Set 23: Section 6.1.1, Confidence Intervals for Normal small samples

Recap from last day:
<u>Case 0</u>: X is normal, σ^2 is known. $\bar{X} \sim \text{Normal}(\mu, \sigma^2/n)$.

$$\bar{x}_{\text{obs}} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

is a $(1-\alpha)100\%$ CI for μ.

<u>Case 1</u>: σ^2 is known. n is large. By the CLT $\bar{X} \approx \text{Normal}(\mu, \sigma^2/n)$. In this case,

$$\bar{x}_{\text{obs}} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

is an approximate $(1-\alpha)100\%$ CI for μ.

<u>Case 2</u>: σ^2 is unknown. n is large. By the CLT $\bar{X} \approx \text{Normal}(\mu, \sigma^2/n)$. In this realistic case,

$$\bar{x}_{\text{obs}} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

is an approximate $(1-\alpha)100\%$ CI for μ where s is the sample standard deviation.

% Confidence	**Critical Value**
90	$Z_{\alpha/2} = Z_{0.05} = 1.645$
95	$Z_{\alpha/2} = Z_{0.025} = 1.96$
99	$Z_{\alpha/2} = Z_{0.005} = 2.576$
99.9	$Z_{\alpha/2} = Z_{0.0005} = 3.29$

If we are given a desired *width* of the confidence interval, we can solve for the unknown, required n.

Example: At a particular location, fifty daily measurements of windspeed (in m/s) are made. It is found that $\overline{x} = 15.9\ m/s$ and $s = 7.7 m/s$. Find a 99% confidence interval for μ, the average daily windspeed. Assume that the measurements are *iid*.

CI's based on the Student distribution: Suppose $X_1, \ldots, X_n$ are iid $\mathrm{Normal}(\mu, \sigma^2)$ where σ is unknown (the realistic case). It can be shown that

$$\frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

where t_{n-1} denotes the t distribution with $n-1$ degrees of freedom. The pdf of $Y \sim t_{n-1}$ is

$$f(y) = \frac{\Gamma(n/2)}{\Gamma((n-1)/2)\Gamma(1/2)} \left(1 + \frac{y^2}{n-1}\right)^{-n/2}$$

Here, the $(1-\alpha)$100% confidence interval for μ is

$$\bar{X} \pm t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

Discussion points:

- **the t_{n-1} distribution is symmetric on $\mathscr{R}$**
- **the t_{n-1} has longer tails than the normal**
- **as $n \to \infty$, $t_{n-1} \to Z \sim \mathrm{Normal}(0,1)$**
- **for $n \geq 30$, you can replace t_{n-1} with Z**
- **the t distribution is intractable; no need to memorize pdf**
- **Table D pp354-5 in the text gives points $t_{n-1,\frac{\alpha}{2}}$**

Example: The following data is collected on the mass (in grams) of adult white mice.

$$14.6, 13.2, 19.5, 10.1, 8.8, 15.5, 16.1$$

Assuming that the weights of mice are normally distributed, find a 95% confidence interval for μ, the mean weight of adult white mice.

From our data, we find $\overline{x}$ and s:

$$\overline{x} \approx 13.97, s \approx 3.66$$