Two, Three and Four-Dimensional Electromagnetics Using Differential Forms

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Abstract

The exterior calculus of differential forms provides a mathematical framework for electromagnetic field theory that combines much of the generality of tensor analysis with the computational simplicity and concreteness of the vector calculus. We review the pedagogical aspects of the calculus of differential forms in providing distinct representations of field intensity and flux density, physically meaningful graphical representations for sources, fields, and fluxes, and a picture of the curl operation that is as simple and intuitive as that of the gradient and divergence. To further highlight the benefits of differential forms notation, we demonstrate the flexibility of the calculus of differential forms in responding to changes in the dimensionality of the underlying manifold on which the calculus is defined. We develop Maxwell's equations in the case of two space dimensions with time (2+1), and in a four-dimensional (4D or spacetime) representation with time included as a differential basis element on an equal footing with the spatial dimensions. The 2+1 case is commonly treated in textbooks using component notation, but we show that Maxwell's equations and the theorems and principles of electromagnetics can be expressed in a fundamentally two-dimensional formulation. In the 4D representation, graphical representations can be given to illustrate four-dimensional fields in a way that provides intuition into the interplay between the electric and magnetic fields in wave propagation. These results illustrate the usefulness of differential forms in providing the physical insight required for engineering applications of electromagnetics.

1. Introduction

Maxwell's equations were originally expressed without the benefit of a compact notation and convenient mathematical system for manipulating field quantities [1, 2]. The subsequent development of the vector calculus provided a powerful and useful tool for working with electromagnetic theory. Tensor analysis is in turn an even more concise notation for electromagnetics, but the full generality of tensors is typically not needed in application problems. The exterior calculus of differential forms provides another framework for working with electromagnetic theory, which combines the simplicity of vector analysis and much of the generality of tensor analysis. While the calculus of differential forms is in many respects similar to the vector calculus, it is actually more general than the vector notation. The purpose of this paper is to explore the ramifications of this increased generality from an educational point of view.

A differential form is by definition any quantity that can be integrated, including differentials. In 1844 Hermann Günter Grassmann published his book *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* [3], in which he developed the idea of an algebra in which the symbols representing geometric entities such as points, lines and planes are manipulated using certain rules. Grassmann introduced what is now called exterior algebra, based upon the exterior product. In the early 1900's, Elie Cartan developed an exterior calculus of differential forms. Since that time, differential forms have received widespread use in the physics and mathematics communities for many problems, including electrodynamics [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

After its early introduction into the engineering community by Deschamps [14], Engle [15], Baldomir [16] and others, the calculus of differential forms has been used in applications to numerical methods [17, 18], boundary conditions [19, 20], Green's functions [21], and anisotropic media [22]. Differential forms have also been considered from an educational point of view [23] and as the basis for monographs [24, 25] and an engineering electromagnetics textbook [26].

Vector analysis provides two types of objects for representing physical quantities: scalar fields and vector fields. In electromagnetics, the electric potential and the electric charge density are scalar fields, and the field intensities **E**, **H** and the flux densities **D**, **B** are vector fields. It is not obvious why field theory requires two vector quantities to represent one physical field. With differential forms, the field intensities are represented by one-forms, or differential forms of degree one, which are naturally integrated over one-dimensional regions in path integrals. The flux densities are two-forms, and are integrated over two-dimensional regions in surface integrals. This increase in generality, in that the space of vector fields expands into two types of differential forms, provides distinct mathematical expressions with different physical interpretations for field intensities and flux densities.

The pictures that one draws for differential forms of different degree are also distinct. Misner, Thorne and Wheeler [5] give graphical representations of one-forms as surfaces (equipotentials in the case of the electric field intensity) and two-forms as tubes (representing electric or magnetic flux). A student acquainted with differential forms can call to mind mental pictures of field intensity surfaces and flux density that reflect the physical behavior and mathematical representation of each.

The derivative operations of the calculus of differential forms used in expressing Maxwell's equations can also be represented graphically. The derivatives of vector calculus are the gradient, curl and divergence. The gradient and divergence lend themselves readily to geometric interpretation, but the curl is more difficult to visualize. This is a major pedagogical hurdle in undergraduate electromagnetics courses. With the calculus of differential forms, the curl can be given a graphical interpretation that is just as intuitive as the gradient and divergence. This representation provides an easy way to help students gain a physical understanding of Ampère's and Faraday's laws.

These pedagogical aspects of the calculus of differential forms have already been treated in the literature and incorporated into textbooks. In this paper, we review this body of work, and add to it another important aspect of the calculus of differential forms: it can be readily extended to formulations of the laws of electromagnetics in dimensions other than three.

It is common in applications to consider systems with an axis of symmetry, so that the dimensionality of the fields and sources are effectively reduced, and Maxwell's equations can be simplified in a way that facilitates their solution. Alternately, the equations of electromagnetics can be formulated in a four-dimensional representation, in such a way that space and time are treated on an equal footing, and the four Maxwell equations combine into two.

The vector calculus cannot readily be used in accomplishing either formulation. In the former case, one typically treats the two-dimensional case as a slice of the three-dimensional case, and thereby avoids the expression of Maxwell's equations in a truly two-dimensional framework. In the latter case, vectors would be extremely unwieldy in describing a higher-dimensional formulation.

These limitations can be overcome by the use of tensor analysis, but as noted above, the full machinery of tensor analysis is typically not required in engineering electromagnetics. The calculus of differential forms is general enough that changes to the dimensionality of the formulation of electromagnetics are more natural than is the case for vector calculus, yet it does not require the index notation of tensor analysis which can hide simple physical intuition.

After reviewing the three-dimensional formulation of electromagnetics, we develop the exterior calculus on a two-dimensional space, and use it express Maxwell's equations for two space dimensions (2+1 formulation). We then consider the case of three space dimensions with time included as a fourth differential basis element in the exterior calculus (4D or spacetime formulation), which leads to a compaction of Maxwell's equation into two equations. We consider the designation 3+1 as referring to the most common expression of Maxwell's equations using a three-dimensional multivariable calculus for the spatial part with the time derivative included explicitly.

The 2+1 formulation is in a sense standard material in electromagnetics textbooks, because it is used in the solution of transverse mode problems for waveguides and many other structures with an axial symmetry. But the usual viewpoint of the 2+1 case is as a slice of the 3D formulation. In this paper, we treat the 2+1 case in a non-embedded way, as if sources and fields were confined to a universe consisting of only two space dimensions plus time ("Flatland" [27]), using the calculus of differential forms.

It is common in mathematical physics to consider field theories of many dimensionalities as models for various physical systems. The spacetime formulation of Maxwell's equations has been given in many papers, monographs, and textbooks (e.g., [5, 11]). These formulations are given almost exclusively using differential forms or tensor notation. Here, we present these formulations from an engineering point of view, and adapt the graphical representations that have been used to illustrate fields in the 3+1 case to provide additional insight into the wave propagation in the spacetime representation.

2. Three-Dimensional Representation

In preparation for considering the two- and four-dimensional representations, we will first review the standard three-dimensional (3+1) representation of Maxwell's equations. The basic differentials are $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$. These differentials are combined using the exterior product, which is antisymmetric, so that

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x \tag{1}$$

$$\mathrm{d}x \wedge \mathrm{d}x = 0 \tag{2}$$

The exterior product of two one-forms is a two-form, which is naturally integrated over a surface. The elementary two-forms are $\mathrm{d}y \wedge \mathrm{d}z$, $\mathrm{d}z \wedge \mathrm{d}x$, and $\mathrm{d}x \wedge \mathrm{d}y$, where we choose the right-cyclic ordering of the coordinates by convention. Because of the antisymmetry of the exterior product, all three-forms can be expressed in the form $f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, where f is some function.

The electric and magnetic field intensities, flux densities, and the source quantities are represented by

differential forms according to

$$\mathcal{E} \;=\; E_1 \mathrm{d}x + E_2 \mathrm{d}y + E_3 \mathrm{d}z \tag{3a}$$

$$\mathcal{H} \;=\; H_1 \mathrm{d}x + H_2 \mathrm{d}y + H_3 \mathrm{d}z \tag{3b}$$

$$\mathcal{D} \;=\; D_1 \mathrm{d}y \wedge \mathrm{d}z + D_2 \mathrm{d}z \wedge \mathrm{d}x + D_3 \mathrm{d}x \wedge \mathrm{d}y \tag{3c}$$

$$\mathcal{B} \;=\; B_1 \mathrm{d}y \wedge \mathrm{d}z + B_2 \mathrm{d}z \wedge \mathrm{d}x + B_3 \mathrm{d}x \wedge \mathrm{d}y \tag{3d}$$

$$\mathcal{J} \;=\; J_1 \mathrm{d}y \wedge \mathrm{d}z + J_2 \mathrm{d}z \wedge \mathrm{d}x + J_3 \mathrm{d}x \wedge \mathrm{d}y \tag{3e}$$

$$\mathcal{Q} \;=\; \rho \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{3f}$$

where $\mathcal{E}$ and $\mathcal{H}$ are one-forms; $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{J}$ are two-forms; and $\mathcal{Q}$ is a three-form. These field quantities, along with the degree of the differential form and the corresponding vector quantity, are given in Table 1. We consider that the differentials have units of lengths, so that the differential form carries the same units as the path, surface, or volume integral of the corresponding vector quantity.

Table 1. The differential forms of electromagnetics, along with their degrees, units, and corresponding vector quantities.

Quantity	Form	Type	Units	Vector/Scalar
Electric Field Intensity	$\mathcal{E}$	one-form	V	**E**
Magnetic Field Intensity	$\mathcal{H}$	one-form	A	**H**
Electric Flux Density	$\mathcal{D}$	two-form	C	**D**
Magnetic Flux Density	$\mathcal{B}$	two-form	Wb	**B**
Electric Current Density	$\mathcal{J}$	two-form	A	**J**
Electric Charge Density	$\mathcal{Q}$	three-form	C	ρ

The path integral of the one-form $\mathcal{E}$ represents potential change along the path. $\mathcal{E}$ can be pictured as parallel surfaces representing equipotentials of the field, such that the value of the integral is given by the number of surfaces pierced by the path. The differential form $\mathrm{d}x$, for example, consists of parallel planes spaced a unit distance apart and perpendicular to the x-axis. The surfaces are oriented in one of the two normal directions, so that the integral is equal to the number of surfaces pierced in the positive direction minus the number of surfaces pierced in the negative direction.
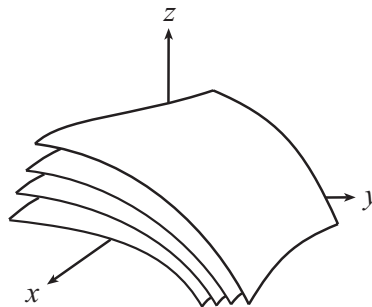

Figure 1. A one-form is represented graphically by surfaces. The integral of the one-form over a path is the number of surfaces pierced by the path. The orientation of the surface together with the direction of the integration path determines the sign of the integral. This one-form represents a conservative field, so that the integral over a closed path is zero, because each surface pierced by the path with a positive orientation is also pierced by the path at another location in the opposite direction.

The two-form $\mathcal{D}$ represents electric flux, and is graphically pictured as a family of tubes. The two-form $\mathrm{d}x \wedge \mathrm{d}y$, for example, consists of two sets of parallel planes perpendicular to the x and y axes, which intersect to form tubes extending in the z direction, as shown in Fig. 2. The integral of a two-form over an area is the number of tubes crossing the area.

The three-form $f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ consists of boxes with sides given by the surfaces corresponding to the one-forms $\mathrm{d}z$, $\mathrm{d}y$, and $\mathrm{d}z$. The density of the boxes is determined by the value of the coefficient function f.
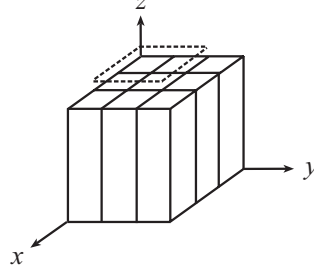

Figure 2. The two-form $\mathrm{d}x \wedge \mathrm{d}y$ consists of two sets of parallel surfaces, one perpendicular to x and the other to y. When integrated over a square in the $x-y$ plane of side 2, the boundary of which is denoted by the dashed line, the value of the integration is four, because four tubes pass through the square.

If a one-form represents a nonconservative field, then in the graphical representation new surfaces are created, allowing a closed path to pierce a nonzero number of surfaces, as shown in Fig. 3(a). This property can be analyzed locally using the exterior derivative operator, which is given by

$$\mathrm{d} = \left[\frac{\partial}{\partial x}\mathrm{d}x + \frac{\partial}{\partial y}\mathrm{d}y + \frac{\partial}{\partial z}\mathrm{d}z \right] \wedge \tag{4}$$

This operator maps a p-form to a $(p+1)$-form. If $\mathcal{E}$ represents a nonconservative field, then the two-form given by $\mathrm{d}\mathcal{E}$ is nonzero.

Stokes' theorem provides a link between integrals of a differential form and its exterior derivative through the relationship

$$\int_M \mathrm{d}\omega = \oint_{\partial M} \omega \tag{5}$$

where M is some region of space and ∂M is its boundary. The dimension of ∂M has to match the degree of ω. If ω is a zero-form, this expression reduces to the fundamental theorem of calculus. If ω is a one-form, this theorem connects the surface integral of the two-form $d\omega$, represented by tubes passing through the surface, to the closed path integral of ω, as shown in Fig. 3. This closed path integral is nonzero only if new surfaces of the one-form are created inside the path. Thus, Stokes' theorem can be interpreted graphically as stating that new surfaces of ω are created by tubes of $d\omega$. If ω is a two-form, then new tubes of ω extend away from cubes of the three-form $d\omega$.

In a three-dimensional space, one-forms and two-forms both have three independent differential elements, so that the spaces of one-forms and two-forms are isomorphic. In vector notation, these two spaces are not explicitly distinguished, but with the calculus of differential forms they are distinct. The different physical meanings of field intensity and flux density are reflected in this distinction, because $\mathcal{E}$ and $\mathcal{H}$ are one-forms, whereas $\mathcal{D}$ and $\mathcal{B}$ are two-forms. Similarly, zero-forms (scalars) and three-forms (densities) are identified in vector notation, but become distinct in the calculus of differential forms.
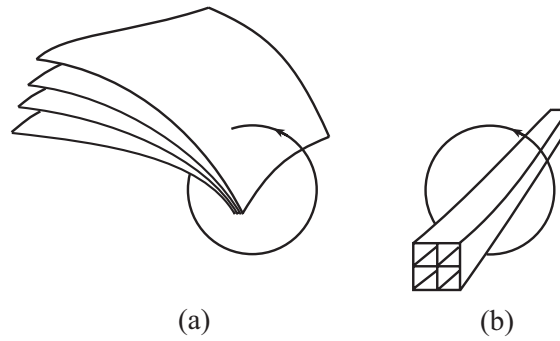
Figure 3. (a) A nonconservative one-form, with new surfaces extending out into space, allowing a nonzero integral over a closed path. (b) The exterior derivative of the one-form is a two-form having tubes where new surfaces of the one-form are created. By Stokes' theorem, the density of the tubes of the two-form is equal to the number of new surfaces of the one-form.

The Hodge star operator is a mapping between the pairs of isomorphic spaces. In the Euclidean metric, we have

$$\star dx = dy \wedge dz, \qquad \star dy \wedge dz = dx$$
$$\star dy = dz \wedge dx, \qquad \star dz \wedge dx = dy$$
$$\star dz = dx \wedge dy, \qquad \star dx \wedge dy = dz$$

The vector quantity that is dual to a one-form α is the same as the vector dual to the two-form $\star\alpha$. Between zero-forms and three-forms, the star operator takes the function f to the three-form $\star f = f dx \wedge dy \wedge dz$. In 3D, the star operator applied twice yields the identity operation, so that $\star\star\alpha = \alpha$ for any form α.

With the differential forms defined in (3), along with the exterior derivative and the Hodge star operator, we can express Maxwell's equations in point form as

$$d\mathcal{E} = -\frac{\partial}{\partial t}\mathcal{B} \tag{6a}$$

$$d\mathcal{H} = \frac{\partial}{\partial t}\mathcal{D} + \mathcal{J} \tag{6b}$$

$$d\mathcal{D} = Q \tag{6c}$$

$$d\mathcal{B} = 0 \tag{6d}$$

with the constitutive relationships

$$\mathcal{D} = \epsilon_0 \star \mathcal{E} \tag{7a}$$

$$\mathcal{B} = \mu_0 \star \mathcal{H} \tag{7b}$$

In vector notation, Gauss's law for the electric field is expressed using the divergence operator. This operation can be pictured in a simple way. Where a vector field has nonzero divergence, there is a source or sink. The vector field represents either flow away from the source, or flow towards a sink. The corresponding picture in differential forms notation is equally intuitive. Boxes of the three-form $\mathcal{Q}$ produce tubes of the two-form $\mathcal{D}$, which are either oriented away from a positive charge or towards a negative charge (Fig. 4).

With vector notation, the curl operation is less intuitive. The magnetic field intensity $\hat{\phi}/\rho$, for example, appears to "curl" around a line current source on the z axis, yet students are often perplexed by

the fact that the curl is actually zero everywhere except at $\rho = 0$. With differential forms, on the other hand, the exterior derivative of a one-form can be given a very simple graphical representation. As shown in Fig. 5, tubes of the two-form $\mathcal{J}$ in Ampère's law produce surfaces of the one-form $\mathcal{H}$. Away from the tubes of the source, no new surfaces of the one-form $\mathcal{H}$ are created, so the curl is zero. This is in exact analogy with the picture given for Gauss's law in Fig. 4, but with the dimensionality of the objects reduced by one.
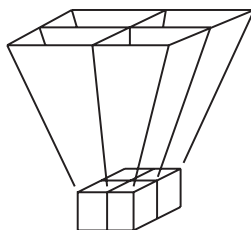
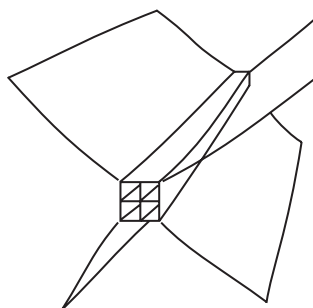**Figure 4**. Boxes of the three-form $\mathcal{Q}$ produce tubes of the two-form $\mathcal{D}$.

Figure 5. Tubes of the two-form $\mathcal{J}$ produce surfaces of the one-form $\mathcal{H}$.

3.   Electromagnetics in Two Space Dimensions

The vector calculus is rather inflexible in responding to changes to the dimensionality of the underlying manifold on which the calculus is defined. The reason for this is that the vector calculus as normally constituted in three space dimensions relies heavily on an isomorphism between two vector spaces. These spaces are first, that spanned by three-dimensional vectors, and second, the space spanned by vector or cross products of pairs of vectors. That these two spaces are actually distinct is most easily seen by considering vectors under a change of variables from a right-handed to a left-handed coordinate system. A vector remains unchanged, regardless of the coordinate system, whereas the cross product of two vectors changes sign.

While in applications this fact is unimportant, because we agree to use always right-handed coordinates, it hints of a deeper issue in regards to the vector calculus. In dimensions other than three, the two spaces that are identified as one within the vector calculus are no longer isomorphic. Thus, the vector calculus has a unique association with three-dimensional spaces.

Because of this, vector expressions for systems with reduced dimensionality often have an unnatural appearance and are rarely given in textbooks. Consider, for example, the surface current density vector field. If the units of this quantity are Ampères per meter, then it should be possible to integrate this vector over a path to find the total amount of current flowing *across* the path. But the total current is not given by the usual path integral of a vector. Rather, it involves a differential normal vector to the path, which is

the cross product of the differential tangent vector and a normal vector for the surface in which the current flows.

In the larger picture of tensor analysis, the two isomorphic vector spaces unfold into distinct spaces, leading to a mathematically more natural calculus, within which the dimensionality of the formulation of Maxwell's equations can be readily increased or decreased. The calculus of differential forms as a subset of tensor analysis is simple enough to allow for easy calculations without tedious index manipulations, but is general enough that it does not rely on this three-dimensional isomorphism, and so does not suffer from some of the limitations of vector calculus, including difficulty in changing the dimensionality of the underlying space from three to two or four.

In this section, we develop Maxwell's equations in the 2+1 representation, and then consider the four-dimensional or spacetime formulation in Sec. 4.

3.1. Exterior Calculus in Two Dimensions

In order to analyze electromagnetic fields in a two-dimensional space, we first develop the exterior calculus in R^2. The basic differential forms in the rectangular coordinate system are

Degree	Basis form(s)	Integration region
zero-forms	1	Point
one-forms	$\mathrm{d}x$, $\mathrm{d}y$	Line
two-forms	$\mathrm{d}x \wedge \mathrm{d}y$	Area

The exterior derivative operator is

$$\mathrm{d} = \left[\frac{\partial}{\partial x}\mathrm{d}x + \frac{\partial}{\partial y}\mathrm{d}y \right] \wedge \tag{8}$$

3.2. 2D Hodge Star Operator

The 2D Hodge star operator applied to zero-forms and two-forms acts as might be expected by analogy with R^3, so that

$$\star_2 1 = \mathrm{d}x \wedge \mathrm{d}y, \quad \star_2 \, \mathrm{d}x \wedge \mathrm{d}y = 1 \tag{9}$$

where we employ the subscript 2 to indicate that the operator here is associated with a two-dimensional underlying space, rather than three-dimensional space as in Eq. (6).

In order to determine the action of the 2D Hodge star operator on one-forms, we must resort to the definition of the operator. For a p-form α, $\star_2 \alpha$ is the unique $(n-p)$-form such that the relationship

$$\alpha \wedge \beta = \langle \star_2 \alpha, \beta \rangle \sigma \tag{10}$$

holds for any $(n-p)$-form β. In this expression, σ is the volume element of the space, which in rectangular coordinates is $\mathrm{d}x \wedge \mathrm{d}y$, and $\langle \cdot, \cdot \rangle$ denotes the inner product induced by the underlying metric, which is in this case the Euclidean metric.

Since $n = 2$, the star operator maps one-forms to one-forms. Considering $\alpha = \mathrm{d}x$, we must have that

$$\mathrm{d}x \wedge \beta = \langle \star_2 \, \mathrm{d}x, \beta \rangle \, \mathrm{d}x \wedge \mathrm{d}y \tag{11}$$

where β is an arbitrary one-form. In the Euclidean metric, $\langle \mathrm{d}x, \mathrm{d}x \rangle = \langle \mathrm{d}y, \mathrm{d}y \rangle = 1$, and $\langle \mathrm{d}x, \mathrm{d}y \rangle = 0$. If we choose $\beta = \mathrm{d}x$, Eq. (11) shows that $\star_2 \, \mathrm{d}x$ cannot have a $\mathrm{d}x$ component, because the left-hand side

vanishes, whereas on the right the inner product of dx with a differential form having a dx component is nonzero. As a consequence, we must have that $\star_2\,dx = a\,dy$, where a is a constant. Choosing $\beta = dy$ leads to

$$dx \wedge dy = a \underbrace{\langle dy, dy \rangle}_{=1} dx \wedge dy \tag{12}$$

which requires that $a = 1$. Proceeding similarly for dy, we find that $\star_2\,dy = -dx$. Summarizing, for one-forms,

$$\star_2 dx = dy, \quad \star_2\,dy = -dx \tag{13}$$

and $\star_2\star_2\alpha = -\alpha$ for a 1-form α. A one-form in 2D is graphically represented as a family of lines. We can see from these expressions that the Hodge star operator maps a one-form to a new one-form represented by orthogonal lines [28].

3.3. Maxwell's Equations

It is straightforward to show that Maxwell's equations in R^3 for sources and materials with a translational invariance in the z direction reduce to the following 2D equations:

$$d\mathcal{E} = -\frac{\partial}{\partial t}\mathcal{B} \tag{14a}$$

$$d\mathcal{H} = \frac{\partial}{\partial t}\mathcal{D} + \mathcal{J} \tag{14b}$$

$$d\mathcal{D} = Q \tag{14c}$$

$$d\mathcal{B} = 0 \tag{14d}$$

with constitutive relations

$$\mathcal{D} = \epsilon\star_2\mathcal{E} \tag{15a}$$

$$\mathcal{B} = \mu\star_2\mathcal{H} \tag{15b}$$

These expressions are of precisely the same form as Maxwell's equations in 3D as given in Eqs. (6), but with the exterior derivative and Hodge star operators interpreted according to Eqs. (8) and (13). We observe that Maxwell's equations are form-invariant with respect to the dimensionality of the underlying space when expressed using the calculus of differential forms.

These relationships for the 2D fields can be obtained from their 3D counterparts by expanding the equations in components, forming the interior product or contraction from the left with $\hat{z}$ to remove the dz differentials, and then applying the definitions of the 2D exterior derivative and Hodge star operators.

3.4. 2D Electric and Magnetic Fields as Differential Forms

The electric field intensity $\mathcal{E}$ remains a one-form, since it must represent a potential field for a charge that is free to move in x and y. As expected, $\mathcal{H}$ and $\mathcal{B}$ are scalar fields, representing the z-components of the 3D field quantities. From Faraday's law, we can see that $\mathcal{B}$ is a two-form, and from the magnetic constitutive relation, $\mathcal{H}$ is a zero-form.

It can be seen that the fields correspond to the three-dimensional TE^z polarization, because the electric field can be viewed as the x and y components of the electric field radiated by a cylindrical current source that is translationally invariant in the z direction. In a two-dimensional space, charges cannot move in the z direction, so the TM^z polarization does not exist in a truly 2D world. Alternately, if magnetic charges were allowed in the model, then the resulting field equations for a z-invariant magnetic current flowing in the x-y plane would be dual to those in Eqs. (14) (although it is interesting to note that when the behavior of Maxwell's equations under reflection of the coordinate system from left-handed to right-handed is rigorously considered, magnetic charges apparently have no place in the formalism [28]).

It is especially interesting to consider the electric flux density. From the action of the 2D Hodge star operator on one-forms in Eq. (13), in 2D the lines of the one-form $\mathcal{D}$ are perpendicular to the lines of $\mathcal{E}$. If D_x^{3D} and D_y^{3D} represent the components of the two-form electric flux density in three dimensions, then $\mathcal{D} = D^{3D} \lrcorner \hat{z} = -D_y^{3D}\mathrm{d}x + D_x^{3D}\mathrm{d}y$. In three dimensions, the tubes of the two-form $\mathcal{D}$ are orthogonal to the surfaces of $\mathcal{E}$. Also, just as the tubes of electric flux begin at positive point charges and end at negative point charge in three dimensions, the lines of $\mathcal{D}$ begin at positive line charges and end at negative line charges

3.5. Twisted Differential Forms

Because the physical direction of the flux is along the lines of the one-form $\mathcal{D}$, rather than perpendicular to it, $\mathcal{D}$ is a twisted form [19]. The one-form $\mathcal{J}$ is also a twisted form, and its components are related to those of the 3D two-form in the same way as was given for the electric flux density.

This difference in orientation between $\mathcal{E}$ and $\mathcal{D}$ is both mathematically and physically reasonable. Integration of any one-form form, whether twisted or nontwisted, counts the number of lines pierced by the integration path. For the electric field intensity, integration over a path gives the potential change *along* the path. For the flux density one-form $\mathcal{D}$, the path integral is the amount of flux that flows perpendicularly *across* the integration path.

3.5.1. Twisted Forms in the 3D Representation

In the three-dimensional representation, one can largely avoid the need to consider twisted quantities explicitly. It is only when reducing to 2D that twisted forms necessarily arise, as in the case of the surface current density at a boundary [19]. But some of the field quantities in the 3D representation of Maxwell's equations are actually twisted differential forms [28, 26], as outlined in Table 2. In the 3D case, a one-form is oriented by a normal direction, whereas a twisted one-form is oriented by a two-form which represents a "screw sense." A two-form is also oriented by a screw sense, and a twisted two-form is oriented by the direction of flow along the tubes of the two-form. As a consequence of this, both $\mathcal{D}$ and $\mathcal{E}$ are oriented by the same direction, and $\mathcal{H}$ and $\mathcal{B}$ are both oriented by the same screw sense. Twisted three-forms are oriented with a sign, which in the case of the electric charge density $\mathcal{Q}$ represents the sign of the charge. The Hodge star operator most naturally maps twisted forms to nontwisted forms and vice versa, so it maps diagonally across Table 2.

A twisted differential form can be oriented in either of two senses. For example, if the magnetic field in 3D above the $z = 0$ plane is $\mathrm{d}x$ and is zero below, then the twisted surface current density one-form is $\mathcal{J}_s = \mathrm{d}x$ and is oriented in the $+\mathrm{d}y$ direction (where $\mathrm{d}y$ is a nontwisted one-form). Alternately, the 3D electric flux density $D^{3D} = \mathrm{d}z \wedge \mathrm{d}x$ corresponds to $\mathcal{D} = -\mathrm{d}x$ in the 2D formulation of this paper, but with orientation also in the $+\mathrm{d}y$ direction. The difference can be accounted for by the fact that the

Table 2. Twisted and nontwisted differential forms in the 3D representation of Maxwell's equations.

<u>Zero-form</u>
ϕ (electric potential)

<u>Twisted zero-form</u>

<u>One-form</u>
(dual to polar vector,
oriented by a direction)
$\mathcal{E}$

<u>Twisted one-form</u>
(dual to axial or pseudovector,
oriented by a screw sense)
$\mathcal{H}$

<u>Two-form</u>
(dual to axial or pseudovector,
oriented by a screw sense)
$\mathcal{B}$

<u>Twisted two-form</u>
(dual to polar vector,
oriented by a direction)
$\mathcal{D}, \mathcal{J}$

<u>Three-form</u>
(oriented by a screw sense)

<u>Twisted three-form</u>
(oriented by a sign)
$\mathcal{Q}$

surface current density is obtained by pullback or boundary restriction, whereas the 2D electric flux density one-form is obtained by removing a differential to lower the dimensionality of the field. In the former case, the 2D orientation can be obtained mathematically using Eq. (39) in [19]. In the notation of Burke [29], if the p-form α_s is the pullback of α to a surface in R^3, then the outer orientation of α_s is the orientation of the nontwisted $(2-p)$-form $\{\alpha_s, \sigma_s\}$ which satisfies

$$\{\alpha_s, \sigma_s\} \wedge n = \{\alpha, \sigma\} \tag{16}$$

where σ_s is the volume element of the surface, σ is the volume element of R^3, and $\{\alpha, \sigma\}$ is the outer orientation of α. For the example given above, $\alpha = H_2 - H_1 = \mathrm{d}x$, $\{\alpha, \sigma\} = \mathrm{d}y \wedge \mathrm{d}z$, and $n = \mathrm{d}z$, from which we have that the outer orientation of $\alpha_s = J_s = \mathrm{d}x$ is $\{\alpha_s, \sigma_s\} = +\mathrm{d}y$, as expected. In the latter case, the 2D orientation can be obtained by pullback or boundary restriction of the 3D outer orientation of D^{3D}. For the example, the outer orientation of $D^{3D} = \mathrm{d}z \wedge \mathrm{d}x$ is $\{D^{3D}, \sigma\} = \mathrm{d}y$. The outer orientation $\{D, \sigma_s\}$ of $D = -\mathrm{d}x$ is the pullback of $\mathrm{d}y$ to the $z = 0$ plane, which is also $+\mathrm{d}y$.

3.6.   Poynting's Theorem

In 2D, the Poynting vector is $\mathcal{S} = \mathcal{E} \wedge \mathcal{H}$, which is a one-form. As with the electric flux density one-form $\mathcal{D}$, the direction of the power flux represented by $\mathcal{S}$ is along the lines of the one-form $\mathcal{S}$. Mathematically, $\mathcal{S}$ is a twisted form, because $\mathcal{H}$ is twisted and causes the product $\mathcal{E} \wedge \mathcal{H}$ to be twisted also. Since the integral of $\mathcal{S}$ along a path that pierces surfaces of $\mathcal{S}$ represents flow of energy *across* the path rather than along it, it is clear that the physical orientation of the power flux represented by $\mathcal{S}$ must be along the lines of the one-form, instead of perpendicular to the lines as is the case with a nontwisted one-form.

Because of this difference, in the vector formulation of the 2D Poynting's theorem the integral of the vector dual to $\mathcal{S}$ is not the usual vector path integral with a differential tangent vector, but involves a differential vector which is normal to the path. Expressed using differential forms, on the other hand,

Poynting's theorem is quite natural:

$$\oint_P \mathcal{S} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_A (\mathcal{H} \wedge \mathcal{B} + \mathcal{E} \wedge \mathcal{D}) - \int_A \mathcal{E} \wedge \mathcal{J} \tag{17}$$

where the closed path P bounds the area A. As with Maxwell's equations, this expression has precisely the same form as in the three-dimensional case. The only difference is that the dimensionality of the integrations and the orders of the differential forms $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ have decreased by one.

3.7. Wave Equation

We now derive the wave equation for the electric field in two-dimensional space. Applying $\mathrm{d}\star_2$ to both sides of Faraday's law,

$$
\begin{aligned}
\mathrm{d}\star_2\mathrm{d}\mathcal{E} &= -\frac{\partial}{\partial t}\mathrm{d}\star_2\mathcal{B} \\
&= -\mu\frac{\partial}{\partial t}\mathrm{d}\mathcal{H}.
\end{aligned}
$$

Substituting for $\mathcal{H}$ using Ampère's law leads to

$$\mathrm{d}\star_2\mathrm{d}\mathcal{E} = -\mu\frac{\partial^2}{\partial t^2}\mathcal{D} - \mu\frac{\partial}{\partial t}\mathcal{J} \tag{18}$$

Applying the Hodge star operator to both sides and using the constitutive relation for $\mathcal{D}$,

$$\star_2\mathrm{d}\star_2\mathrm{d}\mathcal{E} = \frac{1}{c^2}\frac{\partial^2\mathcal{E}}{\partial t^2} - \mu\star_2\frac{\partial\mathcal{J}}{\partial t} \tag{19}$$

where it is important to note the minus sign in $\star_2\mathcal{D} = -\epsilon\mathcal{E}$, which arises due to Eq. (13).

The Laplace–de Rham operator is defined by

$$\Delta\alpha = (-1)^{n(p+1)}\left[(-1)^n\star_2\mathrm{d}\star_2\mathrm{d} + \mathrm{d}\star_2\mathrm{d}\star_2\right]\alpha \tag{20}$$

For $n = 2$, the signs are always positive, so we have

$$\Delta\alpha = \left[\star_2\mathrm{d}\star_2\mathrm{d} + \mathrm{d}\star_2\mathrm{d}\star_2\right]\alpha \tag{21}$$

It is easy to check by calculating in components that this operator acts according to the familiar relationship

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{22}$$

on the components of a differential form.

Using the identity (20) in Eq. (19), and assuming that the charge density two-form $\mathcal{Q}$ is zero, we obtain the wave equation

$$\left[\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathcal{E} = -\mu\star_2\frac{\partial\mathcal{J}}{\partial t} \tag{23}$$

This equation is identical in form to the 3D wave equation, except that the sign of the source term on the right-hand side is different. This difference in sign is accounted for by the relationship $\mathcal{J} = J^{3D}\lrcorner\hat{z} = -J_y^{3D}\mathrm{d}x + J_x^{3D}\mathrm{d}y$ in moving from the 3D to the 2D representation of the current together with the action of the 2D Hodge star operator on one-forms.

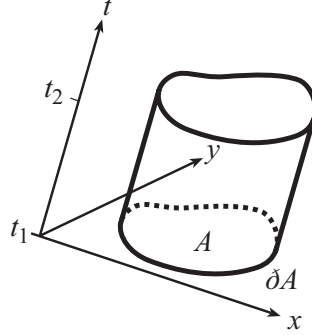# 4. The Four-dimensional Representation of Maxwell's Equations

Figure 6. Faraday's law in 4-space.

In four-dimensional Minkowski space Maxwell's equations assume an extremely compact form [5, 30]. To introduce the four-dimensional representation of Maxwell's equations we first consider the integral form of Faraday's law in the three-dimensional representation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_A \mathcal{B} = - \int_{\partial A} \mathcal{E} \,. \tag{24}$$

We apply Faraday's law to a surface A as depicted in Figure 24. Integrating over the time interval $T = [t_1, t_2]$ and considering the time t as a fourth coordinate in addition to the three spatial coordinates yields

$$\int_A \mathcal{B} \Big|_{t_2} - \int_A \mathcal{B} \Big|_{t_1} + \int_{\partial A \times T} \mathcal{E} \wedge \mathrm{d}t = 0 \,. \tag{25}$$

The domain of integration in the four-space formed by the coordinates x, y, z, t is the two-dimensional surface of the three-dimensional cylinder generated by translating the surface A along the time-axis from t_1 to t_2. In the first and second integrals in the above equation the integration is performed over the top and bottom surfaces of the cylinder. The third integral is performed over the side surface of the cylinder. In Fig. 24 we have chosen the surface A in the xy-plane, but A may be any two-dimensional surface embedded in 3-space.

We now introduce the two-form $\mathcal{F}$ on four-dimensional space,

$$\mathcal{F} = E_x \mathrm{d}x \wedge \mathrm{d}t + E_y \mathrm{d}y \wedge \mathrm{d}t + E_z \mathrm{d}z \wedge \mathrm{d}t \tag{26}$$

$$+ B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y \,.$$

This two-form is called the *Faraday form*. With the Faraday form we can write (25) as

$$\int_{\partial V} \mathcal{F} = 0 \tag{27}$$

where the boundary of the cylinder ∂V is composed by $\partial A \times t_1 \cup \partial A \times t_2 \cup A \times [t_1, t_2]$. We easily can see that in the integral (27) the first three terms of (26), i.e. the electric field terms contribute to the integral over the side wall of the cylinder, whereas the second three terms, namely the magnetic field terms contribute to the integration over the bottom and top surfaces. The above equation simply states that the flux described

by the Faraday form is free of divergence in the four-space. Integration over the boundary of a volume which is extended in time yields Faraday's law. Integration over a space-like three-dimensional volume which is extended in all three space dimensions but not in time shows that the magnetic induction or flux density is free of divergence.

In a similar way we can construct the *Maxwell form* by combining the electric displacement form $\mathcal{D}$ and the magnetic field form $\mathcal{H}$ to a two-form

$$\mathcal{G} = D_x \mathrm{d}y \wedge \mathrm{d}z + D_y \mathrm{d}z \wedge \mathrm{d}x + D_z \mathrm{d}x \wedge \mathrm{d}y \tag{28}$$
$$- H_x \mathrm{d}x \wedge \mathrm{d}t - H_y \mathrm{d}y \wedge \mathrm{d}t - H_z \mathrm{d}z \wedge \mathrm{d}t$$

Before further proceeding in setting up the four-dimensional formulation of Maxwell's theory we consider a distinctive feature of the four-dimensional space. The four-dimensional *Minkowski space* exhibits a metric which is not positive definite. In Minkowski space the distance s between two one-forms

$$\mathcal{A} = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z + A_t \mathrm{d}t \,, \tag{29a}$$

$$\mathcal{B} = B_x \mathrm{d}x + B_y \mathrm{d}y + B_z \mathrm{d}z + B_t \mathrm{d}t \tag{29b}$$

is given by

$$\mathcal{A} \lrcorner \mathcal{B} = \star_4 (\mathcal{A} \wedge \star_4 \mathcal{B}) = A_x B_x + A_y B_y + A_z B_z - A_t B_t / c^2 \tag{30}$$

where c ist the free-space speed of light. This definition follows from special relativity and means that points on the light cone exhibit zero distance in Minkowski space. We therefore have to define a suitable star operator. This requirement is satisfied by defining the four-dimensional Hodge operator for Minkowski space as follows: For zero-forms and four-forms,

$$\star_4 (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge c\,\mathrm{d}t) = -1 \,, \quad \star_4 1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge c\,\mathrm{d}t \tag{31}$$

For one-forms and three-forms,

$$\star_4 (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = -c\,\mathrm{d}t \,, \qquad \star_4 \, c\,\mathrm{d}t = -\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

$$\star_4 (\mathrm{d}y \wedge \mathrm{d}z \wedge c\mathrm{d}t) = -\mathrm{d}x \,, \qquad \star_4 \, \mathrm{d}x = -c\,\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z$$

$$\star_4 (\mathrm{d}z \wedge \mathrm{d}x \wedge c\mathrm{d}t) = -\mathrm{d}y \,, \qquad \star_4 \, \mathrm{d}y = -c\,\mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}x \tag{32}$$

$$\star_4 (\mathrm{d}x \wedge \mathrm{d}y \wedge c\mathrm{d}t) = -\,\mathrm{d}z \,, \qquad \star_4 \, \mathrm{d}z = -c\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y$$

For two-forms,

$$\star_4 (\mathrm{d}y \wedge \mathrm{d}z) = -\mathrm{d}x \wedge c\mathrm{d}t \,, \qquad \star_4 (\mathrm{d}x \wedge c\mathrm{d}t) = \mathrm{d}y \wedge \mathrm{d}z$$

$$\star_4 (\mathrm{d}z \wedge \mathrm{d}x) = -\mathrm{d}y \wedge c\mathrm{d}t \,, \qquad \star_4 (\mathrm{d}y \wedge c\mathrm{d}t) = \mathrm{d}z \wedge \mathrm{d}x \tag{33}$$

$$\star_4 (\mathrm{d}x \wedge \mathrm{d}y) = -\mathrm{d}z \wedge c\mathrm{d}t \,, \qquad \star_4 (\mathrm{d}y \wedge c\mathrm{d}t) = \mathrm{d}x \wedge \mathrm{d}y$$

The four-dimensional Hodge operator satisfies

$$\star_4 \star_4 = \pm 1 \tag{34a}$$

$$\star_4 \star_4 \star_4 \star_4 = 1 \tag{34b}$$

$$\star_4^{-1} = \star_4 \star_4 \star_4 \tag{34c}$$

The exterior derivative is given by

$$\mathrm{d} = \left[\frac{\partial}{\partial x}\mathrm{d}x + \frac{\partial}{\partial y}\mathrm{d}y + \frac{\partial}{\partial z}\mathrm{d}z + \frac{\partial}{\partial t}\mathrm{d}t \right] \tag{35}$$

The Stokes' theorem also holds in four dimension. Applying it to (24) yields

$$\mathrm{d}\mathcal{F} = 0 \tag{36}$$

In four-space the constitutive relations become

$$\mathcal{G} = \sqrt{\frac{\epsilon_0}{\mu_0}} \star_4 \mathcal{F} \tag{37}$$

The factor (ϵ_0/μ_0) is due to the system of units used here and has no physical meaning. Poincaré's lemma states that if the exterior derivative of a p-form is zero, on a simply connected space it can be expressed as an exterior derivative of some $(p-1)$-form. Since the exterior derivative of $\mathcal{F}$ vanishes, we therefore have that

$$\mathcal{F} = \mathrm{d}\mathcal{A} \tag{38}$$

with the four-vector potential form

$$\mathcal{A} = A_x\mathrm{d}x + A_y\mathrm{d}y + A_z\mathrm{d}z + c\,\Phi\mathrm{d}t. \tag{39}$$

where A_x, A_y, A_z are the components of the magnetic vector potential and Φ is the scalar electric potential.

To develop a four-dimensional formulation of Ampère's law and the divergence law for the electric flux density we introduce the four-current differential form

$$\begin{aligned} \mathcal{J}_4 &= -J_x\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t - J_y\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}t \\ &\quad -J_z\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \rho\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned} \tag{40}$$

where J_x, J_y, J_z are the components of the electric current and ρ is the electric charge. The four-dimensional formulation of Ampère's law and the divergence law is

$$\mathrm{d}\mathcal{G} = \mathcal{J}_4 \tag{41}$$

In integral form this can be expressed as

$$\int_{\partial V} \mathcal{G} = \int_V \mathcal{J}_4 \tag{42}$$

From (37) and (41) we obtain

$$\sqrt{\frac{\epsilon_0}{\mu_0}}\mathrm{d}\star_4\,\mathrm{d}\mathcal{A} = \mathcal{J}_4 \tag{43}$$

so that in the four-space formulation the four-current $\mathcal{J}_4$ is the source of the Maxwell form $\mathcal{G}$.

We introduce the *coderivative* as

$$\tilde{\mathrm{d}} = (-1)^k \star_4^{-1} \mathrm{d} \star_4 \tag{44}$$

where k is the degree of the form on which the operator acts. We define the *wave operator*

$$\square = d\tilde{d} + \tilde{d}d. \tag{45}$$

Using the gauge freedom in $\mathcal{A}$ to choose $d\star_4\mathcal{A} = 0$ (Lorenz gauge), we can then write the *wave equation*

$$\square\mathcal{A} = \sqrt{\frac{\mu_0}{\epsilon_0}}\star_4\mathcal{J}_4 \tag{46}$$

In four-dimensional spacetime with the coordinates t, x, y and z for a zero-form f the wave operator is

$$\square f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \tag{47}$$

In order to illustrate how four-dimensional structures can be visualized by projections, Figure 7(a) gives a representation of a four-dimensional cubical space domain which is bounded by eight three-dimensional right parallelipipeds, four of which are depicted in Figures 7. Each of these parallelipipeds is bounded by six two-dimensional surfaces.
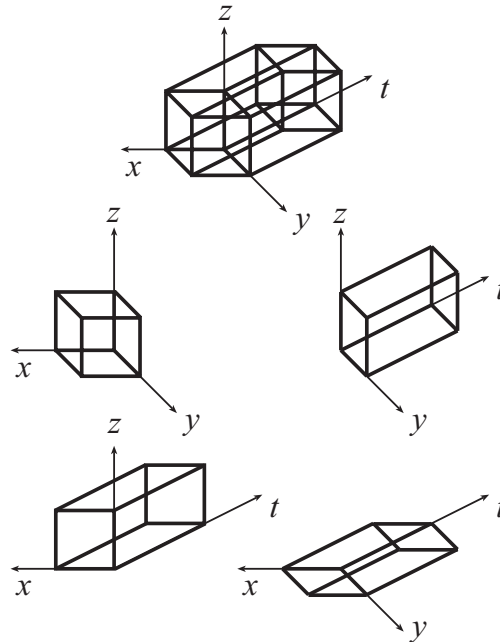

Figure 7. Top: Visualizing a hypercubical element in 4-space by projections. The faces of the hypercube are three-dimensional right parallelipipeds, several of which are shown below in the figure.

4.1.   Four-Dimensional View of Plane Wave Fields

As an example let us discuss the propagation of a plane wave in four-space. We consider a plane wave described by the vector potential form

$$\mathcal{A} = A_x(z,t)dx = u(z,t)dx \tag{48}$$

From this we derive the Faraday form

$$\mathcal{F} = \frac{\partial A_x(z,t)}{\partial t}dt \wedge dx + \frac{\partial A_x(z,t)}{\partial z}dz \wedge dx \tag{49}$$

By looking at the component notation we can see that this Faraday form describes a plane wave with an electric field component in the x-direction and a magnetic field component in the y-direction. The corresponding Faraday form is

$$\mathcal{F} = E_x \mathrm{d}x \wedge \mathrm{d}t + B_y \mathrm{d}z \wedge \mathrm{d}x \tag{50}$$

Inserting the ansatz (48) in the homogeneous wave equation

$$\Box \mathcal{A} = 0 \tag{51}$$

yields

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} = 0 \tag{52}$$

This can be solved by changing variables to $p = z - ct$, $q = z + ct$ to obtain the transformed partial differential equation

$$\frac{\partial^2 u}{\partial p \partial q} = 0 \tag{53}$$

From this relationship, it is apparent that we can express the solution u in the form

$$u = u_1(p) + u_2(q) \tag{54}$$

Finally, transforming back to the original space and time coordinates leads to the general solution

$$u = u_1(z - ct) + u_2(z + ct) \tag{55}$$

for the coefficient of the potential in Eq. (48). Choosing a solution such that $u_2 = 0$, we can obtain the components of the Faraday form using Eqs. (49) as

$$\mathcal{F} = u'(-c\mathrm{d}t \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}x) \tag{56}$$

Using the constitutive relation (37) we can find the Maxwell form,

$$\mathcal{G} = \sqrt{\frac{\epsilon_0}{\mu_0}} u'(\mathrm{d}y \wedge \mathrm{d}z - c\mathrm{d}y \wedge \mathrm{d}t) \tag{57}$$

The 4D Faraday and Maxwell forms provide a graphical representation of the fields associated with wave propagation. $\mathcal{F}$ can be placed in the form

$$\mathcal{F} = u'(-c\mathrm{d}t + \mathrm{d}z) \wedge \mathrm{d}x \tag{58}$$

which shows that this 2-form consists of surfaces perpendicular to the x axis together with surfaces perpendicular to the $-ct + z$ axis. Similarly,

$$\mathcal{G} = \sqrt{\frac{\epsilon_0}{\mu_0}} u' \mathrm{d}y \wedge (-c\mathrm{d}t + \mathrm{d}z) \tag{59}$$

which leads to surfaces perpendicular to the y axis and surfaces perpendicular to the $-ct+z$ axis. Projections of these differential forms are shown in Fig. 8. It can be seen that in these projections, the tubes of both two-forms are oriented in the $ct + z$ direction. If one further projects $\mathcal{F}$ onto a spacelike surface in the x-z plane, only the $B_y \mathrm{d}z \wedge \mathrm{d}x$ term remains. When projected onto a time-like surface in the x-t plane, $E_x \mathrm{d}x \wedge \mathrm{d}t$ is obtained. Similar behavior can be seen for the Maxwell form $\mathcal{G}$.
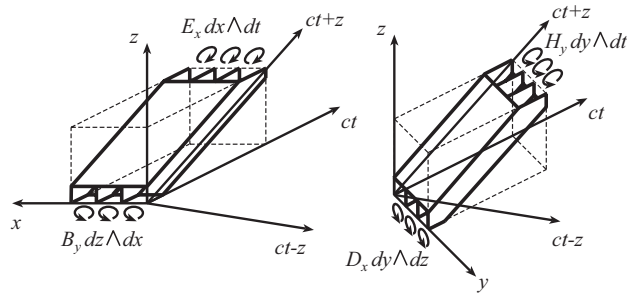
Figure 8. Three-dimensional projections of the 4D Faraday and Maxwell forms for a wave traveling in the z direction. The Faraday form reduces to the electric field intensity or magnetic flux density on timelike or spacelike 2D slices, whereas the Maxwell form reduces to magnetic field intensity and electric flux density.

5. Conclusion

In this paper, we have considered some of the pedagogical aspects of the exterior calculus of differential forms in electromagnetic theory. Relative to the vector calculus, differential forms notation simplifies many calculations and provides additional insight into the behavior of sources and fields through clear and simple graphical representations.

We have also dealt with electromagnetic field quantities and Maxwell's equations when formulated in a two-dimensional space with a time dimension, and in a four-dimensional representation including three space dimensions and a time axis together on an equal footing. These treatments illustrate the utility of the exterior calculus of differential forms in working with electromagnetic field theory in spatial dimensions other than three.

Finally, we remark that although this presentation was made in a way that highlighted differences between the vector calculus and the calculus of differential forms, one should not view users of the two notations as being in competition. Rather, one who is expert in the vector calculus can gain additional insight into the mathematics of electromagnetic fields by reconsidering already known principles in a different light using the alternate mathematical framework of differential forms.

Acknowledgement

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