Assignment 1

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1. (4 points) In November of each year, a walk-in clinic allows people to walk in to get a flu shot. Let X be the number of people who come to the clinic for a flu shot on a randomly selected day (in November). Suppose X has the following distribution:

X	0	1	2	3	4	5
P(X = x)	0.3	0.1	0.2	0.1	0.1	0.2

If at least 3 people walk in for a flu shot on a particular day, what is the probability that there are 4 or fewer who walk in for a flu shot that day?

$$P(X \leq 4 | X \geq 3) = \frac{P((X \leq 4) \cap (X \geq 3))}{P(X \geq 3)}$$
$$= \frac{0.2}{0.4}$$

$$= 0.5$$

2. (6 points) Daily sales at a gas station are thought to be independent of one another with daily mean \$5000 and standard deviation \$700. Approximate the probability that the average daily sales over one year (i.e. 365 days) is greater than \$5,100.

With $n \geq 30$ it is appropriate to use the Normal distribution:

$$X \sim Normal(\mu = 5000, \sigma = 700)$$

$$P(X > 5100) = 1 - P(X \leq 5100)$$

$$= 1 - P(\frac{X - \mu}{\sigma} \leq \frac{5100 - 5000}{700}) = 1 - P(Z \leq 0.142857162)$$

$$\simeq 1 - P(Z \leq 0.14)$$
$$= 1 - 0.5517$$

$$= 0.4483$$

3. (10 points) Suppose that diseased trees are distributed randomly and uniformly throughout a large forest with an average of λ per acre. Let X denote the number of diseased trees in a randomly chosen one-acre forest plot where the range of X is, X = {0, 1, 2, . . . }.

(a) We will use the Poisson distribution with mean λ to model X. Write down the probability mass function (p.m.f.) for X. Why would this distribution be suitable for modelling the number of diseased trees in a randomly chosen one-acre plot of forest?

The Poisson probability mass function is given as follows:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

The poisson distribution models the random occurance of events in any fixed interval, including time and space. In this case, we know the average number of diseased trees λ per one-acre interval, and that each interval is independant.

(b) Suppose that we observe the number of diseased trees on n randomly chosen one-acre parcels, X1, X2, ..., Xn. The random variables X1, X2, ..., Xn can be assumed to be independent. Write down the JOINT probability mass function for X1, X2, ..., Xn. Simplify this expression which is a function of λ and the X'

s. Use this joint distribution for the remainder of Question 1.

$$P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = (\frac{\lambda^{x_1} e^{-\lambda}}{x_1!})(\frac{\lambda^{x_2} e^{-\lambda}}{x_2!})(...)(\frac{\lambda^{x_n} e^{-\lambda}}{x_n!})$$
$$= \frac{\lambda^{\sum i=1^n x_i} e^{-n\lambda}}{\prod_{i=1}^n x_i!}$$

(c) We are going to use the Method of Maximum Likelihood to estimate λ. Write down the Likelihood function L(λ).

The Likelihood function omits multiplicative constants to simplify and focus on our desired parameter λ

$$L(\lambda) = cp(x_i, \lambda), c = \prod_{i=1}^n x_i!$$
$$= L(\lambda) = e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}$$

(d) Write down the Log-likelihood function l(λ).

$$l(\lambda) = -n\lambda + (\sum_{i=1}^n x_i) ln\lambda$$

(e) Write down the Score Function S(λ).

$$S(\lambda) = l'(\lambda) = -n + \frac{\sum_{i=1}^n x_i}{\lambda}$$

(f) Derive the maximum likelihood estimate of λ.

The maximum likelihood is when the previous equality is set to zero and we solve for λ

$$0 = -n + \frac{\sum_{i=1}^n x_i}{\lambda}$$

Solving, we get:

$$\lambda = \frac{1}{n} \sum_{i=1}^n x_i$$

(g) Write down the Information Function I(λ).

$$I(\lambda) = -l''(\lambda) = -(\frac{-\sum_{i=1}^n x_i}{\lambda^2}) = \frac{\sum_{i=1}^n x_i}{\lambda^2}$$

(h) Use the second derivative test to show that you have found a maximum.

$$l''(\lambda) = \frac{-\sum_{i=1}^{n} x_i}{\lambda^2}$$

$l''(\lambda)$ is always negative as n goes to ∞ implies that we have found a maximum.

(i) Suppose that the numbers of diseased trees observed in n = 4 randomly chosen one-acre parcels were: 5, 8, 9, 2. Compute the maximum likelihood estimate of λ using this data.

$$\lambda = \frac{1}{4}\sum_{i=1}^{4} x_i = \frac{1}{4}(5+8+9+2) = \frac{1}{4}(24)$$

$$\lambda = 6$$