Recap :- Relative Likelihood functions

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Section 2.7 :- Invariance property of MLE's.

This is something we are kind of implicitly assumed and used in section 2.2 (Frequency tables). But now we will define it formally.

Invariance Property :

Let $\theta = g(\beta)$ be a 1-1 transformation of β, and, $\hat{\theta}$ be MLE of θ such that $\theta_1 \leq \theta \leq \theta_2$ be a $100p\%$ L.I for θ. Then the MLE of β is

$$\hat{\beta} = g^{-1}(\hat{\theta})$$

and its $100p\%$ LI is $[g^{-1}(\theta_1), g^{-1}(\theta_2)]$, if g is monotone increasing ($g'(\theta) > 0, \forall \theta$)

and, $[g^{-1}(\theta_2), g^{-1}(\theta_1)]$, if g is monotone decreasing ($g'(\theta) < 0, \forall \theta$)

Examples of monotone functions

- $\log_{10}(x)$ is monotone increasing
- $\ln(x)$ " " "
- 10^x and e^x . . . .

☒ The percentile : The $100\alpha^{th}$ percentile of a Continuous R.V., X, is the variate value Q_α satisfying

$$\mathbb{P}(X \leq Q_\alpha) = F(Q_\alpha) = \alpha,$$ where $F(x)$ is the CDF of X.

↳ Median : It is the 50^{th} percentile ($\alpha = 0.5$)

$$0.5 = \mathbb{P}(X \leq Q_{0.5}) = \int_{-\infty}^{Q_{0.5}} f(x;\theta)\, dx.$$

↳ pdf of X

Let's look at an Example,

↳ Example 2.7.1 of the Complete Lecture Notes (pg- 42)

Pareto Distribution pdf:

$$f(x;\theta, x_m) = \frac{\theta\, x_m}{x^{\theta+1}} \; ; \; x \geq x_m, \; \theta > 0$$

Shape (θ)

Scale (x_m)

In this Example, $x_m = 1$, so the pdf is written as,

$$f(x;\theta) = \begin{cases} \theta x^{-(\theta+1)} & ; x \geq 1, \; \theta > 0 \\ 0 & ; \text{otherwise. } (x < 1) \end{cases}$$

a) Here, X = family income,

$X \sim \text{pareto}(\theta)$, $\theta > 0$

$x_1, x_2, \ldots, x_{10}$ iid from the distribution.

Find $\hat{\theta}$.

Let, $\underset{\sim}{x} = (x_1, x_2, \cdots, x_{10})$

Joint pdf : $f(\underset{\sim}{x}; \theta) = \prod_{i=1}^{10} \theta x_i^{-(\theta+1)}$, $\theta > 0$, $x_i \geq 1$ $\forall i = 1(1)10$

$L(\theta) = \theta^{10} \prod_{i=1}^{10} x_i^{-\theta}$, $\theta > 0$, $x_i \geq 1$, $\forall i = 1, 2, \cdots, 10$

$l(\theta) = \log\left[\theta^{10} \prod_{i=1}^{10} x_i^{-\theta}\right]$

$\boxed{\begin{array}{l} \log(AB) = \log(A) + \log(B) \\ \log(A_1, A_2, \cdots, A_k) = \sum_{i=1}^{k} \log(A_i) \end{array}}$

Log Properties

$= 10 \log(\theta) - \theta \sum_{i=1}^{10} \log(x_i)$, $\theta > 0$, $x_i \geq 1$.

Then, $S(\theta) = l'(\theta)$

$= \frac{10}{\theta} - \sum_{i=1}^{10} \log(x_i)$

Evaluate $l'(\hat{\theta}) = 0 \Rightarrow \frac{10}{\hat{\theta}} = \sum_{i=1}^{10} \log(x_i)$

$\Rightarrow \hat{\theta} = 10 / \sum_{i=1}^{10} \log(x_i)$

From the data, $\hat{\theta} = 10/19.154 \approx 0.522$.

Now, $l''(\theta) = -10/\theta^2$, $\theta > 0$

$I(\theta) = 10/\theta^2$, $\theta > 0$

|| Since, $I(\theta) > 0$, $\forall \theta > 0$
$\therefore$ $\hat{\theta}$ is maximum

$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})} = \frac{\prod_{i=1}^{10} \theta x_i^{-\theta}}{\prod_{i=1}^{10} 0.522\, x_i^{-1.522}} = \left(\frac{\theta}{0.522}\right)^{10} \prod_{i=1}^{10} x_i^{1.522-\theta}$

10% L.I : Solve $R(\theta) - 0.1 = 0$

$\Rightarrow$ Using R, we find 10% L.I :

$(0.24, 0.96)$

— x —