Chapter 4

Emission and absorption of radiation by atoms

In this chapter, we first discuss atom–field interactions using quantum mechanical perturbation theory for a classical driving field and then for a quantum mechanical field. In the latter case, spontaneous emission appears as a fully quantum-mechanical phenomenon. We then examine the so-called Rabi model, a model of a "two-level" atom interacting with a strong near-resonant classical field, introduce the rotating wave approximation, and then introduce the fully quantum mechanical Rabi model, better known as the Jaynes–Cummings model. Differences in the evolution predicted by the two-models are drawn out, where we show that the Jaynes–Cummings model predicts behavior that has no semiclassical analog and that depends entirely on the discreteness of photons. Finally, we study an extension of the Jaynes–Cummings model, the dispersive model, for the case where the quantized field is far out-of-resonance with the atomic transition frequency. This will eventually allow us to describe how superpositions of different coherent states of the field, states known as Schrödinger cat states, can be generated from atom–field interactions.

4.1 Atom–field interactions

To begin, let us suppose that the Hamiltonian of an electron bound to an atom in the absence of external fields, in the configuration representation, is given by

$$\hat{H}_0 = \frac{1}{2m}\hat{\mathbf{P}}^2 + V(r) \tag{4.1}$$

where $V(r)$ is the usual Coulomb interaction binding the electron to the nucleus and $r = |\mathbf{r}|$. In the configuration space representation $\hat{\mathbf{P}} = -i\nabla$, $\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle$ and the wave functions are given by $\psi(\mathbf{r}) = \langle \mathbf{r} \mid \psi \rangle$. We assume that energy eigenstates $|k\rangle$ of $\hat{H}_0$, satisfying the time-independent Schrödinger equation

$$\hat{H}_0 \psi_k^{(0)}(\mathbf{r}) = E_k \psi_k^{(0)}(\mathbf{r}), \tag{4.2}$$

where $\langle \mathbf{r} \mid k \rangle = \psi_k^{(0)}(\mathbf{r})$, are known. In the presence of external fields the Hamiltonian is modified to

$$\hat{H}(\mathbf{r}, t) = \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r}, t)]^2 - e\,\Phi(\mathbf{r}, t) + V(r) \tag{4.3}$$

where $\mathbf{A}(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, t)$ are the vector and scalar potentials respectively of the external field and where $-e$ is the electron charge, e taken to be positive. The fields themselves are given by

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\nabla\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t), \end{aligned} \tag{4.4}$$

and are invariant under the gauge transformations

$$\begin{aligned} \Phi'(\mathbf{r}, t) &= \Phi(\mathbf{r}, t) - \frac{\partial \chi(\mathbf{r}, t)}{\partial t}, \\ \mathbf{A}'(\mathbf{r}, t) &= \mathbf{A}(\mathbf{r}, t) + \nabla\chi(\mathbf{r}, t). \end{aligned} \tag{4.5}$$

The time-dependent Schrödinger equation is

$$\hat{H}(\mathbf{r}, t)\Psi(\mathbf{r}, t) = i\,\hbar\frac{\partial \Psi(\mathbf{r}, t)}{\partial t}. \tag{4.6}$$

In order eventually to simplify the form of the atom–field interaction, we define a unitary operator $\hat{R}$ such that $\Psi'(\mathbf{r}, t) \equiv \hat{R}\,\Psi(\mathbf{r}, t)$. We have

$$\hat{H}'\Psi'(\mathbf{r}, t) = i\,\hbar\frac{\partial \Psi'(\mathbf{r}, t)}{\partial t} \tag{4.7}$$

where

$$\hat{H}' = \hat{R}\hat{H}\hat{R}^{\dagger} + i\,\hbar\frac{\partial \hat{R}}{\partial t}\hat{R}^{\dagger}. \tag{4.8}$$

We now choose $\hat{R} = \exp(-ie\chi(\mathbf{r}, t)/\hbar)$ so that (using $\hat{\mathbf{P}} = -i\,\hbar\nabla$)

$$\hat{H}' = \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}']^2 - e\,\Phi' + V(r) \tag{4.9}$$

where $\mathbf{A}'$ and Φ' are given by Eq. (4.5). At this point we make a definite choice of gauge, namely the Coulomb (or radiation) gauge, for which $\Phi = 0$ and $\mathbf{A}$ satisfies the transversality condition $\nabla \cdot \mathbf{A} = 0$. The vector potential $\mathbf{A}$, for no sources near the atom, satisfies the wave equation

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{4.10}$$

This choice of gauge is not relativistically invariant, in contrast to the Lorentz gauge, but the domain of quantum optics is, for the most part, nonrelativistic so that no inconsistency will be introduced. The Coulomb gauge has the advantage that the radiation field is completely described by the vector potential, as is obvious from Eq. (4.3), which, in this gauge, reads

$$\begin{aligned} \hat{H}(\mathbf{r}, t) &= \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r}, t)]^2 + V(r) \\ &= \frac{\hat{\mathbf{P}}^2}{2m} + \frac{e}{m}\,\mathbf{A}\cdot\hat{\mathbf{P}} + \frac{e^2}{2m}\mathbf{A}^2 + V(r). \end{aligned} \tag{4.11}$$

Equation (4.9) now reads

$$\hat{H}'(\mathbf{r},t) = \frac{1}{2m}[\hat{\mathbf{P}} + e(\mathbf{A} + \nabla\chi)]^2 + e\frac{\partial\chi}{\partial t} + V(r). \tag{4.12}$$

The solution of the wave equation (4.10) has the form

$$\mathbf{A} = \mathbf{A}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + c.c. \tag{4.13}$$

where $|\mathbf{k}| = 2\pi/\lambda$ is the wave vector of the radiation. For $|\mathbf{r}|$ of typical atomic dimensions (a few Ångströms) and λ of typical optical wavelengths (a few hundred nanometers in the range 400–700 nm), $\mathbf{k}\cdot\mathbf{r} \ll 1$ so that over the extent of an atom, the vector potential is spatially uniform, $\mathbf{A}(\mathbf{r},t) \simeq \mathbf{A}(t)$. This is the so-called dipole approximation. We now choose the gauge function $\chi(\mathbf{r},t) = -\mathbf{A}(t)\cdot\mathbf{r}$. With this choice,

$$\begin{aligned} \nabla\chi(\mathbf{r},t) &= -\mathbf{A}(t), \\ \frac{\partial\chi}{\partial t}(\mathbf{r},t) &= -\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{r}\cdot\mathbf{E}(t), \end{aligned} \tag{4.14}$$

and thus

$$\hat{H}' = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\,\mathbf{r}\cdot\mathbf{E}(t). \tag{4.15}$$

This equation contains only one interaction term (within the dipole approximation) as opposed to the two terms of Eq. (4.11). We shall work with $\hat{H}'$ in all that follows, the interaction being in what is sometimes called the "length" gauge. The quantity $-e\mathbf{r}$ is the dipole moment: $\mathbf{d} = -e\mathbf{r}$. In general, i.e. for an unspecified representation, the dipole moment is an operator, $\hat{\mathbf{d}}$. We shall denote it as such in what follows. Thus we write

$$\hat{H}' = \hat{H}_0 - \hat{\mathbf{d}}\cdot\mathbf{E}(t) \tag{4.16}$$

where $\hat{H}_0$ is given by Eq. (4.1).

4.2 Interaction of an atom with a classical field

So far, we have not specified the nature of the interacting field and have not even stated whether we consider the field to be classical or quantum mechanical. The derivation leading to Eq. (4.16) is valid for both classical and quantum fields. But eventually we want to demonstrate differences in the way an atom behaves when interacting with classical or quantum fields. With that in mind, we first turn to the case when an atom is driven by a classical sinusoidal electric field.

We assume that the field has the form $\mathbf{E}(t) = \mathbf{E}_0\cos(\omega t)$, ω being the frequency of the radiation, and that this field is abruptly turned on at $t = 0$. The

dipole approximation, where we assume that $\mathbf{k}\cdot\mathbf{r} \ll 1$ over the atom, has already been made. We further assume that the initial state of the atom is $|i\rangle$ where $\hat{H}_0|i\rangle = E_i|i\rangle$. For times $t > 0$ we expand the state vector $|\psi(t)\rangle$ in terms of the complete set of uncoupled atomic states $|k\rangle$:

$$|\psi(t)\rangle = \sum_k C_k(t) e^{-iE_k t/\hbar}|k\rangle, \tag{4.17}$$

where the time-dependent amplitudes $C_k(t)$ satisfy the normalization requirement

$$\sum_k |C_k(t)|^2 = 1. \tag{4.18}$$

Substituting this expansion into the time-dependent Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \left(\hat{H}_0 + \hat{H}^{(\mathrm{I})}\right)|\psi(t)\rangle, \tag{4.19}$$

where $H^{(\mathrm{I})} = -\hat{\mathbf{d}}\cdot\mathbf{E}(t)$, then multiplying from the left by $\langle l|e^{iE_l t/\hbar}$ leads to the set of coupled first-order differential equations for the amplitudes

$$\dot{C}_l(t) = -\frac{i}{\hbar}\sum_k C_k(t)\langle l|\hat{H}^{(\mathrm{I})}|k\rangle e^{i\omega_{lk}t} \tag{4.20}$$

where the $\omega_{lk} = (E_l - E_k)/\hbar$ are the transition frequencies between levels l and k. These equations are, so far, exact and need to be solved subject to the initial condition $C_i(0) = 1$, only state $|i\rangle$ being initially populated. As time goes forward, population will be lost from state $|i\rangle$ and increased in some initially unpopulated state $|f\rangle$, i.e. the amplitude $C_f(t)$ increases. The probability for the atom to make a transition from state $|i\rangle$ to state $|f\rangle$ in time t is given by

$$P_{i\to f}(t) = C_f^*(t)C_f(t) = |C_f(t)|^2. \tag{4.21}$$

The equations for the amplitudes are solvable only for very simple cases. These days, of course, one might solve the set of differential equations numerically, but in the case of a driving field that in some sense is "weak", we can use a time-dependent perturbation theory [1] approach to the problem. "Weak" in this case means that $|\mathbf{E}_0|$ is small, or actually, that $|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle|$ is small. As a matter of bookkeeping, we write the interaction Hamiltonian as $\lambda\hat{H}^{(\mathrm{I})}$, where λ is treated as a number in the range $0 \le \lambda \le 1$. (At the end of the calculations we will always take $\lambda \to 1$.) We then expand the probability amplitude for, say, state $|l\rangle$ in the power series

$$C_l(t) = C_l^{(0)}(t) + \lambda C_l^{(1)}(t) + \lambda^2 C_l^{(2)}(t) + \cdots \tag{4.22}$$

Inserting such expansions into Eqs. (4.20) and equating like powers of λ we obtain, up to second order,

$$\dot{C}_l^{(0)} = 0, \tag{4.23}$$

$$\dot{C}_l^{(1)} = -\frac{i}{\hbar}\sum_k C_k^{(0)} H_{lk}^{(\mathrm{I})}(t) e^{i\omega_{lk}t}, \tag{4.24}$$

$$\dot{C}_l^{(2)} = -\frac{i}{\hbar}\sum_k C_k^{(1)} H_{lk}^{(\mathrm{I})}(t) e^{i\omega_{lk}t}, \tag{4.25}$$

where $H_{lk}^{(\mathrm{I})}(t) \equiv \langle l|\hat{H}^{(\mathrm{I})}(t)|k\rangle$. Note the general pattern that relates the nth order to the $(n-1)$th order:

$$\dot{C}_l^{(n)} = -\frac{i}{\hbar}\sum_k C_k^{(n-1)}(t) H_{lk}^{(\mathrm{I})}(t) e^{i\omega_{lk}t}. \tag{4.26}$$

The essential assumption underlying the perturbation-theory approach is that the driving field is so weak that the atomic populations change very little. That is, if $C_i(0) = 1,\ C_f(0) = 0\ (f \neq i)$ then for $t > 0$, to a good approximation $C_i(t) \approx 1, |C_f(t)| \ll 1\ (f \neq i)$. Thus, in the first-order equation (4.24), the only term surviving the sum on the right-hand side is for $k = i$ yielding

$$\dot{C}_f^{(1)}(t) = -\frac{i}{\hbar} H_{fi}^{(\mathrm{I})}(t) e^{i\omega_{fi}t} C_i^{(0)}(t) \tag{4.27}$$

or

$$C_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{fi}^{(\mathrm{I})}(t') e^{i\omega_{fi}t'} C_i^{(0)}(t'). \tag{4.28}$$

Inserting this result into the second-order equation, Eq. (4.25), we obtain

$$\begin{aligned} C_f^{(2)}(t) &= -\frac{i}{\hbar}\sum_l \int_0^t dt' H_{fl}^{(\mathrm{I})}(t') e^{i\omega_{fl}t'} C_l^{(1)}(t') \\ &= \left(-\frac{i}{\hbar}\right)^2 \sum_l \int_0^t dt' \int_0^{t'} dt'' H_{fl}^{(\mathrm{I})}(t') e^{i\omega_{fl}t'} \\ &\quad \times H_{li}^{(\mathrm{I})}(t'') e^{i\omega_{li}t''} C_i^{(0)}(t''). \end{aligned} \tag{4.29}$$

Equation (4.28) gives the amplitude for a transition from state $|i\rangle$ to state $|f\rangle$ while Eq. (4.29) gives the amplitude for a transition from state $|i\rangle$ to states $\{|l\rangle\}$ then to state $|f\rangle$. The total transition probability from state $|i\rangle$ to state $|f\rangle$ is

$$P_{i\to f}(t) = \left|C_f^{(0)}(t) + C_f^{(1)}(t) + C_f^{(2)}(t) + \cdots\right|^2. \tag{4.30}$$

Now the dipole moment operator $\hat{\mathbf{d}}$ has nonvanishing matrix elements only between states of opposite parity. Thus the first-order correction to the amplitude

of the initial state vanishes:

$$C_i^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{ii}^{(\mathrm{I})}(t') C_i^{(0)}(t') = 0 \tag{4.31}$$

because $H_{ii}^{(\mathrm{I})}(t) = 0$. Therefore, to first order $C_i(t) = C_i^{(0)}(t) = 1$ so that

$$C_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{fi}^{(\mathrm{I})}(t')\, e^{i\omega_{fi}t'}. \tag{4.32}$$

With $H^{(\mathrm{I})} = -\hat{\mathbf{d}}\cdot\mathbf{E}_0\cos\omega t$, and by expanding the cosine in terms of exponentials, this integrates to

$$C_f^{(1)}(t) = \frac{1}{2\hbar}(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi} \times\left\{\frac{\left(e^{i(\omega+\omega_{fi})t}-1\right)}{(\omega+\omega_{fi})} - \frac{\left(e^{-i(\omega-\omega_{fi})t}-1\right)}{(\omega-\omega_{fi})}\right\}, \tag{4.33}$$

where $(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi} = \langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle$. If the frequency of the radiation, ω, is near resonance with the atomic transition frequency ω_{fi}, the second term clearly dominates the first. Therefore we may drop the "antiresonant" first term, making the so-called "rotating wave approximation" (RWA), familiar in the context of magnetic resonance [2]. Thus we have to first order the transition probability

$$P_{i\to f}^{(1)}(t) = \left|C_f^{(1)}(t)\right|^2 = \frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\frac{\sin^2(\Delta t/2)}{\Delta^2} \tag{4.34}$$

where $\Delta = \omega - \omega_{fi}$ is the "detuning" between the radiation field and the atomic transition. When $\Delta \neq 0$, $P_{i\to f}^{(1)}(t)$ maximizes at

$$\left(P_{i\to f}^{(1)}\right)_{\max} = \frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\frac{1}{\Delta^2}. \tag{4.35}$$

For the case of exact resonance, $\Delta = 0$,

$$\left(P_{i\to f}^{(1)}\right)_{\max} = \frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{4\hbar^2}t^2\,. \tag{4.36}$$

For the perturbation expansion to be valid we must have $(P_{i\to f}^{(1)})_{\max} \ll 1$. For the off-resonance case, this places conditions on both $|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|$ and Δ. For the resonant case, Eq. (4.36) is valid only for very short times. In Fig. 4.1 we plot the

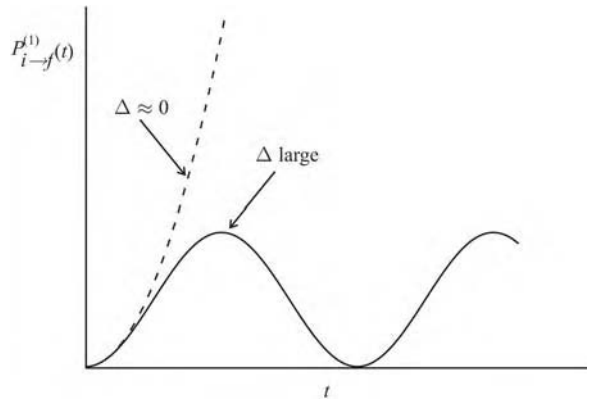


Fig. 4.1. A plot of $P^{(1)}_{i\to f}(t)$ versus time for small and large detunings Δ.

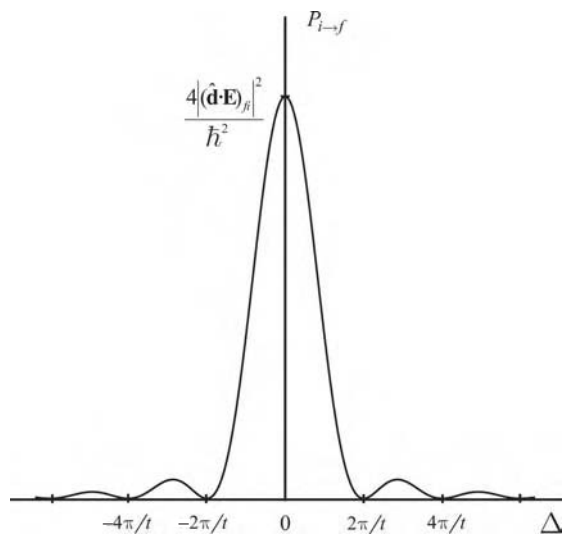


Fig. 4.2. The transition probability $P^{(1)}_{i\to f}(t)$ as a function of Δ.

evolution of the probability distribution $P^{(1)}_{i\to f}(t)$ for both small detuning ($\Delta \approx 0$) and large detuning. The latter is periodic. The transition probability $P^{(1)}_{i\to f}(t)$ is a sharply peaked function of the detuning at $\Delta = 0$ as shown in Fig. 4.2. The width of the peak is proportional to t^{-1} while the height is proportional to t^2. Thus the area under the peak is proportional to t. In fact

$$\int_{-\infty}^{\infty} \frac{\sin^2(\Delta t/2)}{\Delta^2} d(\Delta) = \frac{\pi}{2} t. \tag{4.37}$$

Furthermore, in the limit when $\Delta \approx 0$ and $t \gg 2\pi/\omega_{fi}$, the function in the integrand of Eq. (4.37) may be approximated by a Dirac delta function:

$$\lim_{t\to\infty} \frac{\sin^2(\Delta t/2)}{\Delta^2} = \frac{\pi}{2} t\, \delta(\Delta), \tag{4.38}$$

although the limit $t \to \infty$ is, in fact, actually constrained by the requirement that the right-hand side of Eq. (4.35) be $\ll 1$. In this case the transition probability is

$$P^{(1)}_{i\to f}(t) = \frac{\pi}{2}\frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2} t\,\delta(\omega-\omega_{fi}). \tag{4.39}$$

We define the time-independent transition probability *rate* as

$$W_{i\to f} = \frac{P^{(1)}_{i\to f}}{t} = \frac{\pi}{2}\frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\,\delta(\omega-\omega_{fi}). \tag{4.40}$$

In practice, there will be a broad range of final states $|f\rangle$ accessible from the initial state, and the driving field will not be monochromatic so that a range of frequencies will have to be summed or integrated over to obtain the total transition rate. If $[f]$ represents a set of accessible final states, then the transition rate for a monochromatic field is

$$W_{i\to[f]} = \frac{\pi}{2}\sum_{[f]}\frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\,\delta(\omega-\omega_{fi}). \tag{4.41}$$

This expression is often famously called Fermi's Golden Rule [3].

Now suppose that the light irradiating the atom is from a lamp emitting a broad range of frequencies where there is no phase relationship between the different frequency components. The amplitude of the light will now be frequency dependent so that the transition probability rate induced by all the frequency components must be

$$\frac{P^{(1)}_{i\to f}(t)}{t} = \frac{1}{\hbar^2}\int d\omega \frac{\sin^2(\Delta t/2)}{\Delta^2}F(\omega) \tag{4.42}$$

where

$$F(\omega) \equiv |\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0(\omega)|i\rangle|^2. \tag{4.43}$$

If the function $F(\omega)$ is broadband and varies slowly with ω in comparison to $(\sin^2(\Delta t/2)/\Delta^2)$, then $F(\omega)$ can be replaced by its resonance value $F(\omega_{fi})$ and taken outside the integral so that

$$P^{(1)}_{i\to f}(t) = \frac{\pi}{2\hbar^2}F(\omega_{fi})\,t. \tag{4.44}$$

Thus the transition rate is

$$W_{i\to f} = \frac{\pi}{2\hbar^2}F(\omega_{fi}). \tag{4.45}$$

The spread of frequencies results in the washing out, or dephasing, of the oscillations seen in Fig. 4.1. This happens because of the lack of phase relations between the various frequency components – the light is incoherent. If the atom is driven by a coherent light field, such as from a laser, dephasing does not occur and

the perturbative time-independent transition rates above generally do not adequately describe the dynamics. We postpone a discussion of driving the atom by a coherent laser field until Section 4.4.

4.3 Interaction of an atom with a quantized field

In the preceding discussion, we made no assumption regarding the relative positions of the energy levels i and f. Transitions between the levels occur with some nonzero probability as long as $\mathbf{E}_0 \neq 0$ whether $E_i < E_f$ or $E_i > E_f$. As we will show, when the field is quantized, transitions will occur for the case $E_i > E_f$ even when no photons are present – the so-called spontaneous emission. This is only one of several differences that will appear in the atom–field dynamics in the comparison between cases when the field is quantized and when it is not.

We consider a single free-space field mode of the form given by Eq. (2.130)

$$\hat{\mathbf{E}}(t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \mathbf{e}[\hat{a}e^{-i\omega t} - \hat{a}e^{i\omega t}] \tag{4.46}$$

where the dipole approximation has been made. This operator is in the Heisenberg picture but we are going to be working in the Schrödinger picture where the field operator is

$$\hat{\mathbf{E}} = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \mathbf{e}(\hat{a} - \hat{a}^\dagger). \tag{4.47}$$

The free Hamiltonian $\hat{H}_0$ now must be

$$\hat{H}_0 = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} \tag{4.48}$$

where $\hat{H}_{\text{atom}}$ is just the free-atom Hamiltonian as before and $\hat{H}_{\text{field}}$ is the free-field Hamiltonian $\hbar\omega\,\hat{a}^\dagger\hat{a}$, where the zero-point term has been dropped as it does not contribute to the dynamics. The interaction Hamiltonian becomes

$$\begin{aligned}\hat{H}^{(\mathrm{I})} &= -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} = -i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}(\hat{\mathbf{d}}\cdot\mathbf{e})(\hat{a} - \hat{a}^\dagger) \\ &= -\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0(\hat{a} - \hat{a}^\dagger)\end{aligned} \tag{4.49}$$

where $\boldsymbol{\mathcal{E}}_0 = i(\hbar\omega/2\varepsilon_0 V)^{1/2}\mathbf{e}$.

Because both atomic and field systems are now quantized, the states of the combined system will involve products of states of both systems. Suppose the initial state of the atom–field system is $|i\rangle = |a\rangle|n\rangle$ where $|a\rangle$ is the initial state of the atom and the field contains n photons. The perturbation interaction of the quantized field causes a transition to the state $|f_1\rangle = |b\rangle|n-1\rangle$, where $|b\rangle$ is another atomic state, by the absorption of a photon or to the state

$|f_2\rangle = |b\rangle|n+1\rangle$ by the emission of a photon. The energies of these states are,

$$\text{for} \quad |i\rangle = |a\rangle|n\rangle, \qquad E_i = E_a + n\,\hbar\omega, \tag{4.50a}$$

$$\text{for} \quad |f_1\rangle = |b\rangle|n-1\rangle, \qquad E_{f_1} = E_b + (n-1)\,\hbar\omega, \tag{4.50b}$$

$$\text{for} \quad |f_2\rangle = |b\rangle|n+1\rangle, \qquad E_{f_2} = E_b + (n+1)\,\hbar\omega, \tag{4.50c}$$

where E_a and E_b are the energies of the atomic states $|a\rangle$ and $|b\rangle$ respectively.

The perturbation given by Eq. (4.49) is now time independent. The matrix elements of the interaction are as follows:

$$\begin{aligned}\langle f_1|\hat{H}^{(\mathrm{I})}|i\rangle &= \langle b, n-1|\hat{H}^{(\mathrm{I})}|a, n\rangle \\ &= -(\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ba}\sqrt{n}, \qquad \text{(absorption)}\end{aligned} \tag{4.51}$$

and

$$\begin{aligned}\langle f_2|\hat{H}^{(\mathrm{I})}|i\rangle &= \langle b, n+1|\hat{H}^{(\mathrm{I})}|a, n\rangle \\ &= (\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ba}\sqrt{n+1}, \qquad \text{(emission)}\end{aligned} \tag{4.52}$$

where

$$(\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ab} = \langle a|\hat{\mathbf{d}}|b\rangle\cdot\boldsymbol{\mathcal{E}}_0 \equiv \mathbf{d}_{ab}\cdot\boldsymbol{\mathcal{E}}_0 \tag{4.53}$$

the factor $\langle a|\hat{\mathbf{d}}|b\rangle = \mathbf{d}_{ab}$ being the dipole matrix element between states $|a\rangle$ and $|b\rangle$. In comparison with the semiclassical case, two things are noteworthy here. The absence of photons ($n = 0$) precludes absorption, just as one might expect. This is obviously in agreement with the case of a classical driving field – no field, no transitions. But in the case of emission, according to Eq. (4.52), transitions may occur even when no photons are present. This is spontaneous emission and it has no semiclassical counterpart. If $n > 0$, the emission of an additional photon is called stimulated emission, this process being the essential one for the operation of the laser (or LASER: Light Amplification by Stimulated Emission of Radiation). The rates of emission and absorption are proportional to the moduli squared of the above matrix elements for the respective processes. The ratio of these rates is

$$\frac{\left|\langle f_2|\hat{H}^{(\mathrm{I})}|i\rangle\right|^2}{\left|\langle f_1|\hat{H}^{(\mathrm{I})}|i\rangle\right|^2} = \frac{n+1}{n}, \tag{4.54}$$

a result to be used shortly.

The perturbation method developed previously can still be used with appropriate modifications to accommodate the fact that the field is now quantized. The Schrödinger equation still has the form of Eq. (4.19) but where now $\hat{H}_0$ is given by Eq. (4.48) and $\hat{H}^{(\mathrm{I})}$ by Eq. (4.49). Ignoring all other atomic states except $|a\rangle$

and $|b\rangle$, the state vector can be written as

$$\begin{aligned}|\psi(t)\rangle = & C_i(t)|a\rangle|n\rangle e^{-iE_at/\hbar}e^{-in\omega t} \\ & + C_{f_1}(t)|b\rangle|n-1\rangle e^{-iE_bt/\hbar}e^{-i(n-1)\omega t} \\ & + C_{f_2}(t)|b\rangle|n+1\rangle e^{-iE_bt/\hbar}e^{-i(n+1)\omega t}\end{aligned} \tag{4.55}$$

where, assuming that $|\psi(0)\rangle = |a\rangle|n\rangle$, $C_i(0) = 1$ and $C_{f1}(0) = C_{f2}(0) = 0$. Following the perturbative method used before, we obtain the first-order correction for the amplitudes C_{f1} and C_{f2} associated with the atom being in state $|b\rangle$:

$$\begin{aligned}C_{f1}^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t dt'\langle f_1|\hat{H}^{(\mathrm{I})}|i\rangle e^{i(E_{f1}-E_i)t/\hbar}, \\ C_{f2}^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t dt'\langle f_2|\hat{H}^{(\mathrm{I})}|i\rangle e^{i(E_{f2}-E_i)t/\hbar},\end{aligned} \tag{4.56}$$

where the former is associated with absorption and the latter with emission. The amplitude associated with the atom being in the state $|b\rangle$, regardless of how it got there, is just the sum of the amplitudes in Eqs. (4.56), i.e. $C_f^{(1)} = C_{f1}^{(1)} + C_{f2}^{(1)}$. From Eqs. (4.50) we obtain

$$C_f^{(1)}(t) = \frac{i}{\hbar}(\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ab}\left\{(n+1)^{1/2}\frac{\left[e^{i(\omega+\omega_{ba})t}-1\right]}{(\omega+\omega_{ba})} - n^{1/2}\frac{\left[e^{i(\omega-\omega_{ba})t}-1\right]}{(\omega-\omega_{ba})}\right\} \tag{4.57}$$

where $\omega_{ba} = (E_b - E_a)/\hbar$ and where the first term is due to emission and the second to absorption. If the number of photons is large, $n \gg 1$, then we can replace $\sqrt{n+1}$ by $\sqrt{n}$ in the first term and then Eqs. (4.57) and (4.33) are essentially the same, there being the correspondence between the classical and quantum field amplitudes $(\mathbf{E}_0)_{\mathrm{cl}} \leftrightarrow (2i\mathcal{E}_0\sqrt{n})_{\mathrm{quantum}}$. This correspondence between quantum and classical fields has its limits, one being the case when $n = 0$, as already discussed.

If $|b\rangle$ is the excited state then $\omega_{ba} > 0$ so if $\omega \sim \omega_{ba}$ then the first term of Eq. (4.51) can be dropped, which is again the rotating wave approximation. Of course, if $|a\rangle$ is the excited state, then $\omega_{ba} < 0$ and if $\omega \sim -\omega_{ba}$, the second term of Eq. (4.57) can be dropped, and we notice that the remaining term does not vanish even when $n = 0$, the transition between $|a\rangle$ and $|b\rangle$ taking place by spontaneous emission. Thus the rotating wave approximation carries over into the case where the field and the atom are quantized. It can be shown that Fermi's Golden Rule carries over in a similar fashion.

We conclude this section with a field-theoretic derivation of the Planck distribution law. Suppose we have a collection of atoms interacting resonantly with a quantized field of frequency $\omega = (E_a - E_b)/\hbar$, where $|a\rangle$ and $|b\rangle$ are the atomic states with $E_a > E_b$. We let N_a and N_b represent the populations of atoms in states $|a\rangle$ and $|b\rangle$ respectively. Further, we let W_{emis} represent the transition rate due to photon emission and W_{abs} the transition rate due to photon absorption.

Because the atoms are constantly emitting and absorbing photons, the atomic populations change with time according to

$$\begin{aligned} \frac{dN_a}{dt} &= -N_a W_{\text{emis}} + N_b W_{\text{abs}} \\ \frac{dN_b}{dt} &= -N_b W_{\text{abs}} + N_a W_{\text{emis}}. \end{aligned} \tag{4.58}$$

At thermal equilibrium, we have

$$\frac{dN_a}{dt} = 0 = \frac{dN_b}{dt} \tag{4.59}$$

and thus we obtain

$$N_a W_{\text{emis}} = N_b W_{\text{abs}}. \tag{4.60}$$

But, according to Boltzmann,

$$\frac{N_b}{N_a} = \exp[(E_a - E_b)/kT] = \exp(\hbar\omega/kT), \tag{4.61}$$

and from Eq. (4.57) we have that

$$\frac{N_b}{N_a} = \frac{W_{\text{emis}}}{W_{\text{abs}}} = \frac{n+1}{n}. \tag{4.62}$$

Thus from Eqs. (4.61) and (4.62) it follows that

$$n = \frac{1}{\exp(\hbar\omega/kT) - 1}, \tag{4.63}$$

in agreement with Eq. (2.141) if we replace n by $\bar{n}$ in Eq. (4.63) to incorporate the fact that we cannot assume a definite number of photons.

Let us compare this derivation to the one given by Einstein [4] before quantum electrodynamics was even invented. His derivation is similar to the above but explicitly makes the distinction between spontaneous and stimulated emission. He introduced the coefficients A, B, and C having the following meanings: AN_a is the growth rate of the population in state $|b\rangle$ owing to spontaneous emission from state $|b\rangle$ (A being the rate of spontaneous emission); $BU(\omega)N_a$ is the growth rate of the population of state $|b\rangle$ owing to stimulated emission from $|a\rangle$, $U(\omega)$ being the spectral energy density of the field; and $CU(\omega)N_b$ is the rate of growth of state $|a\rangle$ as the result of absorption by atoms in state $|b\rangle$. Note that the spontaneous

emission term is independent of $U(\omega)$. The rate equations for the populations are now

$$\begin{aligned} \frac{dN_a}{dt} &= -[A + BU(\omega)]N_a + CU(\omega)N_b \\ \frac{dN_b}{dt} &= -CU(\omega)N_b + [A + BU(\omega)]N_a. \end{aligned} \tag{4.64}$$

At long times, the populations reach a steady state and the derivatives on the left-hand sides vanish to yield

$$[A + BU(\omega)]N_a = CU(\omega)N_b. \tag{4.65}$$

Using once again the relation in Eq. (4.61), we are led to

$$U(\omega) = \frac{A}{C\exp(\hbar\omega/kT) - B}. \tag{4.66}$$

But for a thermal field, we must have, by comparison to Eq. (2.151), that $C = B$ and that

$$\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}. \tag{4.67}$$

It is worth comparing the rates of spontaneous emission A and stimulated emission $BU(\omega)$:

$$\frac{A}{BU(\omega)} = \exp(\hbar\omega/kT - 1). \tag{4.68}$$

For a natural source such as the sun, which we approximate as a black body with surface temperature $T \approx 6000$ K, this ratio is about 400 for $\lambda = 400$ nm and about 30 for $\lambda = 700$ nm [5]. Thus at both ends of the visible spectrum spontaneous emission dominates stimulated emission. In the range of visible light, it is only in "unnatural" sources, where there exists a population inversion (i.e. more atoms in the excited state than in the ground), such as in lasers, where stimulated emission dominates spontaneous emission.

Spontaneous emission is a complex phenomenon and we have discussed only its most essential features. For atoms in free space there is an infinity of modes into which the atoms may radiate. Spontaneous emission in this case is well described by the Weisskopf–Wigner theory [6] as an irreversible decay process. A discussion of that theory is outside the scope of this book. However, under certain circumstances, where there may be only one mode into which an atom can radiate, such as in the case of an atom in a cavity, spontaneous emission can

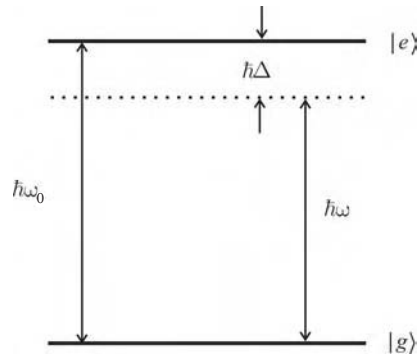


Fig. 4.3. Energy level diagram for a two-level atom acting with a near resonant classical driving field of frequency ω. The resonant frequency between the two atomic levels is ω_0 and the detuning $\Delta = \omega_0 - \omega$.

be reversible, that is, the atom can re-absorb the emitted photon. Such behavior is discussed in Section 4.5.

4.4 The Rabi model

The perturbation theory approach to atom–field interactions assumes that the initial atomic state population is essentially unchanged, that is, the probability amplitude for the atom being in any other state remains small. On the other hand, a strong laser field of frequency near resonance with a pair of atomic levels (assumed of opposite parity) will cause a large population transfer to the near-resonant state but not to any other. In such a case, perturbation theory must be abandoned. Only the two dominant states will be retained and the problem will be solved more "exactly". This is the Rabi model [7], so named because of its original setting in magnetic resonance as studied by Rabi long ago. We study the semiclassical case first.

For definiteness and to follow convention, we label our two atomic states $|g\rangle$ (for ground) and $|e\rangle$ (for excited). The energy difference between these states is characterized by the transition frequency $\omega_0 = (E_e - E_g)/\hbar$. This frequency is close to the frequency ω of the driving laser field as shown in Fig. 4.3. The interaction Hamiltonian we write as

$$\hat{H}^{(\mathrm{I})}(t) = \hat{V}_0 \cos\omega t \tag{4.69}$$

where $\hat{V}_0 = -\hat{\mathbf{d}} \cdot \mathbf{E}_0$. We write the state vector as

$$|\psi(t)\rangle = C_g(t)e^{-iE_g t/\hbar}|g\rangle + C_e(t)e^{-iE_e t/\hbar}|e\rangle. \tag{4.70}$$

From the Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)\,|\psi(t)\rangle, \tag{4.71}$$

where

$$\hat{H} = \hat{H}_0 + \hat{V}_0 \cos\omega t, \tag{4.72}$$

we arrive at the coupled set of equations for the amplitudes C_g and C_e:

$$\begin{aligned} \dot{C}_g &= -\frac{i}{\hbar}\mathcal{V}\cos\omega t e^{-i\omega_0 t}\, C_e \\ \dot{C}_e &= -\frac{i}{\hbar}\mathcal{V}\cos\omega t e^{i\omega_0 t}\, C_g \end{aligned} \tag{4.73}$$

where $\mathcal{V} = \langle e|\hat{V}_0|g\rangle = -\hat{\mathbf{d}}_{eg}\cdot\mathbf{E}_0$, which we have taken to be real. As an initial condition we take all the population to be in the ground state: $C_g(0) = 1$ and $C_e(0) = 0$. In Eqs. (4.73) we expand $\cos\omega t$ in exponentials and retain only those terms oscillating at the frequency $\omega_0 - \omega$ to obtain

$$\begin{aligned} \dot{C}_g &= -\frac{i}{2\hbar}\mathcal{V}\exp[i(\omega-\omega_0)t]C_e \\ \dot{C}_e &= -\frac{i}{2\hbar}\mathcal{V}\exp[-i(\omega-\omega_0)t]C_g. \end{aligned} \tag{4.74}$$

Dropping the terms oscillating at $\omega_0 + \omega$, of course, constitutes the RWA. Eliminating C_g, we have for C_e:

$$\ddot{C}_e + i(\omega-\omega_0)\dot{C}_e + \frac{1}{4}\,\frac{\mathcal{V}^2}{\hbar^2}C_e = 0. \tag{4.75}$$

As a trial solution we set

$$C_e(t) = e^{i\lambda t} \tag{4.76}$$

which leads to the two roots

$$\lambda_\pm = \frac{1}{2}\{\Delta \pm [\Delta^2 + \mathcal{V}^2/\hbar^2]^{1/2}\} \tag{4.77}$$

where $\Delta = \omega_0 - \omega$ is the detuning of the atomic transition frequency and the laser field. Thus the general solution is of the form

$$C_e(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t}, \tag{4.78}$$

where from the initial conditions we must have

$$A_\pm = \mp\frac{1}{2\hbar}\,\mathcal{V}[\Delta^2 + \mathcal{V}^2/\hbar^2]^{-1/2}. \tag{4.79}$$

Finally, then, our solution is

$$\begin{aligned} C_e(t) &= i\frac{\mathcal{V}}{\Omega_R\hbar}\, e^{i\Delta t/2}\sin(\Omega_R t/2) \\ C_g(t) &= e^{i\Delta t/2}\left\{\cos(\Omega_R t/2) - i\frac{\Delta}{\Omega_R}\sin(\Omega_R t/2)\right\}, \end{aligned} \tag{4.80}$$

where

$$\Omega_R = [\Delta^2 + \mathcal{V}^2/\hbar^2]^{1/2} \tag{4.81}$$

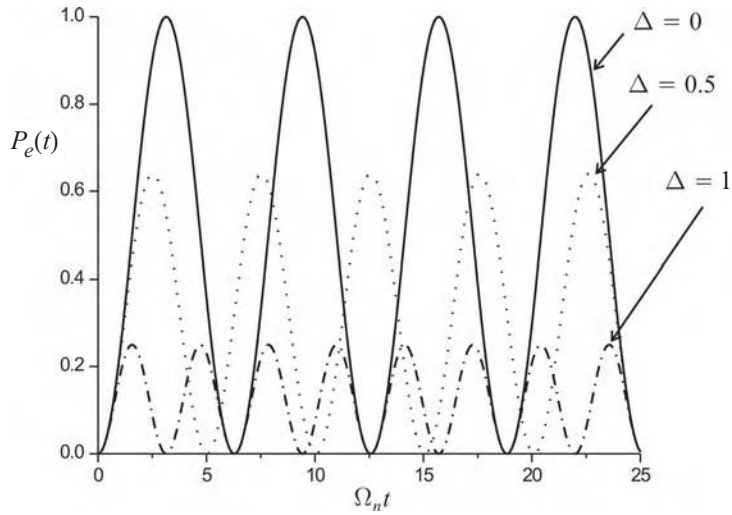


Fig. 4.4. Plots of $P_e(t)$ versus t for various detunings Δ.

is the so-called Rabi frequency. The probability that the atom is in state $|e\rangle$ is

$$\begin{aligned} P_e(t) &= |C_e(t)|^2 \\ &= \frac{\mathcal{V}^2}{\Omega_{\mathrm{R}}^2 \hbar^2} \sin^2(\Omega_{\mathrm{R}} t/2), \end{aligned} \tag{4.82}$$

which is plotted in Fig. 4.4 for various values of Δ. For the case of exact resonance, $\Delta = 0$, we have

$$P_e(t) = \sin^2\left(\frac{\mathcal{V}t}{2\hbar}\right), \tag{4.83}$$

and at the time $t = \pi\hbar/\mathcal{V}$ all the atomic population has been transferred to the excited state.

It is frequently convenient to consider the quantity known as the atomic inversion $W(t)$ defined as the difference in the excited- and ground-state populations:

$$W(t) = P_e(t) - P_g(t), \tag{4.84}$$

which, for the resonant case and with the atom initially in the ground state, is

$$\begin{aligned} W(t) &= \sin^2\left(\frac{\mathcal{V}t}{2\hbar}\right) - \cos^2\left(\frac{\mathcal{V}t}{2\hbar}\right) \\ &= -\cos(\mathcal{V}t/\hbar). \end{aligned} \tag{4.85}$$

Note that for $\Delta = 0$ the Rabi frequency is just $\Omega_{\mathrm{R}} = \mathcal{V}/\hbar$, the oscillation frequency of the atomic inversion. Again, for $t = \pi\hbar/\mathcal{V}$ all the population is transferred to the excited state: $W(\pi\hbar/\mathcal{V}) = 1$. In the parlance of NMR experiments

[8], such transfers are called π-pulses. On the other hand, if $t = \pi\hbar/2\mathcal{V}$ then $W(\pi\hbar/2\mathcal{V}) = 0$ and the population is shared coherently between the excited and ground states with

$$\begin{aligned} C_e(\pi\hbar/2\mathcal{V}) &= \frac{i}{\sqrt{2}}, \\ C_g(\pi\hbar/2\mathcal{V}) &= \frac{1}{\sqrt{2}}, \end{aligned} \tag{4.86}$$

so that

$$|\psi(\pi\hbar/2\mathcal{V})\rangle = \frac{1}{\sqrt{2}}(|g\rangle + i|e\rangle). \tag{4.87}$$

For obvious reasons, the transfer of population from the ground state to that state of Eq. (4.87) is called a $\pi/2$-pulse. Such transfers of populations by π- or $\pi/2$-pulses are standard procedures for manipulating not only spin states in NMR experiments [2] but have become routine for manipulating atomic or ionic states in laser spectroscopy experiments [9].

The results of the pertubation theory may be recovered from the Rabi model either when the size of $\mathcal{V}/2\hbar$ is so small compared with the detuning Δ that it can be neglected in Ω_R or if the radiation field acts only for such a short time that the term $\sin^2(\mathcal{V}t/2\hbar)$ can legitimately be represented by the first term in its expansion. In both cases the depletion of the initial atomic population is small and the perturbation-theory approach remains valid.

4.5 Fully quantum-mechanical model; the Jaynes–Cummings model

We now turn to the quantum electrodynamic version of the Rabi model. In our previous pertubation discussion of an atom interacting with a quantized electromagnetic field, we assumed the field to be a single-mode free field (plane wave). As we just discussed above, a free atom interacts with an infinite number of modes and thus the dynamics is not well described assuming only a single-mode field. On the other hand, it has recently become possible to manufacture environments where the density of modes is significantly different than in free space. We have in mind here small microwave cavities, or in some cases, optical cavities, capable of supporting only a single mode or maybe a few widely spaced (in frequency) modes. Thus in some cases, the ideal single-mode interaction can be realized in the laboratory. We shall discuss some specific examples in Chapter 10 but for now we consider an atom, with levels $|g\rangle$ and $|e\rangle$ as before, interacting with a single-mode cavity field of the form

$$\hat{\mathbf{E}} = \mathbf{e}\left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2}(\hat{a} + \hat{a}^\dagger)\sin(kz) \tag{4.88}$$

where $\mathbf{e}$ is an arbitrarily oriented polarization vector.

The interaction Hamiltonian is now

$$\begin{aligned}\hat{H}^{(\mathrm{I})} &= -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} \\ &= \hat{d}g(\hat{a}+\hat{a}^{\dagger})\end{aligned} \tag{4.89}$$

where

$$g = -\left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2}\sin(kz) \tag{4.90}$$

and where $\hat{d} = \hat{\mathbf{d}}\cdot\mathbf{e}$.

At this point it is convenient to introduce the so-called atomic transition operators

$$\hat{\sigma}_+ = |e\rangle\langle g|, \qquad \hat{\sigma}_- = |g\rangle\langle e| = \hat{\sigma}_+^{\dagger}, \tag{4.91}$$

and the inversion operator

$$\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|. \tag{4.92}$$

These operators obey the Pauli spin algebra

$$\begin{aligned}[\hat{\sigma}_+, \hat{\sigma}_-] &= \hat{\sigma}_3 \\ [\hat{\sigma}_3, \hat{\sigma}_\pm] &= 2\hat{\sigma}_\pm.\end{aligned} \tag{4.93}$$

Only the off-diagonal elements of the dipole operator are nonzero, since by parity consideration $\langle e|\hat{d}|e\rangle = 0 = \langle g|\hat{d}|g\rangle$, so that we may write

$$\begin{aligned}\hat{d} &= d|g\rangle\langle e| + d^*|e\rangle\langle g| \\ &= d\hat{\sigma}_- + d^*\hat{\sigma}_+ = d(\hat{\sigma}_+ + \hat{\sigma}_-)\end{aligned} \tag{4.94}$$

where we have set $\langle e|\hat{d}|g\rangle = d$ and have assumed, without loss of generality, that d is real. Thus the interaction Hamiltonian is

$$\hat{H}^{(\mathrm{I})} = \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}+\hat{a}^{\dagger}) \tag{4.95}$$

where $\lambda = dg/\hbar$.

If we define the level of the energy to be zero halfway between the states $|g\rangle$ and $|e\rangle$ as in Fig. 4.5, then the free atomic Hamiltonian may be written as

$$\hat{H}_{\mathrm{A}} = \frac{1}{2}(E_e - E_g)\hat{\sigma}_3 = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3, \tag{4.96}$$

where $E_e = -E_g = \frac{1}{2}\hbar\omega_0$. The free-field Hamiltonian is, after dropping the zero-point energy term,

$$\hat{H}_{\mathrm{F}} = \hbar\omega\hat{a}^{\dagger}\hat{a}. \tag{4.97}$$

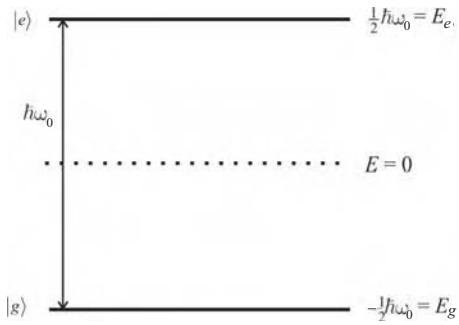


Fig. 4.5. Atomic energy level diagram where the $E = 0$ level is taken halfway between the two levels.

Thus the total Hamiltonian is

$$\begin{aligned}\hat{H} &= \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{F}} + \hat{H}^{(\mathrm{I})} \\ &= \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger).\end{aligned} \tag{4.98}$$

In the free-field case, as we have already shown, the operators $\hat{a}$ and $\hat{a}^+$ evolve as

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}. \tag{4.99}$$

One can show similarly that for the free-atomic case

$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_0 t}. \tag{4.100}$$

Thus we can see that the approximate time dependences of the operator products in Eq. (4.98) are as follows:

$$\begin{aligned}\hat{\sigma}_+\hat{a} &\sim e^{i(\omega_0-\omega)t} \\ \hat{\sigma}_-\hat{a}^\dagger &\sim e^{-i(\omega_0-\omega)t} \\ \hat{\sigma}_+\hat{a}^\dagger &\sim e^{i(\omega+\omega_0)t} \\ \hat{\sigma}_-\hat{a} &\sim e^{-i(\omega+\omega_0)t}.\end{aligned} \tag{4.101}$$

For $\omega_0 \approx \omega$ the last two terms vary much more rapidly than the first two. Furthermore, the last two terms do not conserve energy in contrast to the first two. The term $\hat{\sigma}_+\hat{a}^\dagger$ corresponds to the emission of a photon as the atom goes from the ground to the excited state, whereas $\hat{\sigma}_-\hat{a}$ corresponds to the absorption of a photon as the atom goes from the excited to the ground state. Integrating the time-dependent Schrödinger equation, as in the perturbative case, will lead, for the last two terms, to denominators containing $\omega_0 + \omega$ as compared with $\omega_0 - \omega$ for the first two terms. The reader will not be surprised to learn that we are going to drop the non-energy conserving terms, making the RWA again, so that our Hamiltonian in this approximation is

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger). \tag{4.102}$$

The interaction described by this Hamiltonian is widely referred to as the Jaynes–Cummings model [10].

Before solving for the dynamics for any specific cases we take note of certain constants of the motion. An obvious one is the electron "number"

$$\hat{P}_{\mathrm{E}} = |e\rangle\langle e| + |g\rangle\langle g| = 1, \quad [\hat{H}, \hat{P}_{\mathrm{E}}] = 0 \tag{4.103}$$

valid when no other atomic states can become populated. Another is the excitation number

$$\hat{N}_{\mathrm{e}} = \hat{a}^{\dagger}\hat{a} + |e\rangle\langle e|, \quad [\hat{H}, \hat{N}_{\mathrm{e}}] = 0. \tag{4.104}$$

Using these constants of the motion we may break the Hamiltonian Eq. (4.102) into two commuting parts:

$$\hat{H} = \hat{H}_{\mathrm{I}} + \hat{H}_{\mathrm{II}} \tag{4.105}$$

where

$$\begin{aligned} \hat{H}_{\mathrm{I}} &= \hbar\omega\hat{N}_{\mathrm{e}} + \hbar\left(\frac{\omega_0}{2} - \omega\right)\hat{P}_{\mathrm{E}}, \\ \hat{H}_{\mathrm{II}} &= -\hbar\Delta + \hbar\lambda(\hat{\sigma}_{+}\hat{a} + \hat{\sigma}_{-}\hat{a}^{\dagger}), \end{aligned} \tag{4.106}$$

such that $[\hat{H}_{\mathrm{I}}, \hat{H}_{\mathrm{II}}] = 0$. Clearly, all the essential dynamics is contained in $\hat{H}_{\mathrm{II}}$ whereas $\hat{H}_{\mathrm{I}}$ contributes only overall irrelevant phase factors.

Let us now consider a simple example, with $\Delta = 0$, where the atom is initially in the excited state $|e\rangle$ and the field is initially in the number state $|n\rangle$. The initial state of the atom–field system is then $|i\rangle = |e\rangle|n\rangle$ and is of energy $E_i = \frac{1}{2}\hbar\omega + n\hbar\omega$. State $|i\rangle$ is coupled to (and only to) the state $|f\rangle = |g\rangle|n+1\rangle$ with energy $E_f = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega$. Note that $E_i = E_f$. We write the state vector as

$$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle \tag{4.107}$$

where $C_i(0) = 1$ and $C_f(0) = 0$. Following standard procedures we obtain, from the interaction picture Schrödinger equation $i\hbar d|\psi(t)\rangle/dt = \hat{H}_{\mathrm{II}}|\psi(t)\rangle$ the equations for the coefficients

$$\begin{aligned} \dot{C}_i &= -i\lambda\sqrt{n+1}\,C_f, \\ \dot{C}_f &= -i\lambda\sqrt{n+1}\,C_i. \end{aligned} \tag{4.108}$$

Eliminating C_f we obtain

$$\ddot{C}_i + \lambda^2(n+1)\,C_i = 0. \tag{4.109}$$

The solution matching the initial conditions is

$$C_i(t) = \cos(\lambda t\sqrt{n+1}). \tag{4.110}$$

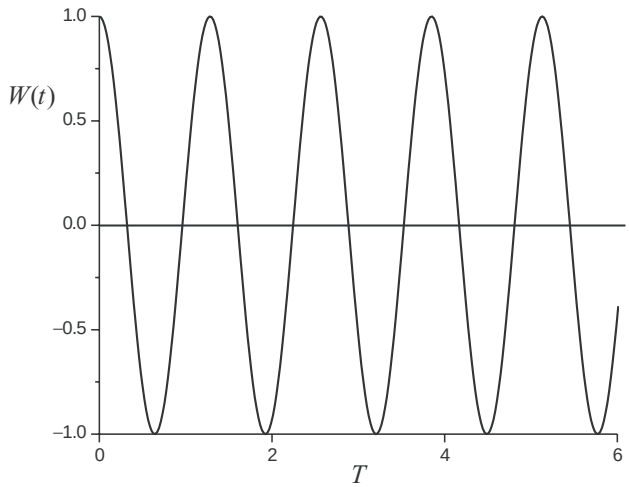


Fig. 4.6. Periodic atomic inversion with the field initially in a number state $|n\rangle$ with $n = 5$ photons.

From Eq. (4.108) we obtain

$$C_f(t) = -i \sin(\lambda t\sqrt{n+1}). \tag{4.111}$$

Thus our solution is

$$\begin{aligned} |\psi(t)\rangle &= \cos(\lambda t\sqrt{n+1})\,|e\rangle|n\rangle \\ &\quad -i \sin(\lambda t\sqrt{n+1})\,|g\rangle|n+1\rangle. \end{aligned} \tag{4.112}$$

The probability that the system remains in the initial state is

$$P_i(t) = |C_i(t)|^2 = \cos^2(\lambda t\sqrt{n+1}) \tag{4.113}$$

while the probability that it makes a transition to the state $|f\rangle$ is

$$P_f(t) = |C_f(t)|^2 = \sin^2(\lambda t\sqrt{n+1}). \tag{4.114}$$

The atomic inversion is given by

$$\begin{aligned} W(t) &= \langle\psi(t)|\hat{\sigma}_3|\psi(t)\rangle \\ &= P_i(t) - P_f(t) \\ &= \cos(2\lambda t\sqrt{n+1}). \end{aligned} \tag{4.115}$$

We may define a quantum electrodynamic Rabi frequency $\Omega(n) = 2\lambda\sqrt{n+1}$ so that

$$W(t) = \cos[\Omega(n)t]. \tag{4.116}$$

Clearly, the atomic inversion for the field initially in a number state is strictly periodic (Fig. 4.6), just as in the semiclassical case of Eq. (4.85) (apart from the minus sign due only to a different initial atomic state) and except for the fact that in

the classical case there must always be a field present initially. But in the quantum-mechanical case there are Rabi oscillations even for the case when $n = 0$. These are the vacuum-field Rabi oscillations [11] and, of course, they have no classical counterpart. They are the result of the atom spontaneously emitting a photon then re-absorbing it, re-emitting it, etc.: an example of reversible spontaneous emission. Such effects can be observed if atoms interact with fields in very high Q cavities. But aside from this, overall, the behavior of the atomic dynamics for a definite number of photons is very much like the semiclassical Rabi model, i.e. it is periodic and regular. Perhaps this is a bit counterintuitive since a number state is the most nonclassical of all the field states. Intuition might suggest that when the field is initially in a coherent state, we should recover the semiclassical, periodic and regular, Rabi oscillations. As we are about to demonstrate, intuition, in this case, fails.

Let us now consider a more general (pure state) solution of the dynamics. We assume the atom is initially in a superposition of states $|e\rangle$ and $|g\rangle$:

$$|\psi(0)\rangle_{\text{atom}} = C_g|g\rangle + C_e|e\rangle, \tag{4.117}$$

and the field is initially in the state

$$|\psi(0)\rangle_{\text{field}} = \sum_{n=0}^{\infty} C_n|n\rangle, \tag{4.118}$$

such that the initial atom–field state is

$$|\psi(0)\rangle = |\psi(0)\rangle_{\text{atom}} \otimes |\psi(0)\rangle_{\text{field}}. \tag{4.119}$$

The solution of the Schrödinger equation is now

$$\begin{aligned} |\psi(t)\rangle = \sum_{n=0}^{\infty} \{&[C_e C_n \cos(\lambda t\sqrt{n+1}) - iC_g C_{n+1}\sin(\lambda t\sqrt{n+1})]\,|e\rangle \\ &+ [-iC_e C_{n-1}\sin(\lambda t\sqrt{n}) + C_g C_n \cos(\lambda t\sqrt{n})]|g\rangle\}\,|n\rangle. \end{aligned} \tag{4.120}$$

In general, this is an entangled state.

For the case of the atom initially in the excited state, where $C_e = 1$ and $C_g = 0$, we may write the solution as

$$|\psi(t)\rangle = |\psi_g(t)\rangle|g\rangle + |\psi_e(t)\rangle|e\rangle \tag{4.121}$$

where $|\psi_g(t)\rangle$ and $|\psi_e(t)\rangle$ are the field components of $|\psi(t)\rangle$ given by

$$\begin{aligned} |\psi_g(t)\rangle &= -i\sum_{n=0}^{\infty} C_n \sin(\lambda t\sqrt{n+1})|n+1\rangle, \\ |\psi_e(t)\rangle &= \sum_{n=0}^{\infty} C_n \cos(\lambda t\sqrt{n+1})|n\rangle. \end{aligned} \tag{4.122}$$

The atomic inversion is

$$\begin{aligned} W(t) &= \langle\psi(t)|\hat{\sigma}_3|\psi(t)\rangle \\ &= \langle\psi_e(t)|\psi_e(t)\rangle - \langle\psi_g(t)|\psi_g(t)\rangle \\ &= \sum_{n=0}^{\infty} |C_n|^2 \cos(2\lambda t\sqrt{n+1}). \end{aligned} \tag{4.123}$$

The result is just the sum of n-photon inversions of Eq. (4.115) weighted with the photon number distribution of the initial field state.

For the coherent state, again that most classical of all quantum states, we have

$$C_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{4.124}$$

and the inversion is

$$W(t) = e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}\cos(2\lambda t\sqrt{n+1}). \tag{4.125}$$

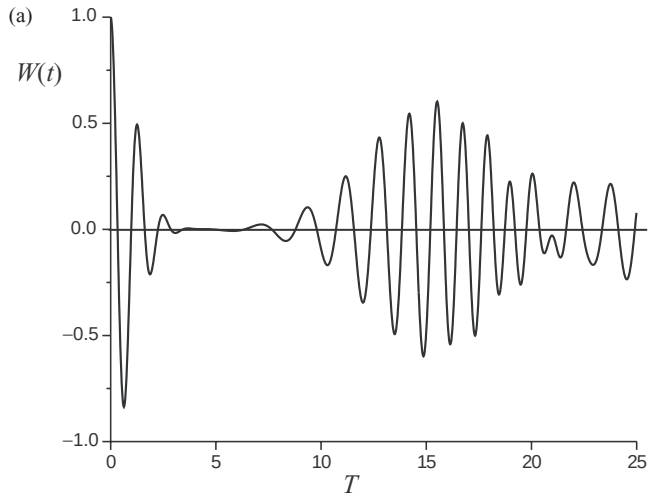
A plot of $W(t)$ versus the scaled time $T = \lambda\, t$ in Fig. 4.7 reveals significant discrepancies between the fully quantized and semiclassical Rabi oscillations. We note first that the Rabi oscillations initially appear to damp out, or collapse. The collapse of the Rabi oscillations was noted fairly early in the study of this "idealized" model interaction [12]. Several years later, perhaps by executing longer runs of a computer program, it was found that after a period of quiescence following the collapse, the Rabi oscillations start to revive [13], although not completely. At longer times one finds a sequence of collapses and revivals, the revivals becoming less distinct as time increases. This collapse and revival behavior of the Rabi oscillations in the fully quantized model is strikingly different than in the semiclassical case where the oscillations have constant amplitude. We must now explain this difference.

First we consider the collapse. The average photon number is $\bar{n} = |\alpha|^2$ so the dominant Rabi frequency is

$$\Omega(\bar{n}) = 2\lambda\sqrt{\bar{n}+1} \approx 2\lambda\sqrt{\bar{n}}, \quad \bar{n} \gg 1. \tag{4.126}$$

But there will be a range of "important" frequencies as a result of the spread of the probabilities $|C_n|^2$ about $\bar{n}$ for photon numbers in the range $\bar{n} \pm \Delta n$; i.e. the frequencies in the range $\Omega(\bar{n} - \Delta n)$ to $\Omega(\bar{n} + \Delta n)$. The collapse time t_c may be estimated from the time-frequency "uncertainty" relation

$$t_c[\Omega(\bar{n} + \Delta n) - \Omega(\bar{n} - \Delta n)] \simeq 1 \tag{4.127}$$



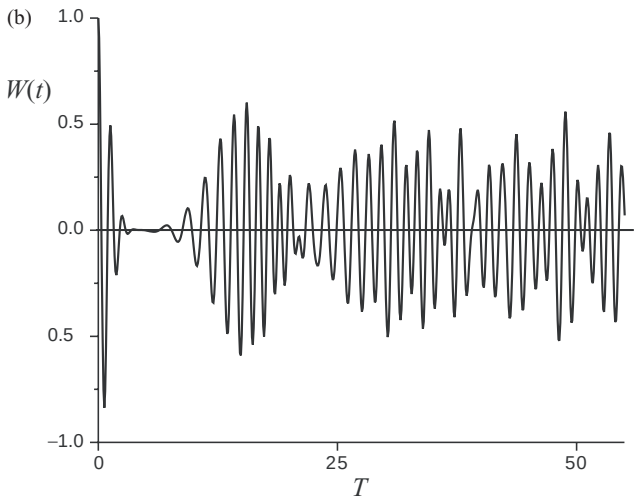


Fig. 4.7. (a) Atomic inversion with the field initially in a coherent state $\bar{n} = 5$. (b) Same as (a) but showing the evolution for a longer time, beyond the first revival. Here, T is the scaled time λt.

where the spread of frequencies is responsible for the "dephasing" of the Rabi oscillations. For the coherent state, $\Delta n = \bar{n}^{1/2}$, and with

$$\begin{aligned}\Omega(\bar{n} \pm \bar{n}^{1/2}) &\simeq 2\lambda[\bar{n} \pm \bar{n}^{1/2}]^{1/2} \\ &= 2\lambda\bar{n}^{1/2}\left[1 \pm \frac{1}{\bar{n}^{1/2}}\right]^{1/2} \\ &\simeq 2\lambda\bar{n}^{1/2}\left(1 \pm \frac{1}{2\bar{n}^{1/2}}\right) \\ &= 2\lambda\bar{n}^{1/2} \pm \lambda \end{aligned} \tag{4.128}$$

it follows that

$$t_c[\Omega(\bar{n}+\bar{n}^{1/2})-\Omega(\bar{n}-\bar{n}^{1/2})] \simeq t_c 2\lambda \simeq 1 \tag{4.129}$$

and thus $t_c \simeq (2\lambda)^{-1}$, which is independent of $\bar{n}$.

The preceding "derivation" of the collapse time is not very rigorous. We shall now give a more rigorous derivation. We expand $[n+1]^{1/2}$ about $\bar{n}$ as

$$[n+1]^{1/2} = [\bar{n}+1]^{1/2} + \frac{1}{2(\bar{n}+1)^{1/2}}(n-\bar{n}) + \cdots \tag{4.130}$$

so that we may approximate the inversion as

$$W(t) \simeq \frac{1}{2}e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \Big[e^{2i\lambda t(\bar{n}+1)^{1/2}} e^{i\lambda n t/(\bar{n}+1)^{1/2}} e^{-i\lambda t\bar{n}/(\bar{n}+1)^{1/2}} + e^{-2i\lambda t(\bar{n}+1)^{1/2}} e^{-i\lambda n t/(\bar{n}+1)^{1/2}} e^{i\lambda t\bar{n}/(\bar{n}+1)^{1/2}} \Big]. \tag{4.131}$$

Note that

$$\sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} e^{in\lambda t/(\bar{n}+1)^{1/2}} = \exp\left[\bar{n}\, e^{i\lambda t/(\bar{n}+1)^{1/2}}\right]. \tag{4.132}$$

For short times t

$$e^{i\lambda t/(\bar{n}+1)^{1/2}} \simeq 1 + i\lambda t/(\bar{n}+1)^{1/2} - \frac{\lambda^2 t^2}{2(\bar{n}+1)} \tag{4.133}$$

and thus

$$e^{\bar{n}e^{i\lambda t/(\bar{n}+1)^{1/2}}} \simeq e^{\bar{n}} e^{i\lambda t\bar{n}/(\bar{n}+1)} e^{-\frac{1}{2}\frac{\lambda^2 t^2}{(\bar{n}+1)}\bar{n}}. \tag{4.134}$$

Putting all of this together we arrive at

$$W(t) \simeq \cos\{2\lambda t(\bar{n}+1)^{1/2}\} \exp\left\{-\frac{1}{2}\frac{\lambda^2 t^2 \bar{n}}{\bar{n}+1}\right\}, \tag{4.135}$$

valid for a short time. The inversion evidently exhibits a Gaussian decay law with a decay time given by

$$t_c = \frac{\sqrt{2}}{\lambda}\sqrt{\frac{\bar{n}+1}{\bar{n}}} \simeq \frac{\sqrt{2}}{\lambda}, \qquad \bar{n} \gg 1 \tag{4.136}$$

which, apart from a numerical constant of the order of unity, agrees with our previous estimate.

Now let us examine the phenomenon of the revivals. $W(t)$ obviously consists of a sum of oscillating terms, each term oscillating at a particular Rabi frequency $\Omega(n) = 2\lambda\sqrt{n+1}$. If two neighboring terms are oscillating 180° out of phase with each other we expect at least an approximate cancellation of these terms. On the other hand, if the neighboring terms are in phase with each other we expect a constructive interference. In fact, this should be so whenever neighboring phases

differ by some multiple of 2π. Since only those important frequencies around $\bar{n}$ will contribute, revivals should occur for times $t = t_R$ such that

$$[\Omega(\bar{n}+1) - \Omega(\bar{n})]\, t_R = 2\pi k, \quad k = 0, 1, 2, \ldots \tag{4.137}$$

holds. Expanding $\Omega(\bar{n})$ and $\Omega(\bar{n}+1)$ we easily arrive at $t_R = (2\pi/\lambda)\bar{n}^{1/2}k$, $(\bar{n} \gg 1)$. More rigorously, using Eq. (4.118) we have

$$W(t) \simeq \cos\left[2\lambda t(\bar{n}+1)^{1/2} + \lambda t\bar{n}/(\bar{n}+1)^{-1/2} - \bar{n}\sin\left(\frac{\lambda t}{(\bar{n}+1)^{1/2}}\right)\right]$$
$$\times \exp\left\{-\bar{n}\left[1 - \cos\left(\frac{\lambda t}{(\bar{n}+1)^{1/2}}\right)\right]\right\}. \tag{4.138}$$

Obviously, the amplitude will be a maximum wherever the time $t = t_R = (2\pi/\lambda)(\bar{n}+1)^{1/2}k \approx (2\pi/\lambda)^{1/2}\bar{n}^{1/2}k$, $(\bar{n} \gg 1)$, in agreement with the previous analysis. In Chapter 10 we discuss two experiments, one in the context of cavity quantum electrodynamics and the other in the context of the center-of-mass motion of a trapped ion, where the predicted collapse and revival of the Rabi oscillations have been observed.

4.6 The dressed states

There are many ways to solve for the dynamics of the Jaynes–Cummings model (see the reviews [14]). In Section 4.5 we have solved the time-dependent Schrödinger equation first for a field containing n photons and then, by simple extrapolation, for the case of a field in a superposition of the number states. Another important way to obtain the dynamics is first to find the stationary states of the Jaynes–Cummings Hamiltonian. For reasons that should become clear shortly, these eigenstates are called the "dressed" states [15].

Consider once again the Jaynes–Cummings model Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \tag{4.139}$$

where we have *not* assumed the resonance condition $\omega = \omega_0$ at this point. In terms of the field number states, the interaction term in $\hat{H}$ causes only transitions of the type

$$|e\rangle|n\rangle \leftrightarrow |g\rangle|n+1\rangle \tag{4.140}$$

or

$$|e\rangle|n-1\rangle \leftrightarrow |g\rangle|n\rangle. \tag{4.141}$$

The product states $|e\rangle|n-1\rangle$, $|g\rangle|n\rangle$, etc., are sometimes referred to as the "bare" states of the Jaynes–Cummings model; they are product states of the unperturbed atom and field. For a fixed n, the dynamics is completely confined to the

two-dimensional space of product states, either ($|e\rangle|n-1\rangle$, $|g\rangle|n\rangle$) or ($|e\rangle|n\rangle$, $|g\rangle|n-1\rangle$). We define the following product states for a given n:

$$\begin{aligned} |\psi_{1n}\rangle &= |e\rangle|n\rangle \\ |\psi_{2n}\rangle &= |g\rangle|n+1\rangle. \end{aligned} \tag{4.142}$$

Obviously $\langle\psi_{1n} \mid \psi_{2n}\rangle = 0$. Using this basis we obtain the matrix elements of $\hat{H}$, $H_{ij}^{(n)} = \langle\psi_{in}|\hat{H}|\psi_{jn}\rangle$, which are

$$\begin{aligned} H_{11}^{(n)} &= \hbar\left[n\omega + \tfrac{1}{2}\omega_0\right], \\ H_{22}^{(n)} &= \hbar\left[(n+1)\omega - \tfrac{1}{2}\omega_0\right], \\ H_{12}^{(n)} &= \hbar\lambda\sqrt{n+1} = H_{21}^{(n)}. \end{aligned} \tag{4.143}$$

Thus in the 2×2 subspace of Eq. (4.142) we obtain the matrix representation of $\hat{H}$:

$$\boldsymbol{H}^{(n)} = \begin{bmatrix} n\omega + \frac{1}{2}\hbar\omega_0 & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{bmatrix}. \tag{4.144}$$

This matrix is "self-contained" since, as we have said, the dynamics connects only those states for which the photon number changes by ± 1. For a given n, the energy eigenvalues of $\boldsymbol{H}^{(n)}$ are as follows:

$$E_{\pm}(n) = \left(n + \frac{1}{2}\right)\hbar\omega \pm \hbar\Omega_n(\Delta) \tag{4.145}$$

where

$$\Omega_n(\Delta) = [\Delta^2 + 4\lambda^2(n+1)]^{1/2} \quad (\Delta = \omega_0 - \omega) \tag{4.146}$$

is the Rabi frequency which now includes the effects of the detuning Δ. Obviously, for $\Delta = 0$ we obtain $\Omega_n(0) = 2\lambda\sqrt{n+1}$, the same quantum electrodynamic Rabi frequencies seen earlier. The eigenstates $|n, \pm\rangle$ associated with the energy eigenvalues are given by

$$\begin{aligned} |n,+\rangle &= \cos(\Phi_n/2)|\psi_{1n}\rangle + \sin(\Phi_n/2)|\psi_{2n}\rangle \\ |n,-\rangle &= -\sin(\Phi_n/2)|\psi_{1n}\rangle + \cos(\Phi_n/2)|\psi_{2n}\rangle \end{aligned} \tag{4.147}$$

where the angle Φ_n is defined through

$$\Phi_n = \tan^{-1}\left(\frac{2\lambda\sqrt{n+1}}{\Delta}\right) = \tan^{-1}\left(\frac{\Omega_n(0)}{\Delta}\right) \tag{4.148}$$

and where

$$\begin{aligned} \sin(\Phi_n/2) &= \frac{1}{\sqrt{2}}\left[\frac{\Omega_n(\Delta) - \Delta}{\Omega_n(\Delta)}\right]^{1/2} \\ \cos(\Phi_n/2) &= \frac{1}{\sqrt{2}}\left[\frac{\Omega_n(\Delta) + \Delta}{\Omega_n(\Delta)}\right]^{1/2}. \end{aligned} \tag{4.149}$$

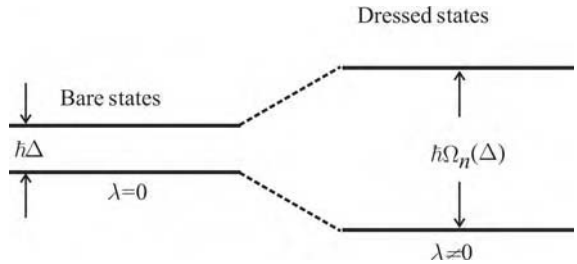


Fig. 4.8. Energy-level splitting due to the interaction of the atom with a quantized field. The split levels on the right are the energy levels of the dressed states.

The states $|n,\pm\rangle$ are often referred to as "dressed states" or as the Jaynes–Cummings doublet. The bare states $|\psi_{1n}\rangle$ and $|\psi_{2n}\rangle$, of energies $E_{1n} = \hbar(\omega_0/2 + \hbar\omega)$ and $E_{2n} = \hbar[-\omega_0/2 + (n+1)\omega]$ respectively, are each further split in energy owing to the interaction as indicated in Fig. 4.8. The splitting of the bare states into the dressed states is a kind of Stark shift, often called the AC, or dynamic, Stark shift. Note that in the limit of exact resonance, $\Delta = 0$, the bare states are degenerate but the splitting of the dressed states of course, remains. In this limit, the dressed states are related to the bare states according to

$$\begin{aligned} |n,+\rangle &= \frac{1}{\sqrt{2}}(|e\rangle|n\rangle + |g\rangle|n+1\rangle) \\ |n,-\rangle &= \frac{1}{\sqrt{2}}(-|e\rangle|n\rangle + |g\rangle|n+1\rangle). \end{aligned} \tag{4.150}$$

To see how the dressed states can be used to obtain the dynamics for rather general initial states, let us consider the specific case of a field prepared in some superposition of number states

$$|\psi_f(0)\rangle = \sum_n C_n|n\rangle \tag{4.151}$$

where an atom, prepared in state $|e\rangle$, is injected into the field. Thus the initial state of the atom–field system is

$$\begin{aligned} |\psi_{\rm af}(0)\rangle &= |\psi_f(0)\rangle|e\rangle \\ &= \sum_n C_n|n\rangle|e\rangle = \sum_n C_n|\psi_{1n}\rangle. \end{aligned} \tag{4.152}$$

From Eqs. (4.147), we obtain $|\psi_{1n}\rangle$ in terms of the dressed states $|n,\pm\rangle$ as

$$|\psi_{1n}\rangle = \cos(\Phi_n/2)|n,+\rangle - \sin(\Phi_n/2)|n,-\rangle \tag{4.153}$$

and thus

$$|\psi_{\rm af}(0)\rangle = \sum_n C_n[\cos(\Phi_n/2)|n,+\rangle - \sin(\Phi_n/2)|n,-\rangle]. \tag{4.154}$$

Since the dressed states $|n, \pm\rangle$ are stationary states of the atom–field system, then the state vector for times $t > 0$ is just given by

$$|\psi_{\rm af}(t)\rangle = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]|\psi_{af}(0)\rangle$$
$$= \sum_n C_n\Big[\cos(\Phi_n/2)|n,+\rangle e^{-iE_+(n)t/\hbar} - \sin(\Phi_n/2)|n,-\rangle e^{-iE_-(n)t/\hbar}\Big]. \tag{4.155}$$

Of course, the entire result may now be recast back into the more familiar "bare" state basis by simply substituting $|n, \pm\rangle$ from Eqs. (4.147). In the limit $\Delta = 0$ we will recover the previous result of Eq. (4.120). The demonstration of this is left as an exercise.

4.7 Density-operator approach: application to thermal states

So far, we have considered only cases where the field and the atom are initially in pure states. In general though, one or both of the subsystems may initially be in mixed states which requires us to seek a solution in terms of the density operator. For example, the field may initially be in a thermal state described by the density operator of Eq. (2.144). In studying the case of a two-level atom interacting with a thermal state described by a density operator, we are afforded yet another way to solve for the dynamics of the Jaynes–Cummings model.

We shall work in the interaction picture and once again assume the resonance condition so that the dynamics is driven by

$$\hat{H}_{\rm I} = \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-). \tag{4.156}$$

If $\hat{\rho}(t)$ is the density operator of the atom–field system at time t, the evolution of the system is given by

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{\rm I}, \hat{\rho}] \tag{4.157}$$

whose solution may be written as

$$\hat{\rho}(t) = \hat{U}_{\rm I}(t)\hat{\rho}(0)\hat{U}_{\rm I}^\dagger(t) \tag{4.158}$$

where

$$\hat{U}_{\rm I}(t) = \exp[-i\hat{H}_{\rm I}t/\hbar]$$
$$= \exp[-i\lambda t(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)]. \tag{4.159}$$

In the two-dimensional atomic subspace, the operators $\hat{\sigma}_\pm$ and $\hat{\sigma}_3$ have the matrix representations

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.160}$$

where we have used the convention

$$\sigma_j = \begin{pmatrix} \langle e|\hat{\sigma}_j|e\rangle & \langle e|\hat{\sigma}_j|g\rangle \\ \langle g|\hat{\sigma}_j|e\rangle & \langle g|\hat{\sigma}_j|g\rangle \end{pmatrix} \quad j = \pm,\ 3. \tag{4.161}$$

In this two-dimensional subspace the evolution operator $\hat{U}_{\rm I}(t)$ may be expanded as

$$\hat{\boldsymbol{U}}_{\rm I}(t) = \begin{pmatrix} \hat{C}(t) & \hat{S}'(t) \\ \hat{S}(t) & \hat{C}'(t) \end{pmatrix} \tag{4.162}$$

where

$$\hat{C}(t) = \cos(\lambda t\sqrt{\hat{a}\hat{a}^\dagger}) \tag{4.163}$$

$$\hat{S}(t) = -i\hat{a}^\dagger \frac{\sin(\lambda t\sqrt{\hat{a}\hat{a}^+})}{\sqrt{\hat{a}\hat{a}^\dagger}} \tag{4.164}$$

$$\hat{C}'(t) = \cos(\lambda t\sqrt{\hat{a}^\dagger\hat{a}}) \tag{4.165}$$

$$\hat{S}'(t) = -i\hat{a}\frac{\sin(\lambda t\sqrt{\hat{a}^\dagger\hat{a}})}{\sqrt{\hat{a}^\dagger\hat{a}}}. \tag{4.166}$$

(The operators $\hat{C}$, $\hat{S}$, etc., here are not to be confused with the cosine and sine operators of the phase introduced in Chapter 2.) The Hermitian adjoint of $\hat{U}_{\rm I}(t)$ of Eq. (4.162) is just

$$\hat{U}_{\rm I}^\dagger(t) = \hat{U}_{\rm I}(-t) = \begin{pmatrix} \hat{C}(t) & -\hat{S}'(t) \\ -\hat{S}(t) & \hat{C}'(t) \end{pmatrix}. \tag{4.167}$$

We now suppose that at $t = 0$ the density operator for the atom–field system factors into field and atomic parts:

$$\hat{\rho}(0) = \hat{\rho}^{\rm F}(0) \otimes \hat{\rho}^{\rm A}(0). \tag{4.168}$$

We further suppose (to work out a particular example) that the atom is initially in the excited state $|e\rangle$ such that $\hat{\rho}^{\rm A}(0) = |e\rangle\langle e|$. The corresponding density matrix for the atom (using the convention of Eq. (4.161)) is

$$\rho^{\rm A}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.169}$$

and thus we may write for the system

$$\hat{\rho}(0) = \begin{pmatrix} \hat{\rho}^{\rm F}(0) & 0 \\ 0 & 0 \end{pmatrix} = \hat{\rho}^{\rm F}(0) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.170}$$

Using Eqs. (4.162), (4.167) and (4.169) in Eq. (4.158) we find that

$$\hat{\rho}(t) = \begin{pmatrix} \hat{C}(t)\hat{\rho}(0)\hat{C}(t) & -\hat{C}(t)\hat{\rho}^{\rm F}(0)\hat{S}'(t) \\ \hat{S}(t)\hat{\rho}^{\rm F}(0)\hat{C}(t) & -\hat{S}(t)\hat{\rho}^{\rm F}(0)\hat{S}'(t) \end{pmatrix}. \tag{4.171}$$

The reduced density operator of the field is found by tracing over the atomic states and thus

$$\hat{\rho}^{\mathrm{F}}(t) = \mathrm{Tr}_{\mathrm{A}}\hat{\rho}(t) = \hat{C}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{C}(t) - \hat{S}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{S}'(t). \tag{4.172}$$

The density matrix elements for the field are

$$\begin{aligned}\hat{\rho}^{\mathrm{F}}_{nm}(t) &\equiv \langle n|\hat{\rho}^{\mathrm{F}}(t)|m\rangle \\ &= \langle n|\hat{C}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{C}(t)|m\rangle - \langle n|\hat{S}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{S}'(t)|m\rangle.\end{aligned} \tag{4.173}$$

On the other hand, tracing over the field states we obtain the reduced density operator of the atom:

$$\hat{\rho}^{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{F}}\hat{\rho}(t) = \sum_{n=0}^{\infty}\langle n|\hat{\rho}(t)|n\rangle. \tag{4.174}$$

The density operator matrix elements are given by

$$\langle i|\hat{\rho}^{\mathrm{A}}(t)|j\rangle = \sum_{n=0}^{\infty}\langle i,n|\hat{\rho}(t)|j,n\rangle = \rho^{\mathrm{A}}_{ij}(t) \tag{4.175}$$

where $i, j = e, g$. The diagonal elements $\rho^{\mathrm{A}}_{ee}(t)$ and $\rho^{\mathrm{A}}_{gg}(t)$ are the populations of the excited and ground states, respectively, and satisfy the condition

$$\rho^{\mathrm{A}}_{gg}(t) + \rho^{\mathrm{A}}_{ee}(t) \ = 1. \tag{4.176}$$

The atomic inversion is given by

$$W(t) = \rho^{\mathrm{A}}_{ee}(t) - \rho^{\mathrm{A}}_{gg}(t) \ = 2\rho^{\mathrm{A}}_{ee}(t) - 1. \tag{4.177}$$

From Eqs. (4.171) and (4.175) we find that

$$\begin{aligned}\rho^{\mathrm{A}}_{ee}(t) &\equiv \sum_{n=0}^{\infty}\langle n|\hat{C}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{C}(t)|n\rangle \\ &= \sum_{n=0}^{\infty}\langle n|\,\hat{\rho}^{\mathrm{F}}(0)|n\rangle\cos^2(\lambda t\sqrt{n+1}).\end{aligned} \tag{4.178}$$

If the field is initially in a pure state

$$|\psi_{\mathrm{F}}\rangle = \sum_{n=0}^{\infty}C_n|n\rangle \tag{4.179}$$

then

$$\hat{\rho}^{\mathrm{F}}(0) = |\psi_{\mathrm{F}}\rangle\langle\psi_{\mathrm{F}}| \tag{4.180}$$

and thus

$$\rho^{\mathrm{A}}_{ee}(t) = \sum_{n=0}^{\infty}|C_n|^2\cos^2(\lambda t\sqrt{n+1}) \tag{4.181}$$

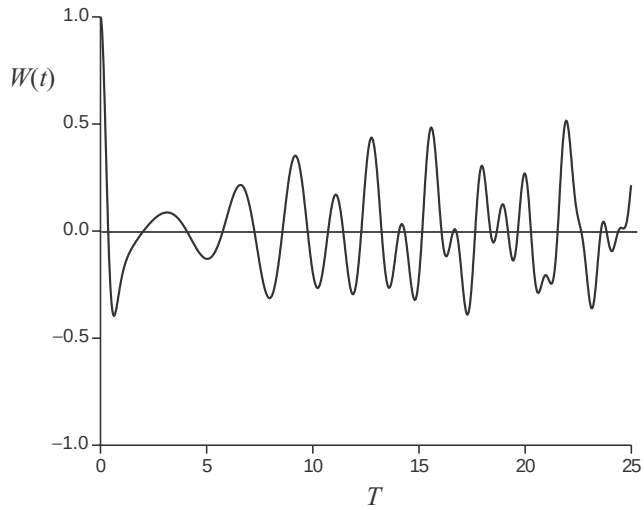


Fig. 4.9. The atomic inversion versus time for the atom initially in the excited state and the field initially in a thermal state with $\bar{n} = 2$. Again, $T = \lambda t$.

which, through Eq. (4.177), yields the atomic inversion found in Eq. (4.123). But suppose the field is initially in a thermal state (a mixed state) where

$$\hat{\rho}^{\mathrm{F}}(0) = \hat{\rho}_{\mathrm{Th}} = \sum_n P_n |n\rangle\langle n| \tag{4.182}$$

where P_n is given by Eq. (2.145). From Eq. (4.178) we ultimately obtain the atomic inversion for an atom resonantly interacting with a thermal field as [16]

$$W(t) = \sum_{n=0}^{\infty} P_n \cos(2\lambda t\sqrt{n+1}). \tag{4.183}$$

In Fig. 4.9 we plot $W(t)$ versus λt for a thermal field containing an average photon number of $\bar{n} = 2$. We leave to the reader an analysis of the observed behavior.

4.8 The Jaynes–Cummings model with large detuning: a dispersive interaction

In the foregoing we have mostly assumed that the detuning $\Delta = \omega_0 - \omega = 0$. An important variation on the original Jaynes–Cummings model is the situation in which the detuning is large enough such that direct atomic transitions do not occur but where nevertheless "dispersive" interactions between a single atom and a cavity field do occur [17]. This version of the Jaynes–Cummings model is important in a number of applications related to fundamental tests of quantum mechanics, some of which are discussed in Chapter 10.

As is shown in Appendix C, the effective atom–field interaction Hamiltonian in the case of large detuning is given by

$$\hat{H}_{\mathrm{eff}} = \hbar\chi[\hat{\sigma}_+\hat{\sigma}_- + \hat{a}^\dagger\hat{a}\,\hat{\sigma}_3], \tag{4.184}$$

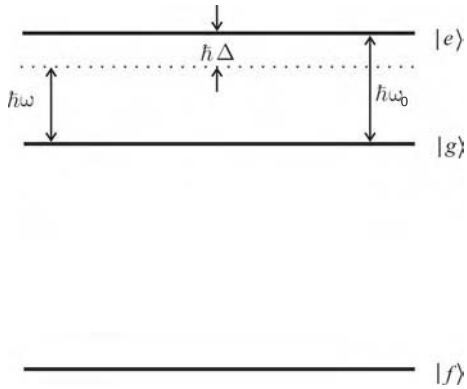


Fig. 4.10. The levels $|e\rangle$ and $|g\rangle$ are out of resonance with the field enough so that there are no direct transitions between them and only the dispersive interaction occurs. The state $|f\rangle$ is so far out of resonance with state $|g\rangle$ and the field that not even a dispersive interaction is present.

where $\chi = \lambda^2/\Delta$. Note that $\hat{\sigma}_+\hat{\sigma}_- = |e\rangle\langle e|$. Suppose the initial state of the atom–field system is $|\psi(0)\rangle = |g\rangle|n\rangle$, that is, the atom is in the ground state and the field is in a number state. Then, according to the interaction Hamiltonian of Eq. (4.184), the state at time $t > 0$ is

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = e^{i\chi nt}|g\rangle|n\rangle \tag{4.185}$$

while for the initial state $|\psi(0)\rangle = |e\rangle|n\rangle$ we have

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = e^{i\chi(n+1)t}|e\rangle|n\rangle. \tag{4.186}$$

Evidently, nothing very interesting happens, just the production of unmeasurable phase factors. On the other hand, for initial coherent states of the field we have, for $|\psi(0)\rangle = |g\rangle|\alpha\rangle$,

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = |g\rangle|\alpha\, e^{i\chi t}\rangle, \tag{4.187}$$

and for $|\psi(0)\rangle = |e\rangle|\alpha\rangle$ we have

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = e^{-i\chi t}|e\rangle|\alpha\, e^{-i\chi t}\rangle. \tag{4.188}$$

We notice that the coherent-state amplitude is rotated in phase space by the angle χt but that the direction of the rotation depends on the state of the atom. Suppose now that the atom is prepared in a superposition of the ground and excited states. For simplicity we take this to be a "balanced" state of the form $|\psi_{\text{atom}}\rangle = (|g\rangle + e^{i\phi}|e\rangle)/\sqrt{2}$, where ϕ is some phase. With the initial state $|\psi(0)\rangle = |\psi_{\text{atom}}\rangle|\alpha\rangle$ we have

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\Big(|g\rangle|\alpha\, e^{i\chi t}\rangle + e^{-i(\chi t-\phi)}|e\rangle|\alpha\, e^{-i\chi t}\rangle\Big). \tag{4.189}$$

This state is a bit more interesting, in fact, much more interesting as, in general, there is entanglement between the atom and the field. If we take $\chi t = \pi/2$ we then have

$$\left|\psi\left(\frac{\pi}{2\chi}\right)\right\rangle = \frac{1}{\sqrt{2}}(|g\rangle|i\,\alpha\,\rangle - i\, e^{i\phi}|e\rangle|-i\,\alpha\,\rangle). \tag{4.190}$$

Notice that, in terms of our phase-space pictures, the two coherent states in Eq. (4.190) are separated by 180°, the maximal separation. Coherent states differing in phase by 180° are also maximally *distinguishable* in the sense that there is essentially no overlap between the two states, at least if $|\alpha|$ is large enough. In fact, this is the case even with $|\alpha|$ as low as $\sqrt{2}$. With very large values of $|\alpha|$, the two coherent states are said to be *macroscopically* distinguishable and, for moderate values, *mesoscopically* distinguishable. The entangled state of Eq. (4.190) might bring to mind the tale of Schrödinger's ill-fated cat [18], suspended in a state of limbo, suspended in an entanglement between life and death and a non-decayed or decayed radioactive *microscopic* atom. Symbolically, the entangled state in Schrödinger's famous "paradox" is thus:

$$|\psi_{\text{atom}-\text{cat}}\rangle = \frac{1}{\sqrt{2}}[|\text{atom not decayed}\rangle|\text{cat alive}\rangle + |\text{atom decayed}\rangle|\text{cat dead}\rangle]. \quad (4.191)$$

The parallel between this state and that of Eq. (4.190) is obvious; the states of the two-level atom play the role of the radioactive atom and the two phase-separated coherent field states that of the cat.

Finally, there is another initial atomic state that is often considered. Suppose there is another atomic state, that we denote $|f\rangle$, of energy $E_f \ll E_g$ as pictured in Fig. 4.10, and of parity opposite that of $|g\rangle$. The cavity is assumed to support no mode resonant with the $f \leftrightarrow g$ transition and we further assume that the state $|f\rangle$ is so far out of resonance with the available cavity field mode that there is no discernable dispersive interaction either. Thus with the atom initially prepared in state $|f\rangle$ and with the field initially in the coherent state $|\alpha\rangle$, the initial product state $|f\rangle|\alpha\rangle$ remains as such; i.e. it does not evolve. Now suppose the atom is prepared in a superposition of the form $|\psi_{\text{atom}}\rangle = (|g\rangle + e^{i\phi}|f\rangle)/\sqrt{2}$. The initial-state atom–field state $|\psi(0)\rangle = |\psi_{\text{atom}}\rangle|\alpha\rangle$ is easily seen to evolve into

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|g\rangle|\alpha\, e^{i\chi t}\rangle + e^{i\phi}|f\rangle|\alpha\rangle). \quad (4.192)$$

In this case, only one component of the initial atomic superposition causes a phase shift in the coherent state, an advantage in certain applications. For $\chi t = \pi/2$ we have

$$\left|\psi\left(\frac{\pi}{2\chi}\right)\right\rangle = \frac{1}{\sqrt{2}}(|g\rangle|-\alpha\rangle + e^{i\phi}|f\rangle|\alpha\rangle), \quad (4.193)$$

another version of the Schrödinger-cat state.

In Chapters 7, 8, and 10 we shall further elaborate on issues related to Schrödinger's cat.

4.9 Extensions of the Jaynes–Cummings model

There are many possible extensions of the original Jaynes–Cummings model involving various types of alternative interactions. Among them are models

involving two-photon transitions, multimode and multilevel models, Raman coupled models, two-channel models, etc. We shall not discuss these models here but we do refer to the various review articles that have appeared [14, 19] and references therein. Some of these extensions will appear in the context of homework problems. Further, it turns out that Jaynes–Cummings types of interaction also occur in the context of the vibrational motion of an ion in an electromagnetic trap. The simplest example of these will be discussed in Chapter 10.

4.10 Schmidt decomposition and von Neumann entropy for the Jaynes–Cummings model

We finish this chapter with a discussion of the Schmidt decomposition and the related von Neumann entropy as they pertain to the Jaynes–Cummings model. As this system is bipartite, a Schmidt decomposition is assured, as discussed in Appendix A. We have already presented the solution of the time-dependent Schrödinger equation in Eq. (4.120), which we rewrite here as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} [a_n(t)|g\rangle|n\rangle + b_n(t)|e\rangle|n\rangle], \tag{4.194}$$

where

$$\begin{aligned} a_n(t) &= C_g C_n \cos(\lambda t\sqrt{n}) - iC_e C_{n-1} \sin(\lambda t\sqrt{n}), \\ b_n(t) &= C_e C_n \cos(\lambda t\sqrt{n+1}) - iC_g C_{n+1} \sin(\lambda t\sqrt{n+1}). \end{aligned} \tag{4.195}$$

But according to the Schmidt decomposition, for any instant in time t we can always find bases $\{|u_i(t)\rangle\}$ for the atom and $\{|v_i(t)\rangle\}$ for the field such that the pure state of the system can be written as

$$|\Psi(t)\rangle = g_1(t)|u_1(t)\rangle|v_1(t)\rangle + g_2(t)|u_2(t)\rangle|v_2(t)\rangle. \tag{4.196}$$

The reduced density matrices of the atom and field in these bases are identical:

$$\begin{aligned} \rho_u(t) &= \begin{pmatrix} |g_1(t)|^2 & 0 \\ 0 & |g_2(t)|^2 \end{pmatrix} \\ \rho_v(t) &= \begin{pmatrix} |g_1(t)|^2 & 0 \\ 0 & |g_2(t)|^2 \end{pmatrix}. \end{aligned} \tag{4.197}$$

In order to find the coefficients $g_1(t)$ and $g_2(t)$ and the eigenvectors $|u_i\rangle$ and $|v_i\rangle$ we first calculate the reduced density operator of the atom in the bare basis specified by $|e\rangle$ and $|g\rangle$ and obtain

$$\rho_u = \begin{pmatrix} \sum_{n=0}^{\infty} |a_n(t)|^2 & \sum_{n=0}^{\infty} a_n(t) b_n^*(t) \\ \sum_{n=0}^{\infty} b_n(t) a_n^*(t) & \sum_{n=0}^{\infty} |b_n(t)|^2 \end{pmatrix}. \tag{4.198}$$

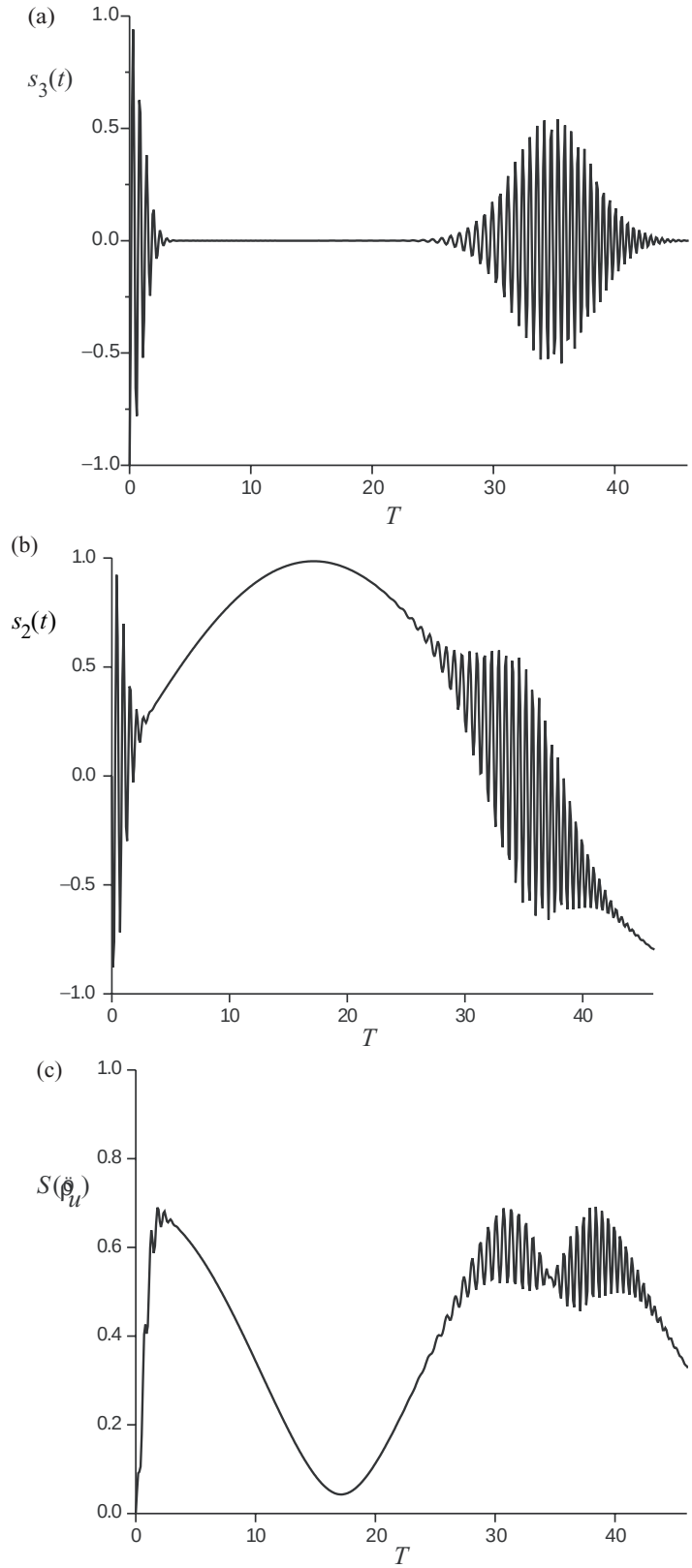


Fig. 4.11. Plots of s_3, s_2 and $S(\hat{\rho}_u)$ against the scaled time $\lambda\, t$. Again $T = \lambda t$.

Then we can use the parametrization of the density matrix in terms of the components of the Bloch vector as in Eq. (A25) to obtain

$$\begin{aligned} s_1(t) &= \sum_{n=0}^{\infty}\Big[a_n(t)b_n^*(t) + b_n(t)a_n^*(t)\Big], \\ s_2(t) &= -i\sum_{n=0}^{\infty}\Big[a_n(t)b_n^*(t) - b_n(t)a_n^*(t)\Big], \\ s_3(t) &= \sum_{n=0}^{\infty}[|a_n(t)|^2 - |b_n(t)|^2]. \end{aligned} \tag{4.199}$$

Then, as discussed in Appendix A, the coefficients $g_1(t)$ and $g_2(t)$ can be expressed in terms of the length of the Bloch vector according to

$$g_1(t) = \frac{1}{2}[1 + |\mathbf{s}(t)|], \quad g_2(t) = \frac{1}{2}[1 - |\mathbf{s}(t)|]. \tag{4.200}$$

Notice that the field mode, previously described in an infinite-dimensional Fock-state basis, is reduced to a "two-level" system in the Schmidt basis. The length of the Bloch vector is a measure of the purity of the atom–field system. For the atom and field in pure states the length of the Bloch vector is unity.

As discussed in Appendix A, the utility of the Schmidt decomposition is that it is easy to obtain an expression for von Neumann entropy, which for each of the subsystems of the Jaynes–Cummings model is

$$S(\hat{\rho}_u) = -g_1(t)\ln g_1(t) - g_2(t)\ln g_2(t) = S(\hat{\rho}_v). \tag{4.201}$$

Whenever $|\mathbf{s}| = 1$ we have $S(\hat{\rho}_u) = 0 = S(\hat{\rho}_v)$. In Fig. 4.11, for an atom initially in the excited state and for the field initially in a coherent state with $\alpha = \sqrt{30}$, we plot (a) the s_3 component of the Bloch vector, (b) the s_2 component, and (c) the von Neumann entropy $S(\hat{\rho}_u)$, all against the scaled time $T = \lambda t$. With our particular initial condition, $s_1 = 0$ for all times. We notice that s_3 undergoes the collapse and revival of the atomic inversion and that s_2 goes close to unity at one point during the quiescent period of the collapse of the former, indicating that the atom and field are nearly in pure states at that time. The von Neumann entropy, of course, is a minimum at that point. This result is perhaps a bit surprising and counter-intuitive. This behavior was first noticed, through a different type of calculation, by Gea-Banacloche [20] and was further examined by Phoenix and Knight [21].

Problems

1. Consider the semiclassical Rabi model (i.e. a two-level atom with a prescribed classical field) in the RWA as described by Eqs. (4.74). Obtain the solution assuming the atom initially in the excited state. Calculate the atomic inversion as a function of time.
2. Using the result of the previous problem, obtain the time-dependent expectation value of the atomic dipole moment operator $\hat{d} = d(\hat{\sigma}_+ + \hat{\sigma}_-)$ for the case of exact resonance. Compare the evolution of the dipole moment with the atomic inversion for the same resonance condition and comment on any similarities or differences.

3. In the fully quantized model of a two-level atom interacting with a quantized field within the RWA, the Jaynes–Cummings model, we have obtained the exact resonance solution for the initial state where the atom is excited and where the field is in a number state $|n\rangle$ (see Eqs. (4.107) to (4.116)). Recall that the Rabi oscillations of the atomic inversion are periodic for this case, just as in the semiclassical model. Use that solution to obtain the expectation value of the atomic dipole moment operator and compare with the result of the previous question. Is the time evolution of the atomic dipole moment in any way similar to that obtained in the semiclassical case? Comment on the result.
4. Obtain the expectation value of the atomic dipole moment as given by the Jaynes–Cummings model in the case where the field is initially in a coherent state. How does the result compare with the two previous cases? You should make a plot of the expectation value of the dipole moment as a function of time.
5. In the text, we obtained the dynamics of the Jaynes–Cummings model assuming exact resonance, $\Delta = 0$. Reconsider the problem for the case where $\Delta \neq 0$. Obtain plots of the atomic inversion and note the effect of the nonzero detuning on the collapse and revivals of the Rabi oscillations. Perform an analysis to obtain the effect of the nonzero detuning on the collapse and revival times.
6. Consider the resonant Jaynes–Cummings model for the initial thermal state as in Section 4.7. Assume the atom is initially in the excited state. Analyse the collapse of the Rabi oscillations and determine the dependence of the collapse time on the average photon number of the thermal field.
7. Consider a simple model of degenerate Raman scattering, pictured in Fig. 4.12 (where $E_g = E_e$), and described by the interaction Hamiltonian $\hat{H}_{\rm I} = \hbar\lambda\hat{a}^\dagger\hat{a}(\sigma_+ + \sigma_-)$, where, as usual, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$.
   (a) Obtain the dressed states for this model.
   (b) Assuming initially the field in a coherent state and the atom in the ground state, obtain the atomic inversion and show that the revivals of the Rabi oscillations are regular and complete.
   (c) Obtain the atomic inversion for an initial thermal state.
8. A resonant two-photon extension of the Jaynes–Cummings model is described by the effective Hamiltonian $H_{\rm eff} = \hbar\eta(\hat{a}^2\hat{\sigma}_+ + \hat{a}^{2\dagger}\hat{\sigma}_-)$, where, for the sake of simplicity, a small Stark shift term has been ignored. This Hamiltonian represents two-photon

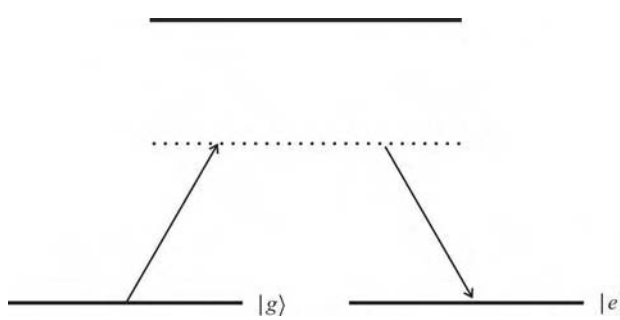


Fig. 4.12. Energy-level diagram for the degenerate Raman coupled model. The broken line represents a "virtual" intermediate state, too far off-resonance from a real level, the upper solid line, to become populated.

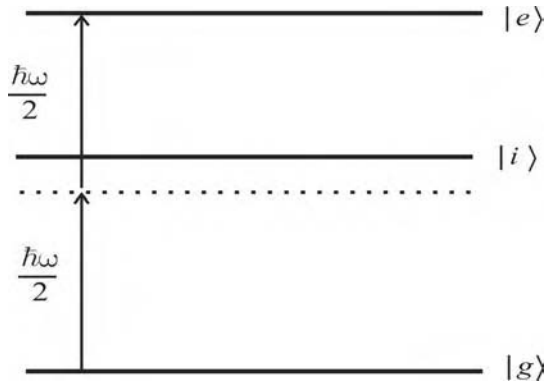


Fig. 4.13. Energy-level diagram for the resonant two-photon process. States $|e\rangle$ and $|g\rangle$ are of like parity whereas the intermediate state $|i\rangle$ is of opposite parity. The broken line represents a virtual atomic level, detuned from state $|i\rangle$.

absorption and emission between atomic levels of like parity. The process is represented by Fig. 4.13, where the broken line represents a virtual intermediate state of opposite parity.

(a) Obtain the dressed states for this system.

(b) Obtain the atomic inversion for this model assuming the atom initially in the ground state and that the field is initially in a number state. Repeat for a coherent state. Comment on the nature of the collapse and revival phenomena for these states.

(c) Obtain the atomic inversion for an initial thermal state.

9. A two-mode variation on the two-photon model of the previous problem is described by the Hamiltonian $H_{\text{eff}} = \hbar\eta(\hat{a}\,\hat{b}\hat{\sigma}_+ + \hat{a}^\dagger\hat{b}^\dagger\hat{\sigma}_-)$, that is, a photon from each mode is absorbed or emitted. Obtain the atomic inversion for the case where both modes are initially in coherent states. Analyse the collapse and revival phenomena.

11. Consider the resonant Jaynes–Cummings model with the atom initially in the excited state and the field initially in a coherent state with $\alpha = \sqrt{30}$.

(a) By taking the trace of the density operator over either the atomic or field states and thus obtaining a reduced density operator (see Appendix A), determine the degree to which the atom and field are entangled as a function of time. Pay particular attention to what happens at about midpoint between the initial collapse of the Rabi oscillations and the first revival. Is the behavior surprising? What connection, if any, is there between the result you have obtained and the results discussed in Section 4.10.

(b) Obtain plots (either contour or 3D) of the Q function (Eq. (3.112)) of the field for various times between time $t = 0$ and the time of the first revival of the Rabi oscillations.

12. Consider the atom–field state of Eq. (4.193) and assume that $\phi = 0$. Suppose that there is some way to determine the state of the atom (such is possible via field ionization, as will be discussed in Chapter 10). For example, if the atom is detected in state $|g\rangle$ then the field state will be reduced to $|-\alpha\rangle = \langle g \mid \psi(\pi)\rangle / |\langle g \mid \psi(\pi)\rangle|$ with a similar result for detection of the atomic state $|f\rangle$. (See Appendix D.) But suppose it were somehow

possible to detect the atom in the superposition states $|S\pm\rangle = (|g\rangle \pm |f\rangle)/\sqrt{2}$. What field states are generated in these cases?

13. Reconsider the problem discussed in Section 4.10 but with the initial state of the atom taken to be the "balanced" superposition $|\psi(0)\rangle_{\text{atom}} = (|e\rangle + |g\rangle)/\sqrt{2}$. Obtain relevant plots of the components of the Bloch vector and of the von Neumann entropy. Contrast the results obtained in this case to the case where the atom is initially in the excited state.

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