Introduction to Methods of Applied Mathematics
or
Advanced Mathematical Methods for Scientists and Engineers

Sean Mauch
http://www.its.caltech.edu/~sean

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Chapter 5

Vector Calculus

5.1 Vector Functions

Vector-valued Functions. A vector-valued function, $\mathbf{r}(t)$, is a mapping $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^n$ that assigns a vector to each value of t.

$$\mathbf{r}(t) = r_1(t)\mathbf{e}_1 + \cdots + r_n(t)\mathbf{e}_n.$$

An example of a vector-valued function is the position of an object in space as a function of time. The function is continous at a point $t = \tau$ if

$$\lim_{t \to \tau} \mathbf{r}(t) = \mathbf{r}(\tau).$$

This occurs if and only if the component functions are continuous. The function is differentiable if

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

exists. This occurs if and only if the component functions are differentiable.

If $\mathbf{r}(t)$ represents the position of a particle at time t, then the velocity and acceleration of the particle are

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2},$$

respectively. The speed of the particle is $|\mathbf{r}'(t)|$.

Differentiation Formulas. Let $\mathbf{f}(t)$ and $\mathbf{g}(t)$ be vector functions and $a(t)$ be a scalar function. By writing out components you can verify the differentiation formulas:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f} \times \mathbf{g}) = \mathbf{f}' \times \mathbf{g} + \mathbf{f} \times \mathbf{g}'$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(a\mathbf{f}) = a'\mathbf{f} + a\mathbf{f}'$$

5.2 Gradient, Divergence and Curl

Scalar and Vector Fields. A *scalar field* is a function of position $u(\mathbf{x})$ that assigns a scalar to each point in space. A function that gives the temperature of a material is an example of a scalar field. In two dimensions, you can graph a scalar field as a surface plot, (Figure 5.1), with the vertical axis for the value of the function.

A *vector field* is a function of position $\mathbf{u}(\mathbf{x})$ that assigns a vector to each point in space. Examples of vectors fields are functions that give the acceleration due to gravity or the velocity of a fluid. You can graph a vector field in two or three dimension by drawing vectors at regularly spaced points. (See Figure 5.1 for a vector field in two dimensions.)

Partial Derivatives of Scalar Fields. Consider a scalar field $u(\mathbf{x})$. The *partial derivative* of u with respect to x_k is the derivative of u in which x_k is considered to be a variable and the remaining arguments are considered to be parameters. The partial derivative is denoted $\frac{\partial}{\partial x_k}u(\mathbf{x})$, $\frac{\partial u}{\partial x_k}$ or u_{x_k} and is defined

$$\frac{\partial u}{\partial x_k} \equiv \lim_{\Delta x \to 0} \frac{u(x_1, \ldots, x_k + \Delta x, \ldots, x_n) - u(x_1, \ldots, x_k, \ldots, x_n)}{\Delta x}.$$

Partial derivatives have the same differentiation formulas as ordinary derivatives.

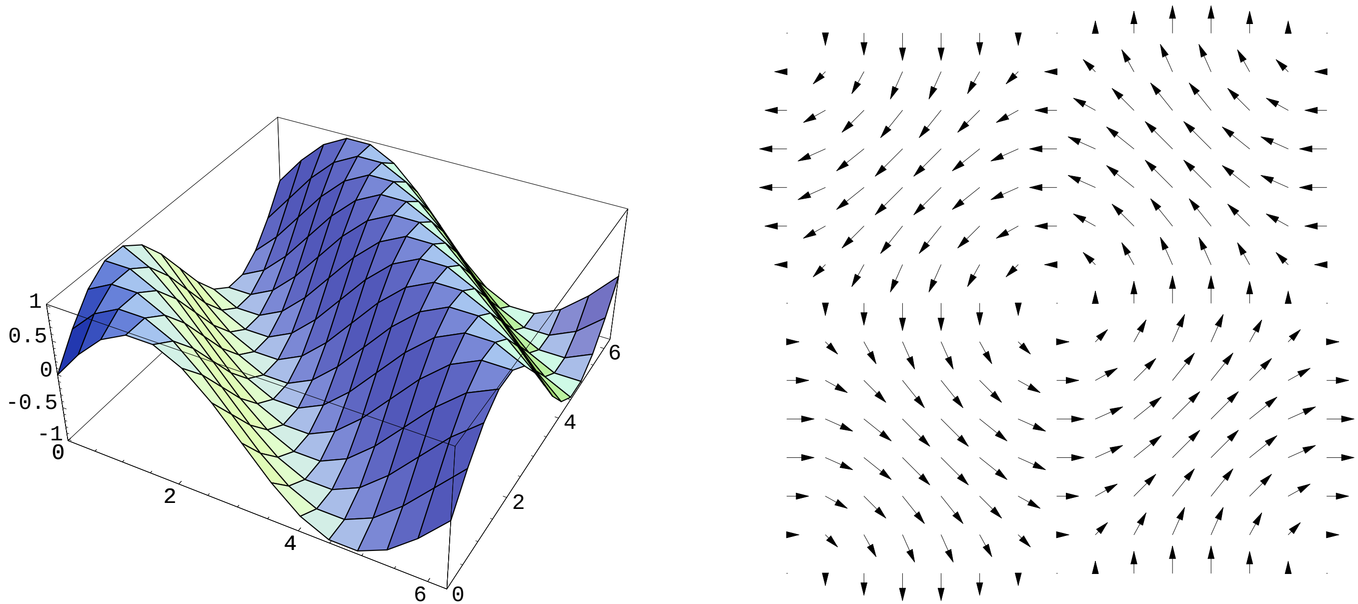

Figure 5.1: A Scalar Field and a Vector Field

Consider a scalar field in $\mathbb{R}^3$, $u(x, y, z)$. Higher derivatives of u are denoted:

$$u_{xx} \equiv \frac{\partial^2 u}{\partial x^2} \equiv \frac{\partial}{\partial x}\frac{\partial u}{\partial x},$$
$$u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\frac{\partial u}{\partial y},$$
$$u_{xxyz} \equiv \frac{\partial^4 u}{\partial x^2 \partial y \partial z} \equiv \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y}\frac{\partial u}{\partial z}.$$

If u_{xy} and u_{yx} are continuous, then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

This is referred to as the *equality of mixed partial derivatives*.

Partial Derivatives of Vector Fields. Consider a vector field $\mathbf{u}(\mathbf{x})$. The partial derivative of $\mathbf{u}$ with respect to x_k is denoted $\frac{\partial}{\partial x_k}\mathbf{u}(\mathbf{x})$, $\frac{\partial \mathbf{u}}{\partial x_k}$ or $\mathbf{u}_{x_k}$ and is defined

$$\frac{\partial \mathbf{u}}{\partial x_k} \equiv \lim_{\Delta x \to 0} \frac{\mathbf{u}(x_1, \ldots, x_k + \Delta x, \ldots, x_n) - \mathbf{u}(x_1, \ldots, x_k, \ldots, x_n)}{\Delta x}.$$

Partial derivatives of vector fields have the same differentiation formulas as ordinary derivatives.

Gradient. We introduce the vector differential operator,

$$\nabla \equiv \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n,$$

which is known as *del* or *nabla*. In $\mathbb{R}^3$ it is

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Let $u(\mathbf{x})$ be a differential scalar field. The *gradient* of u is,

$$\nabla u \equiv \frac{\partial u}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial u}{\partial x_n}\mathbf{e}_n,$$

Directional Derivative. Suppose you are standing on some terrain. The slope of the ground in a particular direction is the *directional derivative* of the elevation in that direction. Consider a differentiable scalar field, $u(\mathbf{x})$. The derivative of the function in the direction of the unit vector $\mathbf{a}$ is the rate of change of the function in that direction. Thus the directional derivative, $D_{\mathbf{a}}u$, is defined:

$$
\begin{aligned}
D_{\mathbf{a}}u(\mathbf{x}) &= \lim_{\epsilon\to 0} \frac{u(\mathbf{x}+\epsilon\mathbf{a}) - u(\mathbf{x})}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{u(x_1+\epsilon a_1,\dots,x_n+\epsilon a_n) - u(x_1,\dots,x_n)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{(u(\mathbf{x}) + \epsilon a_1 u_{x_1}(\mathbf{x}) + \cdots + \epsilon a_n u_{x_n}(\mathbf{x}) + \mathcal{O}(\epsilon^2)) - u(\mathbf{x})}{\epsilon} \\
&= a_1 u_{x_1}(\mathbf{x}) + \cdots + a_n u_{x_n}(\mathbf{x})
\end{aligned}
$$

$$D_{\mathbf{a}}u(\mathbf{x}) = \nabla u(\mathbf{x}) \cdot \mathbf{a}.$$

Tangent to a Surface. The gradient, ∇f, is orthogonal to the surface $f(\mathbf{x}) = 0$. Consider a point $\boldsymbol{\xi}$ on the surface. Let the differential $d\mathbf{r} = dx_1\mathbf{e}_1 + \cdots dx_n\mathbf{e}_n$ lie in the tangent plane at $\boldsymbol{\xi}$. Then

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n = 0$$

since $f(\mathbf{x}) = 0$ on the surface. Then

$$
\begin{aligned}
\nabla f \cdot d\mathbf{r} &= \left(\frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n\right) \cdot (dx_1\mathbf{e}_1 + \cdots + dx_n\mathbf{e}_n) \\
&= \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n \\
&= 0
\end{aligned}
$$

Thus ∇f is orthogonal to the tangent plane and hence to the surface.

Example 5.2.1 *Consider the paraboloid, $x^2 + y^2 - z = 0$. We want to find the tangent plane to the surface at the point $(1, 1, 2)$. The gradient is*

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}.$$

At the point $(1,1,2)$ *this is*

$$\nabla f(1,1,2) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

We know a point on the tangent plane, $(1,1,2)$*, and the normal,* $\nabla f(1,1,2)$*. The equation of the plane is*

$$\nabla f(1,1,2) \cdot (x,y,z) = \nabla f(1,1,2) \cdot (1,1,2)$$
$$\boxed{2x + 2y - z = 2}$$

The gradient of the function $f(\mathbf{x}) = 0$, $\nabla f(\mathbf{x})$, is in the direction of the maximum directional derivative. The magnitude of the gradient, $|\nabla f(\mathbf{x})|$, is the value of the directional derivative in that direction. To derive this, note that

$$D_{\mathbf{a}} f = \nabla f \cdot \mathbf{a} = |\nabla f| \cos\theta,$$

where θ is the angle between ∇f and $\mathbf{a}$. $D_{\mathbf{a}} f$ is maximum when $\theta = 0$, i.e. when $\mathbf{a}$ is the same direction as ∇f. In this direction, $D_{\mathbf{a}} f = |\nabla f|$. To use the elevation example, ∇f points in the uphill direction and $|\nabla f|$ is the uphill slope.

Example 5.2.2 *Suppose that the two surfaces* $f(\mathbf{x}) = 0$ *and* $g(\mathbf{x}) = 0$ *intersect at the point* $\mathbf{x} = \boldsymbol{\xi}$*. What is the angle between their tangent planes at that point? First we note that the angle between the tangent planes is by definition the angle between their normals. These normals are in the direction of* $\nabla f(\boldsymbol{\xi})$ *and* $\nabla g(\boldsymbol{\xi})$*. (We assume these are nonzero.) The angle,* θ*, between the tangent planes to the surfaces is*

$$\boxed{\theta = \arccos\left(\frac{\nabla f(\boldsymbol{\xi}) \cdot \nabla g(\boldsymbol{\xi})}{|\nabla f(\boldsymbol{\xi})|\,|\nabla g(\boldsymbol{\xi})|}\right).}$$

Example 5.2.3 *Let* u *be the distance from the origin:*

$$u(\mathbf{x}) = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_i x_i}.$$

In three dimensions, this is

$$u(x,y,z) = \sqrt{x^2 + y^2 + z^2}.$$

The gradient of u, $\nabla(\mathbf{x})$, is a unit vector in the direction of $\mathbf{x}$. The gradient is:

$$\nabla u(\mathbf{x}) = \left\langle \frac{x_1}{\sqrt{\mathbf{x}\cdot\mathbf{x}}}, \ldots, \frac{x_n}{\sqrt{\mathbf{x}\cdot\mathbf{x}}} \right\rangle = \frac{x_i \mathbf{e}_i}{\sqrt{x_j x_j}}.$$

In three dimensions, we have

$$\nabla u(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle.$$

This is a unit vector because the sum of the squared components sums to unity.

$$\nabla u \cdot \nabla u = \frac{x_i \mathbf{e}_i}{\sqrt{x_j x_j}} \cdot \frac{x_k \mathbf{e}_k}{\sqrt{x_l x_l}} \frac{x_i x_i}{x_j x_j} = 1$$

Figure 5.2 shows a plot of the vector field of ∇u in two dimensions.

Example 5.2.4 *Consider an ellipse. An implicit equation of an ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We can also express an ellipse as $u(x,y) + v(x,y) = c$ where u and v are the distance from the two foci. That is, an ellipse is the set of points such that the sum of the distances from the two foci is a constant. Let $\mathbf{n} = \nabla(u+v)$. This is a vector which is orthogonal to the ellipse when evaluated on the surface. Let $\mathbf{t}$ be a unit tangent to the surface. Since $\mathbf{n}$ and $\mathbf{t}$ are orthogonal,

$$\begin{gathered}\mathbf{n}\cdot\mathbf{t} = 0 \\ (\nabla u + \nabla v)\cdot\mathbf{t} = 0 \\ \nabla u \cdot \mathbf{t} = \nabla v \cdot (-\mathbf{t}).\end{gathered}$$

Since these are unit vectors, the angle between ∇u and $\mathbf{t}$ is equal to the angle between ∇v and $-\mathbf{t}$. In other words: If we draw rays from the foci to a point on the ellipse, the rays make equal angles with the ellipse. If the ellipse were

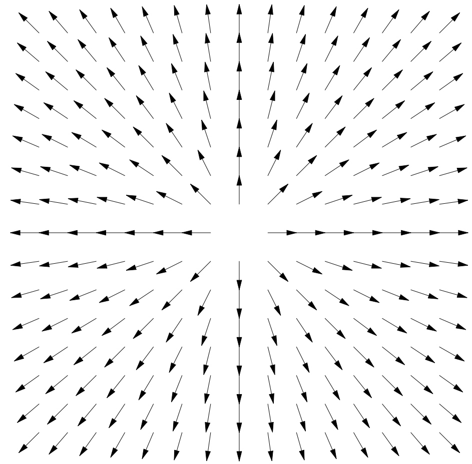

Figure 5.2: The gradient of the distance from the origin.

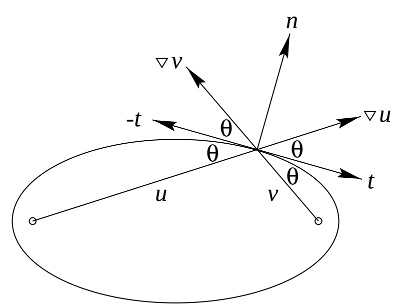

Figure 5.3: An ellipse and rays from the foci.

a reflective surface, a wave starting at one focus would be reflected from the ellipse and travel to the other focus. See Figure 5.3. This result also holds for ellipsoids, $u(x, y, z) + v(x, y, z) = c$.

We see that an ellipsoidal dish could be used to collect spherical waves, (waves emanating from a point). If the dish is shaped so that the source of the waves is located at one foci and a collector is placed at the second, then any wave starting at the source and reflecting off the dish will travel to the collector. See Figure 5.4.

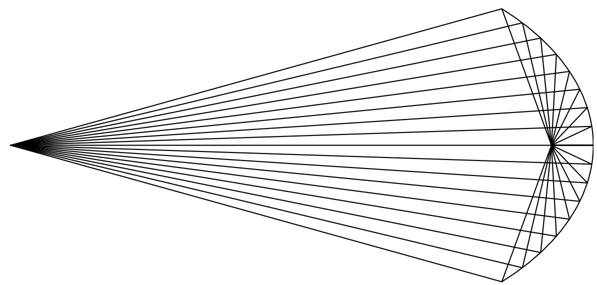

Figure 5.4: An elliptical dish.

5.3 Exercises

Vector Functions

Exercise 5.1
Consider the parametric curve

$$\mathbf{r} = \cos\left(\frac{t}{2}\right)\mathbf{i} + \sin\left(\frac{t}{2}\right)\mathbf{j}.$$

Calculate $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$. Plot the position and some velocity and acceleration vectors.
Hint, Solution

Exercise 5.2
Let $\mathbf{r}(t)$ be the position of an object moving with constant speed. Show that the acceleration of the object is orthogonal to the velocity of the object.
Hint, Solution

Vector Fields

Exercise 5.3
Consider the paraboloid $x^2 + y^2 - z = 0$. What is the angle between the two tangent planes that touch the surface at $(1, 1, 2)$ and $(1, -1, 2)$? What are the equations of the tangent planes at these points?
Hint, Solution

Exercise 5.4
Consider the paraboloid $x^2 + y^2 - z = 0$. What is the point on the paraboloid that is closest to $(1, 0, 0)$?
Hint, Solution

Exercise 5.5
Consider the region R defined by $x^2 + xy + y^2 \leq 9$. What is the volume of the solid obtained by rotating R about the y axis?

Is this the same as the volume of the solid obtained by rotating R about the x axis? Give geometric and algebraic explanations of this.

Hint, Solution

Exercise 5.6
Two cylinders of unit radius intersect at right angles as shown in Figure 5.5. What is the volume of the solid enclosed by the cylinders?

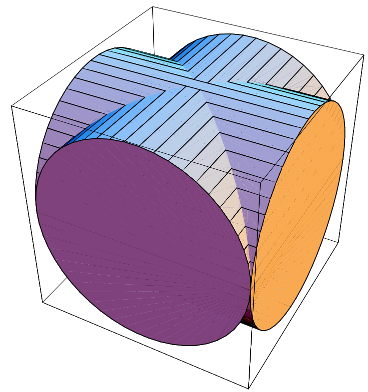

Figure 5.5: Two cylinders intersecting.

Hint, Solution

Exercise 5.7
Consider the curve $f(x) = 1/x$ on the interval $[1 \ldots \infty)$. Let S be the solid obtained by rotating $f(x)$ about the x axis. (See Figure 5.6.) Show that the length of $f(x)$ and the lateral area of S are infinite. Find the volume of S. [1]
Hint, Solution

Exercise 5.8
Suppose that a deposit of oil looks like a cone in the ground as illustrated in Figure 5.7. Suppose that the oil has a

[1] You could fill S with a finite amount of paint, but it would take an infinite amount of paint to cover its surface.

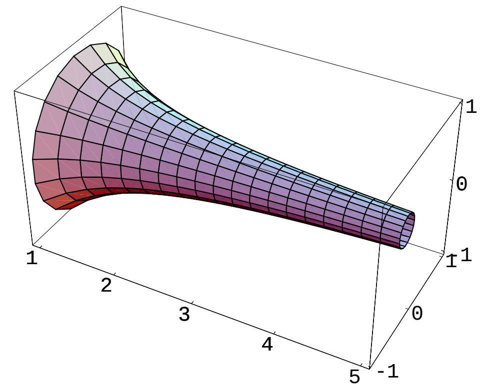

Figure 5.6: The rotation of $1/x$ about the x axis.

density of $800 \mathrm{kg}/\mathrm{m}^3$ and it's vertical depth is $12 \mathrm{m}$. How much work[2] would it take to get the oil to the surface.
Hint, Solution

Exercise 5.9
Find the area and volume of a sphere of radius R by integrating in spherical coordinates.
Hint, Solution

[2] Recall that work = force × distance and force = mass × acceleration.

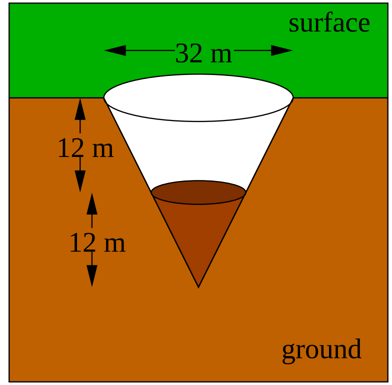

Figure 5.7: The oil deposit.

5.4 Hints

Vector Functions

Hint 5.1

Plot the velocity and acceleration vectors at regular intervals along the path of motion.

Hint 5.2

If $\mathbf{r}(t)$ has constant speed, then $|\mathbf{r}'(t)| = c$. The condition that the acceleration is orthogonal to the velocity can be stated mathematically in terms of the dot product, $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$. Write the condition of constant speed in terms of a dot product and go from there.

Vector Fields

Hint 5.3

The angle between two planes is the angle between the vectors orthogonal to the planes. The angle between the two vectors is

$$\theta = \arccos\left(\frac{\langle 2, 2, -1\rangle \cdot \langle 2, -2, -1\rangle}{|\langle 2, 2, -1\rangle||\langle 2, -2, -1\rangle|}\right)$$

The equation of a line orthogonal to $\mathbf{a}$ and passing through the point $\mathbf{b}$ is $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$.

Hint 5.4
Since the paraboloid is a differentiable surface, the normal to the surface at the closest point will be parallel to the vector from the closest point to $(1, 0, 0)$. We can express this using the gradient and the cross product. If (x, y, z) is the closest point on the paraboloid, then a vector orthogonal to the surface there is $\nabla f = \langle 2x, 2y, -1\rangle$. The vector from the surface to the point $(1, 0, 0)$ is $\langle 1 - x, -y, -z\rangle$. These two vectors are parallel if their cross product is zero.

Hint 5.5
CONTINUE

Hint 5.6
CONTINUE

Hint 5.7
CONTINUE

Hint 5.8
Start with the formula for the work required to move the oil to the surface. Integrate over the mass of the oil.

$$\text{Work} = \int (\text{acceleration}) \, (\text{distance}) \, \mathrm{d}(\text{mass})$$

Here (distance) is the distance of the differential of mass from the surface. The acceleration is that of gravity, g.

Hint 5.9
CONTINUE

5.5 Solutions

Vector Functions

Solution 5.1
The velocity is

$$\mathbf{r}' = -\frac{1}{2}\sin\left(\frac{t}{2}\right)\mathbf{i} + \frac{1}{2}\cos\left(\frac{t}{2}\right)\mathbf{j}.$$

The acceleration is

$$\mathbf{r}' = -\frac{1}{4}\cos\left(\frac{t}{2}\right)\mathbf{i} - \frac{1}{4}\sin\left(\frac{t}{2}\right)\mathbf{j}.$$

See Figure 5.8 for plots of position, velocity and acceleration.

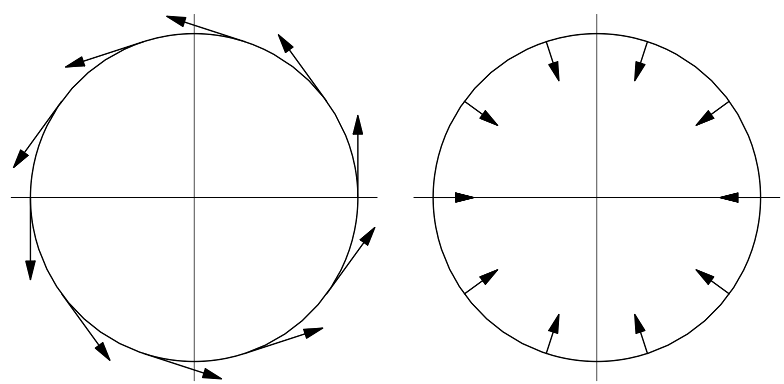

Figure 5.8: A Graph of Position and Velocity and of Position and Acceleration

Solution 5.2
If $\mathbf{r}(t)$ has constant speed, then $|\mathbf{r}'(t)| = c$. The condition that the acceleration is orthogonal to the velocity can be stated mathematically in terms of the dot product, $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$. Note that we can write the condition of constant

speed in terms of a dot product,

$$\sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)} = c,$$

$$\mathbf{r}'(t) \cdot \mathbf{r}'(t) = c^2.$$

Differentiating this equation yields,

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$$

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0.$$

This shows that the acceleration is orthogonal to the velocity.

Vector Fields

Solution 5.3

The gradient, which is orthogonal to the surface when evaluated there is $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$. $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ are orthogonal to the paraboloid, (and hence the tangent planes), at the points $(1, 1, 2)$ and $(1, -1, 2)$, respectively. The angle between the tangent planes is the angle between the vectors orthogonal to the planes. The angle between the two vectors is

$$\theta = \arccos\left(\frac{\langle 2, 2, -1\rangle \cdot \langle 2, -2, -1\rangle}{|\langle 2, 2, -1\rangle||\langle 2, -2, -1\rangle|}\right)$$

$$\boxed{\theta = \arccos\left(\frac{1}{9}\right) \approx 1.45946.}$$

Recall that the equation of a line orthogonal to $\mathbf{a}$ and passing through the point $\mathbf{b}$ is $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$. The equations of the tangent planes are

$$\langle 2, \pm 2, -1\rangle \cdot \langle x, y, z\rangle = \langle 2, \pm 2, -1\rangle \cdot \langle 1, \pm 1, 2\rangle,$$

$$\boxed{2x \pm 2y - z = 2.}$$

The paraboloid and the tangent planes are shown in Figure 5.9.

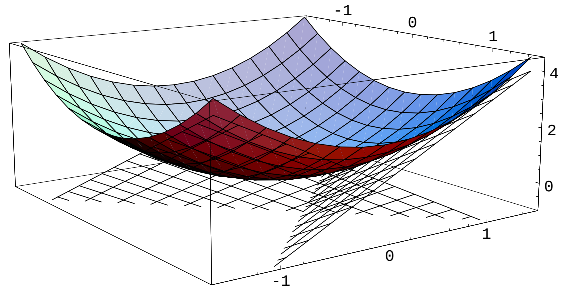

Figure 5.9: Paraboloid and Two Tangent Planes

Solution 5.4

Since the paraboloid is a differentiable surface, the normal to the surface at the closest point will be parallel to the vector from the closest point to $(1, 0, 0)$. We can express this using the gradient and the cross product. If (x, y, z) is the closest point on the paraboloid, then a vector orthogonal to the surface there is $\nabla f = \langle 2x, 2y, -1\rangle$. The vector from the surface to the point $(1, 0, 0)$ is $\langle 1 - x, -y, -z\rangle$. These two vectors are parallel if their cross product is zero,

$$\langle 2x, 2y, -1\rangle \times \langle 1 - x, -y, -z\rangle = \langle -y - 2yz, -1 + x + 2xz, -2y\rangle = \mathbf{0}.$$

This gives us the three equations,

$$\begin{aligned} -y - 2yz &= 0, \\ -1 + x + 2xz &= 0, \\ -2y &= 0. \end{aligned}$$

The third equation requires that $y = 0$. The first equation then becomes trivial and we are left with the second equation,

$$-1 + x + 2xz = 0.$$

Substituting $z = x^2 + y^2$ into this equation yields,

$$2x^3 + x - 1 = 0.$$

The only real valued solution of this polynomial is

$$x = \frac{6^{-2/3}\left(9+\sqrt{87}\right)^{2/3} - 6^{-1/3}}{\left(9+\sqrt{87}\right)^{1/3}} \approx 0.589755.$$

Thus the closest point to $(1, 0, 0)$ on the paraboloid is

$$\left(\frac{6^{-2/3}\left(9+\sqrt{87}\right)^{2/3} - 6^{-1/3}}{\left(9+\sqrt{87}\right)^{1/3}}, 0, \left(\frac{6^{-2/3}\left(9+\sqrt{87}\right)^{2/3} - 6^{-1/3}}{\left(9+\sqrt{87}\right)^{1/3}}\right)^2\right) \approx (0.589755, 0, 0.34781).$$

The closest point is shown graphically in Figure 5.10.

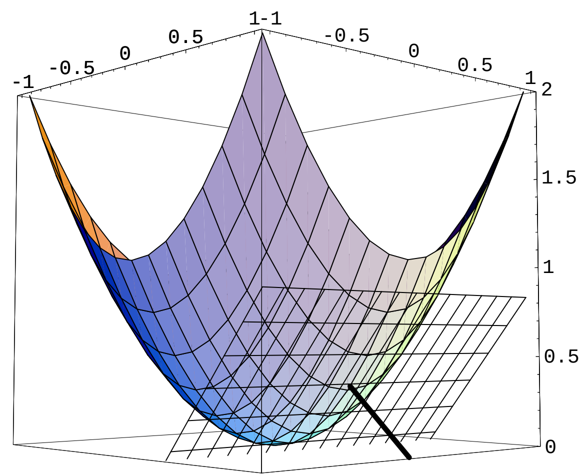

Figure 5.10: Paraboloid, Tangent Plane and Line Connecting $(1, 0, 0)$ to Closest Point

Solution 5.5

We consider the region R defined by $x^2 + xy + y^2 \leq 9$. The boundary of the region is an ellipse. (See Figure 5.11 for the ellipse and the solid obtained by rotating the region.) Note that in rotating the region about the y axis, only the

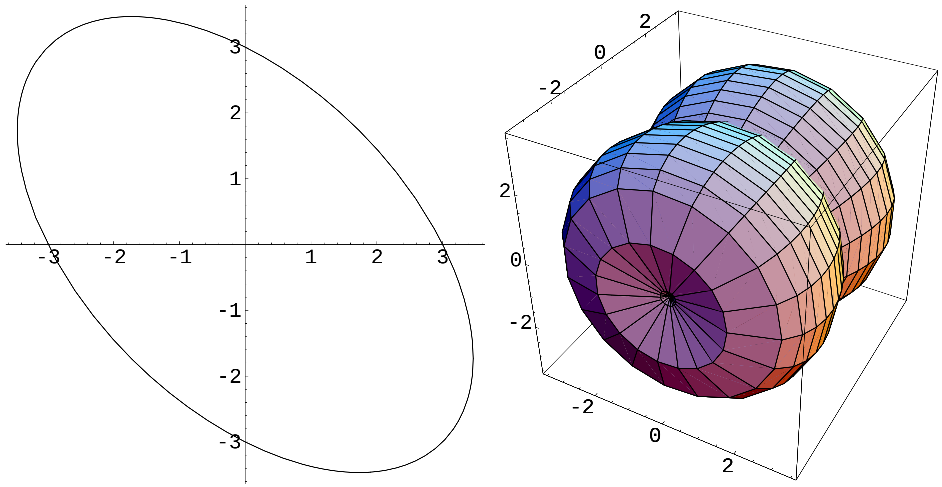

Figure 5.11: The curve $x^2 + xy + y^2 = 9$.

portions in the second and fourth quadrants make a contribution. Since the solid is symmetric across the xz plane, we will find the volume of the top half and then double this to get the volume of the whole solid. Now we consider rotating the region in the second quadrant about the y axis. In the equation for the ellipse, $x^2 + xy + y^2 = 9$, we solve for x.

$$x = \frac{1}{2}\left(-y \pm \sqrt{3}\sqrt{12 - y^2}\right)$$

In the second quadrant, the curve $(-y - \sqrt{3}\sqrt{12 - y^2})/2$ is defined on $y \in [0 \ldots \sqrt{12}]$ and the curve $(-y - \sqrt{3}\sqrt{12 - y^2})/2$ is defined on $y \in [3 \ldots \sqrt{12}]$. (See Figure 5.12.) We find the volume obtained by rotating the

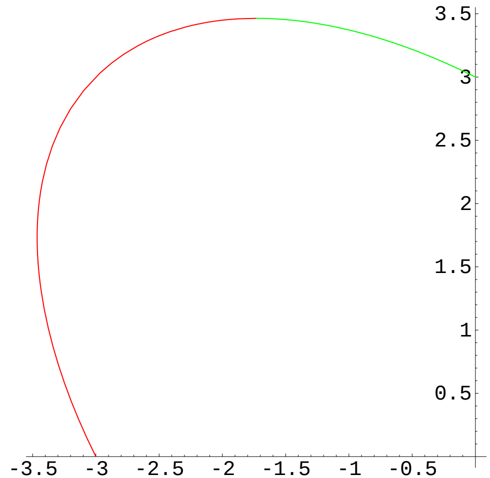

Figure 5.12: $(-y-\sqrt{3}\sqrt{12-y^2})/2$ in red and $(-y+\sqrt{3}\sqrt{12-y^2})/2$ in green.

first curve and subtract the volume from rotating the second curve.

$$V = 2\left(\int_0^{\sqrt{12}} \pi \left(\frac{-y-\sqrt{3}\sqrt{12-y^2}}{2}\right)^2 \mathrm{d}y - \int_3^{\sqrt{12}} \pi \left(\frac{-y+\sqrt{3}\sqrt{12-y^2}}{2}\right)^2 \mathrm{d}y\right)$$

$$V = \frac{\pi}{2}\left(\int_0^{\sqrt{12}} \left(y+\sqrt{3}\sqrt{12-y^2}\right)^2 \mathrm{d}y - \int_3^{\sqrt{12}} \left(-y+\sqrt{3}\sqrt{12-y^2}\right)^2 \mathrm{d}y\right)$$

$$V = \frac{\pi}{2}\left(\int_0^{\sqrt{12}} \left(-2y^2+\sqrt{12}y\sqrt{12-y^2}+36\right) \mathrm{d}y - \int_3^{\sqrt{12}} \left(-2y^2-\sqrt{12}y\sqrt{12-y^2}+36\right) \mathrm{d}y\right)$$

$$V = \frac{\pi}{2}\left(\left[-\frac{2}{3}y^3 - \frac{2}{\sqrt{3}}\left(12-y^2\right)^{3/2} + 36y\right]_0^{\sqrt{12}} - \left[-\frac{2}{3}y^3 + \frac{2}{\sqrt{3}}\left(12-y^2\right)^{3/2} + 36y\right]_3^{\sqrt{12}}\right)$$

$$\boxed{V = 72\pi}$$

Now consider the volume of the solid obtained by rotating R about the x axis? This as the same as the volume of the solid obtained by rotating R about the y axis. Geometrically we know this because R is symmetric about the line $y = x$.

Now we justify it algebraically. Consider the phrase: Rotate the region $x^2 + xy + y^2 \leq 9$ about the x axis. We formally swap x and y to obtain: Rotate the region $y^2 + yx + x^2 \leq 9$ about the y axis. Which is the original problem.

Solution 5.6

We find of the volume of the intersecting cylinders by summing the volumes of the two cylinders and then subracting the volume of their intersection. The volume of each of the cylinders is 2π. The intersection is shown in Figure 5.13. If we slice this solid along the plane $z = \text{const}$ we have a square with side length $2\sqrt{1 - z^2}$. The volume of the intersection of the cylinders is

$$\int_{-1}^{1} 4\left(1 - z^2\right) \, \mathrm{d}z.$$

We compute the volume of the intersecting cylinders.

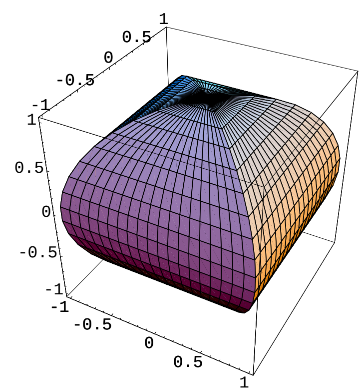

Figure 5.13: The intersection of the two cylinders.

$$V = 2(2\pi) - 2\int_0^1 4\left(1 - z^2\right)\,\mathrm{d}z$$

$$\boxed{V = 4\pi - \frac{16}{3}}$$

Solution 5.7
The length of $f(x)$ is

$$L = \int_1^\infty \sqrt{1 + 1/x^2}\,\mathrm{d}x.$$

Since $\sqrt{1 + 1/x^2} > 1/x$, the integral diverges. The length is infinite.
We find the area of S by integrating the length of circles.

$$A = \int_1^\infty \frac{2\pi}{x}\,\mathrm{d}x$$

This integral also diverges. The area is infinite.
Finally we find the volume of S by integrating the area of disks.

$$V = \int_1^\infty \frac{\pi}{x^2}\,\mathrm{d}x = \left[-\frac{\pi}{x}\right]_1^\infty = \pi$$

Solution 5.8
First we write the formula for the work required to move the oil to the surface. We integrate over the mass of the oil.

$$\text{Work} = \int (\text{acceleration})\,(\text{distance})\,\mathrm{d}(\text{mass})$$

Here (distance) is the distance of the differential of mass from the surface. The acceleration is that of gravity, g. The differential of mass can be represented an a differential of volume time the density of the oil, $800\,\text{kg/m}^3$.

$$\text{Work} = \int 800g(\text{distance})\,\mathrm{d}(\text{volume})$$

We place the coordinate axis so that $z = 0$ coincides with the bottom of the cone. The oil lies between $z = 0$ and $z = 12$. The cross sectional area of the oil deposit at a fixed depth is πz^2. Thus the differential of volume is $\pi\, z^2\, \mathrm{d}z$. This oil must me raised a distance of $24 - z$.

$$W = \int_0^{12} 800\, g\, (24 - z)\, \pi\, z^2\, \mathrm{d}z$$

$$W = 6912000 g\pi$$

$$\boxed{W \approx 2.13 \times 10^8\, \frac{\mathrm{kg\, m^2}}{\mathrm{s^2}}}$$

Solution 5.9

The Jacobian in spherical coordinates is $r^2 \sin\phi$.

$$\begin{aligned}
\text{area} &= \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi\, \mathrm{d}\phi\, \mathrm{d}\theta \\
&= 2\pi R^2 \int_0^{\pi} \sin\phi\, \mathrm{d}\phi \\
&= 2\pi R^2 [-\cos\phi]_0^{\pi}
\end{aligned}$$

$$\boxed{\text{area} = 4\pi R^2}$$

$$\begin{aligned}
\text{volume} &= \int_0^{R} \int_0^{2\pi} \int_0^{\pi} r^2 \sin\phi\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r \\
&= 2\pi \int_0^{R} \int_0^{\pi} r^2 \sin\phi\, \mathrm{d}\phi\, \mathrm{d}r \\
&= 2\pi \left[\frac{r^3}{3}\right]_0^{R} [-\cos\phi]_0^{\pi}
\end{aligned}$$

$$\boxed{\text{volume} = \frac{4}{3}\pi R^3}$$

5.6 Quiz

Problem 5.1
What is the distance from the origin to the plane $x + 2y + 3z = 4$?
Solution

Problem 5.2
A bead of mass m slides frictionlessly on a wire determined parametrically by $\mathbf{w}(s)$. The bead moves under the force of gravity. What is the acceleration of the bead as a function of the parameter s?
Solution

5.7 Quiz Solutions

Solution 5.1

Recall that the equation of a plane is $\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ where $\mathbf{a}$ is a point in the plane and $\mathbf{n}$ is normal to the plane. We are considering the plane $x + 2y + 3z = 4$. A normal to the plane is $\langle 1, 2, 3 \rangle$. The unit normal is

$$\mathbf{n} = \frac{1}{\sqrt{15}} \langle 1, 2, 3 \rangle.$$

By substituting in $x = y = 0$, we see that a point in the plane is $\mathbf{a} = \langle 0, 0, 4/3 \rangle$. The distance of the plane from the origin is $\mathbf{a} \cdot \mathbf{n} = \frac{4}{\sqrt{15}}$.

Solution 5.2

The force of gravity is $-g\mathbf{k}$. The unit tangent to the wire is $\mathbf{w}'(s)/|\mathbf{w}'(s)|$. The component of the gravitational force in the tangential direction is $-g\mathbf{k} \cdot \mathbf{w}'(s)/|\mathbf{w}'(s)|$. Thus the acceleration of the bead is

$$-\frac{g\mathbf{k} \cdot \mathbf{w}'(s)}{m|\mathbf{w}'(s)|}.$$