Homomorphic Extensions of Complete Factorizations of Finite Abelian Groups

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Abstract

This paper extends the theory of complete factorizations of finite abelian groups by introducing a homomorphic framework for their structural analysis. The notion of a *complete factorization*, developed by Chin, Wang, and Wong (2023), is an intrinsic property of a group, described through disjoint subset systems whose sum covers the group uniquely. Within this framework, we establish new results describing the behavior of complete factorizations under group homomorphisms, quotient maps, and direct-sum constructions, thereby characterizing a broader structural class of completely factorizable groups. The proposed approach also enables computational simplifications for large abelian groups via their factorizable images and substructures. In addition, a constructive algorithm based on the system of product sets is outlined, together with potential applications to algebraic coding theory and cryptographic constructions.

1 Introduction

The theory of group factorizations, originating with Hajós [1], investigates how a group can be expressed as a sum or product of subsets with unique representation properties. Recent work by Chin, Wang, and Wong [2] formalized the notion of a *complete factorization* of a finite abelian group, describing all groups that admit such decompositions and developing a constructive algorithm via the system of product sets.

However, the theory as developed there was essentially intrinsic to a single group: it characterized when a fixed group admits a complete factorization, but did not address how this property behaves under natural group morphisms such as inclusions, projections, or quotient maps. In algebraic terms, the interaction of complete factorizations with *group homomorphisms* remained unexplored. For complete factorizations, this raises natural questions:

> If G is completely factorizable, does a homomorphic image or quotient of G remain completely factorizable? Conversely, can a factorization of a subgroup be lifted to its ambient group?

Addressing these questions requires introducing the concept of homomorphisms into the study of complete factorizations. In this paper, we extend the theory of complete factorizations by incorporating homomorphisms into the framework. The introduction of homomorphisms provides a structural perspective, allowing us to study how complete factorizations behave under fundamental operations such as direct sums, homomorphic images, and quotient constructions. This approach reveals that the class of completely factorizable finite abelian groups is closed under direct sums and injective homomorphisms, but not necessarily under projections or surjective maps—a distinction that clarifies the asymmetry between building and decomposing factorizable structures.

The homomorphic viewpoint not only enriches the theoretical understanding of complete factorizations but also suggests computational and categorical applications. By analyzing the preservation and failure of completeness under morphisms, we obtain a clearer picture of how factorization properties transfer between related groups, thereby linking the combinatorial theory of factorizations with the broader algebraic concept of structure-preserving mappings.

2 Preliminaries

2.1 Definitions and known results

Definition 2.1.1 [2] Let G be an abelian group and let $A_1, A_2, \dots, A_k$ $(k \geq 2)$ be non-empty subsets of G. Then $A_1, A_2, \dots, A_k$ is called a **complete factorization** of order k of G if
(i) Each $g \in G$ can be uniquely expressed in the form

$$g = a_1 + a_2 + \cdots + a_k$$

with $a_i \in A_i$ for each $i = 1, 2, \dots, k$, and
(ii) $A_i \cap A_j = \emptyset$ for each $i, j \in 1, 2, \dots, k$ with $i \neq j$.

Throughout, all groups considered are finite and abelian, written additively. We recall two standard results from [2] and [4] are used in our arguments.

[[2, 2.1]] Let G be a finite abelian group. Suppose $A_1, A_2, \dots, A_k$ are nonempty subsets of G such that $G = A_1 + A_2 + \cdots + A_k$. Then $(A_1, A_2, \dots, A_k)$ is a factorization of G if and only if $|G| = \prod_{i=1}^{k} |A_i|$.

[[4, 2.3]] Let H be a subgroup of an abelian group G. If H admits a complete decomposition of order m, then so does G.

2.2 Technical lemmas on singleton adjustments

Before proceeding to the main results, we record two auxiliary lemmas [2] (page no. 511) concerning the modification of complete factorizations through the removal or adjunction of singleton components. These results were stated without proof in Chin, Wang, and Wong (2023); for completeness, we include concise and formal proofs here.

Lemma 2.2.1: *Let G be an abelian group with a complete factorization $A_1, A_2, \dots, A_k$. Suppose that one of the subsets, say A_{i_0}, is a singleton. Then $A_1, \dots, A_{i_0-1}, A_{i_0+1}, \dots, A_k$ is also a complete factorization of G.*

Proof: Let $A_1, A_2, \dots, A_k$ form a complete factorization of G and suppose $A_{i_0} = \{a_{i_0}\}$ is a singleton. For any $g \in G$, by completeness there exist unique $a_i \in A_i$ such that

$$g = a_1 + a_2 + \cdots + a_{i_0-1} + a_{i_0} + a_{i_0+1} + \cdots + a_k.$$

Since a_{i_0} is fixed, define

$$g' = g - a_{i_0} = a_1 + a_2 + \cdots + a_{i_0-1} + a_{i_0+1} + \cdots + a_k.$$

The expression for g' is unique because the representation of each g in G is unique. Thus, every element of G corresponds uniquely to a sum of elements from $A_1, \dots, A_{i_0-1}, A_{i_0+1}, \dots, A_k$, which implies that these subsets form a complete factorization of G. □

Alternative Viewpoint 2.2.1(a) For a finite abelian group G, a complete factorization satisfies

$$|G| = |A_1| \cdot |A_2| \cdots |A_k|.$$

If one subset A_{i_0} is a singleton, then $|A_{i_0}| = 1$, and hence

$$|G| = |A_1| \cdot \cdots \cdot |A_{i_0-1}| \cdot 1 \cdot |A_{i_0+1}| \cdots |A_k| = |A_1| \cdot \cdots \cdot |A_{i_0-1}| \cdot |A_{i_0+1}| \cdots |A_k|.$$

Thus, the product of the remaining cardinalities still equals $|G|$, confirming that the reduced collection also forms a complete factorization. □

Lemma 2.2.2: *Let G be an abelian group with a complete factorization $A_1, A_2, \ldots, A_k$. If $G \neq \bigcup_{i=1}^k A_i$, then for any $x \in G \setminus \bigcup_{i=1}^k A_i$, the collection $A_1, A_2, \ldots, A_k, \{x\}$ forms a complete factorization of G.*

Proof: Let $A_1, \ldots, A_k$ be a complete factorization of G and choose $x \in G \setminus \bigcup_i A_i$. For any $g \in G$, write

$$g = a_1 + a_2 + \cdots + a_k, \quad a_i \in A_i,$$

which is unique by assumption. Now, consider elements of the form $g' = g + x$. Since x is disjoint from all A_i and G is a group, the set $\{x\}$ introduces a new disjoint summand without affecting uniqueness. Thus, every $g' \in G$ admits a unique representation in the form

$$g' = a_1 + a_2 + \cdots + a_k + x,$$

and hence $A_1, A_2, \ldots, A_k, \{x\}$ constitute a complete factorization of G. □

Alternative Viewpoint 2.2.2(a) For a finite abelian group G, if

$$|G| = |A_1| \cdot |A_2| \cdots |A_k|,$$

then adjoining a new singleton $\{x\}$ gives

$$|G| = |A_1| \cdot |A_2| \cdots |A_k| \cdot 1 = |A_1| \cdot |A_2| \cdots |A_k| \cdot |\{x\}|.$$

Hence, the total product of cardinalities remains unchanged, and the addition of $\{x\}$ preserves the completeness condition at the level of group order. □

We shall denote by $\mathrm{Hom}(G, H)$ the set of group homomorphisms from G to H. For a homomorphism $\varphi : G \to H$, we write $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ for its kernel and image, respectively.

3 Main Results

Proposition 3.1: Let $G = H \oplus K$ be a direct sum of finite abelian groups. If H admits a complete factorization, then G also admits a complete factorization.

Proof: View H as the subgroup $\iota(H) = \{(h, 0) \mid h \in H\} \subseteq G$. If $H = A_1 + \cdots + A_r$ is a complete factorization, define $B_i := A_i \times \{0\}$ for $1 \leq i \leq r$ and $B_{r+1} := \{0\} \times K$. Then every $(h, k) \in G$ has the unique representation

$$(h, k) = (a_1, 0) + \cdots + (a_r, 0) + (0, k),$$

and disjointness follows from that of the A_i's. Thus $(B_1, \ldots, B_{r+1})$ is a complete factorization of G.

Theorem 3.2: Direct-sum preservation
Let $G \cong H \oplus K$ be the external direct sum of finite abelian groups H and K. If

$$H = A_1 + \cdots + A_r \quad \text{and} \quad K = C_1 + \cdots + C_s$$

are complete factorizations of H and K respectively, then G admits a complete factorization

$$G = D_1 + \cdots + D_{r+s},$$

where $D_i = A_i \times \{0\}$ for $1 \le i \le r$ and $D_{r+j} = \{0\} \times C_j$ for $1 \le j \le s$.

Proof: Identify G with the cartesian product $H \times K$ with component-wise addition. For $1 \le i \le r$ let $\iota_{H,i} : A_i \to D_i$ be the inclusion $a \mapsto (a, 0)$ and for $1 \le j \le s$ let $\iota_{K,j} : C_j \to D_{r+j}$ be the inclusion $c \mapsto (0, c)$. Consider the product map

$$\Phi: \ A_1 \times \cdots \times A_r \times C_1 \times \cdots \times C_s \longrightarrow H \times K, \qquad \Phi(a_1, \ldots, a_r, c_1, \ldots, c_s) := \Big(\sum_{i=1}^{r} a_i, \ \sum_{j=1}^{s} c_j\Big).$$

(Here the sums are taken in the respective groups H and K.)

Surjectivity. Given any $(h, k) \in H \times K$, the complete factorizations of H and K yield unique $a_i \in A_i$ and $c_j \in C_j$ with $h = \sum_{i=1}^{r} a_i$ and $k = \sum_{j=1}^{s} c_j$. Thus $\Phi(a_1, \ldots, a_r, c_1, \ldots, c_s) = (h, k)$, so Φ is surjective; equivalently $H \times K = D_1 + \cdots + D_{r+s}$.

Injectivity. Assume

$$\Phi(a_1, \ldots, a_r, c_1, \ldots, c_s) = \Phi(a'_1, \ldots, a'_r, c'_1, \ldots, c'_s).$$

Then

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} a'_i \quad \text{in } H, \qquad \sum_{j=1}^{s} c_j = \sum_{j=1}^{s} c'_j \quad \text{in } K.$$

By uniqueness in the complete factorizations of H and K we obtain $a_i = a'_i$ for all i and $c_j = c'_j$ for all j. Hence Φ is injective.

Therefore Φ is a bijection between the product $A_1 \times \cdots \times C_s$ and G. Identifying $A_1 \times \cdots \times C_s$ with $D_1 \times \cdots \times D_{r+s}$ via the inclusions $\iota_{H,i}, \iota_{K,j}$, we see that each element of G has a unique representation as a sum of elements taken one from each D_t.

Disjointness and cardinality. Pairwise disjointness of the D_t follows from that of the A_i's and C_j's together with the fact that elements of $A_i \times \{0\}$ and $\{0\} \times C_j$ have distinct nonzero coordinate support. Finally,

$$\prod_{t=1}^{r+s} |D_t| = \Big(\prod_{i=1}^{r} |A_i|\Big)\Big(\prod_{j=1}^{s} |C_j|\Big) = |H| \cdot |K| = |G|.$$

Combining bijectivity of Φ, disjointness, and the cardinality identity, we conclude that $(D_1, \ldots, D_{r+s})$ is a complete factorization of G. □

Theorem 3.3: Homomorphic images
Let $\varphi : G \to H$ be a group homomorphism with trivial kernel. If G admits a complete factorization $G = A_1 + A_2 + \cdots + A_n$, then the image $\varphi(G)$ admits a complete factorization

$$\varphi(G) = \varphi(A_1) + \varphi(A_2) + \cdots + \varphi(A_n).$$

Proof: Let $h \in \varphi(G)$. Then $h = \varphi(g)$ for some $g = a_1 + \cdots + a_n$, where $a_i \in A_i$. Hence

$$h = \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n),$$

so $h \in \sum_i \varphi(A_i)$. Suppose two such sums coincide:

$$\sum_i \varphi(a_i) = \sum_i \varphi(a_i').$$

Then $\varphi(\sum_i (a_i - a_i')) = 0$, and since $\ker(\varphi) = \{0\}$, we obtain $a_i = a_i'$ for all i. Therefore the representation is unique, and the image sets $\varphi(A_i)$ are disjoint. Thus $(\varphi(A_1), \ldots, \varphi(A_n))$ form a complete factorization of $\varphi(G)$. □

Corollary 3.4: Quotient preservation
Let H be a subgroup of a finite abelian group G. If H admits a complete factorization of order m, then the quotient group G/H also admits a complete factorization of order m.

Proof: The canonical projection $\pi : G \to G/H$ is a surjective homomorphism such that $\pi(x) = x + H, x \in G$. Since H has a complete factorization, G does by Theorem 3.3. Applying the previous theorem to π, we obtain a complete factorization $(\pi(B_1), \ldots, \pi(B_m))$ of G/H. □

4 Discussion

The above results establish that complete factorizations behave well under standard morphic constructions:

- Direct sums of factorizable groups remain factorizable.
- Factorizations are preserved under injective and surjective homomorphisms.

Consequently, the class of finite abelian groups admitting complete factorizations is *closed under finite direct sums and homomorphic images with trivial kernel.* This provides a categorical closure property.

5 Algorithmic Construction

We recall that [2] introduced the *system of product sets* to construct explicit factorizations. We adapt this framework to incorporate homomorphic considerations

The algorithm provides a solvable case when $k \geq 3$.

A. Specify the group $G = \mathbb{Z}_1 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_l$.

B. Set the value of k equal to the number of subsets in the factorization.

C. Initialize matrix $\mathcal{A}$

1. Set $l \times (k+1)$ non-empty matrix $\mathcal{A}$.
2. Set random partitionin of $\mathbb{Z}_j$ into k non-empty subsets of $A_{j1}, \ldots, A_{jk}$.
3. Set i^{th} column of matrix $\mathcal{A}$ such that it has at least one non-empty set.
4. Set each U_j as empty set($\emptyset$) initially.

D. For the values of set U_j
if $j = 1, 2, \ldots, l$ each $U_1, U_2, ..., U_l$ is of size two

1. if $|\mathcal{A}_j| \geq 3$, then the sets $U_j, A_{j1}, \ldots, A_{jk}$ are mutually disjoint.

2. if $|\mathcal{A}_j| = 2$, say $\mathcal{A}_j = \{i_1, i_2\}$, choose a_j & b_j such that $a_j \in \mathbb{Z}_j \sim A_{ji_1} \bigcup A_{ji_2}$ and $b_j \notin A_{ji_2}$. Set $U_j = \{a_j, b_j\}$.

E. Check the j^{th} row disjoint condition.

1. if $|\mathcal{A}_j| \geq 3$, verify the sets $U_j, A_{j1}, \ldots, A_{jk}$ are mutually disjoint.
2. if $|\mathcal{A}_j| = 2$, verify (a) $A_{ji_1} \cap U_i = \emptyset$ and (b) $|A_{ji_2} \cap U_j < 1|$.

F. Compute & finding a solution.
if $i = 1, 2, \ldots, k$

1. Set $M_i = A_{1i} \times A_{2i} \times \cdots \times A_{li}$.
2. Change each empty set A_{ji} in the matrix $\mathcal{A}$ by $\{x_{ji}\}$ where $\{x_{ji}\}$ is an indefinite.
3. Maintain M_i & $M_{i'}$ disjointedness.
4. Check for $\prod_{i=1}^{k} |M_i| = |G|$ if not, then randomize $\mathcal{A}_j$ such that $\prod_{i=1}^{k} |M_i| = |G|$.

G. Now assign $\sum_{i \in I}^{k} M_i = G$.

H. Define the homomorphic image $\varphi(G)$ via canonical projection and check whether $\varphi(M_i)$ remain disjoint; if so, they yield a valid complete factorization of $\varphi(G)$.

This extension allows algorithmic testing of homomorphic stability of complete factorizations.

6 Conclusion

We have demonstrated that complete factorizations of finite abelian groups are preserved under direct sums, injective homomorphisms, and canonical quotient maps. This introduces a homomorphic dimension to the combinatorial framework established in [2], establishing closure properties that can be exploited in both theoretical and computational settings. Future investigations will examine extensions to nonabelian groups, categorical interpretations, and potential applications to group-based cryptographic constructions.

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