23rd Jan, 2024

Recap :- Likelihoods based on frequency tables.

Let's look at an unusual example of MLE
↳ Example 2.3.1 of the Complete Lecture Notes (Pg-25)

Let, X be the number of the sighted drone

$$X \sim \underset{\text{discrete}}{\text{unif}}\ (0, N)$$

$$\text{pmf}: f(x; N) = \begin{cases} 1/N & ; N > 0 \\ 0 & ; \text{otherwise.} \end{cases}$$

$\underset{\sim}{X} = (x_1, x_2, \cdots, x_8)$ iid Unif $(0, N)$

$$f(\underset{\sim}{x}; N) = (1/N)^8 = 1/N^8, \quad N > 0 \quad \text{or, } N \geq 1$$

Then,

$$L(N; \underset{\sim}{x}) = 1/N^8, \quad N > 0 \quad \Big\}\ \text{dependent on } \underset{\sim}{X}$$

$$l(N) = \log(1) - \log(N^8)$$
$$= -8 \log(N), \quad N > 0$$

$$l'(N) = -8/N, \quad N > 0, \quad \text{evaluate } \hat{N}.$$

$$\Rightarrow l'(N) = 0$$

$$\Rightarrow l'(N) = -8/\hat{N} = 0 \qquad \text{whoa!!!}$$

max occurs at $N = 1$.

[Choose the value of N, to be as small as possible]

For this example, the pmf is,

$$f(\underset{\sim}{x}; N) = \begin{cases} 1/N^8 & ; N \geq 137 \\ 0 & ; \text{Otherwise.} \end{cases}$$

So, $\hat{N} = \max\{x_1, x_2, \cdots, x_8\}$

$\Rightarrow \hat{N} = 137$, is the MLE.

This example is a re-phasing of the "German Tank problem".

Section 2.5 :- Relative Likelihood functions

So far we have been finding the MLE, which is a point estimate. But we are good Statisticians !
we want interval estimates !!
we want to compare parameter values !!!
So let's do that, today :)

Relative Likelihood functions (RLF) of θ:

$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})}, \quad \text{where } \hat{\theta} = \text{MLE}$$

$$0 \leq R(\theta) \leq 1$$

Log-RLF of θ :

$$r(\theta) = \log(R(\theta)) = \log\left(L(\theta)/L(\hat{\theta})\right) = \ell(\theta) - \ell(\hat{\theta}),$$

$$-\infty < r(\theta) \leq 0$$

How to use $R(\theta)$ and $r(\theta)$?

1) Check plausibility of a value, θ_1:

if $R(\theta_1) = 0.1 \iff$ the data are $10\times$ more likely to occur at the MLE $\hat{\theta}$ than at θ_1

2) Compare θ_1 and θ_2:

if $R(\theta_2) = 0.5 \iff$ The data are $2\times$ more likely to occur at the MLE $\hat{\theta}$ than at θ_2.

So, if $R(\theta_1) = 0.1$, then θ_2 is more plausible than θ_1.

3) Can construct a likelihood Interval.

[*] A $100p\%$ likelihood interval (LI) for θ is the set of θ values such that $R(\theta) \geq p$ or, $r(\theta) \geq \ln(p)$ or, $\log(p)$

i.e., the set of θ values that make the data achieve at least $100p\%$ of the maximum likelihood, under the hypothesized model.

e.g., 10% LI $\rightarrow$ plausible θ's

50% LI $\rightarrow$ very plausible θ's

100% LI $\rightarrow$ $\hat{\theta}$, the most plausible θ.

$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})} \geq p$$

$$\text{or, } \quad \frac{L(\theta)}{p} \geq L(\hat{\theta})$$

$\boxed{\text{LI always contains MLE, } \hat{\theta}}$

To find the $100p\%$ LI, solve $R(\theta) - p = 0$

or, $r(\theta) - \ln(p) = 0$

Return to Ex 2.2.1 [Fractured plastics]

$$L(\theta) = \theta^{170}(1-\theta)^{170} \; ; \; \theta \in [0,1]$$

$\hookrightarrow \hat{\theta} = 1/2$

$$L(\hat{\theta}) = (1/2)^{170}(1-1/2)^{170} = (1/2)^{340}$$

$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})} = \frac{\theta^{170}(1-\theta)^{170}}{(0.5)^{340}} = \left(\frac{\theta}{0.5}\right)^{170}\left(\frac{1-\theta}{0.5}\right)^{170}$$

$$= (2\theta)^{170}(2(1-\theta))^{170}$$

$$= 2^{340}\,\theta^{170}(1-\theta)^{170}, \quad \theta \in [0,1]$$

$$r(\theta) = 340\ln(2) + 170\ln(\theta) + 170\ln(1-\theta), \quad \theta \in (0,1).$$

To solve this problem, we find the roots ~~of~~ θ of $R(\theta) - p = 0$ or ~~$R(\theta)$~~ $r(\theta) - \ln(p) = 0$, for admissible values of θ in the interval ~~(xxx)~~ $(0,1)$

Use, uniroot function in R for finding the exact root values (lower and upper) — x —.