Math 415 Homework 1

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1 Find all the zero divisors of $\mathbb{Z}/20\mathbb{Z}$.

Notice that the additive identity of $\mathbb{Z}/20\mathbb{Z}$ is $\mathbb{Z} = 1\mathbb{Z}$. Therefore a zero divisor is any a,b $\in$ Z such that ab $\equiv$ 1 mod 20. This is the set of all integers that are coprime to 20 or {[1], [3], [7], [9], [11], [13], [17], [19]}. ([x] means the equivalence class of x modulo 20)

2 Determine U($\mathbb{Z}[i]$) where $\mathbb{Z}[i] \subseteq \mathbb{C}$ is the ring of Gaussian Integer.

Notice 1 $\in \mathbb{Z}[i]$ is its own inverse so 1 $\in$ U($\mathbb{Z}[i]$). Likewise -1 $\in \mathbb{Z}[i]$ is its own inverse so -1 $\in$ U(Z[i]). Notice $i(-i) = 1 = (-i)i$ so i, -i $\in$ U($\mathbb{Z}[i]$).

I claim there are no more units in the Gaussian Integers. Assume x $\in$ U($\mathbb{Z}[i]$) and x $\neq$ 1,-1,i,-i.

So x = a + bi and there exists a y = α + β i such that xy = 1.

3 Let G be a group and let R be a ring. We denote by RG the set of all functions f: G $\to$ R whose support {x $\in$ G | f(x) $\neq$ 0} is finite. Note that if G is finite then this condition is automatically satisfied.

3.1 Show that RG is a ring under the operations defined by

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = \sum_{y \in G} f(y)g(y^{-1}x)$$

for all f,g $\in$ RG and x $\in$ G. Note the sum on the right makes sense as it has only finitely many non-zero terms. The product just defined is usually called the convolutional product and we call RG the group ring.

(f+g)(x) =f(x) +g(x)equation* equation*(fg)(x) =*typeoutnfss@catcodes*

Let a,b,c $\in$ RG

3.1.1 (a + b)(x) = (b + a)(x)

Consider (a + b)(x) = a(x) + b(x) = b(x) + a(x) = (b + a)(x).

3.1.2 (a + b)(x) + c(x) = a(x) + (b + c)(x)

Consider (a + b)(x) + c(x) = a(x) + b(x) + c(x) = a(x) + (b + c)(x).

3.1.3 There is an additive identity.

Consider i: G $\to$ R defined by i(x) = 0. Notice this fuctions support is empty so it is finite. Also for all y $\in$ RG, (y + i)(x) = y(x) + 0 = y(x). So i is the additive identity of RG.

3.1.4 There is an element -a $\in$ RG such that (a + (-a))(x) = 0.

Consider the function α: G $\rightarrow$ R defined by α(x) = -(a(x)). As G is a group any a(x) will have an additive inverse so this is possible. Also α has the same support as a. So (a + α)(x) = a(x) + α(x) = a(x) + (-a(x)) = 0. So this is -a $\in$ RG.

3.1.5 (a(bc))(x) = ((ab)c)(x)

Consider

$$(a(bc))(x) = \sum_{y \in G} a(y)(bc)(y^{-1}x) = \sum_{y \in G} \sum_{z \in G} a(y)b(z)c(z^{-1}y^{-1}x)$$

Consider

$$((ab)c)(x) = \sum_{y \in G} (ab)(y)c(y^{-1}x) = \sum_{y \in G} \sum_{z \in G} a(z)b(z^{-1}y)c(y^{-1}x)$$

Let yz = w. So z = y^{-1} w

So

$$\sum_{y \in G} \sum_{z \in G} a(y)b(z)c(z^{-1}y^{-1}x) = \sum_{y \in G} \sum_{w \in G} a(y)b(y^{-1}w)c(w^{-1}yy^{-1}x) = \sum_{y \in G} \sum_{w \in G} a(y)b(y^{-1}w)c(w^{-1}x)$$

So

$$\sum_{y \in G} \sum_{z \in G} a(z)b(z^{-1}y)c(y^{-1}x) = \sum_{y \in G} \sum_{w \in G} a(y^{-1}w)b(y^{-1}wy)c(y^{-1}x) = \sum_{y \in G} \sum_{w \in G} a(y^{-1}w)b(y^{-1}wy)c(y^{-1}x)$$

Notice these summations happen over the the same set of elements so they are in fact equal.

3.1.6 (a(b + c))(x) = ab(x) + ac(x) and ((b + c)a) = ba(x) + ca(x)

3.2 If R is unital then RG is unital.

Assume R is unital.

4 Question 4

$fg = \frac{1}{4}(23) - \frac{1}{2}(123) + \frac{1}{4}(231) - \frac{1}{2}(132)$