Sets 18 and 19: Section 5.4 (discrete parts only!) Jointly distributed random variables

Joint probability distributions associated with a vector rv $(X_1, \ldots, X_k)$**.**

Example: a trivariate discrete distribution described by the pmf $p(x, y, z)$

	X=1	**X=2**	**X=3**
Y=1	0.10	0.20	0.00
Y=2	0.00	0.05	0.05

$Z = 5$

	X=1	**X=2**	**X=3**
Y=1	0.00	0.30	0.10
Y=2	0.05	0.05	0.10

$Z = 6$

The *marginal* pmf $\mathrm{p}(x) = \Sigma_{y,z}\mathrm{p}(x, y, z)$

The concept of independence applies to rv's.

Definition: Random variables are independent if their joint pmfs factor into their marginal pmfs.

Example: Consider the bivariate pmf given by

	X=1	**X=2**
Y=1	0.4	0.2
Y=2	0.1	0.3

(a) Obtain the marginal pmf for X**.**

(b) **Obtain the marginal pmf for Y.**

(c) **Are X and Y independent?**

Proposition: In the discrete case,

$$\mathrm{E}[g(X_1, \ldots, X_k)] = \sum_{x_1} \cdots \sum_{x_k} g(x_1, \ldots x_k) p(x_1, \ldots x_k)$$

Example: An instructor gives a quiz with two parts. For a randomly selected student, let X and Y be the scores obtained on the two parts respectively. The joint pmf $p(x, y)$ of X and Y:

p(x,y)	**y=0**	**y=5**	**y=10**	**y=15**
x=0	0.02	0.06	0.02	0.10
x=5	0.04	0.15	0.20	0.10
x=10	0.01	0.15	0.14	0.01

(a) **What is the expected total score $\mathrm{E}(X + Y)$?**

(b) **What is the expected maximum score from the two parts?**

(c) **Are X and Y independent?**

(d) **Obtain $\mathrm{P}(Y = 10 \mid X \geq 5)$.**

Example: We return to the discrete distribution described by the pmf $p(x, y, z)$

	X=1	**X=2**	**X=3**
Y=1	0.10	0.20	0.00
Y=2	0.00	0.05	0.05

$Z = 5$

	X=1	**X=2**	**X=3**
Y=1	0.00	0.30	0.10
Y=2	0.05	0.05	0.10

$Z = 6$

Obtain $\mathrm{E}(g)$ **where** $g(x, y, z) = xz$**.**

Problem: The number of customers waiting for the gift-wrap service at department store is a rv X **taking possible values 0, 1, 2, 3 and 4 with corresponding probabilities 0.10, 0.20, 0.30, 0.25 and 0.15. A random customer has 1, 2 or 3 packages for wrapping with probabilities 0.6, 0.3 and 0.1 respectively. Let** Y **be the total number of packages to be wrapped by customers waiting in line.**

(a) Determine $\mathrm{P}(X = 3, Y = 3)$**.**

(b) Determine $\mathrm{P}(X = 4, Y = 11)$**.**

Omit this example

Definition: The *covariance* between the rvs X **and** Y **is given by**

$$\begin{aligned}\mathrm{Cov}(X, Y) &= \mathrm{E}[\ (X - \mathrm{E}(X))(Y - \mathrm{E}(Y))\ ] \\ &= \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)\end{aligned}$$

Interpretation:

- **positive covariance**
  - **large** x**'s occur with large** y**'s**
  - **small** x**'s occur with small** y**'s**
- **negative covariance**
  - **large** x**'s occur with small** y**'s**
  - **small** x**'s occur with large** y**'s**

Correlation is the scaled and preferred version of covariance.

Definition: The *correlation* between the rvs X and Y is given by

$$\rho = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

Notes:

- $-1 \leq \mathrm{Corr}(X, Y) \leq 1$
- ρ **is the population analogue of** r
- **if** $a > 0$**, then** $\mathrm{Corr}(X, aX + b) = 1$
- **if** $a < 0$**, then** $\mathrm{Corr}(X, aX + b) = -1$

Example: Obtain the correlation between X and Y where the joint pmf of X and Y is:

	X=1	**X=2**	**X=3**
Y=1	0.1	0.2	0.3
Y=2	0.0	0.2	0.2

Proposition: If X and Y are independent, then

$$\mathrm{Cov}(X, Y) = 0$$

In addition, $\mathrm{Corr}(X, Y) = 0$ **provided** $\mathrm{Var}(X)$ **and** $\mathrm{Var}(Y)$ **are nonzero. The converse is not true.**

Recall that correlation does not imply causation.

Proposition: $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$

Proposition: More generally,

$$\mathrm{Var}(aX + bY + c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X,Y)$$

Proposition: Even more generally,

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i + c\right) &= \textstyle\sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) + 2\sum_{i<j} a_i a_j \mathrm{Cov}(X_i, X_j) \\
\mathrm{E}\left(\sum_{i=1}^{n} a_i X_i + c\right) &= c + \textstyle\sum_{i=1}^{n} a_i \mathrm{E}(X_i)
\end{aligned}$$

Here is a useful result.

Corollary: Suppose that the rv's $X_1, \ldots, X_n$ are a *random sample.* In other words, the X's are independent and arise from a common distribution with mean μ and variance σ^2. Then the sample mean has the following properties:

- $\mathrm{E}(\bar{X}) = \mu$
- $\mathrm{Var}(\bar{X}) = \sigma^2/n$