8th March '2024

From Last class,

For LRT's of 1-parameter : $H_0 : \mu = \mu_0$

To find $100(1-\alpha)\%$ C.I for μ,

find set of μ_0, such that,

$$p\text{-value} = \mathbb{P}[D \geq d_{obs}(\mu_0) \mid H_0 \text{ true}] \geq \alpha$$

Under H_0, $D \approx \chi^2_{(1)}$. Hence,

$$\mathbb{P}[D \geq d_{obs}(\mu_0) \mid H_0 \text{ true}] \geq \alpha \iff d_{obs}(\mu_0) \leq \chi^2_{\alpha,(1)}$$

Critical Value.

Critical values comes from your test Statistic's sampling Distribution..

Section 5.2 :→ Approximating C.I's (with Likelihood intervals)

$H_0 : \mu = \mu_0$

Recall under H_0, LRS $D \equiv -2r(\mu) \approx \chi^2_{(1)}$

For a significance level α,

$$\left. d_{obs}(\mu_0) \leq \chi^2_{\alpha,(1)} \right\} \text{ characteristics of } 100(1-\alpha)\% \text{ C.I.}$$

$$\Rightarrow -2r(\mu_0) \leq \chi^2_{\alpha,(1)}$$

$$\Rightarrow r(\mu_0) \geq -\frac{1}{2}\chi^2_{\alpha,(1)}$$

$$\left. \Rightarrow R(\mu_0) \geq e^{-\frac{1}{2}\chi^2_{\alpha,(1)}} \right\} \text{ characteristics of } 100 \cdot e^{-\frac{1}{2}\chi^2_{\alpha,(1)}}\% \text{ L.I}$$

C.I %	Equivalent L.I %
90	25.8
95	14.7
96	10
99	3.6

Back to our previous Example

$$\mathbb{P}\left[\chi^2_{(1)} \geq 3.843\right] = 0.05$$

$$D = -2r(\mu) = 3.843$$

$\chi^2_{(1)} = 3.843$ From Table

then, $R(\mu) = e^{-\frac{1}{2}(3.843)}$

$$\Rightarrow r(\mu) = -1.9215$$

$$\Rightarrow e^{r(\mu)} = e^{-1.9215}$$

$$\Rightarrow R(\mu) = 0.147$$

14.7% L.I = 95% C.I

★ Convenient when you have R available to compute L.I's

$$R(\mu) \longleftrightarrow [A, B] \longleftrightarrow \alpha$$

Section 5.3 :-> Another Approximation for C.I's.

For many cases (all of them, in this course), MLE's have an approx. Normal distribution for large sample sizes 'n'.

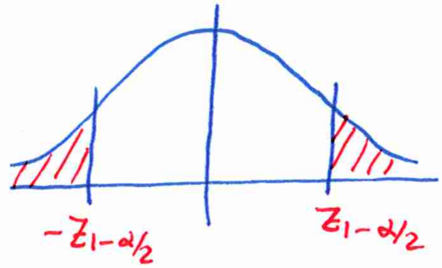
$$\hat{\theta} \approx N\left(\theta_0, I(\hat{\theta})^{-1}\right), \ \theta_0 \text{ is the true value.}$$

$$I(\hat{\theta})^{-1} = 1/I(\hat{\theta})$$

$$\Longleftrightarrow \frac{\hat{\theta} - \theta_0}{\sqrt{I(\hat{\theta})^{-1}}} \text{ or, } \frac{\hat{\theta} - \theta_0}{1/\sqrt{I(\hat{\theta})}} \text{ or } \sqrt{I(\hat{\theta})}\left(\hat{\theta} - \theta_0\right) \approx N(0,1)$$

$$100(1-\alpha)\% \text{ C.I} : \ \hat{\theta} \pm \frac{Z_{1-\alpha/2}}{\sqrt{I(\hat{\theta})}},$$

$H_0: \theta = \theta_0, \quad H_1: \theta \neq \theta_0 \quad \longleftarrow$ 2 tailed test.



$$Z_{\alpha/2} = -Z_{1-\alpha/2}$$

Now, Let's Look at an Example,

Example 5.2.1 from the Complete Lecture Notes.

Here, $X_{obs} = 23$.

$X \sim \text{Binomial}(n=100, \theta)$

MLE of θ, $\hat{\theta} = 23/100 = 0.23$

Method - 1 : $H_0 : \theta = \theta_0$

$X \sim \text{Bin}(100, \theta_0)$, under H_0

If, $n\theta_0 > 5$ and $n(1-\theta_0) > 5$,

then $X \approx \text{Normal}(\mu = n\theta_0, \sigma^2 = n\theta_0(1-\theta_0))$

Let, $D = \left| \dfrac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} \right| = |Z|, \quad Z \sim N(0,1)$

95% C.I

$$P(D \geq d_{obs}) = P(|Z| \geq |Z_{obs}|) \geq 0.05$$

$$\Rightarrow P\left(|Z| \geq \left| \frac{23 - 100\theta_0}{10\sqrt{\theta_0(1-\theta_0)}} \right| \right) \geq 0.05$$

Critical values from table

$$\Leftrightarrow \left| \frac{23 - 100\theta_0}{10\sqrt{\theta_0(1-\theta_0)}} \right| \leq Z_{0.975} = 1.96$$

① Solve for θ_0.

② Approximate $\sqrt{\theta_0(1-\theta_0)}$ with $\sqrt{\hat{\theta}(1-\hat{\theta})}$.

Using (2), $\dfrac{23}{100} \pm 1.96 \sqrt{\dfrac{0.23(1-0.23)}{10}}$

$= [0.148, 0.312]$

Method - 2 : 95% C.I = 14.7% L.I = $[0.155, 0.319]$

Method-3 : $\hat{\theta} \approx \mathcal{N}\left(\theta_0, I(\hat{\theta}_0)^{-1}\right)$

$$L(\theta) = \theta^{x}(1-\theta)^{n-x}$$

$$l(\theta) = x\ln(\theta) + (n-x)\ln(1-\theta)$$

$$l'(\theta) = x/\theta - \frac{n-x}{1-\theta}$$

$$l''(\theta) = -x/\theta^2 - \frac{n-x}{(1-\theta)^2}$$

$$I(\theta) = -l''(\theta) = x/\theta^2 + \frac{(n-x)}{(1-\theta)^2}$$

$$I(\hat{\theta}) = \frac{\cancel{x}\cdot n^2}{x^{\cancel{2}}} + \frac{n-x}{(1-x/n)^2} = \frac{n^2}{x} + \frac{(\cancel{n-x})n^2}{(n-x)^{\cancel{2}}}$$

$$\Rightarrow I(\hat{\theta}) = \frac{n}{\hat{\theta}} + \frac{n}{(1-\hat{\theta})} \quad \cdots \quad [\hat{\theta} = x/n]$$

$$I(\hat{\theta})^{-1} = \left(\frac{n}{\hat{\theta}} + \frac{n}{(1-\hat{\theta})}\right)^{-1}$$

[* Need common denominators before you can take reciprocal!]

$$= \left(\frac{n(1-\hat{\theta}) + n\hat{\theta}}{\hat{\theta}(1-\hat{\theta})}\right)^{-1} = \left(\frac{n}{\hat{\theta}(1-\hat{\theta})}\right)^{-1}$$

$$\Rightarrow I(\hat{\theta})^{-1} = \hat{\theta}(1-\hat{\theta})/n$$

Therefore, $\hat{\theta} \pm 1.96\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$

$= [0.148, 0.312]$ is the 95% C.I.