Dirac Constraint Analysis

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Submitted by

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Contents

Chapter 1

Constraint Analysis and Dirac Bracket

We will try to obtain Hamiltonian description of constrained system through Dirac bracket method.
Consider general transformation from configuration space to phase space as,

$$A\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}, A = \begin{pmatrix} \mathbb{I} & 0 \\ \tilde{b} & b \end{pmatrix} \tag{1.1}$$

where,

$$b = b(q), \tilde{b} = \tilde{b}(q)$$

$$A^{-1} = \frac{1}{det(A)}(adj(A))^T$$

$$= b^{-1}\begin{pmatrix} b & \tilde{b} \\ 0 & \mathbb{I} \end{pmatrix}^T = b^{-1}\begin{pmatrix} b & 0 \\ \tilde{b} & \mathbb{I} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \mathbb{I} & 0 \\ b^{-1}\tilde{b} & b^{-1} \end{pmatrix} \tag{1.2}$$

$\therefore A^{-1}$ exists only if b^{-1} exists,

we have

$$p^i = \tilde{b}^{ij}q_j + b^{ij}\dot{q}_j \tag{1.3}$$

and

$$p^i = \frac{\partial p^i}{\partial q_j} q_j + \frac{\partial p^i}{\partial \dot{q}_j} \dot{q}_j \tag{1.4}$$

$$\therefore b^{ij} = \frac{\partial p^i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \tag{1.5}$$

Now, the transformation is invertible if A^{-1} exists, i.e if b^{-1} exists,

$$\Rightarrow det\left(\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}\right) \neq 0 \tag{1.6}$$

However if for certain system the lagrangian happens to satisfy

$$det\left(\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}\right) = 0 \tag{1.7}$$

then the transformation will be non-invertible, and this will imply that, $det\left(\frac{\partial p^i}{\partial \dot{q}_j}\right) = 0$

Which means that not all velocities can be expressed in terms of independent momenta. This suggests that there exists constraints between dynamical variables.

Consider matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)_{N\times N} \tag{1.8}$$

having rank R<N. I.e there are R number of velocities expressible as coordinates and independent momenta, and remaining N-R velocities cannot be determined, so there are N-R constraint relations.

$$\dot{q}_a = f_a(q_i, p^b); \qquad a, b = 1, 2, ..., R \tag{1.9}$$

$$p^a = g^a(q_i, \dot{q}_b); \qquad a, b = 1, 2, ..., R \tag{1.10}$$

$$p^\alpha = g^\alpha(q_i, p^a); \qquad \alpha = R+1, R+2, ..., N \tag{1.11}$$

$\therefore$ constraint relations will be given by p^α,

Defining quantity φ such tha,

$$\varphi^\alpha = p^\alpha - g^\alpha(q_i, p^a) = 0 \tag{1.12}$$

These constraints will reduce the dimensionality of phase space of the system by N-R. φ^α are also known as primary constraints. Dimensionality of hyperspace which is constrained will be 2N-(N-R) = N+R, denoted as Γ_c. Γ is 2N dimensional phase space of the system.
Two dynamical variables F and G in Γ are said to be weakly equal if F≈G on Γ_c constrained hypersphere, i.e $(F-G)\mid_{\Gamma_c}=0$
Canonical Hamiltonian of this theory will be,

$$H_{can} = p^i\dot{q}_i - L(q_i,\dot{q}_i) = p^a\dot{q}_a + g^\alpha\dot{q}_\alpha - L(q_i,\dot{q}_i) \tag{1.13}$$

$$So, \quad \frac{\partial H_{can}}{\partial \dot{q}_\alpha} = 0 + g^\alpha - \frac{\partial L}{\partial \dot{q}_\alpha} = g^\alpha - p^\alpha \approx 0 \tag{1.14}$$

$$\therefore \frac{\partial H_{can}}{\partial \dot{q}_\alpha} = -\varphi^\alpha = 0 \tag{1.15}$$

On Γ_c, H_{can} is independent of velocities, $H_{can} = H_{can}(q_i, p^a)$.
Now, here hamiltonain is not unique because any linear combination of φ^α can be added to hamiltonian to construct a new hamiltonian, $\because$ φ^α 's are zero,constructing such hamiltonian incorporates primary constraints.

$$H_p = H_{can} + \lambda_\alpha\varphi^\alpha \tag{1.16}$$

λ_α are undetermined lagrange multipliers. So, $H_p \approx H_{can}$
Using poisson brackets, hamilton's equation of motion can be written as follows,

$$\dot{q}_i \approx \{q_i, H_p\} = \frac{\partial H_p}{\partial p_i} = \frac{\partial (H_{can} + \lambda_\alpha\varphi^\alpha)}{\partial p_i} \tag{1.17}$$

and

$$\dot{p}_i \approx \{p_i, H_p\} = -\frac{\partial H_p}{\partial q_i} = -\frac{\partial (H_{can} + \lambda_\alpha\varphi^\alpha)}{\partial q_i} \tag{1.18}$$

To find lagrange multipliers, consider

$$\dot{q}_\alpha \approx \{q_\alpha, H_p\} = \frac{\partial H_p}{\partial p_\alpha} \tag{1.19}$$

$$H_p = H_{can} + \lambda_\alpha\varphi^\alpha = p^i\dot{q}_i + g^\alpha\dot{q}_\alpha + \lambda_\alpha\varphi^\alpha - L(q_i,\dot{q}_i)$$

$$= p^i \dot{q}_i + g^\alpha \dot{q}_\alpha + \lambda_\alpha p^\alpha - \lambda_\alpha g^\alpha - L(q_i, \dot{q}_i)$$

$$\frac{\partial H_p}{\partial p_\alpha} = 0 + 0 + \lambda_\alpha - 0 - 0 = \lambda_\alpha$$

$$\therefore \dot{q}_\alpha \approx \lambda_\alpha \tag{1.20}$$

and these remain undetermined,
Time evolution of any dynamical variable F will be given by,

$$\dot{F} \approx \{F, H_p\} = \{F, H_{can} + \lambda_\alpha \varphi^\alpha\} \tag{1.21}$$

Consider time evolution of constraints φ^α

$$\dot{\varphi^\alpha} \approx \{\varphi^\alpha, H_{can} + \lambda_\beta \varphi^\beta\} \tag{1.22}$$

$$= \{\varphi^\alpha, H_{can}\} + \lambda_\beta \{\varphi^\alpha, \varphi^\beta\} + \{\varphi^\alpha, \lambda_\beta\} \varphi^\beta$$

$$= \{\varphi^\alpha, H_{can}\} + \lambda_\beta \{\varphi^\alpha, \varphi^\beta\} \approx 0$$

$$\therefore \dot{\varphi^\alpha} \approx 0 \tag{1.23}$$

$\therefore$ Constraints do not vary with time

Now from above expression, we may obtain some of the multipliers λ_α, or some new set of constraints (secondary constraints). On demanding time invariance of secondary constraints, we will obtain some multipliers or some set of newer constraints (tertiary constraints). This process is to be carried out till time invariant constraints are obtained.

<u>Primary Constraints</u>: Constraints that are directly obtained from Lagrange's equation of motion

<u>Secondary Constraints</u>: Constraints that are obtained after demanding time invariance of primary constraints.

Tertiary Constraints: Constraints that are obtained after demanding time invariance of secondary constraints.

Now denoting all the constraints of the theory,

$$\varphi^{\overline{\alpha}} \approx 0, \qquad \overline{\alpha} = 1, 2, ...n < 2N \tag{1.24}$$

These constraints can be divided into two distinct classes.

1) First Class Constraints: These constraints have weakly vanishing poisson bracket with all the constraints(including themselves)

$$\psi^{\tilde{\alpha}} \approx 0, \qquad \tilde{\alpha} = 1, 2, ..., n_1 \tag{1.25}$$

2) Second Class Constraints: These constraints at least have one non-vanishing poisson bracket with other constraints.

$$\phi^{\hat{\alpha}} \approx 0, \qquad \hat{\alpha} = 1, 2, ..., 2n_2 \tag{1.26}$$

$[n_1 + 2n_2 = n < N]$

In process of determining all time independent constraints, all lagrange multipliers may be completely determined, or some may still remain undetermined. Number of undetermined multipliers equals number of first class constraints in primary constraints. For obtaining consistent hamiltonian description, we need as many gauge fixing conditions as there are first class constraints. Gauge fixing conditions are denoted as,

$$\chi^{\tilde{\alpha}} \approx 0, \qquad \tilde{\alpha} = 1, 2, ..., n_1 \tag{1.27}$$

These gauge fixing conditions are chosen such that first class constraints get converted to second class constraints, so after gauge fixing is done, we obtain all constraints which are second class in nature, which are denoted as,

$$\phi^{A} \approx 0, \qquad A = 1, 2, ..., 2(n_1 + n_2) < 2N \tag{1.28}$$

However the poisson bracket of constraints with any dynamical variable does not vanish in general, $\{F, \phi^A\} \neq 0$
To overcome this problem, we can construct quantity C^{AB} such that poisson bracket ϕ^A and ϕ^B is

$$C^{AB} = \{\phi^A, \phi^B\} \tag{1.29}$$

where C^{AB} is a non-singular, even dimensional, antisymmetric matrix, $\therefore C^{-1}_{AB}$ exists, such that

$$C^{AD}C^{-1}_{DB} = \delta^A_B = C^{-1}_{BD}C^{DA}$$

Now defining new bracket between two dynamical variables F and G, namely Dirac bracket as

$$\boxed{\{F, G\}_D = \{F, G\} - \{F, \phi^A\}C^{-1}_{AB}\{\phi^B, G\}} \tag{1.30}$$

Consider dirac bracket of dynamical variable F with ϕ^A,

$$\{F, \phi^A\}_D = \{F, \phi^A\} - \{F, \phi^B\}C^{-1}_{BD}\{\phi^D, \phi^A\}$$

$$= \{F, \phi^A\} - \{F, \phi^B\}C^{-1}_{BD}C^{DA} = \{F, \phi^A\} - \{F, \phi^B\}\delta^A_B$$

$$= \{F, \phi^A\} - \{F, \phi^A\} = 0$$

$$\therefore \{F, \phi^A\}_D = 0 \tag{1.31}$$

These Dirac Brackets thus obtained can then be elevated to commutator relations for quantisation of theory. We will now see application of this constraint analysis to constrained systems.

Chapter 2

Application to Relativistic Free Particle

Dynamics of relativistic free particle is given by,

$$m\frac{d^2x^\mu}{d\tau^2} = 0 \tag{2.1}$$

Usual four-velocities and momentum are,

$$u^\mu = \frac{dx^\mu}{d\tau}, \quad and \;\; p^\mu = m\frac{dx^\mu}{d\tau} = mu^\mu, \quad So, \;\; \frac{dp^\mu}{d\tau} = 0 \tag{2.2}$$

Action for relativistic particle will be,

$$S = m\int ds = m\int \sqrt{\eta_{\mu\nu}dx^\mu dx^\nu} = m\int \sqrt{dx_\mu dx^\mu}$$

$$= m\int d\lambda\sqrt{\frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda}} = m\int d\lambda(\dot{x_\mu}\dot{x^\mu})^{\frac{1}{2}} = \int d\lambda L \tag{2.3}$$

where L is lagrangian, and λ parameter is label for trajectory.

$$L = m(\dot{x_\mu}\dot{x^\mu})^{\frac{1}{2}} = m\sqrt{\eta_{\mu\nu}\dot{x^\mu}\dot{x^\nu}} \tag{2.4}$$

Under a transformation $\lambda \to \xi = \xi(\lambda)$

$$\dot{x^\mu} = \frac{dx^\mu}{d\lambda} = \frac{d\xi}{d\lambda}\frac{dx^\mu}{d\xi} \tag{2.5}$$

then action will be

$$S' = m \int d\lambda \left(\frac{d\xi}{d\lambda} \frac{dx^\mu}{d\xi} \right)^{\frac{1}{2}} \left(\frac{d\xi}{d\lambda} \frac{dx_\mu}{d\xi} \right)^{\frac{1}{2}} = m \int d\lambda \left(\frac{d\xi}{d\lambda} \right) \left(\frac{dx^\mu}{d\xi} \frac{dx_\mu}{d\xi} \right)^{\frac{1}{2}}$$

$$= m \int d\xi \left(\frac{dx^\mu}{d\xi} \frac{dx_\mu}{d\xi} \right)^{\frac{1}{2}} = S \tag{2.6}$$

$\therefore$ action is invariant under transformation $\lambda \to \xi = \xi(\lambda)$ (Invariance under diffeomorphism)

Let quantity $P_{\mu\nu}$ be

$$P_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x^\mu} \partial \dot{x^\nu}} \tag{2.7}$$

$$\therefore P_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x^\mu} \partial \dot{x^\nu}} = \frac{m}{\dot{x_\rho}\dot{x^\rho}} \left(\eta_{\mu\nu} - \frac{\dot{x_\mu}\dot{x_\nu}}{\dot{x_\sigma}\dot{x^\sigma}} \right) \tag{2.8}$$

So, $\dot{x^\mu} P_{\mu\nu}$ will be

$$\dot{x^\mu} P_{\mu\nu} = \frac{m}{\dot{x_\rho}\dot{x^\rho}} \left(\dot{x_\nu} - \frac{\dot{x^\mu}\dot{x_\mu}\dot{x_\nu}}{\dot{x_\sigma}\dot{x^\sigma}} \right) = 0 = P_{\mu\nu}\dot{x^\nu} \tag{2.9}$$

$\therefore$ $P_{\mu\nu}$ is projection operator, $\therefore$ we will have,

$$det(P_{\mu\nu}) = det \left(\frac{\partial^2 L}{\partial \dot{x^\mu} \partial \dot{x^\nu}} \right) = 0 \tag{2.10}$$

We will introduce an auxiliary variable g into lagrangian and rewrite it in such a way that if limit for mass is taken to zero, motion of massless relativistic particles can be described(lagrangian won't be zero for m=0).

$$L = \frac{1}{2} \left(g^{-\frac{1}{2}} (\dot{x^\mu}\dot{x_\mu}) + g^{\frac{1}{2}} (m^2) \right) = \frac{1}{2} \sqrt{g} \left(g^{-1} (\dot{x^\mu}\dot{x_\mu}) + m^2 \right) \tag{2.11}$$

g can be thought of as a metric on 1D manifold of trajectory of particle, using the fact that lagrangian is independent of g, the expression for g can be obtained,

$$\frac{\partial L}{\partial g} = 0 = \frac{1}{2}\left(-\frac{1}{2}g^{\frac{1}{2}}g^{-1}(\dot{x^\mu}\dot{x_\mu}) + \frac{1}{2}g^{-\frac{1}{2}}m^2\right)$$

$$\therefore -(\dot{x^\mu}\dot{x_\mu})g^{-1} + m^2 = 0$$

$$\therefore g = \frac{\dot{x^\mu}\dot{x_\mu}}{m^2}, \quad g^{-\frac{1}{2}} = \frac{m}{\sqrt{\dot{x^\mu}\dot{x_\mu}}}, \quad g^{\frac{1}{2}} = \frac{\sqrt{\dot{x^\mu}\dot{x_\mu}}}{m} \tag{2.12}$$

So we can obtain canonical momenta from modified lagrangian,

$$p_\mu = \frac{\partial L}{\partial \dot{x^\mu}} = g^{-\frac{1}{2}}\dot{x_\mu}, \quad p_g = \frac{\partial L}{\partial \dot{g}} = 0 \tag{2.13}$$

$\therefore$ primary constraint for this theory is

$$\varphi^1 = p_g = 0 \tag{2.14}$$

$\therefore$ Canonical Hamiltonian for this theory will be,

$$H_{can} = p_\mu x^\mu + p_g\dot{g} - L = g^{\frac{1}{2}}p_\mu p^\mu - L$$

$$= g^{\frac{1}{2}}p_\mu p^\mu - \frac{1}{2}\left(g^{-\frac{1}{2}}(\dot{x^\mu}\dot{x_\mu}) + g^{\frac{1}{2}}(m^2)\right)$$

$$= g^{\frac{1}{2}}p_\mu p^\mu - \frac{1}{2}\left(g^{\frac{1}{2}}(p^\mu p_\mu) + g^{\frac{1}{2}}(m^2)\right)$$

$$\therefore H_{can} = \frac{g^{\frac{1}{2}}}{2}(p_\mu p^\mu - m^2) \tag{2.15}$$

Equal time poisson brackets for this theory will be,

$$\{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = \{g, g\} = \{p_g, p_g\} = 0$$

$$\{x^\mu, g\} = \{x^\mu, p_g\} = \{p_\mu, g\} = \{p_\mu, p_g\} = 0$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{g, p_g\} = 1 \tag{2.16}$$

Adding primary constraint to canonical momenta to obtain primary hamiltonian,

$$H_p = H_{can} + \lambda_1 \varphi^1 = \frac{g^{\frac{1}{2}}}{2}(p_\mu p^\mu - m^2) + \lambda_1 p_g \tag{2.17}$$

Now evaluating time evolution of primary constraint,

$$\dot{\varphi}^1 \approx \{\varphi^1, H_p\} = \left\{p_g, \frac{g^{\frac{1}{2}}}{2}(p_\mu p^\mu - m^2) + \lambda_1 p_g\right\}$$

$$\therefore \dot{\varphi}^1 \approx \left\{p_g, \frac{g^{\frac{1}{2}}}{2}(p_\mu p^\mu - m^2)\right\} + \{p_g, \lambda_1 p_g\}$$

$$\approx \{p_g, g^{\frac{1}{2}}\}\frac{1}{2}(p_\mu p^\mu - m^2) + 0$$

$$\approx \left(\frac{1}{2g^{\frac{1}{2}}}\right)\{p_g, g\}\frac{1}{2}(p_\mu p^\mu - m^2)$$

$$\dot{\varphi}^1 \approx -\frac{1}{4g^{\frac{1}{2}}}(p_\mu p^\mu - m^2) \tag{2.18}$$

Now demanding that $\dot{\varphi}^1 \approx 0$ will give us secondary constraint,

$$\varphi^2 = \frac{1}{2}(p_\mu p^\mu - m^2) \tag{2.19}$$

evaluating time evolution of this secondary constraint,

$$\dot{\varphi}^2 \approx \{\varphi^2, H_p\} = \left\{\frac{1}{2}(p_\mu p^\mu - m^2), \frac{g^{\frac{1}{2}}}{2}(p_\nu p^\nu - m^2) + \lambda_1 p_g\right\} = 0$$

$$\therefore \dot{\varphi^2} \approx 0 \tag{2.20}$$

The chain of constraints has terminated, from (45) and (50) the constraints we obtained are,

$$\varphi^1 = p_g, \quad \varphi^2 = \frac{1}{2}(p_\mu p^\mu - m^2) \tag{2.21}$$

It is clear that,

$$\{\varphi^1, \varphi^1\} = \{\varphi^2, \varphi^2\} = \{\varphi^1, \varphi^2\} = 0 \tag{2.22}$$

$\therefore$ φ^1 and φ^2 are first class constraints.

Introducing gauge fixing conditions as following,

$$\chi^1 = g - \frac{1}{m^2} \tag{2.23}$$

$$\chi^2 = \lambda - x^0 \tag{2.24}$$

clearly,

$$\{\chi^1, \chi^1\} = \{\chi^2, \chi^2\} = \{\chi^1, \chi^2\} = 0 \tag{2.25}$$

and

$$\{\varphi^1, \chi^1\} = \left\{p_g, g - \frac{1}{m^2}\right\} = -1 = -\{\chi^1, \varphi^1\} \tag{2.26}$$

$$\{\varphi^2, \chi^2\} = \left\{\frac{1}{2}(p_\mu p^\mu - m^2), \lambda - x^0\right\} = -\frac{1}{2}\{p^\mu p_\mu, x^0\}$$

$$-\frac{1}{2}(p^\mu(-\delta^0_\mu) + (-\delta^0_\mu)p_\mu) = p^0$$

$$\therefore \{\varphi^2, \chi^2\} = p^0 = -\{\chi^2, \varphi^2\} \tag{2.27}$$

Now Combining all the constraints

$$\phi^A = (\varphi^1, \varphi^2, \chi^1, \chi^2)$$

$\therefore C^{AB}$ will be

$$C^{AB} = \{\phi^A, \phi^B\} = \begin{pmatrix} \{\varphi^1, \varphi^1\} & \{\varphi^1, \varphi^2\} & \{\varphi^1, \chi^1\} & \{\varphi^1, \chi^2\} \\ \{\varphi^2, \varphi^1\} & \{\varphi^2, \varphi^2\} & \{\varphi^2, \chi^1\} & \{\varphi^2, \chi^2\} \\ \{\chi^1, \varphi^1\} & \{\chi^1, \varphi^2\} & \{\chi^1, \chi^1\} & \{\chi^1, \chi^2\} \\ \{\chi^2, \varphi^1\} & \{\chi^2, \varphi^2\} & \{\chi^2, \chi^1\} & \{\chi^2, \chi^2\} \end{pmatrix} \tag{2.28}$$

$$C^{AB} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & p^0 \\ 1 & 0 & 0 & 0 \\ 0 & -p^0 & 0 & 0 \end{pmatrix} \tag{2.29}$$

$$C^{-1}_{AB} = \frac{1}{det(C^{AB})}(adj(C^{AB}))^T$$

$$det(C^{AB}) = (p^0)^2, and \quad adj(C^{AB}) = \begin{pmatrix} 0 & 0 & -(p^0)^2 & 0 \\ 0 & 0 & 0 & p^0 \\ (p^0)^2 & 0 & 0 & 0 \\ 0 & -p^0 & 0 & 0 \end{pmatrix}$$

$$\therefore C^{-1}_{AB} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{p^0} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{p^0} & 0 & 0 \end{pmatrix} \tag{2.30}$$

So Dirac brackets between fundamental variables can be evaluated,

- $\{x^\mu, x^\nu\}_D = \{x^\mu, x^\nu\} - \{x^\mu, \phi^A\}C^{-1}_{AB}\{\phi^B, x^\nu\}$

$$= 0 - \left(\{x^\mu, \varphi^1\}(1)\{\chi^1, x^\nu\} + \{x^\mu, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, x^\nu\}\right)$$

$$- \left(\{x^\mu, \chi^1\}(-1)\{\varphi^1, x^\nu\} + \{x^\mu, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, x^\nu\}\right)$$

here,

$$\{x^\mu, \varphi^1\} = 0 = \{\varphi^1, x^\nu\}, \quad and \quad \{x^\mu, \chi^2\} = 0 = \{\chi^2, x^\nu\}$$

$$\therefore \{x^\mu, x^\nu\}_D = 0 \tag{2.31}$$

- $\{p^\mu, p^\nu\}_D = \{p^\mu, p^\nu\} - \{p^\mu, \phi^A\}C^{-1}_{AB}\{\phi^B, p^\nu\}$

$$= 0 - \left(\{p^\mu, \varphi^1\}(1)\{\chi^1, p^\nu\} + \{p^\mu, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, p^\nu\}\right)$$

$$- \left(\{p^\mu, \chi^1\}(-1)\{\varphi^1, p^\nu\} + \{p^\mu, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, p^\nu\}\right)$$

here,

$$\{p^\mu, \varphi^1\} = 0 = \{\varphi^1, p^\nu\}, \quad and \quad \{p^\mu, \varphi^2\} = 0 = \{\varphi^2, p^\nu\}$$

$$\therefore \{p^\mu, p^\nu\}_D = 0 \tag{2.32}$$

- $\{g, g\}_D = \{g, g\} - \{g, \phi^A\}C^{-1}_{AB}\{\phi^B, g\}$

$$= 0 - \left(\{g, \varphi^1\}(1)\{\chi^1, g\} + \{g, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, g\}\right)$$

$$- \left(\{g, \chi^1\}(-1)\{\varphi^1, g\} + \{g, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, g\}\right)$$

here,

$$\{g, \chi^1\} = 0 = \{\chi^1, g\}, \quad and \quad \{g, \chi^2\} = 0 = \{\chi^2, g\}$$

$$\therefore \{g, g\}_D = 0 \tag{2.33}$$

- $\{p_g, p_g\}_D = \{p_g, p_g\} - \{p_g, \phi^A\}C^{-1}_{AB}\{\phi^B, p_g\}$

$$= 0 - \left(\{p_g, \varphi^1\}(1)\{\chi^1, p_g\} + \{p_g, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, p_g\}\right)$$

$$- \left(\{p_g, \chi^1\}(-1)\{\varphi^1, p_g\} + \{p_g, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, p_g\}\right)$$

here,

$$\{p_g, \varphi^1\} = 0 = \{\varphi^1, p_g\}, \quad and \quad \{p_g, \varphi^2\} = 0 = \{\varphi^2, p_g\}$$

$$\therefore \{p_g, p_g\}_D = 0 \tag{2.34}$$

- $\{x^\mu, g\}_D = \{x^\mu, g\} - \{x^\mu, \phi^A\}C^{-1}_{AB}\{\phi^B, g\}$

$$= 0 - \left(\{x^\mu, \varphi^1\}(1)\{\chi^1, g\} + \{x^\mu, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, g\}\right)$$

$$- \left(\{x^\mu, \chi^1\}(-1)\{\varphi^1, g\} + \{x^\mu, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, g\}\right)$$

here,

$$\{\chi^1, g\} = 0, \quad \{\chi^2, g\} = 0$$

$$\{x^\mu, \chi^1\} = 0, \quad \{\varphi^2, g\} = 0$$

$$\therefore \{x^\mu, g\}_D = 0 \tag{2.35}$$

- $\{x^\mu, p_g\}_D = \{x^\mu, p_g\} - \{x^\mu, \phi^A\}C^{-1}_{AB}\{\phi^B, p_g\}$

$$= 0 - \left(\{x^\mu, \varphi^1\}(1)\{\chi^1, p_g\} + \{x^\mu, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, p_g\}\right)$$

$$- \left(\{x^\mu, \chi^1\}(-1)\{\varphi^1, p_g\} + \{x^\mu, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, p_g\}\right)$$

here,

$$\{x^\mu, \varphi^1\} = 0, \quad \{x^\mu, \varphi^2\} = 0$$

$$\{\varphi^1, p_g\} = 0, \quad \{\varphi^2, p_g\} = 0$$

$$\therefore \{x^\mu, p_g\}_D = 0 \tag{2.36}$$

- $\{p^\mu, g\}_D = \{p^\mu, g\} - \{p^\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, g\}$

$$= 0 - \left(\{p^\mu, \varphi^1\}(1)\{\chi^1, g\} + \{p^\mu, \varphi^2\} \left(\frac{-1}{p^0} \right) \{\chi^2, g\} \right)$$
$$- \left(\{p^\mu, \chi^1\}(-1)\{\varphi^1, g\} + \{p^\mu, \chi^2\} \left(\frac{1}{p^0} \right) \{\varphi^2, g\} \right)$$

here,

$$\{\chi^1, g\} = 0, \quad \{\chi^2, g\} = 0$$

$$\{p^\mu, \chi^1\} = 0, \quad \{\varphi^2, g\} = 0$$

$$\therefore \{p^\mu, g\}_D = 0 \tag{2.37}$$

- $\{p_\mu, p_g\}_D = \{p_\mu, p_g\} - \{p_\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, p_g\}$

$$= 0 - \left(\{p_\mu, \varphi^1\}(1)\{\chi^1, p_g\} + \{p_\mu, \varphi^2\} \left(\frac{-1}{p^0} \right) \{\chi^2, p_g\} \right)$$
$$- \left(\{p_\mu, \chi^1\}(-1)\{\varphi^1, p_g\} + \{p_\mu, \chi^2\} \left(\frac{1}{p^0} \right) \{\varphi^2, p_g\} \right)$$

here,

$$\{p_\mu, \varphi^1\} = 0, \quad \{\chi^2, p_g\} = 0$$

$$\{p^\mu, \chi^1\} = 0, \quad \{\varphi^2, p_g\} = 0$$

$$\therefore \{p_\mu, p_g\}_D = 0 \tag{2.38}$$

- $$\{g, p_g\}_D = \{g, p_g\} - \{p_\mu, \phi^A\}C^{-1}_{AB}\{\phi^B, p_g\}$$

$$= 0 - \left(\{g, \varphi^1\}(1)\{\chi^1, p_g\} + \{g, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, p_g\}\right)$$

$$- \left(\{g, \chi^1\}(-1)\{\varphi^1, p_g\} + \{g, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, p_g\}\right)$$

here,

$$\{g, \varphi^2\} = 0, \quad \{g, \chi^1\} = 0, \quad \{g, \chi^2\} = 0$$

$$\therefore \{g, p_g\}_D = 1 - \{g, p_g\}(1)\left\{g - \frac{1}{m^2}, p_g\right\} = 1 - 1 = 0$$

$$\therefore \{g, p_g\}_D = 0 \tag{2.39}$$

- $$\{x^\mu, p_\nu\}_D = \{x^\mu, p_\nu\} - \{x^\mu, \phi^A\}C^{-1}_{AB}\{\phi^B, p_\nu\}$$

$$= \delta^\mu_\nu - \left(\{x^\mu, \varphi^1\}(1)\{\chi^1, p_\nu\} + \{x^\mu, \varphi^2\}\left(\frac{-1}{p^0}\right)\{\chi^2, p_\nu\}\right)$$

$$- \left(\{x^\mu, \chi^1\}(-1)\{\varphi^1, p_\nu\} + \{x^\mu, \chi^2\}\left(\frac{1}{p^0}\right)\{\varphi^2, p_\nu\}\right)$$

here,

$$\{x^\mu, \varphi^1\} = 0, \quad \{x^\mu, \chi^1\} = 0, \quad \{\varphi^2, p_\nu\} = 0$$

$$\therefore \{x^\mu, p_\nu\}_D = \delta^\mu_\nu + \{x^\mu, \varphi^2\}\left(\frac{1}{p^0}\right)\{\chi^2, p_\nu\}$$

$$= \delta^\mu_\nu + \left\{x^\mu, \frac{1}{2}(p^\sigma p_\sigma - m^2)\right\}\left(\frac{1}{p^0}\right)\{\lambda - x^0, p_\nu\} = \delta^\mu_\nu + \frac{1}{2}(2p^\mu)\left(\frac{1}{p^0}\right)(-\delta^0_\nu)$$

$$\therefore \boxed{\{x^\mu, p_\nu\}_D = \delta^\mu_\nu - \frac{p^\mu}{p^0}\delta^0_\nu} \tag{2.40}$$

These are the required dirac brackets, if we set the constraint (50) strongly to zero, we will obtain hamiltonian,

$$H = p^0 = \sqrt{\mathbf{p}^2 + m^2} \tag{2.41}$$

This concludes constraint analysis for free relativistic particle.

Chapter 3

Application to Maxwell Theory

Lagrangian density for Maxwell theory is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.1}$$

$F^{\mu\nu}$ is field strength tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.2}$$

where $F^{0i} = -E^i$, and $B^k = -\frac{1}{2}\epsilon^{ijk}F_{jk}$
Maxwell theory is invariant under gauge transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$$

Lagrangian density can also be written as,

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu)$$

$$= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\mu A_\nu \partial^\mu A^\nu)$$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{3.3}$$

Conjugate momenta to A_μ is given by,

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$$

$$= -\frac{1}{2}[\delta^0_\mu \delta^\mu_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial_\mu A_\nu(\delta^\mu_0 \delta^\nu_\mu - \delta^\nu_0 \delta^\mu_\mu)]$$

$$= -\frac{1}{2}[(\partial^0 A^\mu - \partial^\mu A^0) + (\partial_0 A_\mu - \partial_\mu A_0)]$$

$$\therefore \Pi^\mu = -F^{0\mu} \tag{3.4}$$

$$\therefore \Pi^i = E^i = -(\partial_0 A^i + \nabla A_0), \quad \Pi^0 = 0$$

For lagrangian,

$$\mathcal{L} = \frac{1}{4}(-\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\nu A^\mu)$$

Consider following total derivative terms,

$$\partial_\mu(A_\nu \partial^\mu A^\nu) = \partial_\mu A_\nu \partial^\mu A^\nu + A_\nu \partial_\mu \partial^\mu A^\nu$$

$$\partial_\mu(A_\nu \partial^\nu A^\mu) = \partial_\mu A_\nu \partial^\nu A^\mu + A_\nu \partial_\mu \partial^\nu A^\mu = \partial_\mu A_\nu \partial^\nu A^\mu + A_\nu \partial^\mu \partial^\nu A_\mu$$

$$\partial_\nu(A_\mu \partial^\mu A^\nu) = \partial_\nu A_\mu \partial^\mu A^\nu + A_\mu \partial_\nu \partial^\mu A^\nu = \partial_\nu A_\mu \partial^\mu A^\nu + A_\mu \partial^\nu \partial^\mu A_\nu$$

$$\partial_\nu(A_\mu \partial^\nu A^\mu) = \partial_\nu A_\mu \partial^\nu A^\mu + A_\mu \partial_\nu \partial^\nu A^\mu$$

$\therefore \mathcal{L} = \frac{1}{4}(A_\nu \partial_\mu \partial^\mu A^\nu - A_\nu \partial^\mu \partial^\nu A_\mu - A_\mu \partial^\nu \partial^\mu A_\nu + A_\mu \partial_\nu \partial^\nu A^\mu)$ + Total derivative terms

$$\mathcal{L} = \frac{1}{4}(2A_\mu \Box A^\mu - 2A_\mu(\partial^\mu \partial^\nu)A_\nu) = \frac{1}{2}(A_\mu \eta^{\mu\nu} \Box A_\nu - A_\mu(\partial^\mu \partial^\nu)A_\nu)$$

$$\therefore \mathcal{L} = \frac{1}{2} A_\mu P^{\mu\nu} A_\nu \tag{3.5}$$

where $P^{\mu\nu}$ is,

$$P^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu \tag{3.6}$$

Now consider,

$$\partial_\mu P^{\mu\nu} = \partial_\mu(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu) = (\partial^\nu\Box - \Box\partial^\nu) = 0$$

similarly, $\partial_\nu P^{\mu\nu} = 0$
$\therefore P^{\mu\nu}$ is a projection operator,

$$\therefore det\left(\frac{\partial^2\mathcal{L}}{\partial(\partial_0 A_\mu)\partial(\partial_0 A^\mu)}\right) = 0$$

We have primary constraint for this theory as,

$$\varphi^1(x) = \Pi^0(x) \approx 0 \tag{3.7}$$

Canonical hamiltonian for this theory will be,

$$\mathcal{H}_{can} = \Pi^\mu \dot{A}_\mu - \mathcal{L} = -\Pi^i \dot{A}_i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= -\Pi^i.(-\Pi^i - \nabla A_0) + \frac{1}{4}(2(-\mathbf{E}^2 + \mathbf{B}^2))$$

$$= \mathbf{\Pi}^2 + \mathbf{\Pi}.(\nabla A_0) + \frac{1}{2}(-\mathbf{E}^2 + \mathbf{B}^2)$$

$$= \mathbf{\Pi}^2 - \frac{\mathbf{\Pi}^2}{2} + \frac{\mathbf{B}^2}{2} + \mathbf{\Pi}.(\nabla A_0)$$

$$\therefore \mathcal{H}_{can} = \frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) + \mathbf{\Pi}.(\nabla A_0) \tag{3.8}$$

$$H_{can} = \int d^3x \mathcal{H}_{can} = \int d^3x \left(\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) + \mathbf{\Pi}.(\nabla A_0)\right)$$

$$= \int d^3x \left(\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) + \nabla.(\mathbf{\Pi}A_0) - A_0\nabla.\mathbf{\Pi}\right)$$

$$= \int d^3x \left(\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) - A_0 \nabla . \mathbf{\Pi} \right) + \oint ds \mathbf{\Pi} A_0$$

At boundary, the surface integral goes to zero, so we obtain

$$H_{can} = \int d^3x \left(\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) - A_0(\nabla . \mathbf{\Pi}) \right) \tag{3.9}$$

Now primary hamiltonian can be obtained by adding primary constraint,

$$H_p = H_{can} = \int d^3x \lambda_1 \varphi^1$$

$$\therefore H_p = \int d^3x \left(\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) - A_0(\nabla . \mathbf{\Pi}) + \lambda_1 \Pi^0 \right) \tag{3.10}$$

Equal time poisson brackets for this theory are given by,

$$\{A_\mu(x), A^\nu(x')\} = 0 = \{\Pi_\mu(x), \Pi^\nu(x')\}$$

$$\{A_\mu(x), \Pi^\nu(x')\} = \delta_\mu^\nu \delta^3(x - x') \tag{3.11}$$

Now evaluating time evolution of primary constraint,

$$\dot{\varphi}^1(x) \approx \{\varphi^1(x), H_p\}$$

$$= \int d^3x' \left\{ \Pi^0(x), \frac{1}{2}(\mathbf{\Pi}^2(x') + \mathbf{B}^2(x')) - A_0(x')(\nabla . \mathbf{\Pi}(x')) + \lambda_1(x')\Pi^0(x') \right\}$$

$$\approx - \int d^3x' \{\Pi^0(x), A_0(x')\} \nabla . \mathbf{\Pi}(x') \approx - \int d^3x' \delta^3(x - x') \nabla . \mathbf{\Pi}(x') \approx \nabla . \mathbf{\Pi}(x)$$

$$\dot{\varphi}^1(x) \approx \nabla . \mathbf{\Pi}(x) \approx 0 \tag{3.12}$$

$\therefore$ we have secondary constraint of the theory,

$$\varphi^2 = \nabla.\mathbf{\Pi}(x) \tag{3.13}$$

Evaluating time evolution of φ^2

$$\dot{\varphi^2} = \{\varphi^2, H_p\}$$

$$= \int d^3x' \left\{ \{\varphi^2, \frac{1}{2}(\mathbf{\Pi}^2(x') + \mathbf{B}^2(x')) - A_0(x')(\nabla.\mathbf{\Pi}(x')) + \lambda_1(x')\Pi^0(x') \right\} = 0$$

$$\therefore \dot{\varphi^2} = 0 \tag{3.14}$$

The chain of constraints terminates here, $\therefore$ we have obtained two first class constraints (79) and (85),

$$\varphi^1(x) = \Pi^0(x), \quad \varphi^2 = \nabla.\mathbf{\Pi}(x) \tag{3.15}$$

Introducing two gauge fixing conditions,

$$\chi^1(x) = A_0(x) \tag{3.16}$$

$$\chi^2(x) = \nabla.\mathbf{A}(x) \tag{3.17}$$

Evaluating equal time poisson brackets between these constraints and gauge fixing conditions,

$$\{\varphi^1(x), \chi^1(x')\} = \{\Pi^0(x), A_0(x')\} = -\delta^3(x - x') \tag{3.18}$$

$$\{\varphi^2(x), \chi^2(x')\} = \{\nabla.\mathbf{\Pi}(x), \nabla.\mathbf{A}(x')\}$$

$$= (\nabla_x)_i(\nabla_{x'})^j\{\Pi_j(x), A^j(x')\} = (\nabla_x)_i(\nabla_{x'})^j(-\delta_i^j\delta^3(x - x'))$$

$$\{\varphi^2(x), \chi^2(x')\} = -\nabla_x^2\delta^3(x - x') \tag{3.19}$$

Due to gauge fixing conditions, first class constraints are now converted to second class constraints. Collecting the constraints and gauge fixing conditions,

$$\phi^A = (\varphi^1, \varphi^2, \chi^1, \chi^2)$$

matrix $C^{AB}(x,x') = \{\phi^A(x), \phi^B(x')\}$ can be obtained using poisson brackets between constraints,

$$C^{AB}(x,x') = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\nabla_x^2 \\ 1 & 0 & 0 & 0 \\ 0 & \nabla_x^2 & 0 & 0 \end{pmatrix} \delta^3(x-x') \tag{3.20}$$

$$C_{AB}^{-1}(x,x') = \frac{1}{det(C^{AB}(x,x'))}(adj(C^{AB}(x,x')))^T$$

$$det(C^{AB}(x,x')) = \nabla_x^4$$

$$adj(C^{AB}(x,x')) = \begin{pmatrix} 0 & 0 & -\nabla_x^4 & 0 \\ 0 & 0 & 0 & -\nabla_x^2 \\ \nabla_x^4 & 0 & 0 & 0 \\ 0 & \nabla_x^2 & 0 & 0 \end{pmatrix} \delta^3(x-x')$$

$$\therefore C_{AB}^{-1}(x,x') = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nabla_x^{-2} \\ -1 & 0 & 0 & 0 \\ 0 & -\nabla_x^{-2} & 0 & 0 \end{pmatrix} \delta^3(x-x') \tag{3.21}$$

Now dirac brackets between any two dynamical variables F(x) and G(x) can be evaluated,

$$\{F(x), G(x')\}_D = \{F(x), G(x')\} - \int d^3y d^3y' \{F(x), \phi^A(y)\} C_{AB}^{-1}(y,y') \{\phi^B(y'), G(x')\} \tag{3.22}$$

- $\{A_\mu(x), A^\nu(x')\}_D = \{A_\mu(x), A^\nu(x')\}$

$$-\int d^3y d^3y' \{A_\mu(x), \phi^A(y)\} C^{-1}_{AB}(y,y') \{\phi^B(y'), A^\nu(x')\}$$

$$= -\int d^3y d^3y' [\{A_\mu(x), \varphi^1(y)\}(1)\{\chi^1(y'), A^\nu(x')\} + \{A_\mu(x), \varphi^2(y)\}(\nabla_y^{-2})\{\chi^2(y'), A^\nu(x')\}$$

$$+\{A_\mu(x), \chi^1(y)\}(-1)\{\varphi^1(y'), A^\nu(x')\} + \{A_\mu(x), \chi^2(y)\}(-\nabla_y^{-2})\{\varphi^2(y'), A^\nu(x')\}]\delta^3(y-y')$$

here,

$$\{A_\mu(x), \chi^1(y)\} = 0 = \{\chi^1(y'), A^\nu(x')\}$$

$$\{A_\mu(x), \chi^2(y)\} = 0 = \{\chi^2(y'), A^\nu(x')\}$$

$$\therefore \{A_\mu(x), A^\nu(x')\}_D = 0 \tag{3.23}$$

- $\{\Pi_\mu(x), \Pi^\nu(x')\}_D = \{\Pi_\mu(x), \Pi^\nu(x')\}$

$$-\int d^3y d^3y' \{\Pi_\mu(x), \phi^A(y)\} C^{-1}_{AB}(y,y') \{\phi^B(y'), \Pi^\nu(x')\}$$

$$= -\int d^3y d^3y' [\{\Pi_\mu(x), \varphi^1(y)\}(1)\{\chi^1(y'), \Pi^\nu(x')\} + \{\Pi_\mu(x), \varphi^2(y)\}(\nabla_y^{-2})\{\chi^2(y'), \Pi^\nu(x')\}$$

$$+\{\Pi_\mu(x), \chi^1(y)\}(-1)\{\varphi^1(y'), \Pi^\nu(x')\} + \{\Pi_\mu(x), \chi^2(y)\}(-\nabla_y^{-2})\{\varphi^2(y'), \Pi^\nu(x')\}]\delta^3(y-y')$$

here,

$$\{\Pi_\mu(x), \varphi^1(y)\} = 0 = \{\varphi^1(y'), \Pi^\nu(x')\}$$

$$\{\Pi_\mu(x), \varphi^2(y)\} = 0 = \{\varphi^2(y'), \Pi^\nu(x')\}$$

$$\therefore \{\Pi_\mu(x), \Pi^\nu(x')\}_D = 0 \tag{3.24}$$

$$\bullet \quad \{A_\mu(x), \Pi^\nu(x')\}_D = \{A_\mu(x), \Pi^\nu(x')\}$$

$$-\int d^3y d^3y' \{A_\mu(x), \phi^A(y)\} C^{-1}_{AB}(y,y') \{\phi^B(y'), \Pi^\nu(x')\}$$

$$= \delta^\nu_\mu \delta^3(x-x')$$

$$-\int d^3y d^3y' [\{A_\mu(x), \varphi^1(y)\}(1)\{\chi^1(y'), \Pi^\nu(x')\} + \{A_\mu(x), \varphi^2(y)\}(\nabla_y^{-2})\{\chi^2(y'), \Pi^\nu(x')\}$$

$$+\{A_\mu(x), \chi^1(y)\}(-1)\{\varphi^1(y'), \Pi^\nu(x')\} + \{A_\mu(x), \chi^2(y)\}(-\nabla_y^{-2})\{\varphi^2(y'), \Pi^\nu(x')\}]\delta^3(y-y')$$

here,

$$\{A_\mu(x), \chi^1(y)\} = 0, \qquad \{A_\mu(x), \chi^2(y)\} = 0$$

$$\therefore \{A_\mu(x), \Pi^\nu(x')\}_D = \delta^\nu_\mu \delta^3(x-x')$$

$$-\int d^3y d^3y' [\{A_\mu(x), \Pi^0(y)\}(\delta^3(y-y'))\{A_0(y'), \Pi^\nu(x')\}$$

$$+\{A_\mu(x), \nabla_y \mathbf{\Pi}(y)\}(\nabla_y^{-2}\delta^3(y-y'))\{\nabla_y \mathbf{A}(y'), \Pi^\nu(x')\}]$$

$$= \delta^\nu_\mu \delta^3(x-x') - \int d^3y d^3y \delta^0_\mu \delta^3(x-y)(\delta^3(y-y'))\delta^\nu_0 \delta^3(y'-x')$$

$$-\int d^3y d^3y' \delta^i_\mu (\nabla_y)_i \delta^3(x-y)(\nabla_y^{-2}\delta^3(y-y'))\delta^\nu_j (\nabla_{y'})^j \delta^3(y'-x')$$

$$= \delta^\nu_\mu \delta^3(x-x') - \delta^0_\mu \delta^\nu_0 \delta^3(x-x') - \delta^i_\mu \delta^\nu_j (\nabla_x)_i \frac{1}{\nabla_x^2} (\nabla_x)^j \delta^3(x-x')$$

$$\therefore \boxed{\{A_\mu(x), \Pi^\nu(x')\}_D = \left(\delta^\nu_\mu - \delta^0_\mu \delta^\nu_0 \delta^3 + \delta^i_\mu \delta^\nu_j (\nabla_x)_i \frac{1}{\nabla_x^2} (\nabla_x)^j\right) \delta^3(x-x')} \tag{3.25}$$

$\therefore$ we can see that,

$$\{A_i(x), A^j(x')\}_D = \{\Pi_i(x), \Pi^j(x')\}_D = 0 \tag{3.26}$$

$$\{A_i(x), \Pi^j(x')\}_D = \left(\delta_i^j + (\nabla_x)_i \frac{1}{\nabla_x^2} (\nabla_x)^j\right) \delta^3(x - x') \tag{3.27}$$

These are the required dirac brackets for Maxwell's theory. Now we can set the constraints in primary hamiltonian to zero, we get,

$$H_p = \int d^3x \frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) \tag{3.28}$$

This concludes constraint analysis for Maxwell's theory.