Econ 604 Notes

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Contents

1 Noncooperative Bargaining

1.0.1 Model "Divide a dollar"

- Two players simultaneously submit demands $d_i \in [0,1]$
- Continuum of Nash equilibria–this will lead us to look at SPE instead.

1.0.2 Model "Ultimatum Game"

- "sequential" but only one round (P1 makes TIOLI offer to P2 who accepts or rejects)
- **Prop:** has a **unique** SPE in which P1 offers (1,0) and P2 accepts any offer.
- multiple NE in the subgame following the offer, but P1 only has a BR if P2 accepts (1,0).

1.0.3 Model "Finite Alternating-Offer Bargaining"

Regularities:

1. Immediate agreement, efficient outcome
2. Proposer's offer makes responder indifferent
3. Unique SPE
4. Division of surplus depends:
    1. $x_1^*(T,\delta) = \sum_{t=1}^{T}(-\delta)^{t-1} = \frac{1-(-\delta)^T}{1+\delta}$
    2. $\delta \to 1 \Rightarrow$ last mover takes all
    3. $T \to \infty \Rightarrow X^* \to \frac{1}{1+\delta}$
    4. $T \to \infty$ then $\delta \to 1 \Rightarrow$ equal sharing

    Every game in this class has a unique subgame perfect equilibrium, and in it there is immediate agreement in the first round.

    The proposer in round T offers to take everything, and the responder accepts every offer

1.0.4 Thm "infinite alternating-offer bargaining game outcome"

The infinite horizon alternating-offer bargaining game with discount factor $\delta < 1$ has a unique subgame perfect equilibrium, in which Player 1 offers the payoff vector $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ in the first round and Player 2 accepts.

2 Repeated Games

2.1 Finitely Repeated Games

2.1.1 Model "Repeated Games"

- Stage Game $\langle N, A, \pi \rangle$ (Normal form game)
- Players $N = \{1, ..., N\}$
- Action profiles $A = A_1 \times \cdots A_n$
- Stage payoffs $\pi : A \to \mathbb{R}^n$ (uniformly bounded) (i.e., $\pi_i(a)$ is i's payoff from action profile a)
- Periods $t = 1, ..., T$, where $T < \infty$
- Discount factor $\delta > 0$. (Also always have $\delta < 1$ if $T = \infty$)

2.1.2 "Histories and Strategies"

A strategy (in a repeated game) maps from the history of past play into an action for the current period.

- History $h^t \equiv (a^1, ..., a^{t-1})$, $h^1 \equiv \varnothing$
- Possible t-histories $\mathcal{H}^t \equiv A^{t-1}$
- Possible histories $\mathcal{H} \equiv \cup_{t=1}^{T} \mathcal{H}^t$
- A (pure) strategy is a function $s_i : \mathcal{H} \to A_i$
- a correlated (behavioral) mixed strategy is defined in terms of strategy profiles to allow for correlation. So $\sigma : \mathcal{H} \to \Delta A$.

  a behavioral mixed-strategy must specify not only what players play "sequentially", but also in all other possible histories of the game.

2.1.3 Def "Utility"

$$\text{Total Utility} = \tilde{u}_i(\sigma) \equiv \sum_{t=1}^{T} \delta^{t-1} \sum_{h^t \in \mathcal{H}^t} \pi_i(\sigma(h^t)) Pr(h^t|\sigma)$$

$$\text{Average Utility} = u_i(\sigma) \equiv \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} \sum_{h^t \in \mathcal{H}^t} \pi_i(\sigma(h^t)) Pr(h^t|\sigma)$$

For infinite horizon we multiply by $1 - \delta$, which is just the limit as $T \to \infty$ of $\frac{1-\delta}{1-\delta^T}$.

2.2 Infinitely Repeated Games

2.2.1 Def "Non-contingent strategy profile"

σ is a non-contingent strategy profile if

$$\sigma(h^t) = \sigma(\hat{h}^t), \forall h^t, \hat{h}^t \in \mathcal{H}^t, \forall t = 1, ..., T.$$

σ is non-contingent if it specifies that in each period players should play some period-specific mixed action profile regardless of the history.

2.2.2 Prop 13.1 "Non-contingent equilibria"

Suppose $T < \infty$. If σ is a non-contingent strategy profile such that $\sigma(h^t)$ is a stage game NE for all $t = 1, ..., T$ then σ is a SPE.

Any sequence of stage Nash profiles can be supported as the outcome of a SPE. (Watson)

2.2.3 Prop 13.2 "Unique Stage Game Equilibrium"

Suppose $T < \infty$. If α^* is the unique NE in the stage game, then the unique SPE in the repeated game is a non-contingent strategy profile σ^* such that, for all $h \in \mathcal{H}, \sigma^*(h) = \alpha^*$.

2.2.4 Def "profitable one-shot deviation"

A *profitable one-shot deviation* from σ is a strategy $\hat{\sigma}_i \neq \sigma_i$ for some player i such that:

1. there exists a unique history $\hat{h}^t \in \mathcal{H}$ such that $\hat{\sigma}_i(h) = \sigma_i(h), \forall h \neq \hat{h}$ and
2. $\tilde{u}_i(\hat{\sigma}_i|_{\hat{h}}, \sigma_{-i}|_{\hat{h}}) > \tilde{u}_i(\sigma_i|_{\hat{h}})$.

2.2.5 Thm "one-shot deviation principle"

In a finitely or infinitely repeated game, a strategy profile is a SPE if and only if it has no profitable one-shot deviations.

2.2.6 Model "Infinite Cournot Duopoly"

- Two firms simultaneously choose *quantities* in each period
- Demand curve (e.g., $P = 12 - Q$)
- Zero costs
- $\delta < 1$

Proposal:

- on path: produce 1/2 monopoly quantity each
- off path: produce Cournot quantity (i.e., stage game NE)

Q. For what value of δ is this proposal a SPE?

2.2.7 Model "Infinite Bertrand Duopoly"

- Two firms simultaneously choose *prices* in each period
- Unit mass of consumers demand 1 unit at any price ≤ 1
- lowest price gets the market; ties split
- Zero costs
- $\delta < 1$

Proposal:

- on-path: set price = 1 (note: this means market is split)
- off-path: set price = 0

Q. For what value of δ is this proposal a SPE?

2.3 The Folk Theorem

2.3.1 Thm "Nash-Threat Folk Theorem"

If v is feasible and Nash-individually rational, then there exists $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, v is the average utility profile of an SPE given δ.

2.3.2 Def "feasible"

A payoff vector $v \in \mathbb{R}^n$ is *feasible* if v is in the convex hull of $\{\pi(a) : a \in A\}$

2.3.3 Def "Nash-individually rational"

A payoff vector $v \in \mathbb{R}^n$ is *Nash-individually rational* if

$$v_i > \underline{v}_i^{Nash} \equiv \min_{\alpha \in \Delta A \text{ is Nash}} \pi_i(\alpha), \forall i.$$

2.3.3.1 Result Nash-individually rational in pure strategies $\subseteq$ Nash-individually rational in mixed strategies.

2.3.4 Def "Minimax-individually rational"

A payoff vector $v \in \mathbb{R}^n$ is *individually rational* if

$$v_i > \underline{v}_i^{min} \equiv \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{a_i \in A_i} \pi_i(a_i, \alpha_{-i}), \forall i.$$

2.3.4.1 Remark $\underline{v}_i^{min} \leq \underline{v}_i^{Nash}, \forall i$

2.3.5 Thm "Minimax Folk Theorem"

If v is in the interior of the set of feasible and individually rational payoffs, then there exists $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, v is the average utility profile of an SPE given δ.

3 Bayesian Nash Equilibrium

3.1 Bayesian Games

3.1.1 Def "Bayesian Game"

A Bayesian game is a tuple $\langle N, A, u, \Theta, \mu \rangle$ where

- each player i privately observes some signal $\theta_i \in \Theta$
- Nature draws a type vector $\theta = (\theta_1, ..., \theta_n) \in \Theta$ from a common prior measure μ on Θ
- $u_i : A \times \Theta \to \mathbb{R}$

A strategy for player i is a measurable function mapping from Θ_i to ΔA_i. S_i is the space of such functions.

3.1.2 Def "Bayesian Normal Form"

$\langle N, S, \mathbb{E}[u] \rangle$ where $\mathbb{E}[u_i(s)] \equiv \mathbb{E}[u_i(s(\theta), \theta)]$ is player i's expected payoff given strategy profile s and the common prior μ.

3.1.3 Def "Bayesian Nash Equilibrium"

In a Bayesian game, $\langle N, A, u, \Theta, \mu \rangle$, a *Bayesian Nash Equilibrium* is a strategy profile $s \in S = \times_{i=1}^n S_i$ such that for each i and each $s_i' \in S_i$,

$$\mathbb{E}[u_i(s(\theta), \theta)] \geq \mathbb{E}[u_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta)].$$

3.1.4 Observation

A BNE is a NE in the game $\langle N, S, \mathbb{E}[u] \rangle$.

3.1.5 Thm "Bayesian Nash Equilibrium if"

In a Bayesian game, $\langle N, A, u, \Theta, \mu \rangle$, a strategy profile $s \in S = \times_{i=1}^n S_i$ is a *Bayesian Nash Equilibrium* if

$$\mathbb{E}[u_i(s(\theta), \theta | \theta_i)] \geq \mathbb{E}[u_i(a_i, s_{-i}(\theta_{-i}) | \theta_i)], \forall a_i \in A_i, \theta_i \in \Theta_i, i \in N.$$

3.1.5.1 Remark This is also "only if" if Θ is finite and μ has full support.

3.2 Applications

3.2.1 Model "Cournot Duopoly with higher order beliefs"

- 2 firms simultaneously choose quantities facing a demand curve
- Firm 1 has zero costs
- Firm 2 is
  - type $\theta_2 = L$ (i.e., has low costs) with probability $1 - \alpha$
  - type $\theta_2 = H$ (i.e., has high costs) with probability α
- Firm 1's belief about Firm 2's costs:
  - Type $\theta_1 = l$ (i.e., believes $\alpha_l = 1/4$) with probability $1 - \beta$
  - Type $\theta_1 = h$ (i.e., believes $\alpha_h = 3/4$) with probability β

3.2.2 Model "Lemons"

- Seller privately knows his good has quality $\theta \in [0, 1]$
- Buyer believes $\theta \sim U[0, 1]$
- Seller chooses price $p \in [0, 1]$
- Buyer chooses set $E \subseteq [0, 1]$ of prices she will accept

Analysis:

1. $E^* = \emptyset \Rightarrow$ continuum of no-trade equilibria
2. $E^* \neq \emptyset \Rightarrow E^*$ contains its maximum, e^*
3. Seller BR to e^*: $BR_s(e^*|\theta) = \begin{cases} e^* \text{ if } \theta < \frac{e^*}{q} \\ (e^*, 1] \text{ if } \theta > \frac{e^*}{q} \end{cases}$.
4. $E^* \neq \emptyset \Rightarrow e^* = 0$
5. $E^* = \{0\} \Rightarrow$ continuum of almost-no-trade equilibria

3.2.2.1 Term: "Adverse selection" Occurs when asymmetric information leads to undesirable outcomes in transactions.

3.3 Auctions

3.3.1 Primitives

- n bidders (one prize)
- Bidder i's type: $\theta_i \in \Theta_i$
- Each bidder simultaneously submits $b_i \in [0, \infty]$
- Payoff: $u_i(b, \theta) = x_i(b)v_i(\theta) - p_i(b)$
  - $x_i(b)$ is the probability i gets the prize
  - $v_i(b)$ is i valuation of the prize
  - $p_i(b)$ is the price i pays

3.3.1.1 Model "Second-price *Private* Values"

- Private values: $\Theta_i = [0, \bar{\theta}]$ and $v_i(\theta) = \theta_i$
- **Thm:** It is a weakly dominant strategy for each player to bid $b_i(\theta_i) = \theta_i$, regardless of the distribution over Θ.

3.3.1.2 Model "First-price *Private* Values"

- (Confirmed) Guess: symmetric strictly increasing bid strategy, $s : \Theta_i \to [0, \infty)$.

$$s(\theta_i) = \theta_i - \int_0^{\theta_i} \left(\frac{F(x_i)}{F(\theta_i)} \right)^{n-1} dx_i$$

3.3.1.3 Model "Second-price *Common* Values"

- **Claim:** Everyone bidding their true expected values is not an equilibrium (suppose not: bid lower, now you (1) lose if you were close, or (2) profitable deviation??)
- **Claim:** Symmetric equilibrium in which each player bids his expected value conditional on θ_i and $\max_{j \neq i} \theta_j = \theta_i$. (Winner's curse!)

4 Perfect Bayesian Equilibrium

4.1 Perfect Bayesian Equilibrium

"PBE $\subseteq$ NE" -Selcuk

4.1.1 Def "Assessment"

An *assessment* is a pair (β, μ), where

1. β is an independent behavior strategy profile
2. μ is a belief system

4.1.1.1 Term "belief system" For each information set, I, $\mu(\cdot|I)$ is a measure over nodes in I.

4.1.1.2 Term "feasible paths (H), aka complete histories" Histories that reach terminal nodes are called *feasible paths* or sometimes *complete histories.* H is the set of all feasible paths.

- $\phi_\beta(H)$ is the probability of a set of feasible paths, H.
- $\phi_\beta(H_x|H_I)$ is the conditional probability that node x is reached given that information set I is reached. $(H_x \subseteq H_I)$

4.1.2 Def "Weak perfect Bayesian Equilibrium (wPBE)"

An assessment (β, μ) is a *weak perfect Bayesian equilibrium* (wPBE) if at each information set, I, the following are satisfied:

1. Bayesian updating:

If $\underbrace{\phi_\beta(H_I) > 0}_{\text{i.e., on eqm path}}$, then $\mu(x|I) = \phi_\beta(H_x|H_I)$ for each node $x \in I$

2. Sequential rationality:

$$\beta_i(\cdot|H_i) \in arg\max_{\beta_i} \int_{x \in I} \int_{h \in H_x} u_i(h) d\phi_{\hat{\beta}_i, \beta_{-i}}(h|H_x) d\mu(x|I).$$

4.2 Applications

4.2.1 Relevant Models:

Gift Game, Simple Signaling, Simple Reputation (aka centipede or grab game)

4.3 Limitations of PBE

4.3.1 Def "Perfect Bayesian Equilibrium" (informal)

$$PBE = wPBE \cap SPE$$

I.e., PBE requires wPBE in every subgame.

PBE $\not\Rightarrow$ correct beliefs

PBE $\not\Rightarrow$ strategic independence

4.3.2 Def "Consistent Assessments"

In a countable game, an assessment (β, μ) is *consistent* if there exists a sequence of assessments $\{(\beta^k, \mu^k)\}$ such that

1. each behavior strategy β^k assigns strictly positive probability to every action at every information set,
2. each μ^k is derived from ϕ_{β^k} according to Bayes' rule, and
3. (β^k, μ^k) converges uniformly to (β, μ).

4.3.3 Def "Sequential Equilibrium"

In a countable game, an assessment is a *sequential equilibrium* if it is sequentially rational and consistent.

5 Mechanism Design

5.0.1 Quasilinear Environment

- $N = \{1, ..., n\}$ agents
- A space $\mathcal{X}$ of alternatives/outcomes
- A space Θ_i of private info (i.e., types) for each agent
  - Type space $\Theta = \Theta_i \times \cdots \times \Theta_n$
  - Common prior ϕ on Θ
- Agents are risk-neutral
- Agents earn utility $u_i : \mathcal{X} \times \Theta \times \mathbb{R}^n \to \mathbb{R}$ where $u_i(\chi, \theta, \tau) = \underbrace{v_i(\chi, \theta)}_{\text{Payoff fn}} + \tau_i$
  - v is how i values the chosen outcome χ given θ
  - $\tau \in \mathbb{R}^n$ is a vector of monetary transfers
    * $\tau_i > 0$ is a transfer TO player i;
    * $\tau_i < 0$ is a transfer FROM player i;

5.0.2 "Regularities"

We construct a mechanism $\langle M, x, t \rangle$ that defines a game for the agents in which:

- Agents simultaneously report messages: $m = (m_1, ..., m_n) \in M = M_1 \times \cdots \times M_n$
- Outcome function $x : M \to \mathcal{X}$ selects an alternative, χ from $\mathcal{X}$, given m.
- Each agent i receives transfer $t_i(m)$.

5.0.3 "Mechanism Design Problem"

A mechanism design problem is the problem of choosing x and t, subject to some constraints, to maximize some objective function.

In the **principal-agent interpretation**, the objective function is (typically) either

- social welfare $\sum_i v_i(x(m(\theta)), \theta)$ or
- revenue $-\sum_i t_i(m(\theta))$

In the **repeated games interpretation**, the objective function is typically

- a weighted sum of the players' utilities, $\sum_i \lambda_i(x(m(\theta)), \theta) + t_i(m(\theta))$ for some vector of weights $\lambda \in \mathbb{R}^n$.

principal-agent interpretation

utility transfer: money

repeated games interpretation

utility transfer: changes in continuation utility

5.0.4 "Constraints"

1. Incentive compatibility IC (=equilibrium concept)
   - Interim IC (aka Bayesian IC) (weakest)
     - requires that the messages constitute a BNE, where each agent sends a message that maximizes her expected utility conditioning on her own info.
   - Ex post IC (EPIC)
     - requires that the messages constitute and ex post equilibrium, where each agent's message must max her ex post utility for all possible realizations of the other agents' private info (where each other agent sends his eqm message).
   - Dominant strategy IC (DSIC) (strongest)
     - requires each agent's message is a (weakly) dominant strategy - that is, each agent's message must maximize her ex post utility for all possible messages they could send.

2. Individual Rationality (IR)
   - Ex ante IR (opt in: *before* learn type; *before* receive messages)
   - Interim IR (opt in: *after* learn type; *before* receive messages)
   - Ex post IR (opt in: *after* learn type; *after* receive messages)
3. Budget Balance
   - Ex ante BB - requires that the principal pays the agents zero in expectation, though he may pay them a positive or negative amount for different realizations (balances to 0)
     a. $E[\sum_i t_i(\theta)] = 0$
   - (Ex post) No subsidy (no matter type vec, all transfers must sum to 0)
     b. $\sum_i t_i \leq 0, \forall\theta$
   - Ex post BB - principal cannot pay a nonzero amount for any realization of their private info (but agents may pay each other)
     c. $\sum_i t_i = 0$

   In principal-agent interpretation: often interested in a principal with NO BB constraint

   In repeated-games interpretation: prefer no-subsidy constraint (this allows agents to burn money but not bring money from outside the game)

5.0.5 Def "Interim/Bayesian IC" (aka BNE of the Mechanism)

The strategy profile, m^* is a Bayesian Nash equilibrium of the mechanism $\langle M, x, t\rangle$ if

$$E[v_i(x(m^*(\theta)), \theta) + t_i(m^*(\theta))|\theta_i] \geq E[v_i(x(m_i, m^*_{-1}(\theta_{-i})), \theta) + t_i(m_i, m^*_{-i}(\theta_{-i}))|\theta_i], \forall m_i \in M_i, \forall\theta_i \in \Theta_i, \forall i.$$

The mechanism $\langle M, x, t\rangle$ *implements* the outcome rule $\hat{x} : \Theta \to \chi$ *in Bayesian Nash equilibrium* if there is a Bayesian Nash equilibrium m^* of $\langle M, x, t\rangle$ with $x(m^*(\theta)) = \hat{x}(\theta)$ for all θ.

Such a mechanism is interim incentive compatible (IIC) (aka Bayesian Incentive compatible)

5.0.6 Def "Ex post Incentive Compatible (EPIC)"

Strategy profile m^* is an *ex post equilibrium* of $\langle M, x, t\rangle$ if

$$v_i(x(m^*(\theta)), \theta) + t_i(m^*(\theta)) \geq v_i(x(m_i, m^*_{-i}(\theta)), \theta) + t_i(m_i, m^*_{-i}(\theta))$$

For all $m_i \in M_i$, $\theta \in \Theta$, and all i. The mechanism $\langle M, x, t\rangle$ *implements* the outcome rule $\hat{x} : \Theta \to \mathcal{X}$ *in ex post equilibrium* if there is an ex post equilibrium m^* of $\langle M, x, t\rangle$ with $x(m^*(\theta)) = \hat{x}(\theta)$ for all θ.

Such a mechanism is **ex post incentive compatible (EPIC)**.

5.0.7 Def "Dominant strategy incentive compatible (DSIC)"

Strategy profile m^* is a dominant strategy equilibrium of $\langle M, x, t\rangle$ if

$$v_i(x(m^*_i(\theta_i), m_{-i}), \theta) + t_i(m^*_i(\theta_i), m_{-i}) \geq v_i(x(m)), \theta) + t_i(m)$$

for all $m \in M$ and all $\theta \in \Theta$, and all i.

The mechanism $\langle M, x, t\rangle$ implements the outcome rule $\hat{x} : \Theta \to \mathcal{X}$ in dominant strategy equilibrium if there is a dominant strategy equilibrium m^* of $\langle M, x, t\rangle$ with $x(m^*(\theta)) = \hat{x}(\theta)$ for all θ.

Such a mechanism is dominant strategy incentive compatible (DSIC).

"No regrets even if others deviate"

5.0.8 Def "Private Values"

$$v_i(\chi, \theta) = v_i(\chi, \theta_i, \theta'_{-i}), \quad \forall\theta'_{-i}, \forall\theta, \forall i$$

5.0.9 Remark "DSIC vs EPIC"

- Private values $\Rightarrow$ DSIC = EPIC
- Interdependent values $\Rightarrow$ DSIC too strong to do anything

5.0.10 Remark "EPIC vs. BIC/IIC"

IIC:

- Relies on details of ϕ
- Not robust to "higher order beliefs"
- Not robust to "information leakage"

EPIC:

- Does not depend on ϕ at all
- Robust to "higher order beliefs"
- Robust to "information leakage"
- Simpler to work with!

5.0.11 Thm "The revelation principle"

If there exists a mechanism $\langle M, x, t \rangle$ that implements $\hat{x}$ in a (Bayesian, ex post, or dominant strategy) equilibrium m^*, then there exists a mechanism $\langle \Theta, \hat{x}, \hat{t} \rangle$, where $\hat{t}(\hat{\theta}) = t(m^*(\hat{\theta}))$ for all $\hat{\theta}$, with an equilibrium $\hat{m}_i(\theta_i) = \theta_i$ for all θ_i and all i.

We say that $\langle \Theta, \hat{x}, \hat{t} \rangle$ truthfully implements $\hat{x}$.

For direct mechanisms we write $\langle x, t \rangle$.

> "Direct" = we ask you to directly report your types rather than sending messages through an arbitrary message space.
>
> Now we give players a message space equal to their type space

5.0.12 Thm "Groves Mechanism" (Aka VCG (Vickrey-Clarke-Groves) mechanisms.)

Assume private values. If $x^*(\theta)$ maximizes social welfare for each θ, then x^* is truthfully implemented in ex post equilibrium by any direct mechanism $\langle x^*, t \rangle$ satisfying

$$t_i(\theta) = \sum_{j \neq i} v_j(x^*(\theta), \theta_j) + h_i(\theta_{-i})$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function, for all θ and all i.

> $h_i(\theta_{-i})$ CANNOT depend on i's type!
>
> If we impose EPIC, efficient mechanisms MUST be Groves Mechanisms
>
> Intuition: you're going to get your welfare (from the outcome), now you're going to get everyone else's welfare, and some arbitrary amount you can't affect

Example "Public Project"

FINISH - need to work through this example

Example "Externalities"

FINISH - need to work through this example

> 2nd price auction is a Groves mechanism: people report truthfully, has efficient outcome.

5.0.13 Result "Clark Pivot Rule"

$$h_i(\theta_{-i}) = -\max_x \sum_{j\neq i} v_j(x(\theta), \theta_j)$$

Notice:

$$t_i(\theta) = \underbrace{\sum_{j\neq i} v_j(x^*(\theta), \theta_j)}_{\text{social welfare if } i \text{ present}} - \underbrace{\max_{\hat{x}} \sum_{j\neq i} v_j(\hat{x}(\theta), \theta_j)}_{\text{social welfare if } i \text{ absent}}$$

With the Clarke pivot rule, the total amount paid by the player is: (social welfare of others if i were absent) - (social welfare of others when i is present).

This is exactly the externality of player i.

5.0.14 Result "Groves $\iff$ Efficient and EPIC"

In the private values setting, Groves Mechanism $\iff$ efficient outcome AND EPIC

Groves thm proves if, can use envelope thm to show only if.

5.1 Envelope Thm

5.1.1 Def "Equilibrium payoff function"

$U_i(\theta; x, t) = v_i(x(\theta), \theta) + t_i(\theta)$ is agent i's utility in a truthful equilibrium of mechanism $\langle x, t\rangle$.

5.1.2 Lemma "Milgrom and Segal"

If $v_i(\chi, (\cdot, \theta_{-i}))$ is differentiable in θ_i and $\frac{d}{d\theta_i} v_i(\chi, \theta)$ is uniformly bounded for all $\chi \in \mathcal{X}$ and all $\theta \in \Theta$, then in any EPIC mechanism $U_i((\cdot, \theta_{-i}))$ is continuous and piecewise differentiable in θ_i.

5.1.3 Thm "Envelope Thm"

If $v_i(\chi, (\cdot, \theta_{-i}))$ is differentiable in θ_i and $\frac{d}{d\theta_i} v_i(\chi, \theta)$ is uniformly bounded for all $\chi \in \mathcal{X}$ and all θ_{-i}, then, for any direct mechanism $\langle x, t\rangle$ that truthfully implements x in ex post equilibrium, for all θ_{-i},

$$U_i(\theta; x, t) = U_i(\underline{\theta}, \theta_{-i}; x, t) + \int_{\underline{\theta}}^{\theta_i} \left(\frac{\partial}{\partial s_i} v_i(x(\hat{\theta}_i, \theta_{-i}), s_i, \theta_{-i})\right)\Big|_{\hat{\theta}_i = s_i} ds_i.$$

Example: A public project

Example: Allocating an object

5.2 Linear Private Values

5.2.1 Def "Linear Private values"

$$v_i(\theta, \chi) = \theta_i \hat{v}_i(\chi), \text{ where } \hat{v}_i : \mathcal{X} \to \mathbb{R}.$$

5.2.2 Thm "Linear Private Values"

In the linear private values environment, a mechanism $\langle x, t\rangle$ implements x in ex post equilibrium if and only if, for all i

1. $\hat{v}_i(x(\theta))$ is non-decreasing in θ_i for all θ_{-i}, and
2. $U_i(\theta; x, t) = U_i(\underline{\theta}_i \theta_{-i}; x, t) + \int_{\underline{\theta}_i}^{\theta_i} \hat{v}_i(x(s_i, \theta_{-i})) ds_i$ for all θ.

5.3 Bayesian Implementation

5.3.1 Thm "D'Aspremont/Gerard-Varet (AGV) mechanisms"

In the private values environment, if $x(\theta)$ maximizes social welfare for all θ and types are distributed independently, then x is truthfully implemented in Bayesian Nash equilibrium by any mechanism $\langle x, t\rangle$ satisfying

$$t_i(\hat{\theta}) = E\big[\sum_{j\neq i} v_j(x(\hat{\theta}), \hat{\theta}_j)|\hat{\theta}_i\big] + h_i(\hat{\theta}_{-i}),$$

where $h_i : \Theta \to \mathbb{R}$ is an arbitrary function, for all $\hat{\theta}$ and all i.

5.3.2 Def "interim equilibrium utility"

$$\overline{U}_i(\theta_i; x, t) \equiv E[U_i(\theta; x, t)|\theta_i]$$

5.3.3 Thm "Bayesian envelope theorem"

Suppose that $E[v_i(\chi, \theta)|\theta_i]$ is differentiable in θ_i for all $\chi \in \mathcal{X}$. Then, for any direct mechanism $\langle x, t\rangle$ that truthfully implements x in Bayesian Nash equilibrium,

$$\overline{U_i}(\theta; x, t) = \overline{U_i}(\underline{\theta}, \theta_{-i}; x, t) + \int_{\underline{\theta}}^{\theta_i} \frac{\partial}{\partial s_i}\mathbb{E}\big[v_i(x(\hat{\theta}_i, \theta_{-i}), s_i, \theta_{-i})\big]\Big|_{\hat{\theta}_i = s_i} ds_i.$$

5.3.4 Thm "Independent types + BNE $\Rightarrow$ EPBB"

Suppose that the players' types are distributed independently. Then, if x is implementable in Bayesian Nash equilibrium, then x is truthfully implementable in Bayesian Nash equilibrium with ; i.e., $\sum_i t_i(\theta) = 0$ for all θ.

5.3.5 Thm "Linear Private Values implements x in BNE iff"

In the linear private values environment, a mechanism $\langle x, t\rangle$ truthfully implements x in Bayesian Nash equilbrium if, and only if

1. $E[\hat{v}_i(x(\theta))|\theta_i]$ is non-decreasing in θ, and
2. $\bar{U}_i(\theta_i; x, t) = \bar{U}_i(\underline{\theta_i}; x, t) + \int_{\underline{\theta}}^{\theta_i} E[\hat{v}_i(x(s_i, \theta_{-i}))|s_i]ds_i$.

5.4 Individual Rationality

5.4.1 Thm "Myerson-Satterthwaite"

The efficient outcome function x^* is not Bayesian implementable with ex post individual rationality and ex ante budget balance.

5.5 Ex post implementation and Budget Balance

5.5.1 Def "(N-1)-additive separability"

A function $\psi : \Theta \to \mathbb{R}$ is $(N-1)$-additively separable if there exist function $\alpha_i : \Theta_{-i} \to \mathbb{R}, i = 1, ..., N$, such that $\psi(\theta) = \sum_i \alpha_i(\theta_{-i})$ for ϕ-almost all θ.

5.5.2 Thm "EPIC and EPBB"

An outcome rule x is ex post implementable with ex post budget balance only if $\sum_i t_i^*(\theta)$ is $(N-1)$-additively separable, where

$$t_i^*(\theta) = \int_{\underline{\theta}}^{\theta_i} \frac{\partial}{\partial s_i} v_i(x(\hat{\theta}_i, \theta_{-i})), s_i, \theta_{-i}) ds_i - v_i(x(\theta), \theta).$$

For sufficiency, add appropriate monotonicity condition.

5.5.3 Observation "Groves mechanisms are generally not budget balanced"

Recall $t_i(\theta) = \sum_{j \neq i} v_j(x^*(\theta), \theta_j) + h_(\theta_{-i})$. Observe $v_j(x^*(\theta), \theta_j)$ itself generally not $(N-1)$-additively separable.

2 players: usually (N-1)-additively separable!

6 Principal-Agent Contracts with Moral Hazard

6.0.1 Model "Moral Hazard"

- Agent exerts $e \in \{e_L, e_H\}$
- Principal receives revenue $\pi \in [\underline{\pi}, \overline{\pi}]$
- Revenue distributed according to PDF $f(\pi|e)$ (has full support $\forall e$)
  - $f(\pi|e_H)$ FOSD $f(\pi|e_L)$
- Agent risk-averse, principal is risk-neutral
- Agent maxes the *expectation of* $u(w, e) = v(w) - g(e)$
  - $v' > 0, v'' \leq 0$ (equality $\Rightarrow$ risk neutral; inequality $\Rightarrow$ risk-averse)
  - has reservation utility $\bar{u}$ if doesn't work with principal
- Principal maximizes the *expectation of* $\pi - w$

6.0.2 Model "Contractible (i.e., observable) Effort (benchmark case/strawman)"

- Principal makes TIOLI offer comprising BOTH effort $e^* \in \{e_L, e_H\}$ and wage ($w^* : \mathbb{R} \to \mathbb{R}$)
- Effort is perfectly enforced
- Principal solves

$$\max_{e \in \{e_L, e_H\}} \max_{w:[\underline{\pi},\overline{\pi}] \to \mathbb{R}} \int_{\underline{\pi}}^{\overline{\pi}} (\pi - w(\pi)) f(\pi|e) d\pi$$

$$\text{s to } \int_{\underline{\pi}}^{\overline{\pi}} (v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \overline{u} \text{ (IR)}$$

It's possible that the associated cost with e_H is too high to make it worth it, so Principal asks only for e_L

To analyze:

1. fix e (this lets us ignore IC)
2. note that IR constraint binds, so $\lambda_{IR} > 0$
3. FOC for $w(\pi)$ is: $-f(\pi|e) = -\lambda_{IR} v'(w(\pi)) f(\pi|e)$
4. Agent is risk-averse ($v'' < 0$) so there is a constant wage, $\bar{w}_e = v^{-1}(\bar{u} + g(e))$.
   - the constant wage makes the integrals disappear
5. Plug solution from inner problem into outer problem; principal picks effort level to solve the simplified outer max problem.

   Economic lesson: can pay the risk-averse agent less if we pay a constant wage with no risk. Let the risk-neutral principal absorb the risk.

6.0.3 Model "Non-contractible effort, Risk Neutral Agent" (still strawman)

- Principal makes TIOLI offer $w^* : \mathbb{R} \to \mathbb{R}$ (i.e., $w(\pi)$)
- risk neutrality: let $v(w) = w$ (or any constant in front wlog)
- sell the firm to the agent $w^*(\pi) = \pi - L$
- now agent solves the problem the principal wanted to solve, but without the IC constraint:

$$\max_{e \in \{e_L, e_H\}} \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e) d\pi - (L + g(e))$$

- IR binds $\to L = E[\pi|e^*] - (g(e^*) - \overline{u})$
- Now this is the same as the problem with contractible effort!

Just sell the firm to the agent! Then the agent solves the same problem the principal would have solved, but without the constraint. Agent is the "residual claimant"

In this model the principal doesn't do anything really other than collect revenue.

6.0.3.1 Term "High effort offer from principal"

$$\overline{v}(\pi) \equiv v(w(\pi))$$

6.0.4 Model "Non-contractible effort, Risk-averse Agent"

Principal makes TIOLI offer of wage w^* to solve:

$$\max_{e\in\{e_L,e_H\}} \max_{w:[\underline{\pi},\overline{\pi}]\to\mathbb{R}} \int_{\underline{\pi}}^{\overline{\pi}} (\pi - \underbrace{w(\pi)}_{v^{-1}(v(w(\pi)))})f(\pi|e)d\pi$$

subject to

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|e)d\pi - g(e) \geq \overline{u}, \text{ and} \tag{IR}$$

$$e \in \arg\max_{\tilde{e}\in\{e_L,e_H\}} \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|\tilde{e})d\pi - g(\tilde{e}). \tag{IC}$$

Low effort:

- Choose constant wage: $w(\pi) = v^{-1}(g(e_L) + \overline{u})$ to bind IR
  - IR binds because we chose the wage to make this bind.
- Then IC is slack
  - this makes sense because if the agent is getting a constant wage (i.e., effort doesn't affect wage), then of course low effort is better, since it's less costly!

High effort: Rewrite the problem as:

- Principal makes TIOLI offer of utility $\overline{v} : \mathbb{R} \to \mathbb{R}$ to solve:

$$\min_{\overline{v}:[\underline{\pi},\overline{\pi}]\to\mathbb{R}} \int_{\underline{\pi}}^{\overline{\pi}} v^{-1}(\overline{v}(\pi))f(\pi|e_H)d\pi$$

subject to

$$\int_{\underline{\pi}}^{\overline{\pi}} \overline{v}(\pi)f(\pi|e_H)d\pi - g(e) \geq \overline{u}, \text{ and} \tag{IR}$$

$$\int_{\underline{\pi}}^{\overline{\pi}} \overline{v}(\pi)\underbrace{(f(\pi|e_H) - f(\pi|e_L))}_{\text{diff in benefits}}d\pi \geq \underbrace{g(e_H) - g(e_L)}_{\text{diff in costs}}. \tag{IC}$$

- IR Binds (i.e., $\lambda_{IR} > 0$)
  - Proof: Suppose not. Subtract a constant utility from agent (for every possible outcome of π). IC will still be satisfied because the agent's payoff is linear (in utils). And because IR was slack, the principal will be better off while the agent still satisfies IC. This contradicts the assumption that we were at the optimal solution. So, IR binds.
- IC Binds (i.e., $\lambda_{IC} > 0$)
  - Proof: We know $\lambda_{IR} > 0$. Suppose IC doesn't bind. We can't subtract utility without adding back. Note that v^{-1} is convex. So, we can subtract from cases with high utility (which is costly for the principal) and add to cases with low utility. So this makes the principal better off while IR is still satisfied, contradicting the assumption we were at the optimal solution. So, IC binds.

FOC for $\overline{v}(\pi)$:

$$-[v^{-1}]'(\overline{v}(\pi)) = -\lambda_{IR} f(\pi|e_H) - \lambda_{IC}(f(\pi|e_H) - f(\pi|e_L))$$

convert back from $\overline{v}$ and divide by $f(\pi|e_H)$:

$$[v^{-1}]'(v(w(\pi))) = \lambda_{IR} + \lambda_{IC}\left(1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right)$$

use inverse function theorem:

$$\underbrace{\frac{1}{v'(w(\pi))}}_{\text{risk-averse } \Rightarrow v' \text{ str dec}} = \underbrace{\lambda_{IR}}_{>0} + \underbrace{\lambda_{IC}}_{>0}\left(1 - \underbrace{\frac{f(\pi|e_L)}{f(\pi|e_H)}}_{\text{inverse LR, so dec}}\right)$$

So, $w(\pi)$ is strictly increasing in LR_H!

- wages for e_H are driven by posterior "inferences," not output
- wages for e_H not necessarily monotone in output, highly nonlinear
- risk motivates the agent, but compensating him for it is costly

  λ_{IR} is the SAME for ALL the FOCs!

  Imposing risk on the (risk-averse) agent "burns" suruplus; since the principal is less risk-averse than the agent it makes sense for the less risk averse party to take on the risk.

6.0.4.1 Term "Likelihood Ratio"

$$\frac{f(\pi|e_L)}{f(\pi|e_H)}$$

LR_H high $\Rightarrow$ agent probably chose high effort.

6.0.5 Example: Prove the IR constraint must bind (in a standard model).

Proof: Suppose not; then, given the optimal wage schedule w, the IR constraint is slack by the amount $\varepsilon > 0$, and the IC constraints are all satisfied. But then the principal can choose a deviant wage schedule $\hat{w}$ such that $v(\hat{w}(\pi)) = v(w(\pi)) - \varepsilon$ for all $\pi \in [0, \bar{\pi}]$. The IC constraints are all still satisfied, since this has the effect of reducing both the left and right sides of each IC constraint by ε. The IR constraint is still satisfied because by supposition it was slack by ε. The principal's profits are now higher, because since $v' > 0$ it must be that $\hat{w}(\pi) < w(\pi)$ for all $\pi \in [0, \bar{\pi}]$. Hence w could not have been an optimal wage schedule.

6.0.6 Thm "Inverse Function Theorem"

If f is continuously differentiable with nonzero derivative at a then

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

7 Misc

7.0.1 "Algorithm SPE supporting XX"

0. If finite: "No non-Nash SPE. Since only NE can be played in the last period, backward induction shows the same is true for every earlier period. So, the only SPE is such that a stage NE is played in each stage game"
1. If infinite...
2. finish!

7.0.2 "Algorithm BNE"

1. Write in normal form if possible
2. look for dominated strategies
3. split the smaller strategy set into cases: one for each pure strategy plus a case for every possible mixture of pure strats. In each case:
   i. determine other player's best response
   ii. verify whether there is a profitable deviation from the strategy of the case

If need only one BNE:

1. "Probability of winning in increasing in i's type, so i's BR will depend on some cutoff, c_i"
   a. usually symmetric cutoff
   b. Guess: $s^* = \times_{i=1}^2 s_i = \begin{cases} E \text{ if } \theta_i \leq c \\ N \text{ ow} \end{cases}$
2. Solve for cutoff
   a. write expected utility of both strategies
   b. Say, "in equilibrium player i of type $\theta_i = c$ should be indifferent between E and N.
      i. Plug in $\theta_i = c$ and equate interim expected utilities
   c. use the fact that distributed uniformly to get rid of PDFs/CDFs
      i. recall: if $x \sim Unif[0,1]$ then $P(x \leq a) = F(a) = a$
   d. algebra to isolate c (fine to have c in terms of F)
   e. in a pinch: guess a random point well above/below a reasonable cutoff
3. Finally, check that the guess actually is a mutual best response
   a. cases!

7.0.3 "Algorithm SPE"

1. draw out extensive form if possible
2. check one-shot deviation principle if repeated
3. FINISH

7.0.4 "Algorithm wPBE"

Find all Pure Strategy wPBE

0. label beliefs and check for dominated strategies.
   a. be careful to bring dominated strategies back into the wPBE when giving an answer
1. Write out strategy spaces for each player.
2. Split smallest strategy space into cases. E.g.,
   a. Case 1: Firm plays M^{Edu}/C^{NoEdu}
   b. seq rat $\Rightarrow$ Worker plays $E^H/E^M/N^L$
   c. Bayes' $\Rightarrow p = 1/4, q = 1/4$
   d. Check whether the strategy guessed in the case really was seq rat. Get either:
      i. $E[M^{Edu}] = 15/4 \geq E[\text{best alternative}^{Edu}] = 16 \Rightarrow$ CONTRADICTION! or
      ii. $E[M^{Edu}] = 17/4 \geq E[\text{best alternative}^{Edu}] = 16 \Rightarrow$ wPBE!

i. if beliefs unconstrained, still need to be sure that $E[C^{NoEdu}] \geq E[\text{best alternative}^{NoEdu}]$ (e.g., $10r + 5s \geq 4$)

3. All cases analyzed $\Rightarrow$ complete set of wPBE.
   i. don't forget beliefs!

Three games to know:

1. Gift game
2. Simple signaling
3. Simple reputation

For mixed strategies

1. find/rule out all pure strategies
2. cases: (1) all mix (2) i mixes, $-i$ does not, etc.
   a. "since player i is mixing they are indifferent between actions so seq rat must hold."
3. define:
   a. Greek letters for mixtures
   b. The probability of each type given each info set is reached. Will be something like: let $q = P(i$ constrained $\mid$ info set I is reached). Don't use Bayes' yet.
4. Say, i is indifferent when $E_{u_i}[F] = E_{u_i}[G]$. Do for all players. This will give values for
   a. one of the mixtures (i.e. α or β, but usually not both) and
   b. the prob that each info set is reached given (i.e., q)
5. Now use Bayes' to "naively" calculate the value of q and set this equal to the value found above.
   a. This will give you the value of α or β for which there is wPBE.

7.0.5 "Algorithm Construct a mechanism that implements [...] with ex post incentive compatibility"

a welfare-maximizing allocation rule:

0. Define welfare
   a. usually just sum of v_i, but watch out for social costs - this is subtracted
1. Solve for efficient x^*
   a. usually just $\frac{\partial w}{\partial x}$
   b. if w nondifferentiable, logic will give an efficient rule
2. you have x, just need t
3. GROVES!
4. check for additional constraints

an inefficient rule:

1. Do we have linear private values?
   a. if so, figure out what $\hat{v}$ is.
      i. recall: LPV means $v = \theta_i \hat{v}$
   b. if not: compute $\frac{\partial}{\partial \theta_i} v$... (also written $\frac{\partial}{\partial s_i}$)
2. Check monotonicity.
3. write envelope condition. to simplify
   a. split integral into logical cases.
      i. e.g., $U_i(\cdot) = h_i(\theta_{-i}) + \int_{-1}^{\theta_i} x^*(s_i, \theta_{-i}) ds_i = h_i(\theta_{-i}) + \int_{-1}^{-\sum_{j\neq i}\theta_j} 0 ds_i + \int_{-\sum_{j\neq i}\theta_j}^{\theta_i} 1 ds_i = h_i(\theta_{-i}) + \begin{cases} \theta_i + \sum_{j\neq i}\theta_j \text{ if } \theta_i > -\sum_{j\neq i}\theta_j \\ 0 \text{ otherwise} \end{cases}$
4. rearrange to get $t_i = U_i - v_i$

graph for fixed price mechanism... FINISH!

7.0.6 "Algorithm Principal-Agent"

- Recall canonical model:
  - Low effort case: IR binds, IC slack
  - High effort case: IR binds, IC binds

Prove IR Binds

- Low effort
  - canonical model: binds
  - Pick constant wage, $w(\pi) = \bar{w} = v^{-1}(g(e_L) + \bar{u})$ to bind.
- High effort
  - canonical model: binds
  - Suppose not. Subtract a constant utility from agent (for every possible outcome of π). IC will still be satisfied because the agent's payoff is linear (in utils). And because IR was slack, the principal will be better off while the agent still satisfies IC. This contradicts the assumption that we were at the optimal solution. So, IR binds.

Prove IC Binds

- Low effort:
  - canonical model: never binds
  - should never bind?
- High effort:
  - canonical model: binds
  - Suppose IC doesn't bind. We can't subtract utility without adding back (since we likely already know IR binds). Note that v^{-1} is convex. So, we can subtract from cases with high utility (which is costly for the principal) and add to cases with low utility. So this makes the principal better off while IR is still satisfied, contradicting the assumption we were at the optimal solution. So, IC binds.

Prove IR Does Not Bind

- Low effort
  - canonical model: binds
  - final exam, didn't bind bc $g(e_L) = 0$.
    - rearrange IR, using that w is const and $g(e_L) = 0$: $\overline{w} * \int f(\pi|e_L)d\pi \geq 0 \iff \overline{w} \geq 0$, must have $\overline{w} > 0$, so IR cannot bind.
- High effort
  - canonical model: binds
  - for a noncanonical model, if it doesn't bind say:
    - "?? LOOK FOR EXAMPLE FINISH"

Prove IC Does Not Bind (low effort case)

1. Prove w constant.
2. Rearrange IC:
   a. $\int v(w(\pi))[f(\pi|e_L) - f(\pi|e_H)]d\pi \geq g(e_L) - g(e_H)$
3. "Since w const, the LHS = 0 (since pdfs integrate to 1). But, the RHS is strictly negative. So, IC cannot bind."

Prove IC Does Not Bind (high effort case)

??

Prove wage constant (low effort case)

1. "If effort doesn't affect wage, then of course low effort is better for the agent, since it's less costly."
2. "So, if wage is non-constant, there is an improvement for the principal: replace the non-constant w function with a constant, $\bar{w}$ that gives the agent the same expected utility; because the agent is more

risk-averse than the (risk-neutral) principal, this gives a higher expected profit to the firm without affecting IR and still satisfying IC."

7.0.7 "Random tips/facts"

TBD

7.0.8 "Mistakes to avoid"

- summing welfare:
  - WRONG: $w = \sum_i [\theta_i x - x^2] = nx \sum_i \theta_i - nx^2$
  - RIGHT: $w = \sum_i [\theta_i x - x^2] = x \sum_i \theta_i - nx^2$
- in the integral in LPV:
  - don't put θ_i in there!! it's $\hat{v}$, NOT v!!!!!!!!!!. Remember: $\hat{v}$ is literally just x in LPV!!!!!!!!!!!
- FOCs in Principal-Agent: when you do chain rule for $\partial \bar{v}(w)$ remember that $\bar{v}'(w) = 1$!