MATH 2300 Sample Proofs

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VECTOR SPACE PROOFS

1. Prove that for any set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in a vector space V, $\operatorname{span}(S)$ is a subspace of V.

Solution: Let $\mathbf{u}, \mathbf{w} \in \operatorname{span}(S)$, $k \in \mathbb{R}$. Then there exist $c_1, \ldots c_n \in \mathbb{R}$ and $k_1, \ldots, k_n \in \mathbb{R}$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

and

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n.$$

A1)

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) + (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n) \\ &= (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + \ldots + (c_n + k_n)\mathbf{v}_n \in \operatorname{span}(S).\end{aligned}$$

Thus $\operatorname{span}(S)$ is closed under addition.

M1)

$$\begin{aligned}k\mathbf{u} &= k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) \\ &= k(c_1\mathbf{v}_1) + k(c_2\mathbf{v}_2) + \ldots + k(c_n\mathbf{v}_n) \\ &= (kc_1)\mathbf{v}_1 + (kc_2)\mathbf{v}_2 + \ldots + (kc_n)\mathbf{v}_n \in \operatorname{span}(S).\end{aligned}$$

Thus $\operatorname{span}(S)$ is closed under scalar multiplication.

Thus by the subspace theorem, $\operatorname{span}(S)$ is a subspace of V.

2. Prove that if S is a linearly independent set of vectors, then S is a basis for $\operatorname{span}(S)$.

Solution: To be a basis for $\operatorname{span}(S)$, it must be linearly independent and span the space. Certainly the span of S is equal to the entire space, $\operatorname{span}(S)$ (by definition), and S is given to be linearly independent in the question. Thus S is a basis for $\operatorname{span}(S)$.

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3. Show that if A is an $m \times n$ matrix, then the solution set V to the equation $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Solution: A1) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ be two solutions to the equation $A\mathbf{x} = \mathbf{0}$ (that is, $\mathbf{x}_1, \mathbf{x}_2 \in V$). Then $\mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^n$, and

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

Thus $\mathbf{x}_1 + \mathbf{x}_2 \in V$.

M1). Let $\mathbf{x}_1 \in V$, $k \in \mathbb{R}$. Then $k\mathbf{x}_1 \in \mathbb{R}^n$, and

$$\begin{aligned} A(k\mathbf{x}_1) &= k(A\mathbf{x}_1) \\ &= k(\mathbf{0}) = \mathbf{0}. \end{aligned}$$

Thus $k\mathbf{x}_1 \in V$ as well.

Thus by the subspace theorem, V is a subspace of $\mathbb{R}^n$.

4. Prove that any finite set of vectors containing the zero vector is linearly dependent.

Solution: Let $S = \{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then the equation

$$c_1\mathbf{0} + c_2\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

has the solution $c_1 = 1, c_2 = c_3 = \cdots = c_n = 0$, which is not all zeros, and thus the set S is linearly dependent.

5. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v} \in V$ can be expressed as a linear combination of the elements of S in exactly one way.

Solution: Let $\mathbf{v} \in V$. Let $c_1, \ldots, c_n, k_1, \ldots, k_n \in \mathbb{R}$ be such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

and

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n.$$

Then subtracting straight down we get

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \ldots + (c_n - k_n)\mathbf{v}_n.$$

But since S is a basis, S is linearly independent, and thus the only way this could happen is if $c_1 - k_1 = 0$, ..., $c_n - k_n = 0$. Thus $c_1 = k_1, \ldots, c_n = k_n$ and so $\mathbf{v}$ can only be written as a linear combination of the elements of S in one way.

*6. Given that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in some vector space V, prove that:

(a) the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

(b) the set $\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$ is linearly independent.

(c) the set $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}\}$ is linearly independent.

*7. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be such that $\mathbf{u} \bullet \mathbf{v} = 0$. Prove that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set.

*8. Let V be a vector space. Prove that for every $\mathbf{u} \in V$, $0 \cdot \mathbf{u} = \mathbf{0}$.

Solution: Let $\mathbf{u} \in V$. Then

$$\begin{aligned} 0 \cdot \mathbf{u} &= (0 + 0) \cdot \mathbf{u} && \text{(since } 0 + 0 = 0\text{)} \\ &= 0 \cdot \mathbf{u} \oplus 0 \cdot \mathbf{u}. && \text{(axiom } M3\text{)} \end{aligned}$$

By axiom $A5$, there exists $-(0 \cdot \mathbf{u})$. Then,

$$0 \cdot \mathbf{u} \oplus -(0 \cdot \mathbf{u}) = \mathbf{0} \qquad \text{(by } A5\text{)},$$

and

$$\begin{aligned} 0 \cdot \mathbf{u} \oplus -(0 \cdot \mathbf{u}) &= (0 \cdot \mathbf{u} \oplus 0 \cdot \mathbf{u}) \oplus -(0 \cdot \mathbf{u}) && \text{(by above)} \\ &= 0 \cdot \mathbf{u} \oplus (0 \cdot \mathbf{u} \oplus -(0 \cdot \mathbf{u})) && \text{(by } A3\text{)} \\ &= 0 \cdot \mathbf{u} \oplus \mathbf{0} && \text{(by } A5\text{)} \\ &= 0 \cdot \mathbf{u}. && \text{(by } A4\text{)} \end{aligned}$$

Therefore

$$0 \cdot \mathbf{u} = \mathbf{0}.$$

*9. Let V be a vector space. Prove that for every $k \in \mathbb{R}$, $k \cdot \mathbf{0} = \mathbf{0}$.

Solution: Let $k \in \mathbb{R}$. Then

$$\begin{aligned} k \cdot \mathbf{0} &= k \cdot (\mathbf{0} \oplus \mathbf{0}) && \text{(by } A4\text{)} \\ &= k \cdot \mathbf{0} \oplus k \cdot \mathbf{0} && \text{(by } M2\text{)}. \end{aligned}$$

By $A5$, there exists $-(k \cdot \mathbf{0})$. Then,

$$k \cdot \mathbf{0} \oplus -(k \cdot \mathbf{0}) = \mathbf{0}, \qquad \text{(by } A5\text{)}$$

but

$$\begin{aligned} k \cdot \mathbf{0} \oplus -(k \cdot \mathbf{0}) &= (k \cdot \mathbf{0} \oplus k \cdot \mathbf{0}) \oplus -(k \cdot \mathbf{0}) && \text{(by above)} \\ &= k \cdot \mathbf{0} \oplus (k \cdot \mathbf{0} \oplus -(k \cdot \mathbf{0})) && \text{(by } A3\text{)} \\ &= k \cdot \mathbf{0} \oplus \mathbf{0} && \text{(by } A5\text{)} \\ &= k \cdot \mathbf{0}. && \text{(by } A4\text{)} \end{aligned}$$

Therefore

$$k \cdot \mathbf{0} = \mathbf{0}.$$

*10. Let V be a vector space. Prove that for every $\mathbf{u} \in V$, $(-1) \cdot \mathbf{u} = -\mathbf{u}$.

Solution: Let $\mathbf{u} \in V$. Then,

$$\begin{aligned} (-1) \cdot \mathbf{u} \oplus \mathbf{u} &= (-1) \cdot \mathbf{u} \oplus 1 \cdot \mathbf{u} && \text{(by } M5\text{)} \\ &= (-1 + 1) \cdot \mathbf{u} && \text{(by } M3\text{)} \\ &= 0 \cdot \mathbf{u} \\ &= \mathbf{0} && \text{(by part 1).} \end{aligned}$$

Therefore, $-\mathbf{u} = (-1) \cdot \mathbf{u}$.

*11. Let V be a vector space. Prove that if for some $k \in \mathbb{R}$ and $\mathbf{u} \in V$, $k \cdot \mathbf{u} = 0$, then either $k = 0$, or $\mathbf{u} = 0$.

Solution: Let $k \in \mathbb{R}$, $\mathbf{u} \in V$, and assume that $k \cdot \mathbf{u} = \mathbf{0}$. If $k = 0$, then this is true, by part 1, so assume that $k \neq 0$. Then

$$\begin{aligned} \frac{1}{k} \cdot (k \cdot \mathbf{u} &= (\frac{1}{k}k) \cdot \mathbf{u}) && \text{(by } M4\text{)} \\ &= 1 \cdot \mathbf{u} \\ &= \mathbf{u} && \text{(by } M5\text{),} \end{aligned}$$

but

$$\begin{aligned} \frac{1}{k} \cdot (k \cdot \mathbf{u}) &= \frac{1}{k} \cdot \mathbf{0} && \text{(by assumption)} \\ &= \mathbf{0} && \text{(by part 2).} \end{aligned}$$

Therefore $\mathbf{u} = \mathbf{0}$.

*12. Prove that a set of vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others.

*13. Let A be a $m \times n$ matrix. Prove that if both the set of rows of A and the set of columns of A form linearly independent sets, then A must be square.

Solution: Let $r_1, \ldots, r_m \in \mathbb{R}^n$ be the rows of A and let $c_1, \ldots, c_n \in \mathbb{R}^m$ be the columns of A. Since the set of rows is linearly independent, and the rows are elements of $\mathbb{R}^n$, it must be that $m \leq n$. Similarly, since the set of columns is linearly independent, and the columns are elements of $\mathbb{R}^m$, it must be that $n \leq m$. Thus $m = n$.

**14. Let V be the set of 2×2 matrices, together with the operation $\oplus$ defined for any 2×2 matrices A and B as

$$A \oplus B = AB \text{ (the usual matrix multiplication)},$$

and with the standard scalar multiplication for matrices.

(a) Show that the vector space axiom $A4$ holds.

(b) Prove that V is **not** a vector space.

**15. Let

$$V = \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0\}$$

together with the operations defined as follows: for $(a, b), (c, d) \in V$, $k \in \mathbb{R}$,

$$(a, b) \oplus (c, d) = (ac, bd)$$

$$k \cdot (a, b) = (a^k, b^k).$$

(a) Show that the vector space axiom $M3$ holds in this space.

(b) Does the axiom $A4$ hold in this space? If so, find the zero vector and prove it is the zero vector. If not, show that there is no possible zero vector.

**16. Let V be a vector space, and let W_1 and W_2 be subspaces of V. Prove that the set

$$U = \{\mathbf{v} : \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}$$

(that is, U is the set of vectors in BOTH W_1 and W_2). Prove that U is a subspace of V as well.

Solution: A1) Let $\mathbf{u}, \mathbf{v} \in U$. Then $\mathbf{u}, \mathbf{v} \in W_1$ and $\mathbf{u}, \mathbf{v} \in W_2$. Then since $A1$ holds in W_1, $\mathbf{u} + \mathbf{v} \in W_1$, and since $A1$ holds in W_2, $\mathbf{u} + \mathbf{v} \in W_2$ as well. Thus $\mathbf{u} + \mathbf{v} \in U$ and so $A1$ holds.

M1) Let $\mathbf{u} \in U$, and let $k \in \mathbb{R}$. Then $\mathbf{u} \in W_1$ and $\mathbf{u} \in W_2$, and so since $M1$ holds in W_1, $k\mathbf{u} \in W_1$, and since $M1$ holds in W_2, $k\mathbf{u} \in W_2$ as well. Thus $k\mathbf{u} \in U$, and so $M1$ holds.

Thus, but the subspace theorem, U is a subspace of V.

**17. Let W be a subspace of a vector space V, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in W$. Prove then that every linear combination of these vectors is also in W.

Solution: Let $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since W is a subspace (and thus a vector space), since W is closed under scalar multiplication ($M1$), we know that $c_1\mathbf{v}_1, c_2\mathbf{v}_2$, and $c_3\mathbf{v}_3$ are all in W as well. Then since W is closed under addition (A1), we know that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is also in W. Then applying closure under addition (A1) again, we get that

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \in W.$$

**18. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Solution: Assume $r > n$, and assume that for each i, $1 \leq i \leq r$,

$$\mathbf{v}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n}).$$

Let $c_1, c_2, \ldots, c_r \in \mathbb{R}$ be such that

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r = \mathbf{0}.$$

This produces the homogeneous system of equations:

$$\begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{r,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{r,2} \\ \vdots & \vdots & & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{r,n} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The coefficient matrix of this system has n rows and r columns. But $r > n$. Therefore this system is guaranteed to have a parameter, and since it is a homogeneous system and has at least one solution, it therefore has infinitely many solutions. Thus there is at least one solution for the c_i's that is not all zero, and so the set S is linearly dependent.

LINEAR TRANSFORMATION PROOFS

19. Prove that the range of a linear transformation $T : V \to W$ is a subspace of W.

20. Prove that given two linear transformations $T_1 : U \to V$ and $T_2 : V \to W$, the composition $T_2 \circ T_1 : U \to W$ is also a linear transformation.

21. Prove that for any linear transformation $T : V \to W$, $\ker(T)$ is a subspace of W.

22. Prove that If $T_1 : U \to V$ is one-to-one, and $T_2 : V \to W$ is one-to-one, then the composition $T_2 \circ T_1 : U \to W$ is also one-to-one.

23. If $T_1 : U \to V$ is onto, and $T_2 : V \to W$ is onto, then the composition $T_2 \circ T_1 : U \to W$ is also onto.

24. Prove that for any one-to-one linear transformation $T : V \to W$, T^{-1} is also a one-to-one linear transformation.

25. Prove that for any $m \times n$ matrix M, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation.

*26. If $T : V \to W$ is a linear transformation, then prove each of the following:

- If T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$.
- If $\ker(T) = \{\mathbf{0}\}$, then T is one-to-one.

*27. If V is a finite-dimensional vector space, and $T : V \to V$ is a linear operator, then prove that if T is one-to-one, then the range of T is all of V.

*28. Prove that if $T : V \to W$ is an isomorphism between V and W (one-to-one and onto), and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.

*29. Prove that every vector space of dimension n is isomorphic to $\mathbb{R}^n$.

**30. Let $T : V \to W$ be a one-to-one linear transformation. Prove that if $\dim(V) = \dim(W)$ (and both V and W are finite-dimensional), then T is an isomorphism.

Solution: Assume $\dim(V) = \dim(W)$ and that $T : V \to W$ is one-to-one. Assume further (in hopes of a contradiction) that T is not onto. Then there exists $\mathbf{w} \in W$ such that $\mathbf{w} \notin T(V)$. But then since $T(V)$ is a subspace of W, but we know that

$T(V) \neq W$, it follows that $\text{rank}(T) = \dim(T(V)) < \dim(W)$. But then, by the Dimension Theorem,

$$\begin{aligned}
\text{nullity}(T) &= \dim(V) - \text{rank}(T) \\
&= \dim(V) - \dim(T(V)) \\
&> \dim(V) - \dim(W) \\
&= \dim(V) - \dim(V) \\
&= 0.
\end{aligned}$$

Then since $\text{nullity}(T) > 0$, $\ker(T) \neq \{\mathbf{0}\}$, and thus there are two vectors in V that map to the zero vector in W. Thus T is not 1:1, a contradiction.

**31. Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations. Prove that if $T_2 \circ T_1$ is one-to-one, then T_1 must be one-to-one.

Solution: Suppose that $T_2 \circ T_1 : U \to W$ is one-to-one. That is, for all $\mathbf{u}, \mathbf{v} \in U$, if $(T_2 \circ T_1)(\mathbf{u}) = (T_2 \circ T_1)(\mathbf{v})$, then $\mathbf{u} = \mathbf{v}$.

It remains to show that $T_1 : U \to V$ is one-to-one. Let $\mathbf{u}, \mathbf{v} \in U$ and assume that $T_1(\mathbf{u}) = T_1(\mathbf{v})$. Then certainly

$$T_2(T_1(\mathbf{u})) = T_2(T_1(\mathbf{v})),$$

and therefore,

$$(T_2 \circ T_1)(\mathbf{u}) = (T_2 \circ T_1)(\mathbf{v}).$$

Then, because $(T_2 \circ T_1)$ is one-to-one, it follows that $\mathbf{u} = \mathbf{v}$. Therefore T_1 is one-to-one.

Solution: Alternate Proof. Suppose that $T_2 \circ T_1 : U \to W$ is one-to-one. Then $\ker(T_2 \circ T_1) = \{\mathbf{0}\}$. Let $\mathbf{u} \in \ker(T_1)$. Then $T_1(\mathbf{u}) = \mathbf{0}$, and so

$$\begin{aligned}
(T_2 \circ T_1)(\mathbf{u}) &= T_2(T_1(\mathbf{u})) \\
&= T_2(\mathbf{0}) && \text{(def'n of } \mathbf{u}\text{)} \\
&= \mathbf{0}. && \text{(property of linear transformations)}
\end{aligned}$$

But then $\mathbf{u} \in \ker(T_2 \circ T_1)$, and so $\mathbf{u} = \mathbf{0}$. Thus since $\mathbf{u} \in \ker(T_1)$ we have that $\ker(T_1) = \{\mathbf{0}\}$. Thus T_1 is 1:1 as well.

**32. Let $T : V \to W$ be an onto linear transformation. Prove that if $\dim(V) = \dim(W)$, then T is an isomorphism.

Solution: Assume that $\dim(V) = \dim(W)$. Then since T is onto, $T(V) = W$, and so $\text{rank}(T) = \dim(W) = \dim(V)$. But then by the dimension theorem,

$$\begin{aligned} \text{nullity}(T) &= \dim(V) - \text{rank}(T) \\ &= \dim(V) - \dim(V) \\ &= 0. \end{aligned}$$

Therefore $\text{Ker}(T) = \{\mathbf{0}\}$, and therefore by the theorem above, we have that T is 1:1. Therefore T is both 1:1 and onto, and is thus an isomorphism.

**33. Let $T : V \to W$ be a one-to-one linear transformation. Prove that T is an isomorphism between V and $T(V)$.

Solution: T is given to be 1:1. Viewing it as a linear transformation between V and $T(V)$, it is also certainly onto by the definition of $T(V)$. Therefore it is 1:1 and onto, and is thus an isomorphism.

**34. Let E be a fixed 2×2 elementary matrix.

(a) Does the formula $T(A) = EA$ define a one-to-one linear operator on $M_{2,2}$? Prove or disprove.

(b) Does the formula $T(A) = EA$ define an onto linear operator on $M_{2,2}$? Prove or disprove.

Solution: 1-to-1: Let $A, B \in M_{2,2}$ be such that $T(A) = T(B)$. Then $EA = EB$. Since E is an elementary matrix, and all elementary matrices are invertible, E^{-1} exists. Multiplying both sides by E^{-1} we get $E^{-1}EA = E^{-1}EB$, and thus $A = B$. Therefore T is 1:1.

Onto: Let $A \in M_{2,2}$. Then $B = E^{-1}A$ is a matrix in $M_{2,2}$ such that $T(B) = EB = A$. Thus T is onto.

Thus T is 1:1 and onto (and is thus an isomorphism).

**35. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T : V \to W$ be a linear transformation. Show that if $T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_n) = \mathbf{0}_W$, then T is the zero transformation (that is, for every $\mathbf{v} \in V$, $T(\mathbf{v}) = \mathbf{0}_W$).

Solution: Let $\mathbf{v} \in V$. Then since B is a basis for V, there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$.

Then,

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T(c_1\mathbf{v}_1) + \ldots + T(c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) \\ &= c_1\mathbf{0}_W + \ldots + c_n\mathbf{0}_W \\ &= \mathbf{0}_W. \end{aligned}$$

Thus T is the zero transformation.

***36. Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Prove that if for all i, $1 \leq i \leq n$, $T_1(\mathbf{v}_i) = T_2(\mathbf{v}_i)$, then $T_1 = T_2$ (that is, for all $v \in V$, $T_1(v) = T_2(v)$).

Solution: Let $\mathbf{v} \in V$. Then since B is a basis for V, there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$.

Then,

$$\begin{aligned} T_1(\mathbf{v}) &= T_1(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T_1(c_1\mathbf{v}_1) + \ldots + T_1(c_n\mathbf{v}_n) \\ &= c_1T_1(\mathbf{v}_1) + \ldots + c_nT_1(\mathbf{v}_n) \\ &= c_1T_2(\mathbf{v}_1) + \ldots + c_nT_2(\mathbf{v}_n) \\ &= T_2(c_1\mathbf{v}_1) + \ldots + T_2(c_n\mathbf{v}_n) \\ &= T_2(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T_2(\mathbf{v}). \end{aligned}$$

Thus $T_1 = T_2$.

37. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Let V be the set of all eigenvectors corresponding to λ, together with the zero vector. Prove that V is a subspace of $\mathbb{R}^n$.

Solution: Let E be the set of all eigenvectors corresponding to λ, together with the zero vector. Let $\mathbf{u}, \mathbf{v} \in E$, $k \in \mathbb{R}$.

- Closure under addition:

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= A\mathbf{u} + A\mathbf{v} \\ &= \lambda\mathbf{u} + \lambda\mathbf{v} \\ &= \lambda(\mathbf{u}+\mathbf{v}). \end{aligned}$$

  Therefore $\mathbf{u}+\mathbf{v}$ is also in E.

- Closure under scalar mult:

$$\begin{aligned} A(k\mathbf{u}) &= kA(\mathbf{u}) \\ &= k(\lambda\mathbf{u}) \\ &= \lambda(k\mathbf{u}). \end{aligned}$$

  Thus $k\mathbf{u} \in E$ as well.

Therefore by the subspace theorem, E is a subspace of $\mathbb{R}^n$.

38. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

39. Show that for all $\mathbf{u}, \mathbf{v}$ in an inner product space V, and $k \in \mathbb{R}$,

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$$

40. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}-\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$$

41. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}, \mathbf{v}-\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$$

42. Show that in any inner product space V, for all $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.

Solution: Let $\mathbf{v} \in V$. Then

$$\begin{aligned}\langle \mathbf{v}, \mathbf{0} \rangle &= \langle \mathbf{v}, 0 \cdot \mathbf{0} \rangle \\ &= 0 \langle \mathbf{v}, \mathbf{0} \rangle \\ &= 0.\end{aligned}$$

43. Prove each of the following properties about inner product spaces: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V, and all $k \in \mathbb{R}$,

- $||\mathbf{u}|| \geq 0$
- $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- $||k\mathbf{u}|| = |k|||\mathbf{u}||$
- $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (Triangle Inequality)
- $d(\mathbf{u}, \mathbf{v}) \geq 0$
- $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
- $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
- $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$. (Triangle Inequality)

44. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then so are $\frac{1}{||\mathbf{u}||}\mathbf{u}$ and $\frac{1}{||\mathbf{v}||}\mathbf{v}$.

*45. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.

Solution:

$$\begin{aligned}|\lambda I - A| &= |(\lambda I - A)^T| \\ &= |(\lambda I)^T - A^T| \\ &= |\lambda I - A^T|.\end{aligned}$$

Thus A and A^T have the same characteristic polynomials, and so must have the same eigenvalues.

**46. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

Solution: I will solve this problem by proving the contrapositives: that A is not invertible if and only if 0 is an eigenvalue of A.

($\Rightarrow$): Assume A is not invertible. Then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and thus at least one non-zero solution, say $\mathbf{x}_0$. Then

$$A\mathbf{x}_0 = 0 = 0\mathbf{x}_0$$

and thus 0 is an eigenvalue of A with eigenvector $\mathbf{x}_0$.

($\Leftarrow$): Assume 0 is an eigenvalue of A. Then $\det(A - 0I) = \det(A) = 0$. Thus A is not invertible.

**47. Prove that if $B = C^{-1}AC$, then B and A have the same eigenvalues (HINT: Look at the characteristic polynomials of B and A).

Solution:

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - C^{-1}AC| \\
&= |\lambda C^{-1}C - C^{-1}AC| \\
&= |C^{-1}(\lambda C - AC)| \\
&= |C^{-1}(\lambda I - A)C| \\
&= |C^{-1}|\ |\lambda I - A|\ |C| \\
&= |\lambda I - A|\ |C^{-1}|\ |C| \\
&= |\lambda I - A|\ |C^{-1}C| \\
&= |\lambda I - A|\ |I| \\
&= |\lambda I - A|.
\end{aligned}$$

Thus A and B have the same characteristic polynomials, and so must have the same eigenvalues.

**48. Let $\mathbf{v}$ be a nonzero vector in an inner product space V. Let W be the set of all vectors in V that are orthogonal to $\mathbf{v}$. Prove that W is a subspace of V.

Solution: Let $W = \{\mathbf{w} \in V : \langle \mathbf{w}, \mathbf{v} \rangle = 0\}$. Certainly W is non-empty since the zero vector is orthogonal to every vector in V.

Let $\mathbf{a}, \mathbf{b} \in W$, $k \in \mathbb{R}$.

A1. We need to check if $\mathbf{a} + \mathbf{b}$ is orthogonal to $\mathbf{v}$.

$$\begin{aligned}
\langle \mathbf{a} + \mathbf{b}, \mathbf{v} \rangle &= \langle \mathbf{a}, \mathbf{v} \rangle + \langle \mathbf{b}, \mathbf{v} \rangle \\
&= 0 + 0 = 0.
\end{aligned}$$

Thus $\mathbf{a} + \mathbf{b} \in W$.

M1. We need to check if $k\mathbf{a} \in W$.

$$\begin{aligned}
\langle k\mathbf{a}, \mathbf{v} \rangle &= k\langle \mathbf{a}, \mathbf{v} \rangle \\
&= k(0) = 0.
\end{aligned}$$

Thus $k\mathbf{a} \in W$.

Therefore by the subspace theorem, W is a subspace of V.

**49. Prove that for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space, if

$$||\mathbf{u}|| = ||\mathbf{v}||,$$

then $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$.

Solution: Assume that $||\mathbf{u}|| = ||\mathbf{v}||$. Then,

$$\begin{aligned}
\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle &= \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\
&= \langle \mathbf{u}-\mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{v}\rangle \\
&= ||\mathbf{u}||^2 - \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - ||\mathbf{v}||^2 \\
&= ||\mathbf{u}||^2 - ||\mathbf{v}||^2 \\
&= ||\mathbf{u}||^2 - ||\mathbf{u}||^2 \\
&= 0.
\end{aligned}$$

Thus $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$.

**50. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a basis for an inner product space V. Show that the zero vector is the only vector in V that is orthogonal to all of the basis vectors.

Solution: We have a property (proved elsewhere) that for all $\mathbf{v} \in V$ (and thus for all $\mathbf{v} \in B$), $\langle \mathbf{0}, \mathbf{v}\rangle = 0$. But this question is in some sense the opposite of this. Let $\mathbf{w} \in V$ be a vector such that for all $\mathbf{v} \in B$, $\langle \mathbf{w}, \mathbf{v}\rangle = 0$. Then we must prove that $\mathbf{w}$ must have been the zero vector.

Since B is a basis for V, we know that there exist $k_1, \ldots, k_r \in \mathbb{R}$ such that $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$.

Then look at $\langle \mathbf{w}, \mathbf{w}\rangle$:

$$\begin{aligned}
\langle \mathbf{w}, \mathbf{w}\rangle &= \langle \mathbf{w}, k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r\rangle \\
&= \langle \mathbf{w}, k_1\mathbf{v}_1\rangle + \langle \mathbf{w}, k_2\mathbf{v}_2\rangle + \cdots + \langle \mathbf{w}, k_r\mathbf{v}_r\rangle \\
&= k_1\langle \mathbf{w}, \mathbf{v}_1\rangle + k_2\langle \mathbf{w}, \mathbf{v}_2\rangle + \cdots + k_r\langle \mathbf{w}, \mathbf{v}_r\rangle \\
&= k_1(0) + k_2(0) + \cdots + k_r(0) \\
&= 0.
\end{aligned}$$

Thus since $\langle \mathbf{w}, \mathbf{w}\rangle = 0$, by positivity, we know that $\mathbf{w} = \mathbf{0}$.

***51. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V. Prove that

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n\rangle\mathbf{v}_n.$$

Solution: Let $\mathbf{u} \in V$. Since S is a basis, there exist $k_1, \ldots, k_n$ such that $\mathbf{u} = k_1\mathbf{v}_1 + \cdots + k_n\mathbf{v}_n$. Then

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + \cdots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= \langle k_1\mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + \langle k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + k_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle && \text{(since } S \text{ is orthogonal)} \\
&= k_i||\mathbf{v}_i||^2 \\
&= k_i && \text{(since } S \text{ is orthonormal).}
\end{aligned}$$

Thus $\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$.

***52. An $n \times n$ matrix A is said to be **nilpotent** if for some $k \in \mathbb{Z}^+$, A^k is a zero matrix. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

Solution: Let A be a nilpotent matrix and let $k \in \mathbb{Z}^+$ be such that A^k is a zero matrix. Let λ be a eigenvalue of A with eigenvector $\mathbf{x}$. Then

$$\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
A^2\mathbf{x} &= A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda^2\mathbf{x}. \\
&\vdots \\
A^k\mathbf{x} &= \lambda^k\mathbf{x}.
\end{aligned}$$

But A^k is a zero matrix, and so the left hand side is a zero matrix. Thus $\lambda^k\mathbf{x}$ is a zero matrix. However $\mathbf{x}$ being an eigenvector forces $\mathbf{x} \neq \mathbf{0}$, and thus $\lambda^k = 0$, and so $\lambda = 0$. Thus 0 is the only eigenvalue of A.

***53. Let W be any subspace of an inner product space V, $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ an orthonormal basis for W. Let $\mathbf{v} \in V$. Let the vector $\mathbf{v}_0$ be defined as

$$\mathbf{v}_0 = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \cdots + \langle \mathbf{v}, \mathbf{b}_n \rangle \mathbf{b}_n = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \mathbf{b}_i.$$

Certainly $\mathbf{v}_0 \in W$. Prove that $\mathbf{v} - \mathbf{v}_0$ is orthogonal to every vector in W.

Solution: What we need to check is the inner product of $\mathbf{v} - \mathbf{v}_0$ with every vector in W and verify it is 0.

Let $\mathbf{w} \in W$. Then since B is an orthonormal basis, $\mathbf{w} = \sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{b}_i \rangle \mathbf{b}_i$. Then

$$\langle \mathbf{v} - \mathbf{v}_0, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}_0, \mathbf{w} \rangle,$$

and

$$\begin{aligned}
\langle \mathbf{v}_0, \mathbf{w} \rangle &= \langle \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \mathbf{b}_i, \mathbf{w} \rangle && \text{(by def'n of } \mathbf{v}_0) \\
&= \sum_{i=1}^{n} \langle \langle \mathbf{v}, \mathbf{b}_i \rangle \mathbf{b}_i, \mathbf{w} \rangle && \text{(by additivity)} \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \langle \mathbf{b}_i, \mathbf{w} \rangle && \text{(by homogeneity)} \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \langle \mathbf{b}_i, \sum_{j=1}^{n} \langle \mathbf{w}, \mathbf{b}_j \rangle \mathbf{b}_j \rangle \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \sum_{j=1}^{n} \langle \mathbf{b}_i, \langle \mathbf{w}, \mathbf{b}_j \rangle \mathbf{b}_j \rangle && \text{(by additivity)} \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \sum_{j=1}^{n} \langle \mathbf{w}, \mathbf{b}_j \rangle \langle \mathbf{b}_i, \mathbf{b}_j \rangle && \text{(by homogeneity)} \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \langle \mathbf{w}, \mathbf{b}_i \rangle \langle \mathbf{b}_i, \mathbf{b}_i \rangle && \text{(since } i \neq j \implies \langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0) \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \langle \mathbf{w}, \mathbf{b}_i \rangle ||\mathbf{b}_i||^2 \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \langle \mathbf{w}, \mathbf{b}_i \rangle && \text{(since } \mathbf{b}_i \text{ is a unit vector)} \\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \langle \mathbf{w}, \mathbf{b}_i \rangle \mathbf{b}_i \rangle && \text{(by homogeneity)} \\
&= \langle \mathbf{v}, \sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{b}_i \rangle \mathbf{b}_i \rangle && \text{(by additivity)} \\
&= \langle \mathbf{v}, \mathbf{w} \rangle.
\end{aligned}$$

Therefore, for any $\mathbf{w} \in W$,

$$\begin{aligned}
\langle \mathbf{v} - \mathbf{v}_0, \mathbf{w} \rangle &= \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}_0, \mathbf{w} \rangle \\
&= \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle \\
&= 0.
\end{aligned}$$

Thus $\mathbf{v} - \mathbf{v}_0$ is orthogonal to everything in W.