4.4 The Gamma Distribution and Its Relatives

The graph of any normal pdf is bell-shaped and thus symmetric. There are many practical situations in which the variable of interest to the experimenter might have a skewed distribution. A family of pdf's that yields a wide variety of skewed distributional shapes is the gamma family. To define the family of gamma distributions, we first need to introduce a function that plays an important role in many branches of mathematics.

Definition

For $\alpha > 0$, the **gamma function** $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x}\, dx \tag{4.5}$$

The most important properties of the gamma function are the following:

1. For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$ [via integration by parts]
2. For any positive integer n, $\Gamma(n) = (n - 1)!$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

By Expression (4.5) if we let

$$f(x;\, \alpha) = \begin{cases} \dfrac{x^{\alpha - 1} e^{-x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

then $f(x;\alpha) \geq 0$ and $\int_0^\infty f(x;\, \alpha)\, dx = \Gamma(\alpha)/\Gamma(\alpha) = 1$, so $f(x;\, \alpha)$ satisfies the two basic properties of a pdf.

The Family of Gamma Distributions

Definition

A continuous rv X is said to have a **gamma distribution** if the pdf of X is

$$f(x;\, \alpha, \beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha - 1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

where the parameters α and β satisfy $\alpha > 0$, $\beta > 0$. The **standard gamma distribution** has $\beta = 1$, so the pdf of a standard gamma rv is given by (4.6).

Figure 4.26a illustrates the graphs of the gamma pdf $f(x;\, \alpha, \beta)$ (4.7) for several (α, β) pairs, while Figure 4.26b presents graphs of the standard gamma pdf. For the standard pdf, when $\alpha \leq 1$, $f(x;\, \alpha)$ is strictly decreasing as x increases from 0; when $\alpha > 1$, $f(x;\, \alpha)$ rises from 0 at $x = 0$ to a maximum and then decreases. The parameter

β in (4.7) is called the *scale parameter* because values other than 1 either stretch or compress the pdf in the x direction.

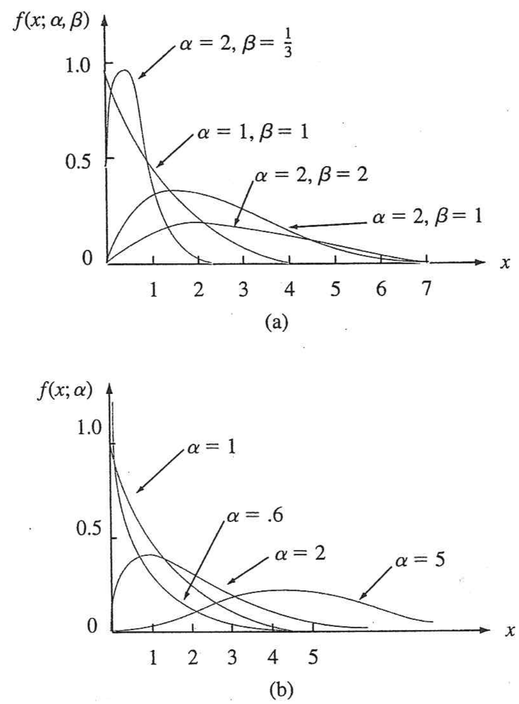

Figure 4.26 (a) Gamma density functions; (b) standard gamma density functions

$E(X)$ and $E(X^2)$ can be obtained from a reasonably straightforward integration, and then $V(X) = E(X^2) - [E(X)]^2$.

Proposition

> The mean and variance of an rv X having the gamma distribution $f(x;\ \alpha,\ \beta)$ are
>
> $E(X) = \mu = \alpha\beta, \quad V(X) = \sigma^2 = \alpha\beta^2$

Computing Probabilities from the Gamma Distribution

When X is a standard gamma rv, the cdf of X,

$$F(x;\ \alpha) = \int_0^x \frac{y^{\alpha - 1}e^{-y}}{\Gamma(\alpha)}\, dy, \qquad x > 0 \tag{4.8}$$

is called the **incomplete gamma function** [sometimes the incomplete gamma function refers to Expression (4.8) without the denominator $\Gamma(\alpha)$ in the integrand].

There are extensive tables of $F(x; \alpha)$ available; in Appendix Table A.4, we present a small tabulation for $\alpha = 1, 2, \ldots, 10$ and $x = 1, 2, \ldots, 15$.

Example 4.21 Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with $\alpha = 2$ sec. Since

$$P(a \le X \le b) = F(b) - F(a)$$

when X is continuous,

$$P(3 \le X \le 5) = F(5; 2) - F(3; 2) = .960 - .801 = .159$$

The probability that the reaction time is more than 4 sec is

$$P(X > 4) = 1 - P(X \le 4) = 1 - F(4; 2) = 1 - .908 = .092$$

■

The incomplete gamma function can also be used to compute probabilities involving nonstandard gamma distributions.

Proposition

Let X have a gamma distribution with parameters α and β. Then for any $x > 0$, the cdf of X is given by

$$P(X \le x) = F(x; \alpha, \beta) = F\left(\frac{x}{\beta}; \alpha\right)$$

where $F(\cdot\,; \alpha)$ is the incomplete gamma function.*

Example 4.22 Suppose the survival time X in weeks of a randomly selected male mouse exposed to 240 rads of gamma radiation has a gamma distribution with $\alpha = 8$ and $\beta = 15$ (data in *Survival Distributions: Reliability Applications in the Biomedical Services,* by A. J. Gross and V. Clark, suggests $\alpha \approx 8.5$ and $\beta \approx 13.3$). The expected survival time is $E(X) = (8)(15) = 120$ weeks, while $V(X) = (8)(15)^2 = 1800$ and $\sigma_X = \sqrt{1800} = 42.43$ weeks. The probability that a mouse survives between 60 and 120 weeks is

$$\begin{aligned} P(60 \le X \le 120) &= P(X \le 120) - P(X \le 60) \\ &= F(120/15; 8) - F(60/15; 8) \\ &= F(8; 8) - F(4; 8) = .547 - .051 = .496 \end{aligned}$$

The probability that a mouse survives at least 30 weeks is

$$\begin{aligned} P(X \ge 30) &= 1 - P(X < 30) = 1 - P(X \le 30) \\ &= 1 - F(30/15; 8) = .999 \end{aligned}$$

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*MINITAB and other statistical packages will calculate $F(x; \alpha, \beta)$ once values of x, α, and β have been specified.

The Exponential Distribution

The family of exponential distributions provides probability models that are very widely used in engineering and science disciplines.

Definition

> X is said to have an **exponential distribution** if the pdf of X is
>
> $$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } \lambda > 0 \qquad (4.9)$$

The exponential pdf is a special case of the general gamma pdf Expression (4.7) in which $\alpha = 1$ and β has been replaced by $1/\lambda$ [some authors use the form $(1/\beta)e^{-x/\beta}$]. The mean and variance of X are then

$$\mu = \alpha\beta = \frac{1}{\lambda}, \qquad \sigma^2 = \alpha\beta^2 = \frac{1}{\lambda^2}$$

Both the mean and standard deviation of the exponential distribution equal $1/\lambda$. Graphs of several exponential pdf's appear in Figure 4.27.

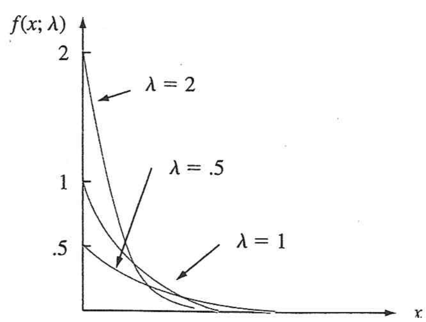

Figure 4.27 Exponential density functions

Unlike the general gamma pdf, the exponential pdf can be easily integrated. In particular, the cdf of X is

$$F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

Example 4.23 Suppose the response time X at a certain on-line computer terminal (the elapsed time between the end of a user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected response time equal to 5 sec. Then $E(X) = 1/\lambda = 5$, so $\lambda = .2$. The probability that the response time is at most 10 sec is

$$P(X \leq 10) = F(10; .2) = 1 - e^{-(.2)(10)} = 1 - e^{-2} = 1 - .135 = .865$$

The probability that response time is between 5 and 10 sec is

$$\begin{aligned} P(5 \leq X \leq 10) &= F(10; .2) - F(5; .2) \\ &= (1 - e^{-2}) - (1 - e^{-1}) = .233 \end{aligned}$$

■

Applications of the Exponential Distribution

The exponential distribution is frequently used as a model for the distribution of times between the occurrence of successive events such as customers arriving at a service facility or calls coming in to a switchboard. The reason for this is that the exponential distribution is closely related to the Poisson process discussed in Chapter 3.

Proposition

Suppose the number of events occurring in any time interval of length t has a Poisson distribution with parameter αt (where α, the rate of the event process, is the expected number of events occurring in 1 unit of time) and that numbers of occurrences in nonoverlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$.

While a complete proof is beyond the scope of the text, the result is easily verified for the time X_1 until the first event occurs:

$$\begin{aligned} P(X_1 \leq t) &= 1 - P(X_1 > t) = 1 - P[\text{no events in } (0, t)] \\ &= 1 - \frac{e^{-\alpha t} \cdot (\alpha t)^0}{0!} = 1 - e^{-\alpha t} \end{aligned}$$

which is exactly the cdf of the exponential distribution.

Example 4.24 Suppose calls are received at a 24-hour "suicide hotline" according to a Poisson process with rate $\alpha = .5$ calls per day. Then the number of days X between successive calls has an exponential distribution with parameter value .5, so the probability that more than 2 days elapse between calls is

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F(2; .5) = e^{-(.5)(2)} = .368$$

The expected time between successive calls is $1/.5 = 2$ days. ■

Another important application of the exponential distribution is to model the distribution of component lifetime. A partial reason for the popularity of such applications is the **"memoryless" property** of the exponential distribution. Suppose component lifetime is exponentially distributed with parameter λ. After putting the component into service, we leave for a period of t_0 hours and then return to find the component still working; what now is the probability that it lasts at least an additional t hours? In symbols we wish $P(X \geq t + t_0 \mid X \geq t_0)$. By the definition of conditional probability,

$$P(X \geq t + t_0 \mid X \geq t_0) = \frac{P[(X \geq t + t_0) \cap (X \geq t_0)]}{P(X \geq t_0)}$$

But the event $X \geq t_0$ in the numerator is redundant, since both events can occur if and only if $X \geq t + t_0$. Therefore,

$$P(X \geq t + t_0 \mid X \geq t_0) = \frac{P(X \geq t + t_0)}{P(X \geq t_0)} = \frac{1 - F(t + t_0; \lambda)}{1 - F(t_0; \lambda)} = e^{-\lambda t}$$

This conditional probability is identical to the original probability $P(X \geq t)$ that the component lasted t hours. Thus, *the distribution of additional lifetime is exactly the same as the original distribution of lifetime,* so at each point in time the component shows no effect of wear. A way to paraphrase this is to say that the distribution of remaining lifetime is independent of current age.

While the memoryless property can be justified at least approximately in many applied problems, in other situations components deteriorate with age or occasionally improve with age (at least up to a certain point). More general lifetime models are then furnished by the gamma, Weibull, or lognormal distributions (the latter two are discussed in the next section).

The Chi-Squared Distribution

Definition

Let v be a positive integer. Then an rv X is said to have a **chi-squared distribution** with parameter v if the pdf of X is the gamma density with $\alpha = v/2$ and $\beta = 2$. The pdf of a chi-squared rv is thus

$$f(x; v) = \begin{cases} \dfrac{1}{2^{v/2}\Gamma(v/2)} x^{(v/2)-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (4.10)$$

The parameter v is called the **number of degrees of freedom** (df) of X. The symbol χ^2 is often used in place of "chi-squared."

The chi-squared distribution is important because it is the basis for a number of procedures in statistical inference. The reason for this is that chi-squared distributions are intimately related to normal distributions (see Exercise 65). We will discuss the chi-squared distribution in more detail in the chapters on inference.