Example 5.6

In the insurance problem of Example 5.1, $p(100, 100) = .10$, while $p_X(100) = .5$ and $p_Y(100) = .25$. Since $.10 \neq (.5)(.25)$, X and Y are not independent. Independence of X and Y requires that *every* entry in the joint probability table be the product of the corresponding row and column marginal probabilities. ■

Example 5.7 *(Example 5.5 continued)*

Because $f(x, y)$ has the form of a product, X and Y would appear to be independent. However, while $f_X(\frac{3}{4}) = f_Y(\frac{3}{4}) = \frac{9}{16}$, $f(\frac{3}{4}, \frac{3}{4}) = 0 \neq \frac{9}{16} \cdot \frac{9}{16}$, so the variables are not in fact independent. To be independent, $f(x, y)$ must have the form $g(x) \cdot h(y)$ *and* the region of positive density must be a rectangle whose sides are parallel to the coordinate axes. ■

Independence of two random variables is most useful when the description of the experiment under study tells us that X and Y have no effect on one another. Then once the marginal pmf's or pdf's have been specified, the joint pmf or pdf is simply the product of the two marginal functions. It follows that

$$P(a \leq X \leq b,\ c \leq Y \leq d) = P(a \leq X \leq b) \cdot P(c \leq Y \leq d)$$

Example 5.8

Suppose that the lifetimes of two components are independent of one another and that the first lifetime X_1 has an exponential distribution with parameter λ_1 while the second, X_2, has an exponential distribution with parameter λ_2. Then the joint pdf is

$$f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

$$= \begin{cases} \lambda_1 e^{-\lambda_1 x_1} \cdot \lambda_2 e^{-\lambda_2 x_2} = \lambda_1 \lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2} & x_1 > 0,\ x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $\lambda_1 = 1/1000$ and $\lambda_2 = 1/1200$, so that the expected lifetimes are 1000 hours and 1200 hours, respectively. The probability that both component lifetimes are at least 1500 hours is

$$\begin{aligned} P(1500 \leq X_1,\ 1500 \leq X_2) &= P(1500 \leq X_1) \cdot P(1500 \leq X_2) \\ &= e^{-\lambda_1(1500)} \cdot e^{-\lambda_2(1500)} \\ &= (.2231)(.2865) = .0639 \end{aligned}$$

■

More Than Two Random Variables

To model the joint behavior of more than two random variables, we extend the concept of a joint distribution of two variables.

Definition

If $X_1, X_2, \ldots, X_n$ are all discrete rv's, the joint pmf of the variables is the function

$$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

If the variables are continuous, the joint pdf of $X_1, \ldots, X_n$ is the function $f(x_1, x_2, \ldots, x_n)$ such that for any n intervals $[a_1, b_1], \ldots, [a_n, b_n]$,

$$P(a_1 \le X_1 \le b_1, \ldots, a_n \le X_n \le b_n)$$
$$= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \ldots dx_1$$

Example 5.9 In a binomial experiment, each trial could result in one of only two possible outcomes. Consider now an experiment consisting of n independent and identical trials, in which each trial can result in any one of r possible outcomes. Let $p_i = P(\text{outcome } i \text{ on any particular trial})$ and define random variables by X_i = the number of trials resulting in outcome i $(i = 1, \ldots, r)$. Such an experiment is called a multinomial experiment, and the joint pmf of $X_1, \ldots, X_r$ is called the multinomial distribution. By using a counting argument analogous to the one used in deriving the binomial distribution, the joint pmf of $X_1, \ldots, X_r$ can be shown to be

$$p(x_1, \ldots, x_r)$$

$$= \begin{cases} \dfrac{n!}{(x_1!)(x_2!) \cdot \ldots \cdot (x_r!)} p_1^{x_1} \cdot \cdots \cdot p_r^{x_r} & x_i = 0, 1, 2, \ldots, \text{ with } x_1 + \cdots + x_r = n \\ 0 & \text{otherwise} \end{cases}$$

As an example, if the allele of each of ten independently obtained pea sections is determined and $p_1 = P(AA)$, $p_2 = P(Aa)$, $p_3 = P(aa)$, X_1 = number of AA's, X_2 = number of Aa's, and X_3 = number of aa's, then

$$p(x_1, x_2, x_3) = \frac{10!}{(x_1!)(x_2!)(x_3!)} p_1^{x_1} p_2^{x_2} p_3^{x_3}, \qquad \begin{array}{l} x_i = 0, 1, \ldots \quad \text{and} \\ x_1 + x_2 + x_3 = 10 \end{array}$$

If $p_1 = p_3 = .25$, $p_2 = .5$, then

$$P(X_1 = 2, X_2 = 5, X_3 = 3) = p(2, 5, 3)$$
$$= \frac{10!}{2!\,5!\,3!}(.25)^2(.5)^5(.25)^3 = .0769$$

The case $r = 2$ gives the binomial distribution, with X_1 = number of successes and $X_2 = n - X_1$ = number of failures. ■

Example 5.10 When a certain method is used to collect a fixed volume of rock samples in a region, there are four resulting rock types. Let X_1, X_2, and X_3 denote the proportion by volume of rock types 1, 2, and 3 in a randomly selected sample (the proportion of rock type 4 is $1 - X_1 - X_2 - X_3$, so a variable X_4 would be redundant). If the joint pdf of X_1, X_2, X_3 is

$$f(x_1, x_2, x_3) = \begin{cases} kx_1x_2(1-x_3) & 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1,\ x_1 + x_2 + x_3 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

then k is determined by

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_3\, dx_2\, dx_1$$
$$= \int_0^1 \left\{ \int_0^{1-x_1} \left[\int_0^{1-x_1-x_2} kx_1x_2(1-x_3)\, dx_3 \right] dx_2 \right\} dx_1$$

This iterated integral has value $k/144$, so $k = 144$. The probability that rocks of type 1 and 2 together account for at most 50% of the sample is

$$P(X_1 + X_2 \le .5) = \iiint\limits_{\substack{0 \le x_i \le 1 \text{ for } i = 1, 2, 3 \\ x_1 + x_2 + x_3 \le 1,\ x_1 + x_2 \le .5}} f(x_1, x_2, x_3)\, dx_3\, dx_2\, dx_1$$
$$= \int_0^{.5} \left\{ \int_0^{.5-x_1} \left[\int_0^{1-x_1-x_2} 144x_1x_2(1-x_3)\, dx_3 \right] dx_2 \right\} dx_1$$
$$= .6066$$

The notion of independence of more than two random variables is similar to the notion of independence of more than two events.

Definition

> The rv's $X_1, X_2, \ldots, X_n$ are said to be **independent** if for *every* subset $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

Thus, if the variables are independent with $n = 4$, then the joint pmf or pdf of any two variables is the product of the two marginals, and similarly for any three variables and all four variables together. Most important, once we are told that n variables are independent, then the joint pmf or pdf is the product of the n marginals.

Example 5.11 If $X_1, \ldots, X_n$ represent the lifetimes of n components, the components operate independently of one another, and each lifetime is exponentially distributed with parameter λ, then

$$f(x_1, x_2, \ldots, x_n) = (\lambda e^{-\lambda x_1}) \cdot (\lambda e^{-\lambda x_2}) \cdot \cdots \cdot (\lambda e^{-\lambda x_n})$$
$$= \begin{cases} \lambda^n e^{-\lambda \Sigma x_i} & x_1 \ge 0,\ x_2 \ge 0, \ldots, x_n \ge 0 \\ 0 & \text{otherwise} \end{cases}$$