Solution of the treat question from Jan 16 | 17th Jan, 2024

[*] Show that $\log(x)$ is monotonically increasing.

↳ The derivative of $\log_a(x)$ (where the log is base a) is $1/x\ln(a)$. Note, $\log(x)$ is only defined when $0 < x < \infty$ (i.e Domain: $\{x \mid 0 < x < \infty\}$), so we never have to worry about the "$1/x$" part of $\frac{1}{x\ln(a)}$ being negative or zero.

The "$1/\ln(a)$" part is a constant, when $a > 1$, $1/\ln(a)$ is greater than 1, so the derivative is always greater than 1 and the function is monotonic increasing.

However, when $0 < a < 1$, $\frac{1}{\ln(a)}$ is negative, so the derivative will always be negative, meaning that the function is monotonic decreasing.

Recap :- $L(\theta)$ and $\hat{\theta}$ for an observed value of R.V., X.

Today :- $L(\theta)$ and $\hat{\theta}$ for more than one observation $x_1, x_2, \ldots, x_n$.

Independent Events and Random Samples [Section 2.4/2.6]

↳ Suppose we have $x_1, x_2, \ldots, x_n$ independent ~~samples~~ observations from a given distribution i.e., these are realizations of a random sample $X_1, X_2, \ldots, X_n$ $\underset{\smile}{iid}$ some distribution with a given pdf/pmf, $f(x;\theta)$.

Joint pmf/pdf is :

$$f(x_1, x_2, \ldots, x_n; \theta) = f(x_1;\theta) * f(x_2;\theta) * \cdots * f(x_n;\theta)$$

$$= \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \prod_{i=1}^{n} C_i \, P_X(x_i;\theta)$$

Likelihood function :

$$L(\theta; \underset{\sim}{x}) = \prod_{i=1}^{n} P_X(x_i;\theta)$$

For Example :- We observe $n = 10$ ~~observation~~ basketball players at tryout. They take shots from the free throw line, until they make one. Assume the player's performances are independent.

Player	0	1	2	3	4	5	6	7	8	9	10
#misses		0	6	2	2	0	5	1	5	4	0

What is the maximized estimated probability of making a shot, $\hat{\theta}$, among this cohort? Would you put them on the team?

↳ Let, X = # of missed shots before the point is scored.

$$X_1, X_2, \cdots, X_{10} \overset{iid}{\sim} \text{geometric}(\theta)$$

$$f(\underset{\sim}{x};\theta) = \prod_{i=1}^{10} \theta(1-\theta)^{x_i} \; ; \; x_i \in \mathbb{N}, \; \theta \in [0,1]$$

or, $x_i \in \{0, 1, \cdots\}$

$$L(\theta) = \prod_{i=1}^{10} \theta(1-\theta)^{x_i} \; ; \; x_i \in \mathbb{N}, \; \theta \in [0,1]$$

No constants to take out!

$$\begin{aligned} \ell(\theta) &= \log\left[\prod_{i=1}^{10} \theta(1-\theta)^{x_i}\right] \\ &= \log(\theta^{10}) + \sum_{i=1}^{10} x_i \log(1-\theta) \\ &= 10\log(\theta) + \sum_{i=1}^{10} x_i \log(1-\theta) \; ; \; \theta \in (0,1) \end{aligned}$$

To find the MLE, we take the derivative:

$$\ell'(\theta) = \frac{10}{\theta} - \sum_{i=1}^{10} x_i \Big/ 1-\theta \; ; \; \theta \in (0,1)$$

$$\ell'(\hat{\theta}) = 0 \implies \frac{10}{\hat{\theta}} - \frac{\sum_{i=1}^{10} x_i}{1-\hat{\theta}} = 0$$

(Because, we know the MLE is the root of the derivative)

$$\implies \hat{\theta} = \frac{10}{10 + \sum_{i=1}^{10} x_i} = \frac{1}{1 + \frac{\sum_{i=1}^{10} x_i}{10}}$$

$$\implies \boxed{\hat{\theta} = \frac{1}{1+\bar{x}}}$$

★ Do 2nd derivative test to check that it is the maximizer

So, using our data, we find that $\sum_{i=1}^{10} x_i = 25$

and $\therefore$ $\bar{x} = 2.5$

Hence, $\hat{\theta} = \frac{1}{3.5} \approx 0.29$

Since the question posed to you was "in words", don't forget to write an answer in words, too

The maximized estimated probability of making a shot from the free throw line is 0.29, based on this group.

Would you put them on the team? Why?

↳ Given that $\hat{\theta} \approx 0.29$, we can estimate the expected # of missed shots. From the Geometric distn, we can find that $E(X) = \frac{1-\theta}{\theta}$

Hence, $\hat{E}(X) = \frac{1-\hat{\theta}}{\hat{\theta}} = \frac{1-1/3.5}{1/3.5} = 2.5$

Hence, anyone with a lower # of missed shots than 2.5, will get added to the team, because that means they are able to score a point in less tries than average.

So, players 1, 3, 4, 5, 7, and 10 made the team.