2nd Feb '2024

☑ Continuing the Fractured Plastics Example (Ex: 2.2.1)

We obtained the joint log-likelihood function,

$$l(\theta,\lambda) = 112\log(1-\lambda\theta) + 88\log(\lambda) + 170\log(\theta) + 58\log(1-\theta) \quad ; \quad \lambda \in (0,1),\ \theta \in (0,1)$$

Also, after doing partial derivations, we got,

$$\frac{\partial}{\partial\theta} l(\theta,\lambda) = \frac{112(-\lambda)}{1-\lambda\theta} + \frac{170}{\theta} - \frac{58}{1-\theta} \quad \cdots\cdots \text{(i)}$$

and,

$$\frac{\partial}{\partial\lambda} l(\theta,\lambda) = \frac{112(-\theta)}{1-\lambda\theta} + \frac{88}{\lambda} \quad \cdots\cdots \text{(ii)}$$

At joint MLE $(\hat{\theta},\hat{\lambda})$, (i) = (ii) = 0,

Therefore, from (ii) we have :

$$-\frac{112\hat{\theta}}{1-\hat{\lambda}\hat{\theta}} + \frac{88}{\hat{\lambda}} = 0$$

Multiply both sides by $\hat{\lambda}(1-\hat{\lambda}\hat{\theta})$

$$\Rightarrow -112\hat{\theta}\hat{\lambda} + 88(1-\hat{\lambda}\hat{\theta}) = 0$$

[Solve for $1-\hat{\lambda}\hat{\theta}$; easy to plug into (i)]

$$\Rightarrow 1-\hat{\lambda}\hat{\theta} = \frac{112\hat{\theta}\hat{\lambda}}{88} \quad (\star)$$

Sub $(\star)$ into (i) :

$$\frac{170}{\hat{\theta}} - \frac{58}{1-\hat{\theta}} - \frac{\cancel{112}\,\cancel{\hat{\lambda}}}{\cancel{112}\,\hat{\theta}\,\cancel{\hat{\lambda}}/88} = 0$$

$$\Rightarrow \frac{170}{\hat{\theta}} - \frac{58}{1-\hat{\theta}} - \frac{88}{\hat{\theta}} = 0$$

$$\Rightarrow \frac{170}{\hat\theta} - \frac{88}{\hat\theta} = \frac{58}{1-\hat\theta}$$

$$\Rightarrow \frac{82}{\hat\theta} = \frac{58}{1-\hat\theta}$$ [Multiply both sides by $\hat\theta(1-\hat\theta)$]

$$\Rightarrow 82(1-\hat\theta) = 58\hat\theta$$

$$\Rightarrow 82 = 82\hat\theta + 58\hat\theta$$

$$\Rightarrow \hat\theta = 82/140 = 41/70$$

$$\Rightarrow \boxed{\hat\theta \approx 0.5857}$$

Now, plug $\hat\theta$ back into $(\star)$ to find $\hat\lambda$

$$\hat\lambda = \frac{88}{112\hat\theta} - \frac{88}{112}\hat\lambda$$

$$\Rightarrow \hat\lambda + \frac{88}{112}\hat\lambda = \frac{88}{112\hat\theta}$$

$$\Rightarrow \hat\lambda\frac{20\cancel{0}}{\cancel{112}} = \frac{88}{\cancel{112}} * \frac{7\cancel{0}}{41}$$

$$\Rightarrow \hat\lambda = \frac{88 * 7}{41 * 20} = 154/205$$

$$\Rightarrow \hat\lambda \approx 0.7512$$

So, our joint MLE is $(\hat\theta, \hat\lambda) = (0.586, 0.751)$

Original value: $\hat\theta = 1/2$

Is this new model a good fit? Check!

	$\hat{E}(X_i)$	observed values
$\hat p_1 = 1-\hat\lambda\hat\theta = 0.56$	$n\hat p_1 = 112$	112
$\hat p_2 = \hat\lambda\hat\theta(1-\hat\theta) = 0.182$	$n\hat p_2 = 36.4$	36
$\hat p_3 = \hat\lambda\hat\theta^2(1-\hat\theta) = 0.107$	$n\hat p_3 = 21.4$	22
$\hat p_4 = \hat\lambda\hat\theta^3 = 0.151$	$n\hat p_4 = 30.2$	30

Much closer in values!

Implies, Good Fit

Section 3.2

Chi-Square Distribution :

Let, X be a Continuous R.V with pdf :-

$$f(x; \vartheta) = \frac{1}{2^{\vartheta/2} \boxed{\Gamma(\vartheta/2)}} x^{\vartheta/2 - 1} e^{-x/2} \quad ; \quad \vartheta \in \mathbb{N}^+, \; x > 0$$

"NU"

Gamma function

$$= C_\vartheta \, x^{\vartheta/2 - 1} e^{-x/2}$$

Then, $X \sim \chi^2_{(\vartheta)}$ "Chi-square distribution" with ϑ degrees of freedom.

Greek letter "Chi"

$$E(x) = \vartheta$$

$$Var(x) = 2\vartheta$$

Properties of $\chi^2_{(\vartheta)}$:-

1) Let, $x_1, x_2, \dots, x_n$ be independent $\chi^2_{(\vartheta)}$ variables,

Where, $i = 1, 2, \dots, n$

Then, $\sum_{i=1}^{n} x_i \sim \chi^2_{(\sum_{i=1}^{n} \vartheta_i)}$ [Sum of independent χ^2 variables is still χ^2]

2) If $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_{(1)}$ [Relationship with Standard Normal Dist]

3) If $Z_1, Z_2, \dots, Z_n \overset{iid}{\sim} N(0,1)$, then $\sum_{i=1}^{n} Z_i^2 \sim \chi^2_{(n)}$

↳ By property (2), $Z_i^2 \sim \chi^2_{(1)}$

By property (1), $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \chi^2_{(1)} =$ [redacted] $\chi^2_{(\sum_{i=1}^{n} 1)} = \chi^2_{(n)}$

End of Chapter 3