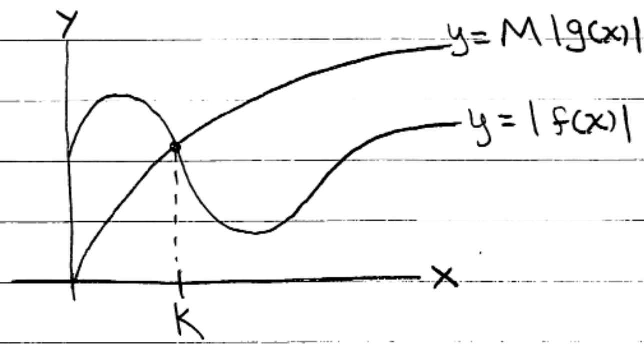
Big-O Notation

Definition: Let $D \subseteq \mathbb{R}^{\text{nonneg}}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. Then we say that $f(x)$ is $O(g(x))$ iff $\exists$ $K, M \in \mathbb{R}^{+}$ such that

$$|f(x)| \le M\,|g(x)| \quad \text{for all real numbers } x > K.$$

Picture:



For all real numbers $x > K$, $|f(x)| \le M|g(x)|$, so $f(x)$ is $O(g(x))$.

(The rate of growth of $|f(x)|$ is at most the rate of growth of $M|g(x)|$)

Notes: (1) If $x > 1$ and $m < n$, then $x^m < x^n$.

(2) $|a+b| \le |a| + |b|$ (Triangle Inequality).

ex: Prove that $f(x) = 2x^3 + 5x^2 - 7x + 1$ is $O(x^3)$.

For all real numbers $x > 1$,

$$\begin{aligned}
|f(x)| &= |2x^3 + 5x^2 - 7x + 1| \\
&\le |2x^3| + |5x^2| + |-7x| + |1| && \text{(by the Triangle Inequality)} \\
&= 2x^3 + 5x^2 + 7x + 1 \\
&< 2x^3 + 5x^3 + 7x^3 + x^3 && (\text{since } x > 1,\ x^2 < x^3,\ x < x^3, \text{ and } 1 < x^3) \\
&= 15x^3 \\
&= 15|x^3|.
\end{aligned}$$

Hence, for all real numbers $x > 1$, $|f(x)| \le 15|x^3|$, which proves that $f(x)$ is $O(x^3)$.

HW: Worksheet