# Machine Learning Background Test

Problem 1: Let $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m \times n}$, $\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{X}_{ij} * \mathbf{Y}_{ij}$. Show that $\mathbf{tr}\left(\mathbf{X}^T \mathbf{Y}\right) = \langle \mathbf{X}, \mathbf{Y} \rangle$, where $\mathbf{tr}\left(\cdot\right)$ is the *trace* operator, based on which we can immediately get $\frac{\partial\, \mathbf{tr}\left(\mathbf{X}^T \mathbf{Y}\right)}{\partial \mathbf{X}} = \mathbf{Y}$ and $\frac{\partial\, \mathbf{tr}\left(\mathbf{X}^T \mathbf{Y}\right)}{\partial \mathbf{Y}} = \mathbf{X}$.

Problem 2: Given $\mathbf{A} \in \mathbb{R}^{m \times n}(m > n)$ and $\mathbf{y} \in \mathbb{R}^m$, also we define for $\mathbf{z} \in \mathbb{R}^m$, $\|\mathbf{z}\|_2^2 = \mathbf{z}_1^2 + \mathbf{z}_2^2 + \cdots + \mathbf{z}_m^2$, please find the best $\mathbf{x} \in \mathbb{R}^n$ to minimize $\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$ (closed solution is expected). This is so called *Least Squares* problem.

Problem 3: Please utilize *gradient descent* idea to write a program to minimize $\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$, where $\mathbf{A} \in \mathbb{R}^{4 \times 2}$ and $\mathbf{y} \in \mathbb{R}^4$ can be randomly generated in MATLAB or Python. Also, please compare the result with closed solution obtained in Problem 2.

Problem 4: Let $\mathbf{X} \in \mathbb{R}^{n \times n}$, please show that if we have $[\mathbf{U}, \mathbf{\Lambda}, \mathbf{V}] = svd(\mathbf{X})$, then $\mathbf{X} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{v}_i^T$ (*svd* is short for *Singular Value Decomposition*).

Problem 5: Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix. Which of the following statements are equivalent to "$\mathbf{A}$ is invertible"? If you think the statement is equivalent, please denote with "T", or a counterexample is expected, **NO** proof is needed.

- The columns of $\mathbf{A}$ span $\mathbb{R}^n$
- The rows of $\mathbf{A}$ are linearly independent
- $trace(\mathbf{A}) \neq 0$
- $\|\mathbf{A}\mathbf{x}\|_2^2 > 0$ for all $\mathbf{x} \neq \mathbf{0}$
- $det(\mathbf{A}) \neq 0$