# Solving the 4-Coloring Problem using a Quantum Circuit Generator

In this paper, we solve the 4-Coloring problem using a quantum algorithm. We go beyond the theory and implement this using Qiskit and run it on a quantum simulator, and so that it can be run on a future quantum computer with longer decoherence time. As the code gets rather long, to avoid making mistakes in quantum uncomputation, we introduce the SafeCircuit class, which when used properly, takes care of uncomputation.

We import qiskit:

In [49]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

# importing Qiskit
from qiskit import BasicAer, IBMQ
from qiskit import QuantumCircuit, ClassicalRegister, QuantumRegister, execu
from qiskit.tools.visualization import plot_histogram
```

We need to extend the n-Controlled Z gate provided in the qiskit tutorial at [https://github.com/Qiskit/qiskit-tutorials (https://github.com/Qiskit/qiskit-tutorials)](https://github.com/Qiskit/qiskit-tutorials) to allow 4 controls. This requires the use of auxiliary qubits.

In [51]:

```python
def n_controlled_Z(circuit, controls, target, aux2=None):
    """Implement a Z gate with multiple controls"""

    # considers nothing about the topology.
    if (len(controls) > 4):
        raise ValueError('The controlled Z with more than 4 ' +
                         'controls is not implemented')
    elif (len(controls) == 1):
        circuit.h(target)
        circuit.cx(controls[0], target)
        circuit.h(target)
    elif (len(controls) == 2):
        circuit.h(target)
        circuit.ccx(controls[0], controls[1], target)
        circuit.h(target)
    elif (len(controls) >= 3):
        if not len(aux2) >= 2: raise Exception("Need auxiliary qubits")

        if (len(controls) == 3):
            circuit.h(target)
            circuit.ccx(controls[0], controls[1], aux2[0])
            circuit.ccx(controls[2], aux2[0], target)
            circuit.ccx(controls[0], controls[1], aux2[0])
            circuit.h(target)

        else:
            raise Exception("This ncz gate is not implemented")

```

In [56]:

```python
def inversion_about_average(circuit, f_in, n, aux2):
    """Apply inversion about the average step of Grover's algorithm."""
    # Hadamards everywhere
    for j in range(n):
        circuit.h(f_in[j])
    # D matrix: flips the sign of the state |000> only
    for j in range(n):
        circuit.x(f_in[j])
    n_controlled_Z(circuit, [f_in[j] for j in range(n-1)], f_in[n-1], aux2)
    for j in range(n):
        circuit.x(f_in[j])
    # Hadamards everywhere again
    for j in range(n):
        circuit.h(f_in[j])
# -- end function
```

Now, let's write a oracle to 2-color a graph of length 3. The full code for Grover's algorithm is in the last cell of this of this document; the function to run can be changed by changing `oracle_func` .

In [70]:

```python
def color_works(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 3

    circuit.cx(f_in[0], aux[0])
    circuit.cx(f_in[1], aux[0])

    circuit.cx(f_in[1], aux[1])
    circuit.cx(f_in[2], aux[1])

    circuit.ccx(aux[0], aux[1], f_out[0])

    # Uncompute
    circuit.cx(f_in[2], aux[1])
    circuit.cx(f_in[1], aux[1])

    circuit.cx(f_in[1], aux[0])
    circuit.cx(f_in[0], aux[0])
```

Here is the SafeCircuit. We provide operations for combining multiple individual circuits.

In [52]:

```python
import copy

class Operation():
    #self.op = None
    #self.dirty = None

    def __init__(self, op, dirty):
        #TODO add an ID for printing
        self.op = op
        self.dirty = dirty

    def __str__(self):
        return self.op[0]+"("+", ".join(repr(e) for e in self.op[1:])+")"+"

    def write(self, cir):
        func = self.op[0] # this might not work, might have to pass cir.func
        getattr(cir, func)(*self.op[1:])

class SafeCircuit():
    written = dict()

    #self.cir = None

    def __init__(self, cir):
        self.cir = cir
        self.l = []
        self.oplist = []
        SafeCircuit.written[self.cir.name] = False # todo some way to hash i

    def __str__(self):
        self.finalize()
        return "\n".join(str(e) for e in self.oplist)
    # could include something if we make sure dirty are not actually used ag
    # this would work for final ones also, because they are never reset

    def add_op(self, *op, dirty=False): # advanced: isoutput should be set o
        self.l.append(Operation(op, dirty))
        # we should be able to get away with only one level, 'dirty'.  Other

    def add_cir(self, subcir): # don't carry through dirty.  The dirty will
        subops = subcir.finalize()
        for op in subops:
            op2 = copy.deepcopy(op)
            op2.dirty = False
            # do these actually need to be uncomputed at the end, or can the
            # I think they all need to be to reverse the central one that is
            self.l.append(op2)

    def blind_concat(self, other):
        raise Exception("Why would you do this?")
        # self.l += copy.deepcopy(other.l)

    def __iter__(self):
        for i in range(len(self.oplist)):
            yield self.oplist[i]

```

```python
    def finalize(self): # just builds the necessary list; you can do this mo
        oplist = []

        for i in range(len(self.l)):
            op = self.l[i].op
            oplist.append(self.l[i])

        for i in range(len(self.l)-1, -1, -1):
            # dirty or not is ignored when executing individually
            if not self.l[i].dirty:
                op = self.l[i].op
                oplist.append(self.l[i])

        self.oplist = oplist
        return oplist

    def dowrite(self):
        for op in self.oplist:
            op.write(self.cir)

    def write(self):
        #TODO check if the self.cir is empty before writing

        if SafeCircuit.written[self.cir.name]:
            raise Exception("You have already written a SafeCircuit to this

        self.finalize()

        SafeCircuit.written[self.cir.name] = True
        self.dowrite()


```

We write various functions, working our way up to 4-coloring a graph.

In [54]:

```python
def color2_works_safe_old(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 4
    # make sure color 0 is vertex 0

    # check connection 0-1

    sc = SafeCircuit(circuit)
    sc.add_op('cx', f_in[0], aux[0])
    sc.add_op('cx', f_in[1], aux[0])


    inv2 = SafeCircuit(circuit)
    inv2.add_op('x', f_in[2])
    inv2.add_op('cx', f_in[2], aux[1], dirty=True)


    print("this is inv2: \n" + "\n".join(str(e) for e in inv2.l))

    sc.add_op('cx', f_in[1], aux[1])
    sc.add_cir(inv2)

    sc.add_op('ccx', aux[0], aux[1], f_out[0], dirty=True) # !

    print("Executing: \n" + "\n".join(str(e) for e in sc.oplist))

    sc.write()
```

In [55]:

```python
def color2_works_safe(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 4
    # make sure color 0 is vertex 0

    # check connection 0-1

    sc = SafeCircuit(circuit)
    sc.add_op('cx', f_in[0], aux[0])
    sc.add_op('cx', f_in[1], aux[0])

    sc.add_op('cx', f_in[1], aux[1])
    sc.add_op('cx', f_in[2], aux[1])
    #print("this is inv2: \n" + "\n".join(str(e) for e in inv2.l))

    sc.add_op('ccx', aux[0], aux[1], f_out[0], dirty=True) # !

    print("Executing: \n" + str(sc)) #"\n".join(str(e) for e in sc.oplist))

    sc.write()
```

In [57]:

```
def color2_works_safe_force0(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 4
    # make sure color 0 is vertex 0

    # check connection 0-1

    sc = SafeCircuit(circuit)
    sc.add_op('cx', f_in[0], aux[0])
    sc.add_op('cx', f_in[1], aux[0])

    sc.add_op('cx', f_in[1], aux[1])
    sc.add_op('cx', f_in[2], aux[1])

    inv2 = SafeCircuit(circuit)
    inv2.add_op('x', f_in[2])
    inv2.add_op('cx', f_in[2], aux[2], dirty=True)
    sc.add_cir(inv2)

    #print("this is inv2: \n" + "\n".join(str(e) for e in inv2.l))

    sc.add_op('ccx', aux[0], aux[1], aux[3])
    sc.add_op('ccx', aux[3], aux[2], f_out[0], dirty=True) # !

    #print("Executing: \n" + str(sc)) #"\n".join(str(e) for e in sc.oplist))

    sc.write()
```

In [58]:

```
def color2_works_safe_forcenot1(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 4
    # make sure color 0 is vertex 0

    # check connection 0-1

    sc = SafeCircuit(circuit)
    sc.add_op('cx', f_in[0], aux[0])
    sc.add_op('cx', f_in[1], aux[0])

    sc.add_op('cx', f_in[1], aux[1])
    sc.add_op('cx', f_in[2], aux[1])

    inv2 = SafeCircuit(circuit)
    inv2.add_op('cx', f_in[2], aux[2])
    inv2.add_op('x', aux[2], dirty=True)
    sc.add_cir(inv2)

    #print("this is inv2: \n" + "\n".join(str(e) for e in inv2.l))

    sc.add_op('ccx', aux[0], aux[1], aux[3])
    sc.add_op('ccx', aux[3], aux[2], f_out[0], dirty=True) # !

    #print("Executing: \n" + str(sc)) #"\n".join(str(e) for e in sc.oplist))

    sc.write()
```

In [53]:

```
def twocolor_sample(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 4
    # make sure color 0 is vertex 0

    # check connection 0-1

    sc = SafeCircuit(circuit)


    s1 = SafeCircuit(circuit)

    s1.add_op('cx', f_in[0], aux[0])
    s1.add_op('cx', f_in[2], aux[0])
    s1.add_op('x', aux[0])

    s1.add_op('cx', f_in[1], aux[1])
    s1.add_op('cx', f_in[3], aux[1])
    s1.add_op('x', aux[1])

    s1.add_op('ccx', aux[0], aux[1], aux[8], dirty=True)
    s1.add_op('x', aux[8], dirty=True)

    sc.add_cir(s1) # now can we reuse 0 and 1?


    s2 = SafeCircuit(circuit)

    s2.add_op('cx', f_in[2], aux[0])
    s2.add_op('cx', f_in[4], aux[0])
    s2.add_op('x', aux[0])

    s2.add_op('cx', f_in[3], aux[1])
    s2.add_op('cx', f_in[5], aux[1])
    s2.add_op('x', aux[1])

    s2.add_op('ccx', aux[0], aux[1], aux[7], dirty=True)
    s2.add_op('x', aux[7], dirty=True)

    sc.add_cir(s2) # now can we reuse 0 and 1?


    sc.add_op('ccx', aux[8], aux[7], f_out[0], dirty=True)

    sc.write()

```

In [71]:

```python

def color4_works_safe(circuit, f_in, f_out, aux, n):
    # see if you can four-color the straight graph of length 2 with some cor

    # check connection 0-1

    #sc = SafeCircuit(circuit)


    s1 = SafeCircuit(circuit)

    s1.add_op('cx', f_in[0], aux[0])
    s1.add_op('cx', f_in[2], aux[0])
    s1.add_op('x', aux[0])

    s1.add_op('cx', f_in[1], aux[1])
    s1.add_op('cx', f_in[3], aux[1])
    s1.add_op('x', aux[1])

    s1.add_op('ccx', aux[0], aux[1], aux[8], dirty=True)
    s1.add_op('x', aux[8], dirty=True)

    # now can we reuse 0 and 1

    #curaux = 2
    ## simulate a fixed 1 connected to point 2 of the graph

    desired_colors=[3,0]

    for node,color in enumerate(desired_colors):
        curaux = 2+node
        print("curaux", curaux)
        assert 2 <= curaux < 7

        for term in range(4):
            if term != color: # and (node == 0 or term != desired_colors[noc
                s3_i = SafeCircuit(circuit)
                bs = bin(term)[2:].zfill(2)
                for i in range(len(bs)):
                    print("node", node, "color", color, "term", term, "bs",
                    if bs[i] == '0':
                        s3_i.add_op('cx', f_in[2*node+i], aux[i])
                        s3_i.add_op('x', aux[i])
                    elif bs[i] == '1':
                        rev5into1 = SafeCircuit(circuit)
                        rev5into1.add_op('x', f_in[2*node+i])
                        rev5into1.add_op('cx', f_in[2*node+i], aux[i], dirty
                        s3_i.add_cir(rev5into1)
                        #print("rev5into1",rev5into1)
                        s3_i.add_op('x', aux[i])

                    else:
                        raise Exception("Internal error converting to binary

                #s3_i.add_op('x', aux[i])

```

```
                s3_i.add_op('ccx', aux[0], aux[1], aux[curaux], dirty=True)
                s3_i.add_op('x', aux[curaux], dirty=True)

                print("node", node, "color", color, "term", term)# "s3_i:\n
                s1.add_cir(s3_i)



    #finalcir1 = SafeCircuit(circuit)
    #finalcir1.add_op('ccx', aux[8], aux[7], aux[1])
    #finalcir1.add_op('ccx', aux[1], aux[2], aux[0])
    #finalcir1.add_op('ccx', aux[8], aux[7], aux[1])
    s1.add_op('ccx', aux[8], aux[2], aux[5])
    s1.add_op('ccx', aux[5], aux[3], f_out[0], dirty=True)
    #sc.add_cir(finalcir1)

    '''
    finalcir2 = SafeCircuit(circuit)
    finalcir2.add_op('ccx', aux[0], aux[3], aux[1])
    finalcir2.add_op('ccx', aux[1], aux[4], f_out[0], dirty=True)
    sc.add_cir(finalcir2)
    '''


    s1.write()



    '''
    circuit.cx(f_in[2], aux[2]) # f_in[2] should be 1, or 0 if x'd.  As if


    circuit.cx(f_in[0], aux[4]) # f_in[0] should be 1, or 0 if x'd.
    '''

    '''
    circuit.ccx(aux[0], aux[1], aux[3])
    circuit.ccx(aux[3], aux[2], aux[5])

    circuit.ccx(aux[5], aux[4], f_out[0]) # !

    circuit.ccx(aux[3], aux[2], aux[5])
    circuit.ccx(aux[0], aux[1], aux[3])
    '''


    #sc.add_op('ccx', aux[0], aux[1], f_out[0], dirty=True) # !


    #print("Executing: \n" + "\n".join(str(e) for e in s1.oplist))


    '''
    circuit.cx(f_in[0], aux[4]) # f_in[0] should be 1, or 0 if x'd.

    circuit.cx(f_in[2], aux[2]) # f_in[2] should be 1, or 0 if x'd.
    '''
```

```
    #circuit.cx(f_in[2], aux[1])
    #circuit.cx(f_in[1], aux[1])


    #circuit.cx(f_in[1], aux[0])
    #circuit.cx(f_in[0], aux[0])

```

In [72]:

```
def color4_works_safe_force0(circuit, f_in, f_out, aux, n):
    # see if you can two-color the straight graph of length 4
    # make sure color 0 is vertex 0

    # check connection 0-1

    sc = SafeCircuit(circuit)


    s1 = SafeCircuit(circuit)

    s1.add_op('cx', f_in[0], aux[0])
    s1.add_op('cx', f_in[2], aux[0])
    s1.add_op('x', aux[0])

    s1.add_op('cx', f_in[1], aux[1])
    s1.add_op('cx', f_in[3], aux[1])
    s1.add_op('x', aux[1])

    s1.add_op('ccx', aux[0], aux[1], aux[8])
    s1.add_op('x', aux[8])

    s1.add_op('cx', f_in[0], aux[2])
    s1.add_op('cx', f_in[1], aux[3])

    finalcir1.add_op('ccx', aux[8], aux[2], aux[5])
    finalcir1.add_op('ccx', aux[5], aux[3], f_out[0], dirty=True)

    sc.add_cir(s1) # now can we reuse 0 and 1?
```

Here is the code for Grover's Algorithm, based on the Qiskit code. To change the function, change `oracle_func` and change `T` to change the number of iterations of Grover's Algorithm.

In [75]:

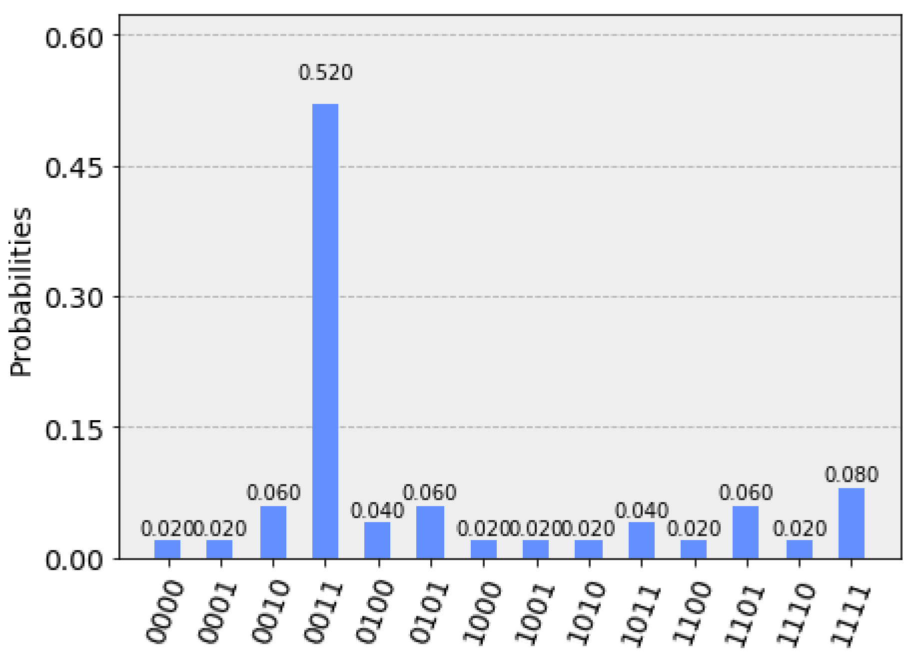
```
# Them

"""
Grover search implemented in Qiskit.

This module contains the code necessary to run Grover search on 3
qubits, both with a simulator and with a real quantum computing
device. This code is the companion for the paper
"An introduction to quantum computing, without the physics",
Giacomo Nannicini, https://arxiv.org/abs/1708.03684.
"""

def input_state(circuit, f_in, f_out, n):
    """(n+1)-qubit input state for Grover search."""
    for j in range(n):
        circuit.h(f_in[j])
    circuit.x(f_out)
    circuit.h(f_out)
# -- end function

# Make a quantum program for the n-bit Grover search.


funcstonumqubits = {color4_works_safe: 4, color2_works_safe: 3}


oracle_func = color4_works_safe #color2_works_safe_force0 #color4_oneseg_sa
n = 3

if oracle_func in funcstonumqubits:
    n = funcstonumqubits[oracle_func]

# n = 4 # remove!

# Exactly-1 3-SAT formula to be satisfied, in conjunctive
# normal form. We represent literals with integers, positive or
# negative, to indicate a Boolean variable or its negation.
exactly_1_3_sat_formula = [[1, 2, -3], [-1, -2, -3], [-1, 2, 3], [1, 2, -3]]

# Define three quantum registers: 'f_in' is the search space (input
# to the function f), 'f_out' is bit used for the output of function
# f, aux are the auxiliary bits used by f to perform its
# computation.
f_in = QuantumRegister(n)
f_out = QuantumRegister(1)
aux = QuantumRegister(9) #len(exactly_1_3_sat_formula) + 1)
aux2 = QuantumRegister(2)

# Define classical register for algorithm result
ans = ClassicalRegister(n)



# Define quantum circuit with above registers
grover = QuantumCircuit()
print(grover.name)
```

```python
grover.add_register(f_in)
grover.add_register(f_out)
grover.add_register(aux)
grover.add_register(aux2)
grover.add_register(ans)

input_state(grover, f_in, f_out, n)

NUMSHOTS = 50

import math
T = int(math.pi/4.0 * math.sqrt(2**n))
print(n, "qubits")
T = 2
print("Performing", T, "iterations of the Grover's Algorithm", "with", repr(


for t in range(T):
    # Apply T full iterations
    #black_box_u_f(grover, f_in, f_out, aux, n, exactly_1_3_sat_formula)
    #example_is010_v0(grover, f_in, f_out, aux, n)
    #color4_works_safe(grover, f_in, f_out, aux, n)
    '''inversion_about_average(grover, f_in, n, aux2)
    oracle_func(grover, f_in, f_out, aux, n)
    '''

    oracle_func(grover, f_in, f_out, aux, n)
    inversion_about_average(grover, f_in, n, aux2)



# Measure the output register in the computational basis
for j in range(n):
    grover.measure(f_in[j], ans[j])

# Execute circuit
backend = BasicAer.get_backend('qasm_simulator')
job = execute([grover], backend=backend, shots=NUMSHOTS)
result = job.result()

# Get counts and plot histogram
counts = result.get_counts(grover)
print(counts)
plot_histogram(counts)
```

```
circuit32
4 qubits
Performing 2 iterations of the Grover's Algorithm with <function color4_works_s
afe at 0x135C1E88> 50 times
curaux 2
node 0 color 3 term 0 bs 00 bs[i] 0
node 0 color 3 term 0 bs 00 bs[i] 0
node 0 color 3 term 0
node 0 color 3 term 1 bs 01 bs[i] 0
node 0 color 3 term 1 bs 01 bs[i] 1
node 0 color 3 term 1
```

```
node 0 color 3 term 2 bs 10 bs[i] 1
node 0 color 3 term 2 bs 10 bs[i] 0
node 0 color 3 term 2
curaux 3
node 1 color 0 term 1 bs 01 bs[i] 0
node 1 color 0 term 1 bs 01 bs[i] 1
node 1 color 0 term 1
node 1 color 0 term 2 bs 10 bs[i] 1
node 1 color 0 term 2 bs 10 bs[i] 0
node 1 color 0 term 2
node 1 color 0 term 3 bs 11 bs[i] 1
node 1 color 0 term 3 bs 11 bs[i] 1
node 1 color 0 term 3
curaux 2
node 0 color 3 term 0 bs 00 bs[i] 0
node 0 color 3 term 0 bs 00 bs[i] 0
node 0 color 3 term 0
node 0 color 3 term 1 bs 01 bs[i] 0
node 0 color 3 term 1 bs 01 bs[i] 1
node 0 color 3 term 1
node 0 color 3 term 2 bs 10 bs[i] 1
node 0 color 3 term 2 bs 10 bs[i] 0
node 0 color 3 term 2
curaux 3
node 1 color 0 term 1 bs 01 bs[i] 0
node 1 color 0 term 1 bs 01 bs[i] 1
node 1 color 0 term 1
node 1 color 0 term 2 bs 10 bs[i] 1
node 1 color 0 term 2 bs 10 bs[i] 0
node 1 color 0 term 2
node 1 color 0 term 3 bs 11 bs[i] 1
node 1 color 0 term 3 bs 11 bs[i] 1
node 1 color 0 term 3
{'0100': 2, '1010': 1, '0101': 3, '1001': 1, '1111': 4, '0001': 1, '1101': 3,
'0010': 3, '1000': 1, '1110': 1, '1011': 2, '1100': 1, '0011': 26, '0000': 1}
```

Out[75]:



In [ ]:

```
```