} x_j^{\theta_j - 1}}{\left(1 + \sum_{j=1}^{k} x_j\right)^{\Theta}},$$
$$0 < x_j, \theta_j > 0, \sum_{j=0}^{k} \theta_j = \Theta. \tag{79}$$

It can be shown that this arises as the joint distribution of $X_j = Y_j/Y_0$ (for $j = 1, \ldots, k$), where $Y_0, Y_1, \ldots, Y_k$ are independent χ^2 random variables with degrees of freedom $2\theta_0, 2\theta_1, \ldots, 2\theta_k$ respectively. This representation can also be used to obtain joint moments of $\mathbf{X}$ easily.

From Equation 49, it can be shown that if $\mathbf{X}$ has a *k-variate inverted Dirichlet* distribution, then $Y_i = X_i / \sum_{j=1}^{k} X_j$ $(i = 1, \ldots, k-1)$ have a $(k-1)$-variate Dirichlet distribution.

Yassaee [219] has discussed numerical procedures for computing the probability integrals of Dirichlet and inverted Dirichlet distributions. The tables of Sobel, Uppuluri, and Frankowski [190,191] will also assist in the computation of these probability integrals. Fabius [78,79] presented some elegant characterizations of the Dirichlet distribution. Rao and Sinha [165] and Gupta and Richards [97] have discussed some characterizations of the Dirichlet distribution within the class of Liouville-type distributions.

Dirichlet and inverted Dirichlet distributions have been used extensively as priors in **Bayesian analysis**. Some other interesting applications of these distributions in a variety of applied problems can be found in the works of Monhor [141], Goodhardt, Ehrenberg, and Chatfield [93] and Lange [126].

9.3 Liouville Distribution

On the basis of a generalization of the Dirichlet integral given by J. Liouville, Marshall, and Olkin [137] introduced the Liouville distribution. A random vector $\mathbf{X}$ is said to have a *multivariate Liouville distribution* if its pdf is proportional to (Gupta and Richards [96])

$$f\left(\sum_{i=1}^{k} x_i\right) \prod_{i=1}^{k} x_i^{a_i - 1}, x_i > 0, a_i > 0, \tag{80}$$

where the function f is positive, continuous, and integrable. If the support of $\mathbf{X}$ is noncompact, it is said to have a *Liouville distribution of the first kind*, while if it is compact, it is said to have a *Liouville distribution of the second kind*. Fang, Kotz, and Ng [80] have presented an alternate definition as $\mathbf{X} \stackrel{d}{=} R\mathbf{Y}$, where $R = \sum_{i=1}^{k} X_i$ has an univariate Liouville distribution (i.e., k=1 in Eq. 50) and $\mathbf{Y} = (Y_1, \ldots, Y_k)^T$ has a Dirichlet distribution independently of R. This stochastic representation has been utilized by Gupta and Richards [98] to establish that several properties of Dirichlet distributions continue to hold for Liouville distributions. If the function $f(t)$ in Equation 50 is chosen to be $(1-t)^{a_{k+1}-1}$ for $0 < t < 1$, the corresponding Liouville distribution of the second kind becomes the Dirichlet distribution; if $f(t)$ is chosen to be $(1+t)^{-\sum_{i=1}^{k+1} a_i}$ for $t > 0$, the corresponding Liouville distribution of the first kind becomes the inverted Dirichlet distribution. For a concise review on various properties, generalizations, characterizations, and inferential methods for the multivariate Liouville distributions, one may refer to Chapter 50 of Reference 124.

10 Multivariate Exponential Distributions

As mentioned earlier, significant amount of work in multivariate distribution theory has been based on bivariate and multivariate normal distributions. Still, just as exponential distribution occupies an important role in univariate distribution theory, bivariate and multivariate exponential distributions have also received considerable attention in the literature from theoretical as well as applied aspects. The volume of Balakrishnan and Basu [31] highlights this point and syntheses all the developments on theory, methods, and applications of univariate and multivariate exponential distributions.

10.1 Freund–Weinman Model

Freund [83] and Weiman [215] proposed a multivariate exponential distribution using the joint pdf

$$p_{\mathbf{X}}(\mathbf{x}) = \prod_{i=0}^{k-1} \frac{1}{\theta_i} e^{-(k-i)(x_{i+1}-x_i)/\theta_i}, \qquad 0 = x_0 \leq x_1 \leq \ldots \leq x_k < \infty,\ \theta_i > 0. \tag{81}$$

This distribution arises in the following reliability context. If a system has k identical components with Exponential(θ_0) lifetimes, and that if ℓ components have failed, the conditional joint distribution of the lifetimes of the remaining $k - \ell$ components is that of $k - \ell$ i.i.d. Exponential(θ_ℓ) random variables, then Equation 51 is the joint distribution of the failure times. The joint density function of **progressively Type-II censored** order statistics from an exponential distribution is a member of Equation 51; see Reference 30. Cramer and Kamps [61] derived an UMVUE in the bivariate normal case. The Freund–Weinman model is symmetrical in $(x_1, \ldots, x_k)$ and hence has identical marginals. For this reason, Block [42] extended this model to the case of nonidentical marginals by assuming that if ℓ components have failed by time $x (1 \leq \ell \leq k-1)$ and that the failures have been to the components $i_1, \ldots, i_\ell$, then the remaining $k - \ell$ components act independently with densities $p^{(\ell)}_{i|i_1,\ldots,i_\ell}(x)$ (for $x \geq x_{i_\ell}$) and that these densities do not depend on the order of $i_1, \cdots, i_\ell$. This distribution, interestingly, has multivariate lack of memory property, that is,

$$p(x_1 + t, \ldots, x_k + t) = \Pr(X_1 > t, \ldots, X_k > t)\, p(x_1, \ldots, x_k).$$

Basu and Sun [37] have shown that this generalized model can be derived from a fatal **shock model**.

10.2 Marshall–Olkin Model

Marshall and Olkin [136] presented a multivariate exponential distribution with joint survival function

$$\Pr(X_1 > x_1, \ldots, X_k > x_k) = \exp\left\{ -\sum_{i=1}^{k} \lambda_i x_i - \sum\sum_{1 \leq i_1 < i_2 \leq k} \lambda_{i_1,i_2} \times \max(x_{i_1}, x_{i_2}) - \ldots - \lambda_{12\ldots k} \times \max(x_1, \ldots, x_k) \right\},\ x_i > 0. \tag{82}$$

This is a mixture distribution with its marginal distributions having the same form and univariate marginals as exponential. This distribution is the only distribution with exponential marginals such that

$$\begin{aligned} &\Pr(X_1 > x_1 + t, \ldots, X_k > x_k + t) \\ &\quad = \Pr(X_1 > x_1, \ldots, X_k > x_k) \\ &\qquad \times \Pr(X_1 > t, \ldots, X_k > t). \end{aligned} \tag{83}$$

Proschan and Sullo [162] discussed the simpler case of Equation 52 with the survival function

$$\begin{aligned} &\Pr(X_1 > x_1, \ldots, X_k > x_k) \\ &\quad = \exp\left\{-\sum_{i=1}^{k} \lambda_{i\,x_i} - \lambda_{k+1} \max(x_1, \ldots, x_k)\right\}, \\ &\quad x_i > 0, \lambda_i > 0, \lambda_{k+1} \geq 0, \end{aligned} \tag{84}$$

which is incidentally the joint distribution of $X_i = \min(Y_i, Y_0)$, $i = 1, \ldots, k$, where Y_i are independent Exponential(λ_i) random variables for $i = 0, 1, \ldots, k$. In this model, the case $\lambda_1 = \cdots = \lambda_k$ corresponds to symmetry and mutual independence corresponds to $\lambda_{k+1} = 0$. While Arnold [14] has discussed methods of estimation of Equation 52, Proschan and Sullo [162] have discussed methods of estimation for Equation 53.

10.3 Block–Basu Model

By taking the absolutely continuous part of the Marshall–Olkin model in Equation 52, Block and Basu [43] defined an absolutely continuous multivariate exponential distribution with pdf

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \frac{\lambda_{i_1} + \lambda_{k+1}}{\alpha} \prod_{r=2}^{k} \lambda_{i_r} \exp\left\{-\sum_{r=1}^{k} \lambda_{i_r}\; x_{i_r}\right. \\ &\quad \left. - \lambda_{k+1} \max(x_1, \ldots, x_k)\right\} \\ &\quad x_{i_1} > \ldots > x_{i_k}, \\ &\quad i_1 \neq i_2 \neq \ldots \neq i_k = 1, \ldots, k, \end{aligned} \tag{85}$$

where

$$\begin{aligned} \alpha = &\sum \cdot \sum_{i_1 \neq i_2 \neq \cdots \neq i_k = 1}^{k} \prod_{r=2}^{k} \lambda_{i_r} / \\ &\left\{\prod_{r=2}^{k} \left(\sum_{j=1}^{r} \lambda_{i_j} + \lambda_{k+1}\right)\right\}. \end{aligned}$$

Under this model, complete independence is present iff $\lambda_{k+1} = 0$ and that the condition $\lambda_1 = \cdots = \lambda_k$ implies symmetry. However, the marginal distributions under this model are weighted combinations of exponentials and they are exponential only in the independent case. It does possess the lack of memory property. Hanagal [102] has discussed some inferential methods for this distribution.

10.4 Olkin–Tong Model

Olkin and Tong [151] considered the following special case of the Marshall–Olkin model in Equation 52. Let W, $(U_1, \ldots, U_k)$, and $(V_1, \ldots, V_k)$ be independent exponential random variables with means $1/\lambda_0$, $1/\lambda_1$, and $1/\lambda_2$ respectively. Let $K_1, \ldots, K_k$ be nonnegative integers such that $K_{r+1} = \cdots K_k = 0$, $1 \le K_r \le \cdots K_1$, and $\sum_{i=1}^{k} K_i = k$. Then, for a given $\mathbf{K}$, let $\mathbf{X}(\mathbf{K}) = (X_1, \ldots, X_k)^T$ be defined by

$$X_i = \begin{cases} \min(U_i, V_1, W), & i = 1, \ldots, K_1 \\ \min(U_i, V_2, W), & i = K_1 + 1, \ldots, K_1 + K_2 \\ \quad\cdots & \quad\cdots \\ \min(U_i, V_r, W), & i = K_1 + \ldots K_{r-1} + 1, \ldots, k. \end{cases} \tag{86}$$

The joint distribution of $\mathbf{X}(\mathbf{K})$ defined above, which is the Olkin–Tong model, is a subclass of the Marshall–Olkin model in Equation 52. All its marginal distributions are clearly exponential with mean $1/(\lambda_0 + \lambda_1 + \lambda_2)$. Olkin and Tong [151] have discussed some other properties as well as some majorization results.

10.5 Moran–Downton Model

The bivariate exponential distribution introduced by Moran [142] and Downton [69] was extended to the multivariate case by Al-Saadi and Young [9] with joint pdf as

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \frac{\lambda_1 \ldots \lambda_k}{(1-\rho)^{k-1}} \exp\left\{ -\frac{1}{1-\rho} \sum_{i=1}^{k} \lambda_{i x_i} \right\} \\ &\quad \times S_k\left(\frac{\rho\lambda_1\, x_1 \ldots \lambda_{k x_k}}{(1-\rho)^k} \right), \; x_i > 0, \end{aligned} \tag{87}$$

where $S_k(z) = \sum_{i=0}^{\infty} z^i/(i!)^k$. Al-Saadi and Young [8,9], Al-Saadi, Scrimshaw, and Young [7], and Balakrishnan and Ng [34] have all discussed inferential methods for this model.

10.6 Raftery Model

Suppose $(Y_1, \ldots, Y_k)$ and $(Z_1, \ldots, Z_\ell)$ are independent Exponential(λ) random variables. Further, suppose $(J_1, \ldots, J_k)$ is a random vector taking on values in $0, 1, \ldots, \ell^k$ with marginal probabilities

$$\begin{aligned} \Pr(J_i = 0) &= 1 - \pi_i \quad \text{and} \\ \Pr(J_i = j) &= \pi_{ij}, i = 1, \ldots, k; j = 1, \ldots, \ell, \end{aligned} \tag{88}$$

where $\pi_i = \sum_{j=1}^{\ell} \pi_{ij}$. Then, the multivariate exponential model discussed by Raftery [163] and O'Cinneide and Raftery [150] is given by

$$X_i = (1 - \pi_i) Y_i + Z_{J_i}, \; i = 1, \ldots, k. \tag{89}$$

The univariate marginals are all exponential. O'Cinneide and Raftery [150] have shown that the distribution of $\mathbf{X}$ defined in Equation 58 is a multivariate phase type distribution.

10.7 Multivariate Weibull Distributions

Clearly, a multivariate Weibull distribution can be obtained by a power transformation from a multivariate exponential distribution. For example, corresponding to the Marshall– Olkin model in Equation 52, we can have a multivariate Weibull distribution with joint survival function of the form

$$\Pr(X_1 > x_1, \ldots, X_k > x_k) \\ = \exp\left\{-\sum_J \lambda_J \max(x_i^{\alpha})\right\}, x_i > 0, \alpha > 0, \tag{90}$$

$\lambda_J > 0$ for $J \in \mathcal{J}$, where the sets J are elements of the class $\mathcal{J}$ of nonempty subsets of $1, \ldots, k$ such that for each i, $i \in J$ for some $J \in \mathcal{J}$. Then, the Marshall–Olkin multivariate exponential distribution in Equation 52 is a special case of Equation 59 when $\alpha = 1$. Lee [127] has discussed several other classes of multivariate Weibull distributions. Hougaard [107, 108] has presented a multivariate Weibull distribution with survival function

$$\Pr(X_1 > x_1, \ldots, X_k > x_k) \\ = \exp\left\{-\left(\sum_{i=1}^{k} \theta_{i\,x_i}^{p}\right)^{\ell}\right\}, \tag{91} \\ x_i \geq 0,\ p > 0,\ \ell > 0,$$

which has been generalized by Crowder [62]. Patra and Dey [156] have constructed a class of multivariate distributions in which the marginal distributions are mixtures of Weibull.

11 Multivariate Gamma Distributions

Many different forms of multivariate gamma distributions have been discussed in the literature since the pioneering paper of Krishnanmoorthy and Parthasarathy [122]. Chapter 48 of Reference 124 provides a concise review of various developments on bivariate and multivariate gamma distributions.

11.1 Cheriyan–Ramabhadran Model

With Y_i being independent Gamma(θ_i) random variables for $i = 0, 1, \ldots, k$, Cheriyan [54] and Ramabhadran [164] proposed a multivariate gamma distribution as the joint distribution of $X_i = Y_i + Y_0$, $i = 1, 2, \ldots, k$.

It can be shown that the density function of **X** is

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \frac{1}{\prod_{i=0}^{k} \Gamma(\theta_i)} \int_0^{\min(x_i)} \\ &\quad \times y_0^{\theta_0 - 1} \left\{ \prod_{i=1}^{k} (x_i - y_0)^{\theta_i - 1} \right\} \\ &\quad \times \exp\left\{ (k-1) y_0 - \sum_{i=1}^{k} x_i \right\}, \\ &\quad 0 \le y_0 \le x_1, \ldots, x_k. \end{aligned} \tag{92}$$

Though the integration cannot be done in a compact form in general, the density function in Equation 61 can be explicitly derived in some special cases. It is clear that the marginal distribution of X_i is Gamma($\theta_i + \theta_0$) for $i = 1, \ldots, k$. The mgf of **X** can be shown to be

$$\begin{aligned} M_{\mathbf{X}}(\mathbf{t}) &= E(e^{\mathbf{t}^T \mathbf{X}}) \\ &= \left(1 - \sum_{i=1}^{k} t_i\right)^{-\theta_0} \prod_{i=1}^{k} (1 - t_i)^{-\theta_i} \end{aligned} \tag{93}$$

from which expressions for all the moments can be readily obtained.

11.2 Krishnamoorthy–Parthasarathy Model

The standard multivariate gamma distribution of Krishnamoorthy and Parthasarathy [122] is defined by its characteristic function

$$\varphi_{\mathbf{X}}(\mathbf{t}) = E\{\exp(i\mathbf{t}^T \mathbf{X})\} = |\mathbf{I} - i\mathbf{R}\mathbf{T}|^{-\alpha}, \tag{94}$$

where **I** is a $k \times k$ identity matrix, **R** is a $k \times k$ correlation matrix, **T** is a Diag($t_1, \ldots, t_k$) matrix, and positive integral values 2α or real $2\alpha > k - 2 \ge 0$. For $k \ge 3$, the admissible nonintegral values $0 < 2\alpha < k - 2$ depend on the correlation matrix **R**. In particular, every $\alpha > 0$ is admissible iff $|\mathbf{I} - i\mathbf{R}\mathbf{T}|^{-1}$ is infinitely divisible, which is true iff the cofactors R_{ij} of the matrix **R** satisfy the conditions $(-1)^{\ell} R_{i_1 i_2} R_{i_2 i_3} \ldots R_{i_\ell i_1} \ge 0$ for every subset $i_1, \ldots, i_\ell$ of $1, 2, \ldots, k$ with $\ell \ge 3$.

11.3 Gaver Model

Gaver [88] presented a general multivariate gamma distribution with its characteristic function

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \left\{ (\beta + 1) \prod_{i=1}^{k} (1 - it_j) - \beta \right\}^{-\alpha}, \quad \alpha, \beta > 0. \tag{95}$$

This distribution is symmetric in $x_1, \ldots, x_k$, and the correlation coefficient is equal for all pairs and is $\beta/(\beta + 1)$.

11.4 Dussauchoy–Berland Model

Dussauchoy and Berland [75] considered a multivariate gamma distribution with characteristic function

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \prod_{j=1}^{k} \left\{ \frac{\phi_j(t_j + \sum_{b=j+1}^{k} \beta_{jb}\ t_b)}{\phi_j(\sum_{b=j+1}^{k} \beta_{jb}\ t_b)} \right\}, \tag{96}$$

where

$$\begin{aligned} &\phi_j(t_j) = (1 - i t_j/a_j)^{\ell_j} \quad \text{for } j = 1, \ldots, k, \\ &\beta_{jb} \geq 0,\, a_j \geq \beta_{jb}\, a_b > 0 \text{ for } j < b = 1, \ldots, k, \\ &\quad \text{and } 0 < \ell_1 \leq \ell_2 \leq \ldots \leq \ell_k. \end{aligned} \tag{97}$$

The corresponding density function $p_{\mathbf{X}}(\mathbf{x})$ cannot be written explicitly in general, but in the bivariate case it can be expressed in an explicit form.

11.5 Prékopa–Szántai Model

Prékopa and Szántai [161] extended the construction of Ramabhadran [164] and discussed the multivariate distribution of the random vector $\mathbf{X} = \mathbf{AW}$, where W_i are independent gamma random variables and $\mathbf{A}$ is a $k \times (2^k - 1)$ matrix with distinct column vectors with 0, 1 entries.

11.6 Kowalczyk–Tyrcha Model

Let $Y \stackrel{d}{=} G(\alpha, \mu, \sigma)$ be a three-parameter gamma random variable with pdf

$$\begin{gathered} \frac{1}{\Gamma(\alpha)\sigma^{\alpha}}\ e^{-(x-\mu)/\sigma}\ (x-\mu)^{\alpha-1}, \\ x > \mu, \sigma > 0, \alpha > 0. \end{gathered} \tag{98}$$

For a given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)^T$ with $\alpha_i > 0$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)^T \in \mathbb{R}^k$, $\sigma = (\sigma_1, \ldots, \sigma_k)^T$ with $\sigma_i > 0$, and $0 \leq \theta_0 < \min(\alpha_1, \ldots, \alpha_k)$, let $V_0, V_1, \ldots, V_k$ be independent random variables with $V_0 \stackrel{d}{=} G(\theta_0, 0, 1)$ and $V_i \stackrel{d}{=} G(\alpha_i - \theta_0, 0, 1)$ for $i = 1, \ldots, k$. Then, Kowalczyk and Tyrcha [125] defined a multivariate gamma distribution as the distribution of the random vector $\mathbf{X}$, where $X_i = \mu_i + \sigma_i(V_0 + V_i - \alpha_i)/\sqrt{\alpha_i}$ for $i = 1, \ldots, k$. It is clear that all the marginal distributions of all orders are gamma and that the correlation between X_i and X_j is $\theta_0/\sqrt{\alpha_i\ \alpha_j}$ $(i \neq j)$. This family of distributions is closed under linear transformation of components.

11.7 Mathai–Moschopoulos Model

Suppose $V_i \stackrel{d}{=} G(\alpha_i, \mu_i, \sigma_i)$ for $i = 0, 1, \ldots, k$, with pdf as in Equation 66. Then, Mathai and Moschopoulos [139] proposed a multivariate gamma distribution of $\mathbf{X}$, where $X_i = \frac{\sigma_i}{\sigma_0}\ V_0 + V_i$ for $i = 1, 2, \ldots, k$. The motivation of this model has also been given by Mathai and Moschopoulos [139]. Clearly, the marginal distribution of X_i is $G(\alpha_0 + \alpha_i, \frac{\sigma_i}{\sigma_0}\ \mu_0 + \mu_i, \sigma_i)$ for $i = 1, \ldots, k$. This family of distributions is closed under shift transformation as well as under convolutions. From the representation of $\mathbf{X}$ above and using the mgf of V_i, it can be easily shown that the mgf of $\mathbf{X}$ is

$$\begin{aligned} M_{\mathbf{X}}(\mathbf{t}) &= E\{\exp(\mathbf{t}^T\mathbf{X})\} \\ &= \frac{\exp\{(\mu + \frac{\mu_0}{\sigma_0}\,\sigma)^T\mathbf{t}\}}{(1-\sigma^T\mathbf{t})^{\alpha_0}\ \prod_{i=1}^{k}\ (1-\sigma_{i}t_i)^{\alpha_i}}, \end{aligned} \tag{99}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)^T$, $\sigma = (\sigma_1, \ldots, \sigma_k)^T$, $\mathbf{t} = (t_1, \ldots, t_k)^T$, $|\sigma_i t_i| < 1$ for $i = 1, \ldots, k$, and $|\sigma^T\mathbf{t}| = |\sum_{i=1}^k\ \sigma_i\ t_i| < 1$. From Equation 67, all the moments of $\mathbf{X}$ can be readily obtained; for example, we have $corr(X_i, X_j) = \alpha_0/\sqrt{(\alpha_0+\alpha_i)(\alpha_0+\alpha_j)}$, which is positive for all pairs.

A simplified version of this distribution when $\sigma_1 = \cdots = \sigma_k$ has been discussed in detail by Mathai and Moschopoulos [139].

11.8 Royen Models

Royen [169, 170] presented two multivariate gamma distributions, one based on "one-factorial" correlation matrix $\mathbf{R}$ of the form $r_{ij} = a_i a_j (i \neq j)$ with $a_1, \ldots, a_k \in (-1, 1)$ or $r_{ij} = -a_i a_j$ and $\mathbf{R}$ is positive semidefinite, and the other relating to the multivariate Rayleigh distribution of Blumenson and Miller [44]. The former, for any positive integer 2α, has its characteristic function as

$$\begin{aligned} \varphi_{\mathbf{X}}(\mathbf{t}) &= E\{\exp(i\mathbf{t}^T\mathbf{x})\} \\ &= |\mathbf{I} - 2i\ \ \mathrm{Diag}(t_1, \ldots, t_k)\mathbf{R}|^{-\alpha}. \end{aligned} \tag{100}$$

12 Some Other General Families

The univariate **Pearson's system** of distributions has been generalized to the bivariate and multivariate cases by a number of authors including Steyn [197] and Kotz [123].

With $Z_1, Z_2, \ldots, Z_k$ being independent standard normal variables and W being an independent **chi-square** random variable with ν **degrees of freedom**, and with $X_i = Z_i\sqrt{\nu/W}$ (for $i = 1, 2, \ldots, k$), the random vector $\mathbf{X} = (X_1, \ldots, X_k)^T$ has a ***multivariate t distribution***. This distribution and its properties and applications have been discussed in the literature considerably, and some tables of probability integrals have also been constructed.

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from the multivariate normal distribution with mean vector ξ and variance-covariance matrix $\mathbf{V}$. Then, it can be shown that the **maximum likelihood estimators** of ξ and $\mathbf{V}$ are the sample mean vector $\bar{\mathbf{X}}$ and the sample variance-covariance matrix $\mathbf{S}$ respectively, and these two are statistically independent. From the reproductive property of the multivariate normal distribution, it is known that $\bar{\mathbf{X}}$ is distributed as multivariate normal with mean ξ and variance-covariance matrix $\mathbf{V}/n$, and that $n\mathbf{S} = \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})^T$ is distributed as ***Wishart distribution*** $W_p(n-1; \mathbf{V})$.

From the multivariate normal distribution in Equation 30, translation systems (that parallel **Johnson's systems** in the univariate case) can be constructed. For example, by performing the logarithmic transformation $\mathbf{Y} = \log\mathbf{X}$, where $\mathbf{X}$ is the multivariate normal variable with parameters ξ and $\mathbf{V}$, we obtain the distribution of $\mathbf{Y}$ to be *multivariate lognormal distribution*. We can obtain its moments, for example, from

Equation 31 to be

$$\begin{aligned}\mu'_{\mathbf{r}}(\mathbf{Y}) &= E\left(\prod_{i=1}^{k} Y_i^{r_i}\right) = E\{\exp(\mathbf{r}^T\mathbf{X})\}\\ &= \exp\left\{\mathbf{r}^T\xi + \frac{1}{2}\,\mathbf{r}^T\mathbf{V}\mathbf{r}\right\}.\end{aligned} \tag{101}$$

Bildikar and Patil [39] introduced a *multivariate exponential-type* distribution with pdf

$$p_{\mathbf{X}}(\mathbf{x};\theta) = h(\mathbf{x})\exp\{\theta^T\mathbf{x} - q(\theta)\}, \tag{102}$$

where $\mathbf{x} = (x_1, \ldots, x_k)^T$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^T$, h is a function of $\mathbf{x}$ alone, and q is a function of $\boldsymbol{\theta}$ alone. The marginal distributions of all orders are also exponential-type. Further, all the moments can be obtained easily from the mgf of $\mathbf{X}$ given by

$$\begin{aligned}M_{\mathbf{X}}(\mathbf{t}) &= E\{\exp(\mathbf{t}^T\mathbf{X})\}\\ &= \exp\{q(\theta + \mathbf{t}) - q(\theta)\}.\end{aligned} \tag{103}$$

Further insightful discussions on this family have been provided by Morris [143], Efron [76], Brown [46], and Jørgensen [109]. A concise review of all the developments on this family can be found in Chapter 54 of Reference 124.

Anderson [11] presented a *multivariate* ***Linnik distribution*** with characteristic function

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \left\{1 + \left(\sum_{i=1}^{m} \mathbf{t}^T\,\Omega_i\,\mathbf{t}\right)^{\alpha/2}\right\}^{-1}, \tag{104}$$

where $0 < \alpha \leq 2$ and $\boldsymbol{\Omega}_i$ are $k \times k$ positive semidefinite matrices with no two of them being proportional.

Kagan [116] introduced two multivariate distributions as follows. The distribution of a vector $\mathbf{X}$ of dimension m is said to belong to the class $\mathcal{D}_{m,k}$ ($k = 1, 2, \ldots, m$; $m = 1, 2, \ldots$), if its characteristic function can be expressed as

$$\begin{aligned}\varphi_{\mathbf{X}}(\mathbf{t}) &= \varphi_{\mathbf{X}}(t_1, \ldots, t_m)\\ &= \prod_{1\leq i_1<\ldots<i_k\leq m} \varphi_{i_1,\ldots,i_k}(t_{i_1}, \ldots, t_{i_k}),\end{aligned} \tag{105}$$

where $(t_1, \ldots, t_m)^T \in \mathbb{R}^m$ and $\varphi_{i_1,\ldots,i_k}$ are continuous complex-valued functions with $\varphi_{i_1,\ldots,i_k}(0, \ldots, 0) = 1$ for any $1 \leq i_1 < \cdots < i_k \leq m$. If Equation 102 holds in the neighborhood of the origin, then $\mathbf{X}$ is said to belong to the class $\mathcal{D}_{m,k}$(loc). Wesolowski [216] has established some interesting properties of these two classes of distributions.

Various forms of multivariate **Farlie–Gumbel–Morgenstern distributions** exist in the literature. Cambanis [50] proposed the general form as one with cdf

$$\begin{aligned}F_{\mathbf{X}}(\mathbf{x}) &= \prod_{i_1=1}^{k} F(x_{i_1})\left[1 + \sum_{i_1=1}^{k} a_{i_1}\{1 - F(x_{i_1})\}\right.\\ &\quad + \sum\sum\nolimits_{1\leq i_1<i_2\leq k} a_{i_1\ i_2}\{1 - F(x_{i_1})\}\{1 - F(x_{i_2})\}\\ &\quad \left. + \cdots + a_{12\ldots k}\prod_{i=1}^{k}\{1 - F(x_i)\}\right].\end{aligned} \tag{106}$$

A more general form was proposed by Johnson and Kotz [113] with cdf

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) &= \prod_{i_1=1}^{k} F_{X_{i_1}}(x_{i_1}) \Bigg[1 + \sum\sum_{1\le i_1<i_2\le k} \\ &\quad \times a_{i_1\,i_2}\{1 - F_{X_{i_1}}(x_{i_1})\}\{1 - F_{X_{i_2}}(x_{i_2})\} \\ &\quad + \cdots + a_{12\ldots k} \prod_{i=1}^{k}\{1 - F_{X_i}(x_i)\}\Bigg]. \end{aligned} \tag{107}$$

In this form, the coefficients a_i were taken to be 0 so that the univariate marginal distributions are the given distributions F_i. The density function corresponding to Equation 104 is

$$\begin{aligned} p_{\mathbf{X}}(\mathbf{x}) &= \prod_{i=1}^{k} p_{X_i}(x_i) \Bigg[1 + \sum\sum_{1\le i_1<i_2\le k} \\ &\quad \times a_{i_1\,i_2}\{1 - 2F_{X_{i_1}}(x_{i_1})\}\{1 - 2F_{X_{i_2}}(x_{i_2})\} \\ &\quad + \cdots + a_{12\ldots k} \prod_{i=1}^{k}\{1 - 2F_{X_i}(x_i)\}\Bigg]. \end{aligned} \tag{108}$$

The multivariate Farlie–Gumbel–Morgenstern distributions can also be expressed in terms of survival functions instead of the cdf's as in Equation 104.

Lee [128] suggested a *multivariate Sarmanov distribution* with pdf

$$p_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{k} p_i(x_i)\{1 + R_{\phi_1,\ldots,\phi_k,\Omega_k}(x_1,\ldots,x_k)\}, \tag{109}$$

where $p_i(x_i)$ are specified density functions on $\mathbb{R}$,

$$\begin{aligned} &R_{\phi_1,\ldots,\phi_k,\Omega_k}(x_1,\ldots,x_k) \\ &\quad = \sum\sum_{1\le i_1<i_2\le k} \omega_{i_1\,i_2}\,\phi_{i_1}(x_{i_1})\phi_{i_2}(x_{i_2}) \\ &\qquad + \cdots + \omega_{12\ldots k} \prod_{i=1}^{k} \phi_i(x_i), \end{aligned} \tag{110}$$

and $\Omega_k = \{\omega_{i_1\,i_2},\ldots,\omega_{12\ldots k}\}$ is a set of real numbers such that the second term in Equation 106 is nonnegative for all $\mathbf{x} \in \mathbb{R}^k$.

13 Related Articles

Continuous Parametric Distributions
Discrete Multivariate Distributions
Multivariate Distributions

References

[1] Adrian, R. (**1808**). Research concerning the probabilities of errors which happen in making observations, etc. *The Analyst; or Math. Museum*, *1*(4), 93–109.

[2] Ahsanullah, M. and Wesolowski, J. (**1992**). *Bivariate Normality via Gaussian Conditional Structure*. Report, Rider College, Lawrenceville, N.J.

[3] Ahsanullah, M. and Wesolowski, J. (**1994**). Multivariate normality via conditional normality. *Stat. & Probab. Lett.*, *20*, 235–238.

[4] Aitkin, M.A. (**1964**). Correlation in a singly truncated bivariate normal distribution. *Psychometrika*, *29*, 263–270.

[5] Akesson, O.A. (**1916**). On the dissection of correlation surfaces. *Ark. Matematik, Astron. Fys.*, *11*(16), 1–18.

[6] Albers, W. and Kallenberg, W.C.M (**1994**). A simple approximation to the bivariate normal distribution with large correlation coefficient. *J. Multivariate Anal.*, *49*, 87–96.

[7] Al-Saadi, S.D., Scrimshaw, D.F., and Young, D.H. (**1979**). Tests for independence of exponential variables. *J. Stat. Comput. Simul.*, *9*, 217–233.

[8] Al-Saadi, S.D. and Young, D.H. (**1980**). Estimators for the correlation coefficients in a bivariate exponential distribution. *J. Stat. Comput. Simul.*, *11*, 13–20.

[9] Al-Saadi, S.D. and Young, D.H. (**1982**). A test for independence in a multivariate exponential distribution with equal correlation coefficient. *J. Stat. Comput. Simul.*, *4*, 219–227.

[10] Amos, D.E. (**1969**). On computation of the bivariate normal distribution. *Math. Comput.*, *23*, 655–659.

[11] Anderson, D.N. (**1992**). A multivariate Linnik distribution. *Stat. Probab. Lett.*, *14*, 333–336.

[12] Anderson, T.W. (**1955**). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Am. Math. Soc.*, *6*, 170–176.

[13] Anderson, T.W. (**1958**). *An Introduction to Multivariate Statistical Analysis*. John Wiley & Sons, New York.

[14] Arnold, B.C. (**1968**). Parameter estimation for a multivariate exponential distribution. *J. Am. Stat. Assoc.*, *63*, 848–852.

[15] Arnold, B.C. (**1983**). *Pareto Distributions*. International Cooperative Publishing House, Silver Spring, Md.

[16] Arnold, B.C. (**1990**). A flexible family of multivariate Pareto distributions. *J. Stat. Plann. Inference*, *24*, 249–258.

[17] Arnold, B.C., Beaver, R.J., Groeneveld, R.A., and Meeker, W.Q. (**1993**). The nontruncated marginal of a truncated bivariate normal distribution. *Psychometrika*, *58*, 471–488.

[18] Arnold, B.C. Castillo, E., and Sarabia, J.M. (**1993**). Multivariate distributions with generalized Pareto conditionals. *Stat. Probab. Lett.*, *17*, 361–368.

[19] Arnold, B.C., Castillo, E., and Sarabia, J.M. (**1994a**). A conditional characterization of the multivariate normal distribution. *Stat. Probab. Lett.*, *19*, 313–315.

[20] Arnold, B.C., Castillo, E., and Sarabia, J.M. (**1994b**). Multivariate normality via conditional specification. *Stat. Probab. Lett.*, *20*, 353–354.

[21] Arnold, B.C., Castillo, E., and Sarabia, J.M. (**1995**). General conditional specification models. *Commun. Stat.–Theory Methods*, *24*, 1–11.

[22] Arnold, B.C., Castillo, E., and Sarabia, J.M. (**1999**). *Conditionally Specified Distributions*, 2nd ed. Springer-Verlag, New York.

[23] Arnold, B.C. and Pourahmadi, M. (**1988**). Conditional characterizations of multivariate distributions. *Metrika*, *35*, 99–108.

[24] Azzalini, A. (**1985**). A class of distributions which includes the normal ones. *Scand. J. Stat.*, *12*, 171–178.

[25] Azzalini, A. and Capitanio, A. (**1999**). Some properties of the multivariate skew-normal distribution. *J. R. Stat. Soc., Ser. B*, *61*, 579–602.

[26] Azzalini, A. and Dalla Valle, A. (**1996**). The multivariate skew-normal distribution. *Biometrika*, *83*, 715–726.

[27] Balakrishna, N. and Jayakumar, K. (**1997**). Bivariate semi-Pareto distributions and processes. *Stat. Pap.*, *38*, 149–165.

[28] Balakrishnan, N. , ed. (**1992**). *Handbook of the Logistic Distribution*. Marcel Dekker, New York.

[29] Balakrishnan, N. (**1993**). Multivariate normal distribution and multivariate order statistics induced by ordering linear combinations. *Stat. Probab. Lett.*, *17*, 343–350.

[30] Balakrishnan, N. and Aggarwala, R. (**2000**). *Progressive Censoring: Theory Methods and Applications*. Birkhäuser, Boston, Mass.

[31] Balakrishnan, N. and Basu, A.P. , eds. (**1995**). *The Exponential Distribution: Theory, Methods and Applications*. Gordon & Breach Science Publishers, Newark, N.J.

[32] Balakrishnan, N., Johnson, N.L., and Kotz, S. (**1998**). A note on relationships between moments, central moments and cumulants from multivariate distributions. *Stat. Probab. Lett.*, *39*, 49–54.

[33] Balakrishnan, N. and Leung, M.Y. (**1998**). Order statistics from the Type I generalized logistic distribution. *Commun. Stat.–Simul. Comput.*, *17*, 25–50.

[34] Balakrishnan, N. and Ng, H.K.T (**2001**). Improved estimation of the correlation coefficient in a bivariate exponential distribution. *J. Stat. Comput. Simul.*, *68* 173–184.

[35] Barakat, R. (**1986**). Weaker-scatterer generalization of the *K*-density function with application to laser scattering in atmospheric turbulence. *J. Opt. Soc. Am.*, *73*, 269–276.

[36] Basu, D. (**1956**). A note on the multivariate extension of some theorems related to the univariate normal distribution. *Sankhyā*, *17*, 221–224.
[37] Basu, A.P. and Sun, K. (**1997**). Multivariate exponential distributions with constant failure rates. *J. Multivariate Anal.*, *61*, 159–169.
[38] Bhattacharyya, A. (**1943**). On some sets of sufficient conditions leading to the normal bivariate distributions. *Sankhyā*. *6*, 399–406.
[39] Bildikar, S. and Patil, G.P. (**1968**). Multivariate exponential-type distributions. *Ann. Math. Stat.*, *39*, 1316–1326.
[40] Birnbaum, Z.W., Paulson, E., and Andrews, F.C. (**1950**). On the effect of selection performed on some coordinates of a multi-dimensional population. *Psychometrika*, *15*, 191–204.
[41] Blaesild, P. and Jensen, J.L. (**1980**). *Multivariate Distributions of Hyperbolic Type*. Research Report No. 67, Department of Theoretical Statistics, University of Aarhus, Aarhus, Denmark.
[42] Block, H.W. (**1977**). A characterization of a bivariate exponential distribution. *Ann. Stat.*, *5*, 808–812.
[43] Block, H.W. and Basu, A.P. (**1974**). A continuous bivariate exponential extension. *J. Am. Stat. Assoc.*, *69*, 1031–1037.
[44] Blumenson, L.E. and Miller, K.S. (**1963**). Properties of generalized Rayleigh distributions. *Ann. Math. Stat.*, *34*, 903–910.
[45] Bravais, A. (**1846**). Analyse mathématique sur la probabilité des erreurs de situation d'un point. *Mémoires Acad. R. Sci., Paris*, *9*, 255–332.
[46] Brown, L.D. (**1986**). Fundamentals of Statistical Exponential Families. IMS Lecture Notes No. 9, Hayward, Calif.
[47] Brucker, J. (**1979**). A note on the bivariate normal distribution. *Commun. Stat.–Theory Methods*, *8*, 175–177.
[48] Cain, M. (**1994**). The moment-generating function of the minimum of bivariate normal random variables. *Am. Stat.*, *48*, 124–125.
[49] Cain, M. and Pan, E. (**1995**). Moments of the minimum of bivariate normal random variables. *Math. Sci.*, *20*, 119–122.
[50] Cambanis, S. (**1977**). Some properties and generalizations of multivariate Eyraud-Gumbel-Morgenstern distributions. *J. Multivariate Anal.*, *7*, 551–559.
[51] Cartinhour, J. (**1990a**). One-dimensional marginal density functions of a truncated multivariate normal density function. *Commun. Stat.–Theory Methods*, *19*, 197–203.
[52] Cartinhour, J. (**1990b**). Lower dimensional marginal density functions of absolutely continuous truncated multivariate distributions. *Commun. Stat.–Theory Methods*, *20*, 1569–1578.
[53] Charlier, C.V.L and Wicksell, S.D. (**1924**). On the dissection of frequency functions. *Ark. Matematik, Astron. Fys.*, *18*(6), 1–64.
[54] Cheriyan, K.C. (**1941**). A bivariate correlated gamma-type distribution function. *J. Indian Math. Soc.*, *5*, 133–144.
[55] Chou, Y.-M. and Owen, D.B. (**1984**). An approximation to percentiles of a variable of the bivariate normal distribution when the other variable is truncated, with applications. *Commun. Stat.–Theory Methods*, *13*, 2535–2547.
[56] Cirel'son, B.S., Ibragimov, I.A., and Sudakov, V.N. (**1976**). "Norms of Gaussian Sample Functions". *Proceedings of the Third Japan–U.S.S.R. Symposium on Probability Theory*, Lecture Notes in Mathematics 550. Springer-Verlag, New York, pp. 20–41.
[57] Coles, S.G. and Tawn, J.A. (**1991**). Modelling extreme multivariate events. *J. R. Stat. Soc., Ser. B*, *52*, 377–392.
[58] Coles, S.G. and Tawn, J.A. (**1994**). Statistical methods for multivariate extremes: an application to structural design. *Appl. Stat.*, *43*, 1–48.
[59] Connor, R.J. and Mosimann, J.E. (**1969**). Concepts of independence for proportions with a generalization of the Dirichlet distribution. *J. Am. Stat. Assoc.*, *64*, 194–206.
[60] Cook, R.D. and Johnson, M.E. (**1986**). Generalized Burr–Pareto-logistic distributions with applications to a uranium exploration data set. *Technometrics*, *28*, 123–131.
[61] Cramer, E. and Kamps, U. (**1997**). The UMVUE of $P(X < Y)$ based on type-II censored samples from Weinman multivariate exponential distributions. *Metrika*, *46*, 93–121.
[62] Crowder, M. (**1989**). A multivariate distribution with Weibull connections. *J. R. Stat. Soc., Ser. B*, *51*, 93–107.
[63] Daley, D.J. (**1974**). Computation of bi- and tri-variate normal integrals. *Appl. Stat.*, *23*, 435–438.
[64] David, H.A. and Nagaraja, H.N. (**1998**). "Concomitants of Order Statistics". In *Handbook of Statistics– 16: Order Statistics: Theory and Methods*, N. Balakrishnan and C.R. Rao, eds. North Holland, Amsterdam, pp. 487–513.
[65] Day, N.E. (**1969**). Estimating the components of a mixture of normal distributions. *Biometrika*, *56*, 463–474.
[66] de Haan, L. and Resnick, S.I. (**1977**). Limit theory for multivariate sample extremes. *Zeit. Wahr. Verw. Geb.*, *40*, 317–337.
[67] Dehuevels, P. (**1978**). Caracterisation complete des lois extremes multivariees et de la convergence des types extremes. *Publ. Inst. Stat, Univ. Paris*, *23*, 1–36.
[68] Dirichlet, P.G.L (**1839**). Sur une nouvelle méthode pour la determination des intégrales multiples. *Liouville J. Math., Ser. I*, *4*, 164–168.
[69] Downton, F. (**1970**). Bivariate exponential distributions in reliability theory. *J. R. Stat. Soc., Ser. B*, *32*, 408–417.

[70] Drezner, Z. and Wesolowsky, G.O. (**1989**). An approximation method for bivariate and multivariate normal equiprobability contours. *Commun. Stat.–Theory Methods*, *18*, 2331–2344.

[71] Drezner, Z. and Wesolowsky, G.O. (**1990**). On the computation of the bivariate normal integral. *J. Stat. Comput. Simul.*, *35*, 101–107.

[72] Dunnett, C.W. (**1958**). Tables of the Bivariate Normal Distribution with Correlation $1/\sqrt{2}$. Deposited in UMT file [abstract in *Math. Comput.*, *14*, (**1960**), 79].

[73] Dunnett, C.W. (**1989**). Algorithm AS251: multivariate normal probability integrals with product correlation structure. *Appl. Stat.*, *38*, 564–579.

[74] Dunnett, C.W. and Lamm, R.A. (**1960**). Some Tables of the Multivariate Normal Probability Integral with Correlation Coefficient 1/3. Deposited in UMT file [abstract in *Math. Comput.*, *14*, (**1960**), 290-291].

[75] Dussauchoy, A. and Berland, R. (**1975**). A Multivariate Gamma Distribution Whose Marginal Laws are Gamma. In *Statistical Distributions in Scientific Work*, Vol. *1*, G.P. Patil, S. Kotz, and J.K. Ord, eds. D. Reidel, Dordrecht, The Netherlands, pp. 319–328.

[76] Efron, B. (**1978**). The geometry of exponential families. *Ann. Stat.*, *6*, 362–376.

[77] Ernst, M.D. (**1997**). *A Multivariate Generalized Laplace Distribution*, Preprint. Department of Statistical Science, Southern Methodist University, Dallas, Tex.

[78] Fabius, J. (**1973a**). Two characterizations of the Dirichlet distribution. *Ann. Stat.*, *1*, 583–587.

[79] Fabius, J. (**1973b**). Neutrality and Dirichlet Distributions. *Transactions of the Sixth Prague Conference in Information Theory*. Academia, Prague, pp. 175–181.

[80] Fang, K.T., Kotz, S., and Ng, K.W. (**1989**). *Symmetric Multivariate and Related Distributions*. Chapman & Hall, London.

[81] Fraser, D.A.S and Streit, F. (**1980**). A further note on the bivariate normal distribution. *Commun. Stat.–Theory Methods*, *9*, 1097–1099.

[82] Fréchet, M. (**1951**). Généralizations de la loi de probabilité de Laplace. *Ann. Inst. Henri Poincaré*, *13*, 1–29.

[83] Freund, J. (**1961**). A bivariate extension of the exponential distribution. *J. Am. Stat. Assoc.*, *56*, 971–977.

[84] Galambos, J. (**1985**). A new bound on multivariate extreme value distributions. *Ann. Univ. Sci. Budapest, Rolando Eötvös, Nomin Sect. Math.*, *27*, 37–40.

[85] Galambos, J. (**1987**). *The Asymptotic Theory of Extreme Order Statistics*, 2nd ed. Krieger, Malabar, Fla.

[86] Galton, F. (**1877**). Typical Laws of Heredity in Man. Address to Royal Institute of Great Britain.

[87] Galton, F. (**1888**). Co-relations and their measurement chiefly from anthropometric data. *Proc. R. Soc. London*, *45*, 134–145.

[88] Gaver, D.P. (**1970**). Multivariate gamma distributions generated by mixture. *Sankhyā, Ser. A*, *32*, 123–126.

[89] Genest, C. and MacKay, J. (**1986**). The joy of copulas: Bivariate distributions with uniform marginals. *Am. Stat.*, *40*, 280–285.

[90] Genz, A. (**1993**). Comparison of methods for the computation of multivariate normal probabilities. *Comput. Sci. Stat.*, *25*, 400–405.

[91] Ghurye, S.G. and Olkin, I. (**1962**). A characterization of the multivariate normal distribution. *Ann. Math. Stat.*, *33*, 533–541.

[92] Gomes, M.I. and Alperin, M.T. (**1986**). Inference in a multivariate generalized extreme value model– asymptotic properties of two test statistics. *Scand. J. Stat.*, *13*, 291–300.

[93] Goodhardt, G.J., Ehrenberg, A.S.C., and Chatfield, C. (**1984**). The Dirichlet: a comprehensive model of buying behaviour (with discussion). *J. R. Stat. Soc., Ser. A*, *147*, 621–655.

[94] Gumbel, E.J. (**1961**). Bivariate logistic distribution. *J. Am. Stat. Assoc.*, *56*, 335–349.

[95] Gupta, P.L. and Gupta, R.C. (**1997**). On the multivariate normal hazard. *J. Multivariate Anal.*, *62*, 64–73.

[96] Gupta, R.D. and Richards, D.St.P. (**1987**). Multivariate Liouville distributions. *J. Multivariate Anal.*, *23*, 233–256.

[97] Gupta, R.D. and Richards, D.St.P. (**1990**). The Dirichlet distributions and polynomial regression. *J. Multivariate Anal.*, *32*, 95–102.

[98] Gupta, R.D. and Richards, D.St.P. (**1999**). The covariance structure of the multivariate Liouville distributions. Preprint.

[99] Gupta, S.D. (**1969**). A note on some inequalities for multivariate normal distribution. *Bull. the Calcutta Stat. Assoc.*, *18*, 179–180.

[100] Gupta, S.S., Nagel, K., and Panchapakesan, S. (**1973**). On the order statistics from equally correlated normal random variables. *Biometrika*, *60*, 403–413.

[101] Hamedani, G.G. (**1992**). Bivariate and multivariate normal characterizations: a brief survey. *Commun. Stat.–Theory Methods*, *21*, 2665–2688.

[102] Hanagal, D.D. (**1993**). Some inference results in an absolutely continuous multivariate exponential model of Block. *Stat. & Probab. Lett.*, *16*, 177–180.

[103] Hanagal, D.D. (**1996**). A multivariate Pareto distribution. *Commun. Stat.–Theory Methods*, *25*, 1471–1488.

[104] Helmert, F.R. (**1868**). Studien über rationlle Vermessungen, im Gebiete der höheren Geodäsie. *Zeitschr. Math. Phys.*, *13*, 73–129.

[105] Holmquist, B. (**1988**). Moments and cumulants of the multivariate normal distribution. *Stochastic Anal. Appl.*, *6*, 273–278.

[106] Houdré, C. (**1995**). Some applications of covariance identities and inequalities to functions of multivariate normal variables. *J. Am. Stat. Assoc.*, *90*, 965–968.

[107] Hougaard, P. (**1986**). A class of multivariate failure time distributions. *Biometrika*, *73*, 671–678.

[108] Hougaard, P. (**1989**). Fitting a multivariate failure time distribution. *IEEE Trans. Reliab.*, *R-38*, 444–448.

[109] Jørgensen, B. (**1997**). *The Theory of Dispersion Models*. Chapman & Hall, London.

[110] Joe, H. (**1990**). Families of min-stable multivariate exponential and multivariate extreme value distributions. *Stat. & Probab. Lett.*, *9*, 75–81.

[111] Joe, H. (**1994**). Multivariate extreme value distributions with applications to environmental data. *Can. J. Stat.*, *22*, 47–64.

[112] Joe, H. (**1996**). *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.

[113] Johnson, N.L. and Kotz, S. (**1975**). On some generalized Farlie-Gumbel-Morgenstern distributions. *Commun. Stat.*, *4*, 415–427.

[114] Johnson, N.L., Kotz, S., and Balakrishnan, N. (**1995**). *Continuous Univariate Distributions*, Vol. *2*, 2nd ed. John Wiley & Sons, New York.

[115] Jupp, P.E. and Mardia, K.V. (**1982**). A characterization of the multivariate Pareto distribution. *Ann. Stat.*, *10*, 1021–1024.

[116] Kagan, A.M. (**1988**). New classes of dependent random variables and generalizations of the Darmois-Skitovitch theorem to the case of several forms. *Teor. Veroyatnost. Primenen.*, *33*, 305–315 (in Russian).

[117] Kagan, A.M. (**1998**). Uncorrelatedness of linear forms implies their independence only for Gaussian random vectors. Preprint.

[118] Kagan, A.M. and Wesolowski, J. (**1996**). Normality via conditional normality of linear forms. *Stat. Probab. Lett.*, *29*, 229–232.

[119] Kamat, A.R. (**1958a**). Hypergeometric expansions for incomplete moments of the bivariate normal distribution. *Sankhyā*, *20*, 317–320.

[120] Kamat, A.R. (**1958b**). Incomplete moments of the trivariate normal distribution. *Sankhyā*, *20*, 321–322.

[121] Kendall, M.G. and Stuart, A. (**1963**). *The Advanced Theory of Statistics*, Vol. *1*. Griffin, London.

[122] Krishnamoorthy, A.S. and Parthasarathy, M. (**1951**). A multivariate gamma-type distribution. *Ann. Math. Stat.*, *22*, 549–557; Correction 31, 229.

[123] Kotz, S. (**1975**). Multivariate Distributions at Cross Roads. In *Statistical Distributions in Scientific Work*, Vol. *1*, G.P. Patil, S. Kotz, and J.K. Ord, eds. D. Reidel, Dordrecht, The Netherlands, pp. 247–270.

[124] Kotz, S., Balakrishnan, N., and Johnson, N.L. (**2000**). *Continuous Multivariate Distributions*, Vol. *1*, 2nd ed. John Wiley & Sons, New York.

[125] Kowalczyk, T. and Tyrcha, J. (**1989**). Multivariate gamma distributions, properties and shape estimation. *Statistics*, *20*, 465–474.

[126] Lange, K. (**1995**). Application of the Dirichlet distribution to forensic match probabilities. *Genetica*, *96*, 107–117.

[127] Lee, L. (**1979**). Multivariate distributions having Weibull properties. *J. Multivariate Anal.*, *9*, 267–277.

[128] Lee, M-L.T. (**1996**). Properties and applications of the Sarmanov family of bivariate distributions. *Commun. Stat.–Theory Methods*, *25*, 1207–1222.

[129] Lin, J.-T. (**1995**). A simple approximation for the bivariate normal integral. *Probab. Eng. Inf. Sci.*, *9*, 317–321.

[130] Lindley, D.V. and Singpurwalla, N.D. (**1986**). Multivariate distributions for the life lengths of components of a system sharing a common environment. *J. Appl. Probab.*, *23*, 418–431.

[131] Lipow, M. and Eidemiller, R.L. (**1964**). Application of the bivariate normal distribution to a stress vs. strength problem in reliability analysis. *Technometrics*, *6*, 325–328.

[132] Lohr, S.L. (**1993**). Algorithm AS285: Multivariate normal probabilities of star-shaped regions. *Appl. Stat.*, *42*, 576–582.

[133] Ma, C. (**1996**). Multivariate survival functions characterized by constant product of the mean remaining lives and hazard rates. *Metrika*, *44*, 71–83.

[134] Malik, H.J. and Abraham, B. (**1973**). Multivariate logistic distributions. *Ann. Stat.*, *1*, 588–590.

[135] Mardia, K.V. (**1962**). Multivariate Pareto distributions. *Ann. Math. Stat.*, *33*, 1008–1015.

[136] Marshall, A.W. and Olkin, I. (**1967**). A multivariate exponential distribution. *J. Am. Stat. Assoc.*, *62*, 30–44.

[137] Marshall, A.W. and Olkin, I. (**1979**). *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York.

[138] Marshall, A.W. and Olkin, I. (**1983**). Domains of attraction of multivariate extreme value distributions. *Ann. Probab.*, *11*, 168–177.

[139] Mathai, A.M. and Moschopoulos, P.G. (**1991**). On a multivariate gamma. *J. Multivariate Anal.*, *39*, 135–153.

[140] Mee, R.W. and Owen, D.B. (**1983**). A simple approximation for bivariate normal probabilities. *J. Qual. Technol.*, *15*, 72–75.
[141] Monhor, D. (**1987**). An approach to PERT: Application of Dirichlet distribution. *Optimization*, *18*, 113–118.
[142] Moran, P.A.P (**1967**). Testing for correlation between non-negative variates. *Biometrika*, *54*, 385–394.
[143] Morris, C.N. (**1982**). Natural exponential families with quadratic variance function. *Ann. Stat.*, *10*, 65–80.
[144] Mukherjea, A. and Stephens, R. (**1990**). The problem of identification of parameters by the distribution of the maximum random variable: solution for the trivariate normal case. *J. Multivariate Anal.*, *34*, 95–115.
[145] Nadarajah, S., Anderson, C.W., and Tawn, J.A. (**1998**). Ordered multivariate extremes. *J. R. Stat. Soc., Ser. B*, *60*, 473–496.
[146] Nagaraja, H.N. (**1982**). A note on linear functions of ordered correlated normal random variables. *Biometrika*, *69*, 284–285.
[147] National Bureau of Standards. (**1959**). *Tables of the Bivariate Normal Distribution Function and Related Functions*. Applied Mathematics Series, 50.
[148] Nayak, T.K. (**1987**). Multivariate Lomax distribution: properties and usefulness in reliability theory. *J. Appl. Probab.*, *24*, 170–171.
[149] Nelsen, R.B. (**1998**). *An Introduction to Copulas*, Lecture Notes in Statistics 139. Springer-Verlag, New York.
[150] O'Cinneide, C.A. and Raftery, A.E. (**1989**). A continuous multivariate exponential distribution that is multivariate phase type. *Stat. Probab. Lett.*, *7*, 323–325.
[151] Olkin, I. and Tong, Y.L. (**1994**). Positive dependence of a class of multivariate exponential distributions. *SIAM J. Control Opt.*, *32*, 965–974.
[152] Olkin, I. and Viana, M. (**1995**). Correlation analysis of extreme observations from a multivariate normal distribution. *J. Am. Stat. Assoc.*, *90*, 1373–1379.
[153] Owen, D.B. (**1956**). Tables for computing bivariate normal probabilities. *Ann. Math. Stat.*, *27*, 1075–1090.
[154] Owen, D.B. (**1957**). The Bivariate Normal Probability Distribution. Research Report SC 3831–TR, Sandia Corporation.
[155] Owen, D.B. and Wiesen, J.M. (**1959**). A method of computing bivariate normal probabilities with an application to handling errors in testing and measuring. *Bell Syst. Tech. J.*, *38*, 553–572.
[156] Patra, K. and Dey, D.K. (**1999**). A multivariate mixture of Weibull distributions in reliability modeling. *Stat. Probab. Lett.*, *45*, 225–235.
[157] Pearson, K. (**1901**). Mathematical contributions to the theory of evolution– VII. On the correlation of characters not quantitatively measurable. *Philos. Trans. R. Soc. London, Ser. A*, *195*, 1–47.
[158] Pearson, K. (**1903**). Mathematical contributions to the theory of evolution– XI. On the influence of natural selection on the variability and correlation of organs. *Philos. Trans. R. Soc. London, Ser. A*, *200*, 1–66.
[159] Pearson, K. (**1931**). *Tables for Statisticians and Biometricians*, Vol. *2*. Cambridge University Press, London.
[160] Pickands, J. (**1981**). Multivariate Extreme Value Distributions. In *Proceedings of the 43rd Session of the International Statistical Institute*. International Statistical Institute, Amsterdam, pp. 859–879.
[161] Prékopa, A. and Szántai, T. (**1978**). A new multivariate gamma distribution and its fitting to empirical stream flow data. *Water Resour. Res.*, *14*, 19–29.
[162] Proschan, F. and Sullo, P. (**1976**). Estimating the parameters of a multivariate exponential distribution. *J. Am. Stat. Assoc.*, *71*, 465–472.
[163] Raftery, A.E. (**1984**). A continuous multivariate exponential distribution. *Commun. Stat.–Theory Methods*, *13*, 947–965.
[164] Ramabhadran, V.R. (**1951**). A multivariate gamma-type distribution, *Sankhyā*, *11*, 45–46.
[165] Rao, B.V. and Sinha, B.K. (**1988**). A characterization of Dirichlet distributions. *J. Multivariate Anal.*, *25*, 25–30.
[166] Rao, C.R. (**1965**). *Linear Statistical Inference and its Applications*, John Wiley & Sons, New York.
[167] Regier, M.H. and Hamdan, M.A. (**1971**). Correlation in a bivariate normal distribution with truncation in both variables. *Aust. J. Stat.*, *13*, 77–82.
[168] Rinott, Y. and Samuel-Cahn, E. (**1994**). Covariance between variables and their order statistics for multivariate normal variables. *Stat. Prob. Lett.*, *21*, 153–155.
[169] Royen, T. (**1991**). Expansions for the multivariate chi-square distribution. *J. Multivariate Anal.*, *38*, 213–232.
[170] Royen, T. (**1994**). On some multivariate gamma-distributions connected with spanning trees. *Ann. Inst. Stat. Math.*, *46*, 361–372.
[171] Ruben, H. (**1954**). On the moments of order statistics in samples from normal populations. *Biometrika*, *41*, 200–227; Correction 41, ix.
[172] Ruiz, J.M., Marin, J., and Zoroa, P. (**1993**). A characterization of continuous multivariate distributions by conditional expectations. *J. Stat. Plann. Inference*, *37*, 13–21.
[173] Sarabia, J.-M. (**1995**). The centered normal conditionals distribution. *Commun. Stat.–Theory Methods*, *24*, 2889–2900.

[174] Satterthwaite, S.P. and Hutchinson, T.P. (**1978**). A generalization of Gumbel's bivariate logistic distribution. *Metrika*, *25*, 163–170.

[175] Schervish, M.J. (**1984**). Multivariate normal probabilities with error bound. *Appl. Stat.*, *33*, 81–94; Correction 34, 103–104.

[176] Shah, S.M. and Parikh, N.T. (**1964**). Moments of singly and doubly truncated standard bivariate normal distribution. *Vidya*, *7*, 82–91.

[177] Sheppard, W.F. (**1899**). On the application of the theory of error to cases of normal distribution and normal correlation. *Philos. Trans. R. Soc. London, Ser. A*, *192*, 101–167.

[178] Sheppard, W.F. (**1900**). On the calculation of the double integral expressing normal correlation. *Trans. Cambridge Philos. Soc.*, *19*, 23–66.

[179] Shi, D. (**1995a**). Fisher information for a multivariate extreme value distribution. *Biometrika*, *82*, 644–649.

[180] Shi, D. (**1995b**). Multivariate extreme value distribution and its Fisher information matrix. *Acta Math. Appl. Sinica*, *11*, 421–428.

[181] Shi, D. (**1995c**). Moment estimation for multivariate extreme value distribution. *Appl. Math. - JCU*, *10B*, 61–68.

[182] Shi, D., Smith, R.L., and Coles, S.G. (**1992**). Joint versus marginal Estimation for Bivariate Extremes. Technical Report No. 2074, Department of Statistics, University of North Carolina, Chapel Hill, N.C.

[183] Sibuya, M. (**1960**). Bivariate extreme statistics, I. *Ann. Inst. Stat. Math.*, *11*, 195–210.

[184] Šidák, Z. (**1967**). Rectangular confidence regions for th means of multivariate normal distribution. *JASA*, *62*, 626–633.

[185] Šidák, Z. (**1968**). On multivariate normal probabilities of rectangles: their dependence on correlations. *Ann. Math. Stat.*, *39*, 1425–1434.

[186] Siegel, A.F. (**1993**). A surprising covariance involving the minimum of multivariate normal variables. *J. Am. Stat. Assoc.*, *88*, 77–80.

[187] Smirnov, N.V. and Bol'shev, L.N. (**1962**). *Tables for Evaluating a Function of a Bivariate Normal Distribution*. Izdatel'stvo Akademii Nauk SSSR, Moscow.

[188] Smith, P.J. (**1995**). A recursive formulation of the old problem of obtaining moments from cumulants and vice versa. *Am. Stat.*, *49*, 217–218.

[189] Smith, R.L. (**1990**). Extreme Value Theory. In *Handbook of Applicable Mathematics*, Vol. *7*. John Wiley & Sons, New York, pp. 437–472.

[190] Sobel, M., Uppuluri, V.R.R, and Frankowski, K. (**1977**). Dirichlet Distribution—Type 1, *Selected Tables in Mathematical Statistics*, Vol. *4*, American Mathematical Society, Providence, R.I.

[191] Sobel, M., Uppuluri, V.R.R, and Frankowski, K. (**1985**). Dirichlet Integrals of Type 2 and their Applications, *Selected Tables in Mathematical Statistics*, Vol. *9*, American Mathematical Society, Providence, R.I.

[192] Somerville, P.N. (**1954**). Some problems of optimum sampling. *Biometrika*, *41*, 420–429.

[193] Song, R. and Deddens, J.A. (**1993**). A note on moments of variables summing to normal order statistics. *Stat. Probab. Lett.*, *17*, 337–341.

[194] Sowden, R.R. and Ashford, J.R. (**1969**). Computation of the bivariate normal integral. *Appl. Stat.*, *18*, 169–180.

[195] Stadje, W. (**1993**). ML characterization of the multivariate normal distribution. *J. Multivariate Anal.*, *46*, 131–138.

[196] Steck, G.P. (**1958**). A table for computing trivariate normal probabilities. *Ann. Math. Stat.*, *29*, 780–800.

[197] Steyn, H.S. (**1960**). On regression properties of multivariate probability functions of Pearson's types. *Proc. R. Acad. Sci., Amsterdam*, *63*, 302–311.

[198] Stoyanov, J. (**1987**). *Counter Examples in Probability*. John Wiley & Sons, New York.

[199] Sun, H.-J. (**1988**). A Fortran subroutine for computing normal orthant probabilities of dimensions up to nine. *Commun. Stat.–Simul. Comput.*, *17*, 1097–1111.

[200] Sungur, E.A. (**1990**). Dependence information in parametrized copulas. *Commun. Stat.–Simul. Comput.*, *19*, 1339–1360.

[201] Sungur, E.A. and Kovacevic, M.S. (**1990**). One dimensional marginal density functions of a truncated multivariate normal density function. *Commun. Stat.–Theory Methods*, *19*, 197–203.

[202] Sungur, E.A. and Kovacevic, M.S. (**1991**). Lower dimensional marginal density functions of absolutely continuous truncated multivariate distribution. *Commun. Stat.–Theory Methods*, *20*, 1569–1578.

[203] Symanowski, J.T. and Koehler, K.J. (**1989**). A Bivariate Logistic Distribution with Applications to Categorical Responses. Technical Report No. 89-29, Iowa State University, Ames, Iowa.

[204] Takahashi, R. (**1987**). Some properties of multivariate extreme value distributions and multivariate tail equivalence. *Ann. Inst. Stat. Math.*, *39*, 637–649.

[205] Takahasi, K. (**1965**). Note on the multivariate Burr's distribution. *Ann. Inst. Stat. Math.*, *17*, 257–260.

[206] Tallis, G.M. (**1963**). Elliptical and radial truncation in normal populations. *Ann. Math. Stat.*, *34*, 940–944.

[207] Tawn, J.A. (**1990**). Modelling multivariate extreme value distributions. *Biometrika*, *77*, 245–253.

[208] Teichroew, D. (**1955**). Probabilities Associated with Order Statistics in Samples from Two Normal Populations with Equal Variances. *Chemical Corps Engineering Agency*. Army Chemical Center, Md.

[209] Tiago de Oliveria, J. (**1962**/**1963**). Structure theory of bivariate extremes; extensions. *Estudos Matemática, Estadística Econometria*, *7*, 165–195.

[210] Tiao, G.C. and Guttman, I.J. (**1965**). The inverted Dirichlet distribution with applications. *J. Am. Stat. Assoc.*, *60*, 793–805; Correction 60, 1251–1252.

[211] Tong, Y.L. (**1970**). Some probability inequalities of multivariate normal and multivariate *t*. *J. Am. Stat. Assoc.*, *65*, 1243–1247.

[212] Urzúa, C.M. (**1988**). A class of maximum-entropy multivariate distributions. *Commun. Stat.–Theory Methods*, *17*, 4039–4057.

[213] Volodin, N.A. (**1999**). Spherically symmetric logistic distribution. *J. Multivariate Anal.*, *70*, 202–206.

[214] Watterson, G.A. (**1959**). Linear estimation in censored samples from multivariate normal populations. *Ann. Math. Stat.*, *30*, 814–824.

[215] Weinman, D.G. (**1966**). A Multivariate Extension of the Exponential Distribution. Ph.D. Dissertation, Arizona State University, Tempe, Ariz.

[216] Wesolowski, J. (**1991**). On determination of multivariate distribution by its marginals for the Kagan class. *J. Multivariate Anal.*, *36*, 314–319.

[217] Wesolowski, J. and Ahsanullah, M. (**1995**). Conditional specification of multivariate Pareto and student distributions. *Commun. Stat.–Theory Methods*, *24*, 1023–1031.

[218] Wolfe, J.H. (**1970**). Pattern clustering by multivariate mixture analysis. *Multivariate Behav. Res.*, *5*, 329–350.

[219] Yassaee, H. (**1976**). Probability integral of inverted Dirichlet distribution and its applications, *Compstat*, *76*, 64–71.

[220] Yeh, H.-C. (**1994**). Some properties of the homogeneous multivariate Pareto(IV) distribution. *J. Multivariate Anal.*, *51*, 46–52.

[221] Young, J.C. and Minder, C.E. (**1974**). Algorithm AS76. An integral useful in calculating non-central *t* and bivariate normal probabilities. *Appl. Stat.*, *23*, 455–457.

[222] Zelen, M. and Severo, N.C. (**1960**). Graphs for bivariate normal probabilities. *Ann. Math. Stat.*, *31*, 619–624.

