Set 7: Independence, Section 3.4.1

Example: 30% of all households have an annual income above $40,000. Of these households, 15% use credit cards for groceries. Of households with an annual income below $40,000, 20% use credit cards for groceries. Suppose that a randomly selected household uses credit cards for grocery purchases. What is the probability that this household has an annual income above $40,000?

Independence: In some cases, knowing the outcome of one event does not provide any information for a second event, or affect the probability of the second event occurring.

Example: Roll two dice. A={first roll 5}, and B={second roll 4}. Suppose I know that A has occurred. Does this change the likelihood that B will occur?

Since the occurrences of A and B do not affect each other, we say that A and B are *independent events*.

Two events are independent if the occurrence or nonoccurrence of one event does not affect the probability of the other event.

Formally, and this is how you are required to prove independence, events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Notes:

- A **and** B **are independent events if** $P(B|A) = P(B)$**.**

- **If** $P(B|A) = P(B)$**, then** $P(A|B) = P(A)$**.**

- **if** A **and** B **are independent events, then** A **and** $\overline{B}$ **are independent,** $\overline{A}$ **and** B **are independent, and also** $\overline{A}$ **and** $\overline{B}$ **are independent.**

Example: To test a new migraine remedy, 100 people were selected. Some were given the remedy, and the others were given nothing. After two hours, all 100 people were asked whether or not they felt better. The results are summarized in the following table.

	Felt better	**Did not feel better**
Remedy	**6**	**24**
Nothing	**14**	**56**

Suppose A **is the event that a randomly selected person took the remedy, and** B **is the event that a randomly selected person felt better. Are** A **and** B **independent events?**

We say that a collection of events are mutually independent if for every subset of this collection, $A_1, A_2, \ldots, A_k$, we have

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\ldots P(A_k)$$

Much like conditional events, we can use tree diagrams to model problems where the events are independent in the same way as we did for conditional events.

Example: For a short multiple-choice quiz to answer. Question 1 has 3 possible responses. Question 2 has 5 possible responses. Question 3 has 6 possible responses. Suppose I answer each question by randomly selecting a response. What is the probability that I will get at least two questions correct?

Example: If it is known that at least two of the three questions are correct, what is the probability that all three questions are correct?

Example: What is the probability of getting question two and three correct?

Example: Suppose that $P(A) = 0.5$, $P(B) = 0.4$, and that A and B are independent events. What is $P(A \cup B)$?

Example: A boiler has five identical relief valves. The probability that any particular valve will open on demand is 0.95. Assuming independent operation of the valves, find the probability that at least one valve opens, and the probability that at least one valve fails to open.

Basic combinatorial results:

Proposition: The number of *permutations* of n distinct objects is $n! = n(n-1)(n-2)\cdots 1$

Example: We can permute symbols A, B and C in $3! = 6$ **ways.**

Definition: $0! = 1$**.**

Proposition: The number of permutations of r objects chosen from n distinct objects is
$n^{(r)} = n!/(n-r)!$

Example: We can permute two of the symbols A, B, C, D and E in $5^{(2)} = 5!/(5-2)! = 120/6 = 20$ **ways.**

Proposition: The number of *combinations* of r objects chosen from n distinct objects is

$$\binom{n}{r} = \frac{n^{(r)}}{r!} = \frac{n!}{r!(n-r)!}$$

Example: We can choose two of the symbols A, B, C, D and E in $\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$ **ways.**