1. (10 marks) During WWII, London was hit by numerous flying bombs. The following data are from an area in South London of 48 km^2. The area was divided into 512 squares with sides of length 1/4 km. The number of hits was recorded for each square. That is, the dataset $x_1, x_2, \ldots, x_{512}$ is obtained, where x_i denotes the number of hits in the i^{th} square. The following table summarizes the number of squares that had no hits, 1 hit, 2 hits, etc.

Number of hits	0	1	2	3	4	5	6	7
Number of squares	207	188	103	8	3	2	0	1

Source: R.D. Clarke. An Application of the Poisson distribution. *Journal of the Institute of Actuaries*, 72:48, 1946; Table 1 on page 481.

If London was hit in a completely random manner, a Poisson distribution should fit the data. Suppose we model the dataset as a realization of a random sample from a Poisson(λ) distribution, where λ describes the average number of hits per square.

(a) (1 mark) Write down the joint probability mass function (pmf) for $X_1, \ldots, X_{512}$, and simplify this expression. Use this joint distribution for the remainder of the question.

(b) (1 mark) Write down the Likelihood function $L(\lambda)$.

(c) (1 mark) Write down the Log-likelihood function $\ell(\lambda)$. Simplify your expression.

(d) (1 mark) Write down the Score function $S(\lambda)$.

(e) (2 marks) Derive an expression for the maximum likelihood estimate (MLE) of λ. Then, use the data to find the value of the MLE ($\hat{\lambda}$).

(f) (2 marks) Write down the Information function $I(\lambda)$ and use the second derivative test to show how that you have found a maximum. (Hint: You may need to find $I(\hat{\lambda})$).

(g) (2 marks) Compare the observed relative frequencies with the corresponding probabilities for the Poisson distribution with λ estimated as in part (e). Comment on the results.

(Hint: a relative frequency of an event is its frequency/total observations i.e., f_i/n)