Econ 605 Notes

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Contents

1 Macro Variables, Capital, and Wealth; Productivity

1.1 Accounting

1.1.1 "GDP"

Product = Income = Expenditure

Product = Income:

$$Y = wL + RK$$

Product = Expenditure:

$$Y = C + I + G + NX$$

> RK = Dividends & Interest + Retained Earnings
>
> Imputed rent is part of GDP (specifically, part of C)

1.1.2 "Income accounting"

Household:

- $Y - T$ = after-tax income is either saved or consumed:
- $S_p = Y - T - C$

Government:

- $G = C_G + I_G$
- $S_G = T - G$

1.1.3 Def "Deflating Nominal Variables"

$$X_t = \frac{X_t^{nom}}{P_{X,t}}$$

where P_X is the deflator or price index which converts base year dollars into t-year dollars for variable X.

> Must deflate $Y^{nom}, C^{nom}, I^{nom}$ with the same price index to maintain additive relationship between real and nominal variables.

Typically choose $P_{C,t}$:

Given $Y_t^{nom} = C_t^{nom} + I_t^{nom}$, we have that $Y_t = C_t + I_t \iff Y_t = \frac{Y_t^{nom}}{P_{C,t}}, C_t = \frac{C_t^{nom}}{P_{C,t}}$, and $I_t = \frac{I_t^{nom}}{P_{C,t}}$

Unless otherwise stated: $K \neq \frac{K^{nom}}{P_I}$. Instead: $K = \frac{K^{nom}}{P_C} = \frac{P_I}{P_C} X$

1.1.4 "p_X"

$$p_X \equiv \text{price of capital} = \frac{P_I}{P_C} = \frac{apples}{bricks}$$

So,

$$K = p_X X$$

> The reason we can use K instead of X in APF: equal inflation rates in investment and consumption make the quantity of capital proportionate to the market value of businesses.

Will also see $p_X = \frac{1}{q}$ where q_t = efficiency units of capital that can be purchased with one apple of investment expenditure

> "p" is price; "P" is deflator

1.1.5 Def "Stocks, Flows, and Rates"

- Stocks: 'dollars variables'
  - e.g., A, K, X
- Flows: 'dollars per year'
  - e.g., Y, C, I
- Rates: 'inverse time'
  - e.g., $d, r, g_X \equiv \frac{\frac{dX}{dt}}{X}$

  flow = quantity measured as bricks per quarter

  stock = total number of bricks

1.1.6 "Dollars, Apples, and Bricks; Units of Measurement"

- Prices
  - Consumer Price Index: P_C: $\frac{\$}{apples}$
  - Investment Price Index: P_I: $\frac{\$}{bricks}$
- Stocks
  - Wealth: K, A: apples
  - Capital (quantity): X: bricks
- Flows (Nominal and Real):
  - $Y^{nom}, C^{nom}, I^{nom}$: $\frac{\$}{year}$
  - Y, C, I: $\frac{apple}{year}$
- Dollars: current year, or nominal, dollars; units of $Y^{nom}, C^{nom}, I^{nom}$
- Apples: base year dollars; units of real Y, C, I, and wealth, K, A
- Bricks: base year investment dollars; units of X, $\frac{I^{nom}}{P_I}$, where P_I is an investment price index that tracks inflation in new capital goods.

  current year = nominal!

1.1.7 "Wealth"

Wealth is the quantity of apples the households can currently buy if they sell everything they own.

Nominal wealth:

$$K^{nom} = P_I * X$$

Real wealth is the number of apples that can be bought with nominal wealth:

$$K = \frac{K^{nom}}{P_C} = \frac{P_I}{P_C} X$$

1.1.8 "K vs X"

- K = market value of capital
- X = quantity of capital
- $K = P_X * X = \frac{P_I}{P_C} X = \frac{X}{q}$ (where q is efficiency units of capital that can be purchased with one apple of investment expenditure)

1.1.9 Def "Capital Accumulation Equation"

Discrete:

$$K_{t+1} - K_t = I_t - dK_t = I_t - (\delta - g_{p,t})K_t$$

Continuous:

$$\dot{K}_t = I_t - dK_t$$

Sometimes given as:

$$\dot{X}_t = q_t I_t - \delta X_t$$

where q_t is given a functional form

Intensive (Solow)

$$\dot{k}_t = sf(k_t) - (g_A + n + d)k_t$$

($g_A = 0$ in basic Solow)

Intensive (Solow) Annual

$$\frac{k_{t+1} - k_t}{1} = sf(k_t) - (n+d)k_t \iff k_{t+1} = sf(k_t) + (1-n-d)k_t$$

This is what we (iteratively) plot to solve the Solow model!

1.1.10 "Depreciation rate"

$$d \equiv \underbrace{\delta}_{\text{wear + tear}} - \underbrace{g_p}_{\text{capital gain}} = \text{depreciation rate on MARKET VALUE of capital}$$

In Solow ISTC: $d = \delta + g_q$

1.1.11 "Relationship between capital stock and past investment history"

Market value of capital (K) is the sum of undepreciated parts of past real investment expenditures. I.e.,

$$K_{t+1} = \sum_{\tau \geq 0} I_{t-\tau}(1-d)^{\tau}$$

1.1.12 "Household Wealth"

$$A_t = K_t + D_t$$

where

- A_t = household financial assets at date t
- K_t = market value of private business assets
- D_t = market value of government debt

1.1.13 "Debt Accumulation Equation"

$$D_{t+1} - D_t = \underbrace{r_t D_t}_{\text{interest due}} + \underbrace{(G_t - T_t)}_{\text{primary deficit}}$$

* In-flow: $r_t D_t + G_t$ * Stock: B_t * Out-flow: T_t

1.1.14 "Wealth Accumulation Equation"

$$\begin{aligned} A_{t+1} - A_t &= K_{t+1} - K_t + D_{t+1} - D_t && (1)\\ &= I_t - dK_t + r_t D_t + (G_t - T_t) && (2)\\ &= Y_t - C_t - dK_t + r_t D_t - T_t && (3)\\ &= \underbrace{Y_t + r_t D_t}_{\text{inflow}} - \underbrace{(C_t + T_t + dK_t)}_{\text{outflow}} && (4) \end{aligned}$$

- Stock: $A_t = K_t + D_t$

  Government debt is a form of wealth

1.2 Measuring Productivity

1.2.1 Def "Productivity"

GDP per hour worked.

1.2.2 "Labor supply"

Labor supply is the weighted sum of hours worked.

Separate workers into categories i by education, age, sex, health etc.

$N_{i,t}$ total hours worked by category i at date t:

$$L_t = h_{0,t}N_{0,t} + h_{1,t}N_{1,t} + ... + h_{I,t}N_{I,t}$$

1.2.3 "Growth Decomposition"

$\alpha_K = \frac{RX}{Y}$ and $\alpha_L = \frac{wL}{Y}$. Makes sense because numerator is total quantity of Labor or Capital times the factor price and the denominator is total income (i.e., total output)

$$g_Y = \alpha_K g_X + (1 - \alpha_K) g_L + g_Z$$

$$g_{Y/N} = \underbrace{\alpha_K g_{X/N}}_{\text{capital intensity}} + \underbrace{(1 - \alpha_K) g_h}_{\text{labor quality}} + \underbrace{g_Z}_{\text{TFP growth}}$$

2 (Neoclassical) Production

2.0.1 "(Non)Zero Profits"

$$Y \equiv wL + RK + \Pi$$

If competitive, $\Pi = 0$ since:

$$\begin{aligned}
Y &= wL + RK + \Pi && (5)\\
Y &= MPL * L + MPK * K + \Pi && (6)\\
Y &= (1-\alpha)\frac{Y}{L}L + \alpha\frac{Y}{K} * K + \Pi && (7)\\
Y &= (1-\alpha+\alpha)Y + \Pi && (8)\\
0 &= \Pi && (9)\\
& && (10)
\end{aligned}$$

If not competitive, $MPK \neq R$ and/or $MPL \neq L$. E.g., say $R = (1-\lambda)\alpha\frac{Y}{K}$. Then

$$\begin{aligned}
Y &= wL + RK + \Pi && (11)\\
Y &= \lambda\frac{Y}{L}L + (1-\lambda)\alpha\frac{Y}{K} * K + \Pi && (12)\\
Y &= (\lambda + \alpha - \alpha\lambda)Y + \Pi && (13)\\
(1-\alpha)(1-\lambda)Y &= \Pi > 0 && (14)\\
& && (15)
\end{aligned}$$

2.0.2 "Constant Returns to Scale (CRS)"

Implies $\alpha_K = 1 - \alpha_L$

2.0.3 "Competitive economy: main assumptions"

- Large number of agents (households and firms)–any agent takes prices as given/individual choices have negligible effect on prices.
- Complete markets (every good is traded).
- Centralized markets (Prices determined through market clearing)
- Common information
- Usually assume closed economy, often assume zero government debt/balanced budget

  Maximized profit is zero $\Rightarrow$ free entry condition holds automatically

2.0.4 "Neoclassical production function assumptions"

1. $F \in C^2$
2. Positive marginal products ($\frac{\partial F}{\partial K} > 0$ $\frac{\partial F}{\partial L} > 0$)
3. Decreasing marginal products ($F_{KK} < 0$ $F_{LL} < 0$)
4. Constant returns to scale ($F(\lambda K, \lambda L) = \lambda F(K, L), \forall \lambda > 0$)
5. Concave (1-4 imply this)

These assumptions guarantee:

- A representative firm operating the aggregate production function $Y = F(K, L)$ is without loss of generality

- In equilibrium, capital and labor are fully employed and factor prices are positive
- Factor prices are equal to marginal products

  In HW we also proved: $F_{KL} > 0$ (i.e., as labor increases, so does capital, and vice versa.)

2.0.5 "Inada Conditions"

Inada conditions guarantee an interior solution for any pair of positive factor prices R, w:

- $\lim_{K\to 0} \frac{\partial F}{\partial K} = \infty$, $\lim_{K\to\infty} \frac{\partial F}{\partial K} = 0$
- $\lim_{L\to 0} \frac{\partial F}{\partial L} = \infty$, $\lim_{L\to\infty} \frac{\partial F}{\partial L} = 0$
- $F(0, L) = 0$

2.0.6 "Aggregate Production Function"

The APF is an indirect objective function $F(K, L)$ corresponding to the solution to the output maximization problem

$$F(K, L) = \max\{K_1^\alpha L_1^{1-\alpha} + \cdots K_N^\alpha L_N^{1-\alpha}\}$$

with aggregate labor and capital constraints

$$K_1 + \cdots K_N \leq K, L_1 + \cdots + L_N \leq L$$

and feasibility constraints on inputs of individual firms

$$0 \leq K_1 \leq K, ..., 0 \leq K_N \leq K \text{ and } 0 \leq L_1 \leq L, ..., 0 \leq L_N \leq L$$

Algorithm Aggregate Production function

The APF will usually end up being either Cobb-Douglas or $AK + BL$, so you'll often actually solve a problem like the following (OLG with K as land):

$$F(K, L_t) = \max_{k_i} \sum_i^{L_t} k_i^\alpha \text{ s. to feasibility constraints}$$

Which gives:

$$\mathcal{L} = \sum_i^{L_t} k_i^\alpha + \lambda_t[\sum_i^{L_t} k_i^\alpha - K]$$

So

$$\partial k_i : \alpha k_i = \lambda_t, \underline{\forall i}$$

Which implies

$$\sum_i^{L_t} k_i = kL_t \Rightarrow k_t = \frac{K}{L_t}$$

So, then you can plug this "solution" into the the original max problem but remove the max operator to get:

$$F(K, L_t) = \sum_i^{L_t} k_i^\alpha = \sum_i^{L_t} (\frac{K}{L_t})^\alpha = K^\alpha L_t^{-\alpha} L_t = K^\alpha L_t^{1-\alpha}$$

2.0.7 Def "Production side equilibrium - N firms"

A production side equilibrium is a list of factor demands $(K_i, L_i)|_{i=1}^N$ and factor prices (R, w) satisfying

1. Profit maximization (for all firms! don't omit a firm here!):

i. Final Goods firms solve

$$(K_i, L_i) \in \arg \max_{K \geq 0, L \geq 0} \{F(K,L) - RK - wL)\} \text{ given } (R, w), \forall i = 1, ...N.$$

ii. Capital Leasing firms solve

$$\max_{\mathcal{K}} (R_t - (r_t + d))\mathcal{K}, \text{ which gives } R_t = r_t + d$$

iii. Capital Goods capital-producing firms solve

$$\max_{\mathcal{I}} (p_{K,t} - 1)\mathcal{I}, \text{ which gives } p_{K,t} = 1$$

2. Free entry (or other condition governing number of firms):

$$\max_{K \geq 0, L \geq 0} \{F_i(K,L) - RK - wL\} = 0, \forall i = 1, ...N.$$

   Zero profits are automatic as a result of profit maximization and market clearing with constant returns to scale.

3. Market clearing (labor market, feasibility, FINISH!!!)

- $\sum_i K_i \leq K$
- $\sum_i L_i \leq L$
- $(K - \sum_i K_i)R = 0$
- $(L - \sum_i L_i)w = 0$

NO asset or goods market clearing here!! that's a concept of households, not production!

Caution: first step should be to define profit for each type of firm given the context of the question.

2.0.8 Def "Production side equilibrium"

If you already have the APF you can use the APF instead of N firms. Both are valid, just different. You'll see both definitions in different contexts and should be able to write both definitions.

2.0.9 "Prop: Production side equilibrium allocation maximizes output"

Production side equilibrium allocation maximizes output

2.0.10 "Prop: if F_i neoclassical"

Let F_i satisfy the neoclassical assumptions for all i. Then

1. Any solution to the output maximization problem is a production side equilibrium
2. any production side equilibrium involves factor demands $(K_i, L_i)|_{i=1}^N$ that solve the output maximization problem.
3. Market-clearing factor prices are equal to the partial derivatives of the aggregate production function.
4. R and w also equal to the Lagrange multipliers of the output maximization problem

2.0.11 "Euler theorem for homogenous functions"

$$F(K,L) = F_K K + F_L L$$

3 Investment

3.0.1 "Present Value"

Definition: present value is future wealth expressed in units of current wealth (units: stock, apples)

3.0.2 "Financial Rate of return"

$$\text{Rate of return} = \frac{\text{dividend + capital gains}}{\text{Value of asset}} = \frac{MPX_{t+\Delta t} - \delta p_{X,t+\Delta t} + (p_{X,t+\Delta t} - p_{X,t})/\Delta t}{p_{X,t}}$$

The financial rate of return is defined as the sum of dividend per dollar of asset plus capital gain/loss per dollar of asset.

Discrete

$$\underbrace{\frac{1+r_{t+1}}{1-d}}_{\text{after depreciation interest}} = \overbrace{\frac{R_{t+1}}{p_t}}^{\text{div}} + \overbrace{\frac{p_{t+1}}{p_t}}^{\text{cap gain}}$$

note: units of LHS and RHS match: both are rates!

if your interpretation is from an asset pricing perspective you might see this (same equation, just rearranged) from prod-side eqm:

$$\mathcal{K}\left(\frac{\overbrace{(R_{t+1}+p_{t+1})(1-d)}^{\text{undepreciated dividends + cap gains (not in rates tho)}}}{\underbrace{1+r_{t+1}}_{\text{discount to PV}}} - \underbrace{p_t}_{\text{purchase price}}\right) = 0$$

Continuous

$$r_t + d = \overbrace{\frac{R_t}{p_t}}^{\text{div}} + \overbrace{\frac{\dot{p}_t}{p_t}}^{\text{cap gain}}$$

notice the connection to Jorgenson's: usually $p_t = 1, \forall t$ so have $r_t + d = \frac{R_t}{1} + 0$

3.0.3 "Investment Decision"

The optimal investment decision equates the firm's financial rate of return (on its stock of bricks) to the market rate of return:

$$r_{t+\Delta t} =^{set} \text{Financial Rate of return}$$

3.0.4 "Present Value Discount factor"

N periods forward, discrete:

$$m_{t,t+N\Delta t} = \Pi_{j=1}^{N} \frac{1}{(1 + r_{t+j\Delta t} * \Delta t)}$$

between s, t, continuous:

$$m_{t,s} = e^{-\int_t^s r(\xi)d\xi}$$

when r is constant:

$$m_{t,s} = e^{-r(s-t)}$$

Present Value:

$$PV(\{y_t\}) = \int_t^\infty m_{t,s} y_s ds = \underbrace{\frac{y_t}{\bar{r} - n}}_{\text{On BGP profits grow at rate } n}$$

m always has a growth rate of $-r$

3.0.5 "Jorgenson's Formula"

$$\frac{MPX_t}{p_{X,t}} = MPK_t = \underbrace{r_t + d_t}_{\text{user cost}}$$

Competitive economy: $MPK = R$

- $R > r + d$ - capital leasing business can make positive economic profits for any K (demand for capital assets is infinite $\Rightarrow$ asset market does not clear)
- $R < r + d$ - renting capital is preferred to owning (no one wants to own capital $\Rightarrow$ asset market does not clear)

  Caution: in Endogenous Growth we start using $MPK \neq R = r + d \ldots$.

3 Interpretations:

1. Optimality condition that describes the demand curve for investment expenditure:

$$MPK_t\Big(I(r) + (1-d)K_{t-\Delta t}, L_D\Big) = r + d$$

2. Financial no-arbitrage condition:

$$r = MPK - d$$

3. Profit maximization/free entry condition for capital leasing businesses whose profit per dollar of capital is the difference between the rental fee and user cost:

$$\max_K \{(R - (r+d))K\} = 0$$

3.0.6 "Market value of a firm"

The *Market value of a firm* is the amount of wealth created by its future economic activity:

$$V_t = PV(\{R_t K_t\}) - PV(\{I_t\})$$

Proposition Market value of a firm equals the resale value of its current assets plus the present value of its future economic profits:

$$V_t = K_t + PV(\{\pi_t\})$$

3.0.7 "Law of motion for (corporate bonds) debt"

$$\dot{B}_t = r_t B_t - b_t$$

- B_t = market value of the current stock of debt (pre-determined)
- b_t = current debt repayment flow: $b_t > 0$ represents paying down the debt, $b_t < 0$ represents more borrowing

3.0.8 "Investment Financing Constraint"

Decision variables: $D_t, \dot{N}_t, \dot{B}_t$

$$I_t = \underbrace{R_t K_t - D_t}_{\text{internal financing}} + \underbrace{\dot{N}_t P_t}_{\text{equity financing}} + \underbrace{\overbrace{\dot{B}_t - r_t B_t}^{=-b_t}}_{\text{Debt financing}}$$

- D_s = future dividend flow
- N_t = current number of shares (pre-determined)
- P_t = current share price
- $M_t = P_t N_t$ = current market value of outstanding shares

3.0.9 Thm "Modigliani-Miller"

$$V_t = M_t + B_t$$

Means:

1. Dividend plan $\{D_t\}$ does not affect the market value of the firm.
2. Value of the firm does not depend on "capital structure" (i.e. the debt/equity mix used for financing), just on the total value of equity and debt.

4 Solow

4.0.1 "Solow Algorithm"

1. Derive LOM for k
   1. replace I with $I = S = sY$ instead of $I = Y - C$
2. Find steady state: k_*. To show it's actually steady state:
   1. show k_* is unique and positive
   2. Alternatively: show exponent on k_t in LOM is < 1 (e.g., α). This implies LOM is concave
   3. If asked about stability: "Solow's steady state is globally stable since productivity growth fueled by investment eventually stops"
3. Plot
   1. just put k_{t+1} on Y axis instead of bothering to convert from continuous to discrete time...

4.0.2 "Uses of Solow Model"

- Make predictions about trajectories for macro variables Y, C, I, K, w, R, r.
- Easy to take to the data
- Analyze "what-if" scenarios
- Understand economic forces behind the co-evolution of macro variables

  Central conclusion of the Solow model: if the returns that capital commands in the market are a rough guide to its contributions to output, then variations in the accumulation of physical capital do not account for a significant part of either worldwide economic growth or cross-country income differences

4.0.3 "Solow 5 Key Equations"

1. Capital Accumulation $\dot{K} = I - dK$
2. Household Saving: $S = sY$ and $C = (1 - s)Y$
3. Goods Market Clearing: $Y = C + I$
4. Labor Supply: $L_t = L_0 e^{nt}$
5. Production Function (Typically Cobb-Douglass): $Y = ZK^\alpha L^{1-\alpha}$

4.0.4 "Solow Saving/Consumption Rule"

$$S_t = sY_t$$

and

$$C_t = Y_t - S_t = (1 - s)Y_t$$

4.0.5 "Solow Market Clearing"

Goods: $Y_t = C_t + I_t$. Implies: $S_t = I_t$

(and don't forget: $S_t = sY_t$, so $I_t = sY_t$)

4.0.6 "Solow Solution/Properties in the long run"

- Capital-labor ratio $k_t = k_*$ Stays constant
- Output per worker $y_t = Y_t/L_t = Zk_*^\alpha = y_*$ Stays constant
- Labor force $L_t = L_0 e^{nt}$ Given, grows at rate n
- GDP $Y_t = y_* L_t = Zk_*^\alpha L_0 e^{nt}$ Grows at rate n
- Capital stock, $K_t = k_* L_t = k_* L_0 e^{nt}$ Grows at rate n

4.0.7 "Solow Wage and interest rate: properties (given Cobb-Douglas)"

- $w_t = (1-\alpha)\frac{Y_t}{L_t} = (1-\alpha)y_t$
- $R_t = \alpha\frac{Y_t}{K_t} = \alpha\frac{Y_t/L_t}{K_t/L_t} = \alpha\frac{y_t}{k_t} = \alpha Z k^{\alpha-1}$
- $r_t = R_t - d = \alpha Z k^{\alpha-1} - d$

So, Wage positively depends on capital per worker and interest rate negatively depends on capital per worker.

4.0.8 "Solow: TFP and long-run productivity"

Output per worker rises more than proportionately with TFP:

$$y_* = \frac{\bar{Y}}{L} = Zk_*^{\alpha} = Z[\frac{sZ}{n+d}]^{\frac{\alpha}{1-\alpha}}$$

- Direct: A TFP increase makes production more efficient
- Indirect: A TFP increase raises MPK and demand for investment, and this stimulates capital accumulation, so $k_*(Z)$ rises

  Capital accumulation amplifies the effect of TFP change on productivity and consumption

4.0.9 "Globally Stable"

Solow's steady state is globally stable. This means no matter the shock, Solow will go back to the steady state.

> No matter where the economy starts in terms of its initial capital stock, it approaches the steady state capital-labor ratio, k_* in the long run

Productivity growth fueled by investment eventually stops

4.0.10 "Solow Long-Run Interest Rate"

Variable	Expression	Properties in the long run
Capital-labor ratio, k	$k_t = k_*$	Stays constant
Output per worker, y	$y_t = Y_t/L_t = Zk_*^{\alpha} = y_*$	Stays constant
Labor force	$L_t = L_0 e^{nt}$	Given, grows at rate n
GDP, Y	$Y_t = y_* L_t = Zk_*^{\alpha} \cdot L_0\, e^{nt}$	Grows at rate n
Capital stock, K	$K_t = k_* L_t = k_* \cdot L_0\, e^{nt}$	Grows at rate n

$$\bar{r} = \alpha\frac{g_A + n + d}{s} - d$$

4.0.11 "Solow Golden Rule"

Golden rule savings rate = The rate of savings that maximizes household consumption. Any savings level above this golden rule is inefficient.

Find Golden rule by taking the derivative with respect to s. E.g.,

$$\begin{aligned} k_G &\in \arg\max[f(k) - (n+d)k] \iff & (16)\\ 0 &= f'(k_G) - (n+d) & (17)\\ f'(k_G) &= n+d & (18)\\ \alpha k_G^{\alpha-1} &= n+d & (19)\\ k_G &= [\frac{\alpha}{n+d}]^{\frac{1}{1-\alpha}} & (20) \end{aligned}$$

Generally, when s is the golden rule savings rate, $MPk = n + d + g_A$. This makes sense because the benefit of increasing capital comes in the form of higher MPk, but this higher level of steady state capital requires more future maintenance (a larger $n + d + g_A$.

In basic Solow the golden rule occurs when $s = \alpha$. So $s > \alpha$ leads to inefficient overaccumulation (Pareto improvement = throw away capital).

4.0.12 "Solow with Tech Change"

In basic Solow, k, y const, $g_Y = g_L = g_k = n$.

With Cobb-Douglas type of tech change doesn't matter:

$$Y_t = \underbrace{K_t^\alpha (A_t L_t)^{1-\alpha}}_{\text{labor-augmenting}} = \underbrace{A_t^{1-\alpha} K_t^\alpha L_t^{1-\alpha}}_{\text{Neutral}} = \underbrace{(A_t^{\frac{1-\alpha}{\alpha}} K_t)^\alpha L_t^{1-\alpha}}_{\text{capital-augmenting}}$$

re-interpret k_t as $k = \frac{K}{AL}$. Then all analysis still holds, but since $g_{AL} = g_A + n$ we add g_A:

$$\dot{k} = sf(k) - (g_A + n + d)$$

4.0.13 "Constant Growth Path + BGP"

A trajectory in which Y, C, I, K grow at constant (possibly zero and possibly different) exponential rates.

Prop If constant growth equilibrium in neoclassical economy with $C_t, I_t > 0$ then we have a BGP with $g_Y = g_K = g_I = g_C$.

Productivity growth fueled by investment eventually stops (this means $\dot{k} = 0$ in long-run)

4.0.14 "Uzawa Thm"

If a one-sector model with technological change has a constant growth equilibrium, then technological change has to be labor augmenting.

Uzawa is a necessary condition for constant growth long-run equilibrium

Uzawa is sufficient if there is a constant growth rate (s) as there is in the standard Solow model

4.0.15 "Investment-specific technological change (ISTC)"

An improvement in the economy's ability to produce capital goods out of investment expenditure:

$$\dot{X}_t = q_t I_t - \delta X_t$$

where q_t = efficiency units of capital that can be purchased with one apple of investment expenditure at time t.

$$q_t = \frac{P_{C,t}}{P_{I,t}} = \frac{1}{p_{X,t}}$$

makes capital goods cheaper: as q_t grows, p_X falls

free entry makes price of existing capital p_X equal to its supply cost, $\frac{1}{q_t}$. When supply cost drops, so does the market value of existing capital, K_t.

5 OLG

5.0.1 "OLG Algorithm"

UMP

1. Combine constraints
2. Take FOCs to get Euler Equation
   a. alternatively, and usually easier: sub a_{t+1} in for c_1, c_2 and take a single FOC wrt a_{t+1}. Then just rearrange FOC and you've solved the problem
   b. won't always work if problem is nonstandard
3. Plug c_1 into combined constraint and solve for $c_{1,t}, c_{2,t+1}$ in terms of w_t.
4. Solve for a_{t+1} (canonical sol is $a_{t+1} = \frac{\beta}{1+\beta} w_t$)

Get LOM for k_{t+1} / find steady state

0. Solve for a_{t+1} in terms of c_1, c_2, w
1. Asset market clearing - this is: $a_{t+1} N_t = K_{t+1}$ (Total Asset Demand = Tomorrow's stock of capital)
2. divide by L_{t+1}, plug in expression for a_{t+1} (get this expression from the UMP!)
3. simplify to get LOM for k_{t+1}
4. solve for (usually unique) steady state

Find $\bar{r}$

1. Start with Jorgenson's
2. You should have k_* by this point. Just plug in and simplify.

Pareto-Inefficient?

1. Compare $\frac{1+\bar{r}}{1+g_Y} = 1$. $\alpha \in (0, \alpha)$ means the eqm you found is Pareto-inefficient because of overaccumulation!

5.0.2 "OLG Facts"

- Solow-type aggregate dynamics of k: unique and globally stable steady state (assuming log preferences)
- Neoclassical production - standard except that it's discrete - can be labor-augmenting: $Y_t = F(K_t, (1+g)^t L_t)$
- Saving: $s_t = \underbrace{r_t a_t}_{\text{capital income}} + \underbrace{w_t l_t}_{\text{labor income}} - c_t$
- LOM for wealth: $a_{t+1} = (1 + r_t)a_t + w_t l_t - c_t = a_t + s_t$ (where a_t = stock; s_t = flow)
- $L_t = (1 + n)^t \iff \frac{L_{t+1}}{L_t} = 1 + n$
- In the canonical model, agents over-save in their youth, leading to a reduction in MPK and a corresponding suppression of the long-term interest rate, $\bar{r}$. Most model extensions correct these inefficiencies by justifying the reason for savings.

  Demand for assets by generation t is $a_{t+1} N_t$!!

5.0.3 "(Canonical) OLG Household Problem"

$$\max_{c_{1,t}, c_{2,t+1}, a_{t+1}} \Big(u(c_{1,t}) + \beta u(c_{2,t+1}) \Big)$$

- $a_{t+1} = (1 + r_t) \underbrace{a_t}_{\text{initial assets}} + w_t \underbrace{l_1}_{=1} - c_{1,t}$
- $a_{t+2} = (1 + r_{t+1}) a_{a+1} + w_{t+1} \underbrace{l_2}_{=0} - c_{2,t+1}$
- $a_t = 0$, given
- $a_{t+2} \geq 0$ (must bind though since utility maximizer wouldn't die with extra cash)

Simplifies to

- $c_{1,t} = w_t - a_{t+1}$
- $c_{2,t+1} = (1 + r_{t+1})a_{t+1}$

Combine to get:

$$PVLC_t = c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} = w_t = PVLR_t$$

5.0.4 "OLG Consumption Euler Equation"

$$\frac{u'(c_{2,t+1})}{u'(c_{1,t})} = \beta(1 + r_{t+1})$$

Conclusion: optimal consumption profile equalizes marginal utility of wealth across periods

5.0.5 "OLG Market Clearing"

Asset Market:

- Start with $K_t = A_t$ and $K_{t+1} = A_{t+1}$.
- Note that $A_{t+1} - A_t = S_{1,t} + S_{2,t}$, where
- the old consume all of their wealth and capital income:
  - $S_{2,t} = r_t K_t - c_{2,t} L_{t-1} = -K_t = -A_t$
- the young will own all the assets next period:
  - $S_{1,t} = a_{t+1} L_t = A_{t+1} = K_{t+1}$.
- Which gives the following final form:

$$a_{t+1} L_t = K_{t+1}$$

If government debt:

$$\underbrace{a_{t+1} L_t}_{\text{total asset demand}} = \underbrace{K_{t+1} + D_{t+1}}_{\text{total asset supply}}$$

Goods Market (Implied by AMC - observe that $S_1 + S_2 = I$)):

$$Y_t - C_t = I_t$$

5.0.6 Def "OLG Equilibrium"

The equilibrium consists of quantities $\{c_{1,t}, c_{2,t+1}, a_{t+1}, C_t, L_t, K_t, Y_t, I_t\}, \forall t$ and prices $\{w_t, R_t, r_t\}, \forall t$ such that

1. Households Maximize Utility:
   - $a_{t+1} = \arg\max_{a'} \Big(u(w_t - a') + \beta u\big((1 + r_{t+1})a'\big)\Big)$
   - GIVEN (w_t, r_{t+1})
2. Production-side equilibrium:
   - **Final goods producers** maximize profits taking K_t, L_t as given: $R_t = MPK_t, w_t = MPL_t, r_t = R_t - d, Y_t = F(K_t, (1 + g)^t L_t)$
   - **Capital leasing firms** choose K_t taking (R_t, r_t) as given
3. All markets clear:
   - Asset: $a_{t+1} L_t = K_{t+1}$
   - AMC $\Rightarrow$ GMC

5.0.7 "Canonical OLG Solution to household problem"

$$a_{t+1} = \frac{\beta}{1 + \beta} w_t$$

5.0.8 “OLG Lifetime Budget Constraint”

$$PVLC_t = c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = w_t = PVLR_t$$

Note: PVLR does NOT include interest on savings! Lifetime Resources (LR) includes interest earnings on savings, but PVLR ignores this interest because it's discounted to PV!

5.0.9 “OLG LOM for Capital”

Aggregate capital accumulation equation is standard: $K_{t+1} - K_t = I_t + dK_t$

intensive capital accumulation equation is usually implicit:

- $a_{t+1} = \hat{a}(w_t, r_{t+1})$ which depends on k_t, k_{t+1}, t. So we get
- $\Phi(k_t, k_{t+1}; t) = 0$

which means multiple equilibria are possible... log preferences fix this!

5.0.10 “OLG Intensive Form”

$$k_t = \frac{K_t}{(1+g)^t L_t}; y_t = \frac{Y_t}{(1+g)^t L_t}$$

5.0.11 “OLG LOM for Capital, Log Preferences”

$$\begin{aligned}
K_{t+1} &= a_{t+1} L_t && (21)\\
&= \frac{\beta}{1+\beta} w_t L_t && \text{(Sol to HH problem)}\\
&= \frac{\beta}{1+\beta}(1-\alpha) Y_t && (22)\\
& && (23)
\end{aligned}$$

So,

$$\begin{aligned}
k_{t+1} &= \frac{\beta(1-\alpha)}{1+\beta} \frac{Y_t}{(1+g)^{t+1} L_{t+1}} && (24)\\
&= \frac{\beta(1-\alpha)}{1+\beta} \frac{Y_t}{(1+g)^{t+1} L_{t+1}} \frac{(1+g)^t L_t}{(1+g)^t L_t} && (25)\\
&= \frac{\beta(1-\alpha)}{1+\beta} \frac{y_t}{(1+g)(1+n)} && (26)\\
&= \frac{\beta(1-\alpha)}{1+\beta} \frac{y_t}{1+g+n+gn} && (27)\\
&\approx \frac{\beta(1-\alpha)}{1+\beta} \frac{y_t}{1+g+n+0} && (28)\\
&\approx \frac{\beta(1-\alpha)}{1+\beta} \frac{y_t}{1+g_Y} && \text{(On BGP)}\\
&= \frac{\beta(1-\alpha)}{1+\beta} \frac{k_t^\alpha}{1+g_Y} && (29)\\
& && (30)
\end{aligned}$$

5.0.12 “OLG Golden Rule”

In canonical model, $\bar{r} = g_Y$ (since $g_Y = n$ and using Jorgenson's). $\bar{r} < g_Y \Rightarrow$ oversaving - Pareto improvement is to throw away capital.

As in Solow, the golden rule capital-labor ratio k_G is the capital-labor ratio that maximizes the long-run consumption-labor ratio, c.

> Economic intuition for Pareto inefficiency: lack of resources in old age and no opportunities to borrow drive excessive saving.

5.0.13 "OLG and Efficiency"

The OLG model has infinitely many Pareto efficient consumption plans.

$\bar{r} < g_Y$ is Pareto-inefficient. Note that competitive equilibrium results in $\bar{r} < g_Y$, so competitive equilibrium is inefficient. Just consume excess capital today and everyone is better off.

> an increase in r results in a DECREASE in investment!!!!!!!!! (higher prices $\Rightarrow$ lower demand)

5.0.14 "OLG with Government Debt"

$D_{t+1} = (1 + r_t)D_t + G_t - T_t$

$G_t > T_t$ means the government has a deficit

LOM for debt: $D_{t+1} - D_t = r_t D_t + G_t - T_t$

> Government debt does not depreciate

5.0.15 "Service Fee"

$r_t D_t$ is the *service fee* on debt.

5.0.16 "OLG with bequests"

$$U = u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta(1 + n)\varphi U_{t+1}$$

where φ = intergenerational discount factor (aka altruism)

Recursive structure makes this an infinite horizon problem:

$$U_t = \sum_{s=t}^{\infty} (\beta\varphi(1 + n))^{t-s}[u(c_{1,s}) + u(c_{2,s+1})]$$

Choice vars:

- $\{c_{1,t}, c_{2,t+1}\}_{t=0}^{\infty}$
- $\{a_{t+1}\}_{t=0}^{\infty}$ = assets carried into old age by generation t
- $\{b_{t+1}\}_{t=0}^{\infty}$ = bequests PER PERSON

Variables known to decision-maker:

- b_0
- $\{w_t, r_{t+1}\}_{t=0}^{\infty}$

Simplification:

- set $g = 0$ so that $g_Y = n, c_{1,s} = \bar{c}_1, c_{2,s+1} = \bar{c}_2$

So, max problem is subject to:

$$a_{s+1} = b_s + w_s - c_{1,s}$$
$$c_{2,s+1} = (1 + r_{s+1})a_{s+1} - (1 + n)b_{s+1}$$

Take FOCs and get:

1. (inverted??) Euler equation:

$$\frac{u'(c_{1,s})}{u'(c_{2,s+1})} = \beta(1 + r_{s+1})$$

2. Intergenerational Euler Equation:

$$\frac{u'(c_{1,s})}{u'(c_{1,s+1})} = \beta\varphi(1 + r_{s+1}), \forall s \geq t$$

*notice both consumptions are subscripted with the "1"

Optimal bequest equalizes the marginal utility of date-t wealth across the current and the future generation.

6 Optimal Control

6.0.1 "Optimal Control Terms"

- $\bar{k}$: max debt constraint or min wealth constraint IN TERMINAL STATE
- Γ: law of motion for capital/wealth/the state variable: $\Gamma(k_t, c_t, t) = \dot{k}$
- λ_t: marginal benefit of one unit of k_t
- $\lambda_t k_t$ = agent's wealth measured in units of utility (i.e., potential to receive future utility)

6.0.2 "Hamiltonian"

$\mathcal{H} = u + \lambda\Gamma$ is the Hamiltonian

FONC interpretation:

- λ_t = marginal benefit of one unit of k_t
- $\lambda_t k_t$ = agent's wealth measured in "utils" (i.e., potential to receive future utility)

6.0.3 "Standard dynamic optimization problem"

- State: k_t; Control: c_t
- Solution is a decision rule: $c_t = \hat{c}(k_t)$
- Equality constraints describe the LOM for the state: $\Gamma(k_t, c_t, t) = k_{t+1} - k_t$

6.0.4 "End Point (Transversality) Conditions"

- Constrained end-point (original problem)
  - $\lambda_T(k_T - \bar{k}) = 0$
- Salvage Value
  - $S(k_T)$ - endpoint constraint replaced by some known function for the salvage value
- Free end-point
  - $\lambda_T = 0$
- Infinite horizon
  - $\lim_{t\to\infty} \mathcal{H}(\hat{k}_t, \hat{c}_t, \hat{\lambda}_t, t) = 0$ - hard to solve this one
  - but, if you have an objective with exponential decay (e.g., $u(k, c, t) = e^{-\rho t} f(k, c)$) we can use: $\lim_{t\to\infty} \hat{\lambda}_t \hat{k}_t = 0$.

Usually have:

- Planner's problem: $\lim_{t\to\infty} \lambda k = 0$
- HH problem: $\lim_{t\to\infty} e^{\overline{r}(0,t)} a_t = 0$

7 Ramsey

7.0.1 "Ramsey Algorithm"

Planner's Problem

0. Change of variables: $L_t = H * l_t$ and $C_t = H * x_t$
1. Transition K to k
2. Solve Planner's Problem
    1. Derive EE
    2. Get two DEs:
        1. $\frac{\dot{c}_t}{c_t} = \alpha k^{\alpha-1} - d - \rho$
        2. $\dot{k}_t = k_t^\alpha - c_t - (n+d)k_t$

Household Problem

1. assert equilibrium (assume $a_{it} = a_t$ which implies $c_it = c_t = \frac{C_t}{L_t}$)
2. Derive EE (should look like planner's problem, but with ε_u replacing θ)
    1. May need to provide LOM for $\dot{A}$ given only a LOM for $\dot{K}$. This requires AMC.
3. Use AMC to convert A to K and then derive LOM for $\dot{k}$
4. Should now have two DEs: 1. $\frac{\dot{c}_t}{c_t} = \alpha k^{\alpha-1} - d - \rho$ 2. $\dot{k}_t = k_t^\alpha - c_t - (n+d)k_t$

Phase Diagram

1. Derive isoclines.
    1. CAUTION: If you have a general $f'(k)$, plug Cobb-Douglas in (on scratch paper) and solve for k_* to be sure you know which way the vertical isocline moves.
    2. $\dot{c} = 0 \Rightarrow k = (\frac{\alpha}{d+\rho})^{\frac{1}{1-\alpha}}$ (hint: make sure exponent is positive)
    3. $\dot{k} = 0 \Rightarrow c_t = k_t^\alpha - (n+d)k_t$
2. Draw new isoclines and arrows; DON'T DRAW TRAJECTORY YET!

Time Paths if Surprise change:

1. This is rare. Make sure you read the question properly. If so, jump automatically to new SA, otherwise will never get there.
2. Exception: a nondistortionary tax implemented without pre-announcement won't change anything, so no jumps.

Time Paths if Pre-announced change:

1. At t_C:
    i. c_{t_C}: check for
        i. Jump: set $\lambda_{t_C-0} = \lambda_{t_C+0}$ to see if c_{t_C} jumps (rare that it does)
            a. jumps if param changes make u_c, Γ_c discontinuous (e.g., pre-announced sales tax jump or other param change that make marginal utility jump). *(Japan example)*
                - Compare λ_{t_C-0} and λ_{t_C+0} to be safe
        ii. Kink: if $\dot{c}$ jumps, KINK (get direction from PD)
    ii. k_{t_C}:
        i. Jump: assert no jump
        ii. Kink: if $\dot{k}$ jumps, KINK (get direction from PD)
2. Between t_C and t_A, governed by OLD arrows
    i. If one isocline moves before the other draw only the intermediate arrows (use pencil)
3. At t_A:
    i. c_{t_A}: check for
        i. Jump: USE LOGIC (also ask: where should you be at t_{c-0} to reach the new stable arm?)
        ii. Kink: rare, since usually jumps; if $\dot{c}$ jumps, KINK (get direction from PD)
    ii. k_{t_A}:
        i. Jump: assert no jump

ii. Kink: if $\dot{k}$ jumps, KINK (get direction from PD)
    a. NOTE: if c jumps, KINK!

c and λ always jump/stay together so long as utility is strictly increasing (it always is!)

- k, w nearly always move together
- k, r nearly always move opposite

Convexity/Concavity of trajectory on $t \in [t_A, t_C]$

1. start with EE: $\dot{c} = \frac{1}{\theta} c_t (r_t - \rho)$ where $r_t = f'(k_t) - d$
2. Differentiate wrt t
3. Look at signs of everything
    - Should know $\dot{c}$ and $\dot{k}$ based on arrows. If not, ask:
        - $\dot{c}$: what happens to your consumption while you await the change?
        - $\dot{k}$: what happens to your wealth while you await the change?
    - $\dot{r}$: this is opposite of $\dot{k}$ (think IS-LM curve)
    - sign of $r_t - \rho$?
        - if to the left of vertical isocline: $r_t < \rho$
        - if to the right of vertical isocline: $r_t > \rho$

7.0.2 "Ramsey Facts"

- representative dynasty
    - the only heterogeneity this model can handle is in "income," but even then, income ratios must stay constant over time
- no welfare thm guarantees equilibrium will be Pareto-optimal, but it is optimal nonetheless
- x_t: household consumption - CONTROL! but we replace this with C_t and then c_t.
- K_t: wealth - STATE! replace with k_t
- H: total number of households
- l_t: household size (exogenous)
- $L_t = 1 * l_t * H$ (1 = labor supply per person)
- $C_t = Hx_t$
- $c_t = \frac{C_t}{e^{gt} L_t}$
- $k_t = \frac{K_t}{e^{gt} L_t}$
- $f(k_t) = y_t = \frac{Y_t}{e^{gt} L_t}$
- ρ: time discounting (impatience) <- I think this is $1 - \beta$ where β is discounting in 607...
- θ: IES intertemporal elasticity of substitution
- $\theta = -\frac{u_c(c)}{c \cdot u_{cc}(c)} = -\frac{d \ln c}{d \ln(u'(c))}$
- $\frac{\dot{c}}{c} + g$: growth rate of per-capita consumption
- $\vec{\alpha} = (\rho, \theta, n, d, g)$ ALL ARE EXOGENOUS
- $PD(\vec{\alpha})$: phase diagram for comparative dynamics
- λ: MU of wealth
- U = household utility, u = individual utility

The solution is $\hat{c}(k_t) = SA(k_t; \alpha)$.

7.0.3 "Ramsey Assumptions"

- Identical preferences ⇒ there exists a representative household ⇒ *behavior depends only on aggregate wealth* K_t * Conditions for existence of representative household without assuming identical preferences are only slightly less restrictive than assuming identical preferences * Requires that household wealth be proportional and income ratios must stay constant over time
- Large number of households
- Infinite horizon (care about offspring just as much as you care about yourself)
- Time-separable utility (utility does not depend on history of past consumption)
- Inelastic labor supply (*Uzawa still applies* so we'll have BGP)

7.0.4 "Ramsey Household Utility"

$$U(x, l) = l \cdot u(\frac{x}{l})$$

where x represents household expenditure and l is the size of the household.

Functional form for utility of the INDIVIDUAL: $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$

Note: $\lim_{\theta \to 1} \frac{c^{1-\theta}-1}{1-\theta} = \ln c$

7.0.5 "Ramsey Market Clearing"

Asset:

$$A_t = K_t$$

Implies Goods market (just like in OLG):

$$Y_t = C_t + I_t$$

In planner's problem we only use goods market condition!

In HH problem we use AMC.

7.1 Planner's Problem

7.1.1 "Ramsey Planner's Problem"

The planner's problem is to max discounted utility for each household.

Start with:

$$\max_{x_t} \int_t^\infty e^{-\rho t} H \cdot l_t \cdot u(\frac{x_t}{l_t}) dt \text{ s to } \dot{K} = F(K_t, e^{gt} L_t) - x_t H - dK_t$$

Replace $x_t H$ with C_t and $l_t H$ with L_t:

$$\max_{C_t} \int_t^\infty e^{-\rho t} L_t \cdot u(\frac{C_t}{L_t}) dt$$

Then convert to intensive form:

$$\max_{c_t} \int_t^\infty e^{-\rho t} e^{nt} u(c_t) dt$$

which, using CRRA parametrization and dropping constants that aren't useful gives

$$\underbrace{V(k_0, t)}_{\text{indirect utility}} = \max_{c_t} \int_t^\infty e^{-\rho t} e^{nt} \frac{(c_t e^{gt})^{1-\theta}}{1-\theta} dt$$

subject to:

1. $\dot{k}_t = f(k_t) - c_t - (n+d)k_t$
2. $k_0 > 0$ given
3. $\lim_{t\to\infty} \lambda_t k_t$ (transversality condition with exponential discounting)
4. implicitly: $k_t, c_t \geq 0$ - will never bind due to Inada so we usually ignore them

Standard Hamiltonian with CRRA preferences:

$$\mathcal{H} = e^{-\rho t} e^{nt} e^{g(1-\theta)t)} \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t[f(k_t) - c_t - (n+d+g)k_t]$$

$\mathcal{H} = U + \lambda \dot{k}$ gives:

- $\frac{\partial \mathcal{H}}{\partial c_t} = 0$
- $\frac{\partial \mathcal{H}}{\partial k_t} = -\dot{\lambda}$
- $\frac{\partial \mathcal{H}}{\partial \lambda_t} = \dot{k_t}$

  The reason it works: no conflict between agents' interests. Since all agents agree, planner acts on behalf of representative household by prescribing how much to save/consume in every period.

  MU of consumption $= MU_t = \lambda_t =$ MU of wealth!

7.1.2 "Necessary condition for Ramsey integral convergence:"

$$g(1-\theta) + n - \rho < 0$$

On the BGP, TVC is satisfied when $\bar{r} > g_Y$

  this condition is also necessary for an interior solution

7.1.3 "Euler Equation for Consumption (Solution to the Ramsey Problem)"

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - (\rho + \theta g))$$

Often written as

$$\underbrace{\frac{\dot{c}_t}{c_t} + g}_{\substack{\text{growth rate per} \\ \text{capita consumption}}} \qquad = \frac{1}{\theta}(r_t - \rho)$$

  Higher ρ higher means $\frac{\dot{c}_t}{c_t}$ is lower (makes sense bc higher discounting means you don't want growth of consumption to be as large (you would rather consume today))

7.1.4 "Intertemporal elasticity of substitution"

$\theta =$ IES, i.e. elasticity of the marginal utility ratio with respect to consumption growth.

- Higher θ means the agent dislikes volatile consumption (i.e. they really really want consumption to be smooth and are very inelastic to changes in the interest rate).
  - θ measures sensitivity to changes in consumption
- $\theta \in (0, \infty)$

7.1.5 "Phase Diagram Equations (steady state)"

$$\dot{k} = 0 \iff c_t = f(k_t) - (n+d+g)k_t$$

$$\dot{c} = 0 \iff k_t = k_*, \text{ where } f'(k_*) = d + \rho + \theta g$$

$$c_* = f(k_*) - (n+d+g)k_*$$

7.1.6 "Ramsey Key Equations"

LOM for capital:

$$\dot{K} = I - dK = Y - C - dK$$

LOM for capital, intensive form:

$$\dot{k}_t = \underbrace{f(k_t) - c_t}_{\substack{\text{actual investment} \\ \text{per effective worker}}} - \underbrace{(n + d + g)k_t}_{\substack{\text{replacement investment} \\ \text{per effective worker}}}$$

Production Function/Total Output

$$Y = F(K, e^{gt}L) = C_t + I_t$$

7.1.7 "Ramsey Golden Rule vs Steady State Capital"

golden rule level of capital is the level that maximizes consumption:

$$k_G = \arg\max_{c^*} = \arg\max f(k^*) - (n + d + g)k^* \Rightarrow f'(k_G) = n + d + g$$

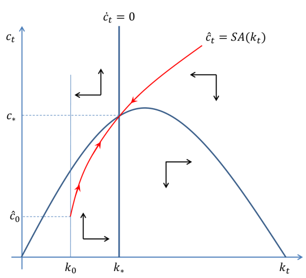
Recall from planner's problem: $f'(k_*) = d + \rho + \theta g$. So

$$f'(k_*) > f'(k_G) \iff \rho > n + (1 - \theta)g$$

7.1.8 "Ramsey Isoclines"

- $\dot{k} = 0$: $c_t = f(k_t) - (n + d + g)k_t$
- $\dot{c} = 0$: $r = \rho + \theta g$ or $k_t = k_*$, where $f'(k_*) = d + \rho + \theta g$

  Usually, $g = 0$, especially if you get the EE from the consumer max problem.



- consumption is rising when steady state capital is low since low capital means high MPK
- capital is rising when steady state consumption is low because we are accumulating capital instead of consuming it

7.1.9 "Ramsey Parameters"

There are only 5 parameters that shift the isoclines:

1. ρ: discounting
   - High ρ: low $\frac{\dot{c}_t}{c_t}$ (makes sense bc higher discounting means you don't want growth of consumption to be as large (you would rather consume today))
2. θ: Intertemporal elasticity of substitution (IES)
   - High θ: agent dislikes volatile consumption (i.e. want consumption to be smooth and are inelastic to changes in the interest rate)
3. n: affects just $\dot{k} = 0$ because it affects replacement investment
4. d: affects just $\dot{k} = 0$ because it affects replacement investment
5. g: affects both because it affects both $\overline{r}$ and replacement investment

7.1.10 "Ramsey Comparative Dynamics"

1. Unanticipated Permanent Change
   i. c, λ must jump to new SA (otherwise will never get there)
   ii. Exception: a nondistortionary tax implemented without pre-announcement won't change anything, so no jumps.
2. Anticipated (Pre-Announced) Permanent Change
   i. At t_C
      - c, λ typically continuous. Two exceptions:
        - capital jumps (won't happen) or
        - param changes make u_c, Γ_c discontinuous (e.g., pre-announced sales tax jump or other param change that make marginal utility jump). *(Japan example)*
          * Need to compare $\lambda_{t_C - 0}$ and $\lambda_{t_C + 0}$ to be safe
   ii. At t_A
      - Plausible behavior includes: (1) staying, (2) jumping up, (3) jumping down. Often can't tell which of 2 or 3 is correct.

IMPORTANT:

- k, w nearly always move together
- k, r nearly always move opposite

  c and λ always jump/stay together so long as utility is strictly increasing (it always is!)

Jump or no jump?

	$t = t_A$	$t = t_C$
Surprise change	N/A	c_t, λ_t typically jump
Pre-announced change	c_t, λ_t typically jump	c_t, λ_t typically continuous*

*Pre-announced change makes c_t, λ_t jump at the time of change if:

- Both c_t, λ_t jump if k_t jumps at time t_C for an exogenous reason (e.g. pre-announced confiscation of wealth)
- c_t jumps if parameter changes make u_c, Γ_c discontinuous (e.g. pre-announced jump in sales tax rate or a parameter change making marginal utility jump)

remember: moving to a steady state happens asymptotically, so any rate of change will be declining.

7.1.11 "Assorted explanations of consumption behavior at t_A"

All of the following are pre-announced (at t_A) changes (at t_C):

- Pre-announced increase in tax on final goods
  - c_t jumps up
    * The anticipated rise in tax weakens the incentive to accumulate wealth (because the effective interest rate is lowered). At TA, the household anticipates that its long-run desired wealth is now lower and raises consumption right away.
- Pre-announced decrease in n
  - c_t jumps up
  - know there will be fewer mouths to feed soon, adjust consumption right away
- Temporary decrease in wage tax
  - IDK.....
- permanent increase in asset tax, delayed implementation
  - pset 6
- ADD MORE HERE INCREMENTALLY! FINISH

7.2 "Ramsey - Competitive Equilibrium/Household Expectations"

7.2.1 "Ramsey Household Facts"

- c_{it}: per-capita consumption (not consumption per effective worker)
- g: tech growth. We usually assume $g = 0$ - no technology growth in the consumer problem so the Euler equations of the Household and Planner coincide.
- $\varepsilon_u = \left|\frac{u''(c_{it})c_{it}}{u'(c_{it})}\right|$: elasticity of Marginal Utility growth wrt consumption growth. If ε_u rises, then consumption growth is less sensitive to changes in the interest rate. In other words, consumers prefer smooth consumption despite time-varying interest rates.
  - Same thing as θ in the planner's problem!
- $\Phi(K)$: an expectation/belief function that allows households to forecast future factor prices

7.2.2 Def "Representative Household flow budget constraint"

$$\dot{A}_t = r_t A_t + w_t L_t - C_t$$

7.2.3 "Ramsey Household Utility Maximization"

$$\max_{c_{it}} \int_0^\infty e^{-\rho t} L_{it} u(c_{it}) dt \text{ s. to:}$$

1. $\underbrace{\dot{A_{it}}}_{\text{rate of change of assets}} = \overbrace{\underbrace{r_t A_{it} + w_t L_t}_{\text{Income}} - \underbrace{c_{it} L_{it}}_{\text{Consumption}}}^{\text{Savings flow}}$ (aka household flow budget constraint) DON'T USE $\dot{K}$ here!!
2. A_{i0}: given
3. $\lim_{T\to\infty} e^{-\bar{r}(0,T)} A_{it} \geq 0$ ($= 0 \iff PVLC = PVLR$)

- Individual State Variable: A_{it}
- Aggregate state variables: can use r_t, w_t or K_t, L_t. Both will give the same answer, but we use K_t, L_t in this class.

7.2.4 "Euler Equation for Consumption (Solution to the Ramsey Problem)"

$$\frac{\dot{c_{it}}}{c_{it}} = -\frac{u'(c_{it})}{u''(c_{it})c_{it}}(r_t - \rho) = \underbrace{\frac{1}{\varepsilon_u}(r_t - \rho)}_{\text{sign flipped due to abs val}}$$

where $\varepsilon_u = \left|\frac{u''(c_{it})c_{it}}{u'(c_{it})}\right|$

This is identical to the Euler equation in the planner's problem if $g = 0$ and $\theta = \varepsilon_u$.

7.2.5 Def "Competitive (Walrasian) Equilibrium"

Competitive equilibrium is a list of quantities $\{C_t, Y_t, I_t, K_t, A_t\}$ and prices $\{w_t, r_t, R_t\}$ such that

1. $\{C_t, A_t\}$ solves the household problem given $\{w_t, r_t\}$
2. Production side equilibrium holds for all t
   1. see production side equilibrium! very long but needs to be included here I think!
3. Market clearing for all t
   1. $A_t = K_t \Rightarrow Y_t = C_t + I_t$

Household i is the representative household, so $c_t = c_{it}, C_t = c_{it}L_{it}; A_t = A_{it}$.

7.2.6 Def "Ramsey: Production Side Equilibirum"

A list of quantities $\{C_t, Y_t, K_t\}$ and prices $\{w_t, r_t, R_t\}$ satisfying, for all t:

1. Aggregate production function: $Y_t = F(K_t, L_t)$
2. Profit maximization and factor market clearing
    1. Final goods sector: $(K, L) \in \arg\max_{(K,L)}(F(K, L) - R_t K - w_t L)$ given (w_t, R_t)
    2. Value maximization and asset market clearing, capital leasing: $K_t \in \arg\max_{A_D}(\int_t^\infty e^{-\bar{r}(t,s)-d(s-t)} R_s A_D ds - A_D)$ given (r_t, R_t)
3. Free entry (automatic given profit maximization and market clearing)
    1. $\max_{(K,L)}(F(K, L) - R_t K - w_t L) = 0$

7.2.7 "Ramsey - differences between Planner and Household Problems"

If we have (1) production side equilibrium, and (2) Asset market clearing ($A = K$), then planner and household have the same (1) utility, (2) LOM for wealth ($\dot{K}$), (3) transversality condition.

Careful: if we don't have asset market clearing and/or production side equilibrium, HH would have different resources and the stable arms would not coincide.

- Planner's Solution describes equilibrium *quantities.* Prices are chosen to decentralize planner's allocation as competitive equilibrium
- Unique Pareto Optimum ⇒ will also be a competitive equilibrium
- TVC: Prod-side eqm and Asset market clearing make it so TVCs are identical

Planner's Hamiltonian:

$$\mathcal{H} = e^{-\rho t} e^{nt} e^{g(1-\theta)t)} \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t[f(k_t) - c_t - (n + d + g)k_t]$$

Household's Hamiltonian:

$$\mathcal{H}_i = e^{-\rho t} L_{it} u(c_t) + \mu_{it}[r_t A_{it} + w_t L_{it} - c_{it} \underbrace{L_{it}}_{\text{num HH members}}]$$

7.2.8 Def "Rational Expectations Equilibrium (REE)"

Refinement of WE - we add an expectation/belief function $\Phi(K) = c_t, C_t = \Phi(K_t)L_t$ that allows households to forecast future factor prices

Explains how households know future factor prices.

REE is also WE.

Definition of equilibrium: REE is a list of quantities, $\{C_t, Y_t, I_t, K_t, A_t\}$, prices $\{w_t, r_t, R_t\}$ and $\Phi(K)$ satisfying

1. Utility maximization (*different from WE*) The consumption decision rule $c_i(a, K_t; \Phi(K_t))$ solves the household problem given the information households have at date t and their beliefs about behavior of others, $c_t = \Phi(K_t)$

Details: Households form factor price forecasts (function-to-function mappings) based of date-t state K_t and their beliefs Φ:

$$\{w_{t+\tau}^e\} = \mathcal{W}(K_t; \Phi(K_t)), \qquad \{r_{t+\tau}^e\} = \mathcal{R}(K_t; \Phi(K_t)), \text{all } \tau > 0.$$

Utility maximization problem:

$$c_i(a, K_t; \Phi) \in \underset{\{c_{it}\}}{\operatorname{argmax}} \int_0^\infty e^{-\rho t} L_{it} u(c_{it}) dt$$

$$\int_t^\infty \exp[-\bar{r}^e(t,s)] \cdot c_{is} \cdot L_{is} ds \leq a \cdot L_{it} + \int_t^\infty \exp[-\bar{r}^e(t,s)] \cdot w_s^e \cdot L_{is} ds$$

Definition of equilibrium – continued

2. Production-side equilibrium (no change)

3. Rational expectations (consistency) conditions on Φ (new)

Households' own consumption decision (c_i) based on *equilibrium* beliefs is consistent with their beliefs about others' consumption (Φ) for any initial condition $K_0 > 0$ and all $t \geq 0$:

$$c_i\left(\frac{K_t}{L_t}, K_t; \Phi(K_t)\right) = \Phi(K_t)$$

where K_t solves

$$\dot{K}_t = F(K_t, L_t) - \Phi(K_t)L_t - dK_t \text{ given } K_0 > 0, L_t = L_0 e^{nt}.$$

4. Asset market clearing (no change), follows from RE condition 3

WE equipped with a belief equal to the stable arm is a REE.

7.3 Government and fiscal policy in the Ramsey model

7.3.1 "Government Budget Balance"

$$G_t = \tau_{A,t} \cdot r_t A_t + \tau_{w,t} \cdot w_t L_t + \tau_{c,t} \cdot C_t + T_t$$

Where:

- $\tau_{A,t}$ = tax on assets (or, equivalently, on capital income)
- $\tau_{w,t}$ = tax on labor income
- $\tau_{c,t}$ = tax on consumption (sales tax)
- T_t = lump-sum tax if positive, or transfer if negative

If taxes are collected in excess of G then excess is distributed back to households via transfers, T_t

7.3.2 "Ramsey Household problem with taxes"

same as before, but now:

$$\dot{A}_{it} = (1 - \tau_{A,t}) \cdot r_t A_{it} + (1 - \tau_{w,t}) \cdot w_t L_{it} + \underbrace{(1 + \tau_{c,t})}_{\text{Note "+"}} \cdot c_{it} L_{it} + T_t$$

Caution: common mistake is to plug in A_t instead of A_{it}... this mistake will usually show no effect of taxes

Instead, need to (1) solve HH problem, (2) aggregate decisions

7.3.3 “Which taxes distortionary?”

Nondistortionary = behavior of the household agrees with planner’s optimum despite taxes being present in the household’s budget constraint (Planner doesn’t care about taxes–so we’re talking just about households).

In general need to solve and do phase diagrams to solve. But, should also memorize the following:

- asset (wealth/capital income) tax: distortionary! New EE is
  $-\frac{\dot{\mu}_{it}}{\mu_{it}} = \underbrace{(1-\tau_A)r_t}_{\text{after-tax interest rate}}$
- wage tax: NOT distortionary (since labor supply is GIVEN–not a choice variable)
- sales tax: NOT distortionary IF constant rate (bc when differentiating over time, the term will disappear–it won’t affect the relative price of consumption)
- lump sum tax: NEVER distortionary

Alternative definition: a tax system is nondistortionary if and only if the competitive equilibrium is Pareto optimal.

7.3.4 “Government in Ramsey”

Assume: G_t does not depend on aggregate state.

Non-distortionary AND budget balanced ⇒ isoclines for HH and Planner coincide

7.3.5 “Proposition: nondistortionary taxes = reduction in initial wealth”

If taxes are non-distortionary, the effect of taxes is just like a reduction in initial wealth

7.3.6 “Government Debt in Ramsey”

Households can hold wealth in the form of K or D.

Asset market clearing:

$$A_t = K_t + D_t$$

Asset market clearing requires (no arbitrage between the two assets):

$$r_t = r_{K,t} = r_{D,t}$$

LOM for Gov Debt:

$$\dot{D}_t = r_t D_t + G_t - T_t$$

Government cannot roll over debt indefinitely in Ramsey! No Ponzi!

7.3.7 Thm “Ricardian Equivalence theorem”

If taxes are non-distortionary, then

1. D_t has no effect on the household lifetime budget constraint, and
2. The solution to the household problem with taxes corresponds to the Pareto optimal quantities

Implication: If taxes are non-distortionary, the amount and timing of intergenerational transfers implemented through D_t (e.g. Social Security, Medicare) does not affect the economy’s consumption and interest rate trajectory. Sorta makes sense: dynastic households receiving a stimulus today will save it instead of spending so that their progeny will be able to pay off the debt later.

No jumps in Ramsey if tax is nondistortionary.

8 Endogenous Growth (EG)

8.1 Endogenous Growth Intro

8.1.1 "EG Algorithm"

Household Problem

1. setup Hamiltonian with $V = K + P_A A$ as the only state, subject to $\dot{V}$ (don't need to get into intensive form!)
   1. Note: you'll probably need to combine $\dot{K}$ and $\dot{A}$ to derive LOM for V!
2. This gives 5 differential equations, 5 algebraic equations. Can't solve, so guess BGP and check.

Planner's Problem

1. setup Hamiltonian with two states (A, K) and two constraints ($\dot{A}, \dot{K}$)
   a. Nonstandard problems usually have 1 state. In this case, can often write the Hamiltonian with the resource constraint plugged into the utility function before taking FOCs (e.g., Y in place of C if there is no capital)
2. eliminate μ_K to get EE
3. eliminate μ_K to get $\frac{\dot{\mu}_A}{\mu_A} = -\frac{z}{w}\frac{\partial Y}{\partial A}$.
   a. recall: social value of idea, $P_A^* = \frac{\mu_A}{\mu_K}$
4. derive LOM for P_A (use fact that $g_{P_A} = g_{\mu_K} - g_{\mu_A}$)
   a. $\dot{P}_A^* = r_t P_A^* - \underbrace{\frac{z}{w}}_{=1} P_A^* \frac{\partial Y}{\partial A} = r_t P_A^* - \frac{\partial Y}{\partial A}$

Derive $\overline{r}$

1. Start with Euler
2. Invoke BGP
   a. DON'T MAKE THE MISTAKE OF SAYING $\dot{c} = 0$!!
      i. ($c = C/L$ will not be constant on the growth path but grow at $g_C - n$)
3. By BGP, $g_c = g_{C/L} = g_Y - n = g_A$
4. Derive g_A
   a. trick is to assert g_A is const, divide $\dot{A}$ by A, then log-diff

Important to have memorized

- $MPK \neq R = \alpha p = \alpha MPx_j = r + d$
- $\dot{V} = rV + wL - cL$
- $V = K + P_A A$
- $\dot{P}_A = -\pi + rP_A$ or for planner: $\dot{P}_A^* = -\frac{\partial Y}{\partial A} + rP_A^*$
- $P_A^* = PV(\{\frac{\partial Y}{\partial A}\}) + PV(\{P_A^* * \frac{\partial \dot{A}}{\partial A}\})$
- $g_Y = g_A + n$
- $MPL_A = \frac{\partial \dot{A}}{\partial L_A} * P_A$ (don't forget extra P_A term on MPL_A just because it's not there on MPL_Y!!!!!!!!!!!!)

8.1.2 "Jorgenson's in EG"

In CE, the noncompetitive sector means that

$$MPK \neq R$$

Instead:

$$R = \alpha p(x) = \alpha MPK = \alpha^2 \frac{Y}{K} = r + d$$

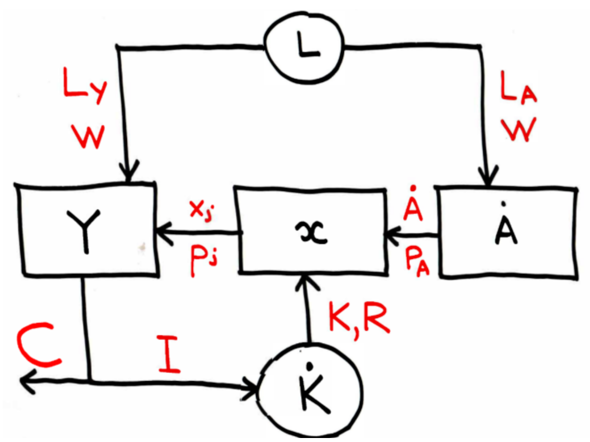
Of course, we'll still have $MPK = R = r + d$ in the planner's solution

8.1.3 "Endogenous Growth Facts"

- A: patent number (index)
- $P_{A,t}$: price of idea
- $M = P_A A$ market value of patents
- φ: Knowledge externality. Standing on shoulders or fishing.
- λ: Labor externality. Stepping on toes or synergy.
- θ: CRRA coefficient - same as in Ramsey - shows up only in planner's problem
- ρ: same as in Ramsey, part of EE
- $x_j = \frac{K}{A}, \forall j = 1, ..., A$. Symmetry $\Rightarrow \sum x_j = Ax_j = Ax = K$
- $\bar{r} = \rho + \theta g_A$ (in standard planner's problem)
- $w = (1-\alpha)Y$ (by profit max in final goods)
- $wL_A = P_A \dot{A}$ (by free entry in research sector)

Marginal products of labor are equal across sectors in standard models, but be careful to not assert that $w = MPL_Y = MPL$ without checking things first...

> Mechanism of growth: $A \uparrow \Rightarrow x_j = \frac{K}{A} \downarrow \Rightarrow MPx_j \uparrow \Rightarrow Y \uparrow$ (don't get this backwards!)
>
> wages always equal; interest rate on assets always equal; but capital income rates not equal - physical capital will be higher (since d added to R and $\frac{\dot{P}_A}{P_A}$ subtracted from patent return rate)!
>
> EG model will have inefficient equilibrium - but this time not related to oversaving, rather having to do with production of ideas and wrong incentives



8.1.4 "EG Market Clearing"

Asset:

$$V = K + P_A A$$

Goods (implied by AMC):

$$Y = C + I \neq GDP$$

Capital:

$$x_1 + \cdots + x_A = Ax = K$$

Labor:

$$L = L_A + L_Y$$

8.1.5 "EG Final Goods Sector (1/4)"

$$Y = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^\alpha = L_Y^{1-\alpha}(x_1^\alpha + \cdots x_A^\alpha) = L_Y^{1-\alpha} \sum_{j=1}^{A} \left(\frac{K}{A}\right)^\alpha = L_Y^{1-\alpha} A \left(\frac{K}{A}\right)^\alpha = K^\alpha (AL_Y)^{1-\alpha}$$

Max problem is

$$\max_{L_Y, x_j} \left(L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^\alpha - wL_Y - \sum_{j}^{A} p_j x_j \right)$$

Competitive, so $w = MPL = (1-\alpha)\frac{Y}{L_Y}$ and $p_j = MPx_j = \left(\frac{L_Y}{x_j}\right)^{1-\alpha}$

8.1.6 "EG Capital Goods Bundling Sector (2/4)"

A monopolists own one of $j = \{1, ..., A\}$ ideas, purchased at price $P_{A,t}$.

Two max problems:

1. determines price at which to buy a patent
2. determines rental fee on each bundle

1. Determining price at which to buy a patent:

$$\text{Lifetime Profit} = \max_{\{x_t, p_t\}} PV(\{\pi_t\}) - P_{A,t} \tag{1}$$

Free-entry $\Rightarrow$ Lifetime Profit $= 0$ (so $PV(\{\pi_t\}) = P_{A,t}$)

$$\dot{P}_{A,t} = \frac{d}{dt} PV(\{\pi_t\}) = \frac{d}{dt} \int_t^\infty e^{-\overline{r}(t,s)} \pi_s ds = \pi_t + r_t P_{A,t}$$

2. Determining rental fee on each bundle:

Inverse demand curve (for each j):

$$p_t(x) = MPx_j = \frac{\partial Y}{\partial x_j} = \frac{\partial}{\partial x_j}\left(L^{1-\alpha}(x_1^\alpha + \cdots + x_A^\alpha)\right) = \alpha\left(\frac{L_{Y,t}}{x_j}\right)^{1-\alpha}$$

Monopolists' problem:

$$\pi_t = \max_x \{p_t(x)x - R_t x\} \tag{2}$$

$$MR = p(x) + xp'(x) = p + (\alpha - 1)p = R = MC$$

Gives:

$$p(x) = \underbrace{\frac{1}{\alpha}}_{\text{Markup}} \underbrace{R}_{\text{MC}}$$

8.1.7 "Monopolist's markup on bundle price"

$$p(x) = \underbrace{\frac{1}{\alpha}}_{\text{Markup}} \underbrace{R}_{\text{MC}}$$

8.1.8 "Research Sector (3/4)"

Representative household:

$$\dot{a} = z_t l_A$$

where

- $\dot{a}$ = flow of ideas
- z_t = research productivity / idea TFP (measurable in data). Assume that $z_t = \gamma(A, L_A)$
- l_A is what research firms choose, taking z and w as given

$$\text{Profit} = P_{A,t}\dot{a}_t - w_t l_{A,t} = (P_{A,t}z_t - w_t)l_{A,t}$$

So, the only prices (P_A and w) consistent with positive and finite labor demand are those such that

$$P_A z = w \iff 0 < l_A < \infty$$

> Relative size of the production and research sectors is the free variable that makes the MPL rise/fall to ensure we have a wage that makes workers indifferent between w_A and w_L

8.1.9 "EG Physical Capital Goods Producing Sector (4/4)"

Standard neoclassical and we still have that $R_t = r_t + d$, BUT:

$$R \neq MPK$$

Since there is a markup from the capital bundlers! (instead $R = p\alpha$)

8.1.10 "EG LOM for capital"

$$\dot{K} = Y - C - dK$$

8.1.11 "EG Key Identity"

$$\underbrace{P_A\dot{A}}_{\text{agg value of research output}} = \underbrace{wL_A}_{\text{researcher wage}}$$

8.1.12 "Patents as financial assets"

A patent is like a financial asset which means there is a no arbitrage condition:

$$r_t = \frac{\overbrace{\pi_t}^{\text{Dividend}} + \overbrace{\dot{P}_{A,t}}^{\text{cap gains}}}{\underbrace{P_{A,t}}_{\text{purchase price}}} \iff \underbrace{\dot{P}_A = P_A r - \pi}_{\text{usually use in this form}}$$

> If you have P_A to invest, you can either invest at the risk-free rate, r in the capital market, or buy a patent and establish a startup!
>
> Memorize this, because differentiating PA, t wrt t requires Leibniz rule twice!

Also, define $M_t = P_{A,t}A_t$. Then

$$\dot{M} = \dot{P}_A A + P_A\dot{A} = \dot{P}_A A + wL_A = (-\pi + P_A r)A + wL_A = rM - \pi A + wL_A$$

Which gives

$$r_t = \frac{\overbrace{\pi_t A_t - w_t L_{A,t}}^{\text{dividend}} + \overbrace{\dot{M}_t}^{\text{cap gains}}}{M_t}$$

8.1.13 "EG w National Accounts"

$$GDP = \underbrace{Y + P_A\dot{A}}_{\text{Product}} = Y + wL_A = \underbrace{RK + \pi A + wL_Y + wL_A}_{\text{Income}} = wL + RK + M \quad \underbrace{(r - \frac{\dot{P}_A}{P_A})}_{\text{Jorg for patents}}$$

8.1.14 "Jorgenson's for patents"

With standard capital d is added to R since depreciation is negative.

$$\text{Financial return on market value capital} = R = r + d$$

With patents, $\frac{\dot{P}_A}{P_A}$ is subtracted since patent values generally appreciate:

$$\text{Financial return on market value of patent} = r - \frac{\dot{P}_A}{P_A}$$

Recall market value of patent is M. So, $M(r - \frac{\dot{P}_A}{P_A})$ is the return on patents. This is handy for the GDP accounting:

$$GDP = wL + RK + M(r - \frac{\dot{P}_A}{P_A})$$

8.1.15 "EG Labor Share and GDP"

Labor Share of GDP (not Y) is $> 1 - \alpha$!

Makes sense because labor share in final goods sector is standard (inputs are split between K and L), but labor share in the research sector is 100%. So, the average contribution of labor to GDP should be higher than $1 - \alpha$.

So, our estimates for α in previous models don't work here; α with this model is lower than with other models!

8.1.16 "EG Euler Thm"

Since PF is CRS/Homogeneous of degree 1:

$$Y = L_Y^{1-\alpha}(Ax)^\alpha = \frac{\partial Y}{\partial L_Y}L_Y + \frac{\partial Y}{\partial x_1}x_1 + \cdots + \frac{\partial Y}{\partial x_A}x_A = wL_Y + px_1 + \cdots + px_A = wL_Y + Axp$$

8.1.17 "EG Labor/Capital/IP Shares of Output"

- Recall that $wL_Y = (1-\alpha)Y$ to see that $Axp = \alpha Y$.
- Plug in the fact that $p = \frac{R}{\alpha}$ to get: share of physical capital $\frac{RK}{Y} = \alpha^2 < \alpha$ like in competitive models!

Overall:

$$Y = (1-\alpha)Y + \alpha^2 Y + \alpha(1-\alpha)Y = wL_Y + RK + \underbrace{\pi A}_{\text{IP Dividends}}$$

Memorize: $\pi A = \alpha(1-\alpha)Y$

8.1.18 "EG Externalities"

Assume functional form:

$$z_t = \gamma(A, L_A) = \eta A^\varphi L_A^{\lambda - 1}$$

So,

$$\dot{A} = \gamma(A, L_A) * L_A = \eta A^\varphi L_A^\lambda$$

where

- $\varphi = 0, \lambda = 1$ means no external effect on γ
- $\varphi > 0$: "standing on shoulders"
- $\varphi < 0$: "fishing out the pond of ideas"
- $\lambda > 0$: "synergy"
- $\lambda < 0$: "stepping on toes"

  Bigger = better for both the idea externality φ and the labor externality, λ

8.1.19 "EG $\bar{r}$"

$$\bar{r} = \rho + \theta g_A$$

Caution: this is not solved yet because g_A is endogenous!

8.1.20 "Profit in Endogenous Growth"

$$\pi_t = \alpha(1-\alpha)\frac{Y}{A}$$

Also:

$$\pi_t = \frac{1}{A}(Y - RK - wL_Y)$$

Also:

$$\pi_t = rP_A - \dot{P}_A$$

Also:

$$P_A = PV(\{\pi_t\}) = \int_t^\infty e^{-\bar{r}(t,s)}\pi_s ds = \underbrace{\frac{\pi_t}{\bar{r}-n}}_{\text{On BGP profits grow at rate } n}$$

8.1.21 "Share of workers in research sector"

$$\frac{s_R}{1-s_R} = \frac{L_A}{L_Y}$$

8.2 Household Problem

8.2.1 "EG Household Problem"

Household Total Financial Wealth = Market Value of businesses = physical capital and IP:

$$\underbrace{V}_{\text{mkt value of businesses}} \overbrace{=}^{\text{by AMC}} \underbrace{K + \overbrace{M}^{=P_A A}}_{\text{wealth of rep HH}}$$

UMP:

$$\max_{C_t} \int_0^\infty e^{-\rho t} L_t u(\frac{C_t}{L_t})dt$$

Subject to:

$$\dot{V}_t = r_t V_t + w_t L_t - C_t = \underbrace{I_t - dK_t}_{\dot{K}} + \underbrace{w_t L_{A,t} + \pi_t A_t + r_t P_{A,t} A_t}_{\dot{M}}$$

And transversality:

$$\lim_{t\to\infty} e^{-\bar{r}(0,t)} V_t = 0 \iff PVLR_t = PVLC_t, \forall t$$

- Individual state: V (note: if you don't use V, you'd have three individual states: K, A, P_A!)
- Aggregate states: R, w (or, L, L_A, right?)
- Control: C

8.2.2 "EG 5 Equations and Unknowns"

(Reduced) list of endogenous variables:

$$(Y_t, A_t, K_t, C_t, L_{Y,t}, L_{A,t}), (P_{A,t}, w_t, r_t)$$

Five differential equations:

$$\dot{A}_t = z_t L_{A,t}$$

$$\dot{P}_{A,t} = -\pi_t + r_t P_{A,t} = -\alpha(1-\alpha)\frac{Y}{A} + r_t P_{A,t}$$

$$\dot{K}_t = Y_t - C_t - dK_t \qquad \text{(LOM for Capital - check Y or GDP)}$$

$$\dot{C}_t = C_t(n + \frac{1}{\theta}(r_t - \rho)) \qquad \text{(Euler)}$$

$$\dot{L}_t = nL_t \qquad \text{(given, exogenous)}$$

Five Algebraic Equations:

$$r_t = \alpha^2 \frac{Y_t}{K_t} - d$$

$$Y_t = K_t^{\alpha}(A_t L_{Y,t})^{1-\alpha}$$

$$w_t = z_t P_{A,t} \qquad \text{(Profit Max: Research)}$$

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \qquad \text{(Profit Max: Final Goods)}$$

$$L = L_A + L_Y \qquad \text{(Labor Mkt Clearing)}$$

This is too hard to solve, so look for a BGP to simplify!

8.2.3 "EG BGP"

PF is identical to that in Solow, so everything in Solow BGP applies exactly the same here!

Variable	Growth rate
$r, R, p, s_R = \frac{L_{A,t}}{L_t}$	0
$L_t, x_t, \pi_t, P_{A,t}$	n
w, A	g_A
$Y_t, K_t, C_t, I_t, M_t = P_{A,t}A_t$	$g_A + n$

$s_R = const$ - share of labor supply in the research sector

Derive g_{P_A}:

$$P_A \dot{A} = wL_A \iff P_A \frac{\dot{A}}{A} = \frac{wL_A}{A} \iff g_{P_A} + 0 = g_w + n - g_A \iff g_{P_A} = n$$

Derive g_V:

$$V = P_A A + K \underbrace{\iff}_{\text{dont forget this!!}} g_V = g_{P_A} + g_A = g_K \Rightarrow g_V = g_A + n$$

8.3 Planner's Problem

8.3.1 "EG Planner's Problem"

- States: A, K
- Controls: C, L_A, L_Y
- TVC: each state has its own TVC!

Assume $z_t = \eta$ (no externalities), use CRRA utility, $\theta > 0$:

$$\max_{C, L_A} \int_0^\infty e^{-\rho t} \frac{1}{1-\theta} L_t \left(\left[\frac{C_t}{L_t} \right]^{1-\theta} - 1 \right) dt$$

subject to

$$\dot{K} = K^\alpha (A(L - L_A))^{1-\alpha} - C_t - dK_t$$

and

$$\dot{A} = zL_A$$

with $K_0 > 0, A_0 > 0$, given and transversality conditions:

$$\lim_{t\to\infty} \mu_{K,t} K_t = 0 \text{ and } \lim_{t\to\infty} \mu_{A,t} A_t = 0$$

Hamiltonian and FOCs are standard. They give:

1. Standard Euler: $\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(r_t^* - \rho + \theta n)$
2. Social value of an idea $= P_A^* = \frac{\mu_A}{\mu_K}$ (marginal effect of additional A on lifetime wealth
3. Planner's wage: $w^* = MPL_Y = z\frac{\mu_A}{\mu_K} = zP_A^*$
   a. This implies: $\frac{z}{w} P_A^* = 1$
4. Planner's interest rate: $r^* = -\frac{\dot{\mu}_K}{\mu_K}$ as in Ramsey, but r^* is no longer the market interest rate–rather the planner's interest rate (although, turns out they are the same)!
5. $\frac{\dot{\mu}_A}{\mu_A} = -\frac{z}{w}\frac{\partial Y}{\partial A}$ which (I think) helps only if you need to answer social value of idea questions. . .

Derive LOM for P_A:

1. Start with the fact that $\frac{\dot{P}_A}{P_A} = \frac{\dot{\mu}_A}{\mu_A} - \frac{\dot{\mu}_K}{\mu_K}$

$$\dot{P}_A^* = r_t P_A^* - \underbrace{\frac{z}{w} P_A^*}_{=1} \frac{\partial Y}{\partial A} = r_t P_A^* - \frac{\partial Y}{\partial A}$$

8.4 Planner and market equilibrium comparison

8.4.1 "EG Planner and Market Comparison"

Social Planner	Competitive Equilibrium	Comparison
g_A	g_A	Same
$\bar{r} = \rho + \theta g_A$	$\bar{r} = \rho + \theta g_A$	Same
$R = \alpha \frac{Y^*}{K^*} = \bar{r} + d = MPK$	$R = \alpha^2 \frac{Y}{K} = \bar{r} + d \neq MPK$	Jorgenson's holds in both, but $R \neq MPK$ in CE
$\frac{K^*}{Y^*} = \frac{\alpha}{\bar{r}+d}$	$\frac{K}{Y} = \frac{\alpha^2}{\bar{r}+d}$	Planner has higher capital intensity than CE
$MPL_Y = (1-\alpha)\frac{Y^*}{L^*} = \eta P_A^*$	$MPL_Y = (1-\alpha)\frac{Y}{L} = \eta P_A$	Planner has higher labor productivity
$\frac{L_A^*}{L_Y^*} = \frac{g_A}{\bar{r}-n}$	$\frac{L_A}{L_Y} = \frac{\alpha g_A}{\bar{r}-n}$	More researchers in planner solution
$P_{A,t}^* = PV\left(\{\frac{\partial Y_s}{\partial A_s}\}_{s=t}^\infty\right) + PV\left(\{\dot{P_{A,s}} \cdot \frac{\partial \dot{A_s}}{\partial A_s}\}_{s=t}^\infty\right)$	$P_{A,t} = PV\left(\{\pi_s\}_{s=t}^\infty\right)$	If no externalities, $P_{A,t}^* > P_{A,t}$

If no externalities:

- $L_A^* > L_A \Rightarrow$
  - $L_Y * < L_Y$

  - $A^* > A$
- $w^* = zP_A^* > w$ (despite this, the planner still underpays researchers relative to their full value??)
- $x^* > x \iff \frac{K^*}{A^*} > \frac{K}{A}$ doesn't directly imply $K^* > K$, but still true...
- $\frac{Y^*}{A^*} > \frac{Y}{A}$

With externalities, all bets are off (e.g., a negative externality will induce the planner to choose a smaller L_A^*).

$g_A^* = g_A$ even though levels are different ($A^* > A$)

8.4.2 "Value of ideas"

Two distortions that affect the market price of an idea relative to its social value:

1. monopoly and
2. externality in the research sector

$$P_A = PV(\{\pi_t\})$$

$$P_A^* = \frac{\mu_A}{\mu_K} = \underbrace{PV(\{\frac{\partial Y}{\partial A}\})}_{\text{Mkt value of future output}} + \underbrace{PV(\{P_A^* * \frac{\partial \dot{A}}{\partial A}\})}_{\text{social value of future ideas (0 if no externalities)}}$$

If no externalities:

$$\frac{\partial Y}{\partial A} = (1-\alpha)\frac{Y}{A} > \alpha(1-\alpha)\frac{Y}{A} = \pi \Rightarrow P_A^* = \frac{1}{\alpha}P_A \Rightarrow P_A^* > P_A$$

If externalities exist, we could have the opposite since the LOM for P_A^* has an extra term (which depends on externalities) measuring the impact of A on $\dot{A}$:

$$\text{Social value of idea} = \text{PV added future final output} + \underbrace{\text{Social value added research output}}_{\lesseqgtr 0}$$

With externalities:

$$MPL_A^* \neq w_* = MPL_Y$$

Careful: MPL_A doesn't really exist I don't think... must use the fact that $P_A\dot{A} = wL_A \iff w = \frac{P_A\dot{A}}{L_A}$

8.4.3 "Private and social marginal benefits from research"

Private marginal benefit:

$$w_A = P_A z$$

Social marginal benefit:

$$MPL_A^* = \underbrace{P_A^* z}_{\text{direct}} + \underbrace{P_A^* L_A \overbrace{\frac{\partial z}{\partial L_A}}^{\text{idea TFP}}}_{\text{indirect gain/loss in A from } L_A}$$

Marginal social product of an idea:

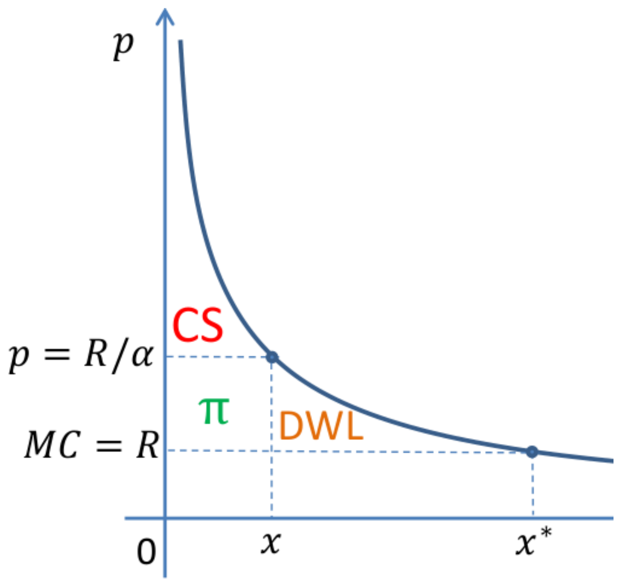
$$MPA^* = \underbrace{\frac{\partial Y}{\partial A}}_{\text{social dividend}} + \underbrace{P_A^* \frac{\partial \dot{A}}{\partial A}}_{\text{gain/loss from knowledge externality}}$$

- Market value of idea: $\frac{\pi_t}{r-n} = \frac{\alpha(1-\alpha)Y/A}{r-n}$
- Social value of idea: $\frac{\alpha(1-\alpha)Y/A}{r-n-\underline{\varphi g_A}}$ (can be bigger or smaller than mkt value depending on sign of φ)

8.4.4 "Consequences of monopoly"

Monopolists choose an output level below that of a competitive market and charge a markup. Consequences:

1. IP priced below its social value: $P_A = \alpha P_A^*$ (if no externalities - otherwise this may not hold!!).
2. Capital income rate is lower than MPK^*, incentives to save are weakened, which lowers long-run labor productivity: $y = \alpha y^*$



9 Misc

9.0.1 "Common Tricks"

- gn small and along BGP g_Y is known:
  - $(1+n)(1+g) = 1+g+n+gn \approx 1+g_Y$ (used in OLG, maybe elsewhere)

9.0.2 "BGP"

On the balanced growth path, all components of expenditure have to grow at the same rate as Y. Specifically: $g_Y = g_K = g_I = g_C$. Usually $g_Y = g_A + n$.

9.0.3 Def "Euler equation"

An Euler equation relates choice in one period to choice in another period.

9.0.4 "Combining stocks vs flows"

- Not valid: $stock + flow$
- Valid: $stock + flow * \Delta t$ or $\frac{stock}{\Delta t} + flow$

  In OLG model with a single period we don't see Δt because it's equal to 1, but it's still there.

This is a "gotcha" mostly only in discrete time because in continuous time the dot notation usually makes it clear.

9.0.5 Def "rate of change over time interval"

$$\frac{X_t - X_{t-1}}{\Delta t}$$

9.0.6 Def "Annual growth rate"

$$g_{X,t} = \frac{X_t - X_{t-1}}{X_{t-1}}$$

9.0.7 "Relate growth rates to rates of change"

The growth rate is approximately equal to the logged rate of change:

$$g_{X,t} = \frac{X_t - X_{t-1}}{X_{t-1}} \approx \ln(X_t) - \ln(X_{t-1})$$

unlike the rate of change, growth rate does not depend on units of measurement

9.0.8 Def "Average (annual) growth rate"

Between any two dates, a, b:

$$g_X = \frac{\ln(X_b) - \ln(X_a)}{b - a}$$

9.0.9 Def "Annualized and instantaneous growth rate"

$$g_{X,t} = \frac{\text{Rate of change over time interval}}{\text{Initial value of } X} = \frac{(X_t - X_{t-\Delta t})/\Delta t}{X_{t-\Delta t}}$$

Taking $\lim_{\Delta t \to 0}$ gives the instantaneous growth rate:

$$g_{X,t} = \frac{1}{X_t}\frac{dX_t}{dt} = \frac{\dot{X}_t}{X_t}$$

9.0.10 "Arithmetic with growth rates"

- $g_{XY} = g_X + g_Y$
- $g_{X/Y} = g_X - g_Y$
- $g_{X^\alpha} = \alpha g_X$

9.0.11 "MPK"

$$MPK \equiv \frac{\partial Y}{\partial K} \neq \frac{\partial F}{\partial K} \text{ (e.g., if } Y = ZF(K, L))$$

9.0.12 "Movement of variables"

- k, w nearly always move together
- k, r nearly always move opposite

9.0.13 "Random tips/facts"

- In OLG: Consumption of the old does not change, because the old consume only the market value of the undepreciated capital stock plus the current period capital income.
- With non-distortionary taxes, competitive equilibrium is Pareto optimal. The Pareto optimum is the solution to a planner's problem. The Pareto optimum trajectories are independent of government debt level and the timing of taxes.
  - C, Y, I do not respond to nondistortionary taxes. Households save the entire tax cut amount - in the form of government debt - expecting to use the entire principal and interest on government bonds to pay higher future taxes.

9.0.14 "Mistakes to avoid"

OLG

- writing asset market clearing wrong:
  - remember, if government debt is present that's an asset and should be on the total asset supply side
- aggregating consumption
  - $C_t = c_{1,t}L_t + c_{2,t}L_{t-1}$. Don't use Euler here. . . .

Ramsey

- don't use $\dot{K}$ for the household problem. . . . . . need to convert to $\dot{A}$.
- don't solve the planner's when you should solve household and vice-versa

Endogenous Growth

- In Household Problem: don't think $MPK = R = r + d$. $R = r + d$, but $R \neq MPK$!
- In planner's problem: remember that now $MPK = R$!!

Misc

- NO asset or goods market in production-side equilibrium!! that's a concept of households, not production!
- get the user cost right - can think of this two ways:
  1. from firm perspective: $finish$
  2. from user cost perspective: $finish$
- don't take derivatives to get growth rates in discrete time!
  - remember: 1 + growth rate = growth factor: $1 + g_Y = 1 + \frac{Y_{t+1} - Y_t}{Y_t} = 1 + \frac{Y_{t+1}}{Y_t} - 1 = \frac{Y_{t+1}}{Y_t}$