Section - 3.1

30th Jan' 2024

MLE's for 2-paramater Likelihoods

In 1-D : find one parameter, say θ.

In k-D : have k parameters ; $\underset{\sim}{\theta} = (\theta_1, \theta_2, \cdots, \theta_k)$

Here, instead of finding $\hat{\theta}$ (the value), we want $\hat{\underset{\sim}{\theta}}$, the joint MLE $\quad \hat{\underset{\sim}{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$

Let's start with $K=2$,

Let's look at the Example 3.1.1, of the Complete Lecture Notes (Pg ~ 48)

↳ Here, $X \sim N(\mu, \delta^2)$ where, μ and δ^2 are unknown. We have to find the joint MLE, $(\hat{\mu}, \hat{\delta^2})$.

The pdf of X : $f(x;\mu,\delta^2) = \frac{1}{\sqrt{2\pi}\,\delta} e^{-\left\{\frac{1}{2\delta^2}(x-\mu)^2\right\}}$, $\delta^2 > 0$

The joint pdf :

$$f(x_1, x_2, \cdots, x_n; \mu, \delta^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\delta} e^{-\left\{\frac{1}{2\delta^2}(x_i-\mu)^2\right\}} \; ; \; \delta^2 > 0$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\delta}\right)^n \prod_{i=1}^{n} e^{-\left\{\frac{1}{2\delta^2}(x_i-\mu)^2\right\}} \; ; \; \delta^2 > 0$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\delta}\right)^n e^{-\left\{\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}} \; ; \; \delta^2 > 0$$

Then, the joint likelihood,

$$L(\mu, \delta^2) = \frac{1}{\delta^n} e^{-\left\{\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}} \quad ; \delta^2 > 0$$

The joint log-likelihood,

$$l(\mu, \delta^2) = \ln\left[\frac{1}{\delta^n} e^{-\left\{\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}}\right]$$

$$= -\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \ln(\delta^n)$$

$$= -\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2 - n\ln(\delta) \quad ; \delta^2 > 0$$

Multivariate Calculus !!!

We need partial Derivatives,

(1) $$\frac{\partial l}{\partial \mu} = \frac{\partial}{\partial \mu}\left[-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$= \frac{1}{\delta^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$(x_1-\mu)^2 + (x_2-\mu)^2 + \cdots + (x_n-\mu)^2$$

$$-2(x_1-\mu) - 2(x_2-\mu) \cdots -2(x_n-\mu)$$

$$= -2\sum_{i=1}^{n}(x_i-\mu)$$

(2) $$\frac{\partial l}{\partial \delta} = \frac{\partial}{\partial \delta}\left[-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] - \frac{\partial}{\partial \delta}\left[n\ln(\delta)\right]$$

$$= -\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 \frac{\partial}{\partial \delta}(\delta^{-2}) - n/\delta$$

$$= \frac{1}{\delta^3}\sum_{i=1}^{n}(x_i-\mu)^2 - n/\delta$$

We know that at the joint MLE $(\hat{\mu}, \hat{\delta}^2)$; (1) = (2) = 0

$\therefore$ From (1), we get : $\frac{1}{\hat{\delta}^2}\left[\sum x_i - n\hat{\mu}\right] = 0$

Solve for $\hat{\mu}$.

$$\Rightarrow \sum_{i=1}^{n} x_i = n\hat{\mu} \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

or, $\boxed{\hat{\mu} = \bar{x}}$ ⟷ Sample mean

Then, from (2) ; we get,

$$\frac{1}{\hat{\delta}^3}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 - \frac{n}{\hat{\delta}} = 0$$

$$\Rightarrow \frac{1}{\cancel{\hat{\delta}^3}\ \hat{\delta}^2}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 = \frac{n}{\cancel{\hat{\delta}}}$$

$$\Rightarrow \hat{\delta}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 \Rightarrow \boxed{\hat{\delta}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

(Putting, $\hat{\mu} = \bar{x}$ here)

Thus, $(\hat{\mu}, \hat{\delta}^2) = \left(\bar{x}, \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$ is the joint MLE of (μ, δ^2)

★ See notes for 2nd derivative test if you are interested.

$$S(\theta) = \nabla l(\theta) \qquad = \left(\frac{\partial l}{\partial \theta_1}, \frac{\partial l}{\partial \theta_2}, \dots, \frac{\partial l}{\partial \theta_k}\right)$$

↑ Gradient Operator