15th March '2024

Continuing with One-Sample Models. ($N(\mu, \sigma^2)$ data)

Example 6.2.2 (Continued) .....

$\hookrightarrow$ $H_0 : \mu = 3000$

$$D = n \ln\left[1 + \frac{1}{n-1} T^2\right], \quad T = \frac{\hat{\mu} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

We know,

$$\hat{\mu} = \bar{y} = 3496.9$$
$$\mu_0 = 3000$$
$$s = 1224.116$$
$$n = 10$$

Therefore, $t_{obs} = \dfrac{(3496.9 - 3000)}{1224.116/\sqrt{10}} = 1.2836$

$$\mathbb{P}[D \geq d_{obs}] = \mathbb{P}(|T| \geq t_{obs})$$
$$= 2\,\mathbb{P}(t_{(9)} \geq 1.2836) > 0.2$$

95% C.I : [\$2621.22, \$4372.58]

In class Exercise: Write a conclusion for this test! Include p-value, estimated parameters, and the 95% C.I.

We have no evidence against the Null Hypothesis, given a p-value > 0.2. The data are consistent with the hypothesis that the mean Co-op monthly salary is \$3000. (95% C.I = [2621.22, 4372.58], $\hat{\mu}$ = \$3496.9, s = \$1224.116)

Section 6.2.3 :→ Hypothesis tests and C.I's for σ^2.

Y_i = Monthly salary of i^{th} Co-op student in work term # 1.

$i = 1, 2, \cdots, n$

$\boxed{n = 352}$

$Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$ → We don't know !

Test, $H_0 : \sigma^2 = 750^2$

$$D = 2\left[\ell(\hat{\mu}, \hat{\sigma}^2) - \ell(\tilde{\mu}, 750^2)\right]$$

$\boxed{\text{Note:- } \hat{\mu} = \tilde{\mu} = \bar{y}}$

$$\ell(\mu, \sigma^2) = -\frac{n}{2}\ln(\sigma^2) + \sum_{i=1}^{n}(y_i - \bar{y})^2 / 2\sigma^2$$

Basic Model : $k = 2$.

Hypothesized Model :- $q = 1$.

$\hat{\sigma}^2 = \$ 778.98^2$; $\hat{\mu} = \tilde{\mu} = \$3149$; $\sigma_0^2 = 750^2$

Plug these into D.

$$\hookrightarrow \hat{\sigma}^2 = \sum_i (y_i - \bar{y})^2 / n = \frac{n-1}{n} s^2$$

$d_{obs} = 1.038$; p-value $= P[D \geq d_{obs}]$

$= P[\chi^2_{(1)} \geq 1.038]$

$= 0.3083$ (using R).

We have no evidence against

$H_0 : \sigma^2 = 750^2$.

To construct a C.I, pivotal quantity is very useful.

Definition :- A pivotal quantity, Q, is

(i) A function of the data.

(ii) A monotone function of the unknown parameter θ.

and, (iii) The sampling distribution of Q does not depend on θ_T, the true value of θ.

- $Z = \dfrac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$ is a pivotal quantity for μ (when σ^2 known)

- $T = \dfrac{\bar{y} - \mu_0}{S/\sqrt{n}} \sim t_{(n-1)}$ is a pivotal quantity for μ (when σ^2 unknown)

- $\dfrac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$ is a pivotal quantity for σ^2

where, $S = \sum_{i=1}^{n} (y_i - \mu)^2 / \sqrt{n-1}$ (sample S.D)

Note, In case of χ^2-distribution,

$$\boxed{\chi^2_{1-\alpha/2} \neq -\chi^2_{\alpha/2}}$$

Not same as,
$-Z_{1-\alpha/2} = Z_{\alpha/2}$
$\Rightarrow Z_{1-\alpha/2} = -Z_{\alpha/2}$