Sets 17: Section 5.3, The Gamma and Exponential Distribution

Definition: A rv X has a $\mathrm{Gamma}(\alpha, \beta)$ distribution, $\alpha > 0$, $\beta > 0$, if it has pdf

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \qquad x > 0$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$

Notes:

- **closed form integral only for special cases of** α, β
- **contrast the range (**$x > 0$**) with the normal**
- **asymmetric**
- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$**,** $\Gamma(1) = 1$**,** $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(n) = (n-1)!$ **for positive integer n**

Proposition: If $X \sim \mathrm{Gamma}(\alpha, \beta)$**, then**

- $\mathrm{E}(X) = \alpha\beta$
- $\mathrm{Var}(X) = \alpha\beta^2$

The Exponential(λ) **distribution is a special case of the** Gamma(α, β) **where** $\alpha = 1$ **and** $\beta = 1/\lambda$**.**

Definition: A rv X **has an** Exponential(λ) **distribution,** $\lambda > 0$**, if it has pdf**

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0$$

Notes:

- $\mathrm{E}(X) = \alpha\beta = 1(1/\lambda) = 1/\lambda$
- $\mathrm{Var}(X) = \alpha\beta^2 = 1(1/\lambda)^2 = 1/\lambda^2$
- **cdf** $F(x) = 1 - e^{-\lambda x}$ **for** $x > 0$

The *memoryless* property: Let X **be the lifespan of a lightbulb in hours and** $X \sim$ Exponential(λ)**. The probability that a used lightbulb (that has already lasted** a **hours) will last an additional** b **hours is given by:**

$\mathrm{P}(X > a + b \mid X > a) =$

Problem: Let X **be the lifespan of a lightbulb in hours.** X **has an exponential distribution with** $\lambda = 0.01$**.**

(a) What is the probability that X **is at most 100 hours?**

(b) What is the probability that X **exceeds the mean lifespan by more than two standard deviations?**

(c) What is the median lifespan?

Relationship between Poisson and Exponential dist'ns:
Let $X_T \sim \text{Poisson}(\lambda T)$ **be the number of events in T time units,** λ **the average rate and**

$$Y \equiv \textbf{waiting time until the first event}$$

Then the cdf of Y **is given by**

$$\begin{aligned}
\text{P}(Y \leq y) &= 1 - \text{P}(Y > y) \\
&= 1 - \text{P}(\textbf{zero events in [0,y]}) \\
&= 1 - \text{P}(X_y = 0) \quad \text{ where } X_y \sim \text{Poisson}(\lambda y) \\
&= 1 - (\lambda y)^0 e^{-\lambda y}/0! \\
&= 1 - e^{-\lambda y}
\end{aligned}$$

which implies $Y \sim \text{Exponential}(\lambda)$