Set 6: Section 3.4

Conditional probability (an important topic):

Problem: Suppose that I roll a die and tell you that the result is even. What is the probability that the outcome is a 6?

The conditional probability of A given B is the probability that A will occur given that B has occurred:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) \neq 0$.

Note that if we rearrange the equation for conditional probability, we get the following:

$$P(A \cap B) = P(A|B)P(B)$$

Problem: A bag contains 5 red marbles and 5 green marbles. Select two marbles from the bag, one at a time, without replacement. If I know that the first marble is red what is the probability that the second marble will also be red?

If I know that the first marble is green, what is the probability that the second marble will be red?

Recall: $P(E) = \frac{n(E)}{n(S)}$**, for equally likely simple events.**

The outcomes in $A|B$ **are precisely those in which** A **occurs, and** B **also occurs. The number of outcomes in** $A|B$ **is** $n(A \cap B)$**.**

If we know that B **occurs, then the total number of possible outcomes will be the number of outcomes in** B**, which is** $n(B)$**. This gives us:**

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

Note that for the conditional probability $P(A|B)$ **to be defined, then the conditioning event must not be** $\emptyset$**. Dividing the top and bottom by** $n(S)$**:**

$$P(A|B) = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$$

Example: Consider the results of the following survey of 250 **single-crop farms:**

	Wheat	Corn	Soy
Alberta	69	15	16
Saskatchewan	61	65	24

Suppose that a single-crop farm is selected at random. If the farm is in Alberta, what is the probability the farm grows soy?

Example: If a farm which grows soy is selected, what is the probability that the farm is in Alberta.

Note: $P(A \mid B) \neq P(B \mid A)$

Example: Suppose 80% of all Canadians exercise one or more days a week, and also, that 20% of all Canadians exercise at five or more days a week. If we randomly select a Canadian who exercises at least one day a week, what is the probability that this Canadian exercises five or more days a week?

Example: Suppose that 30% of all students drive to school, 50% take the bus, and 20% walk. Of those who drive, 20% are usually late for their first class of the day. Of those who take the bus, 10% are usually late for their first class of the day. Of those who walk, 15% are usually late for their first class of the day. What is the probability that a randomly selected student is regularly late for their first class?

What is the probability that this student walks to school?

We say that a collection of events $A_1, A_2, \ldots, A_k$ are exhaustive if they completely cover the sample space. That is, if these events are exhaustive, then $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

Law of Total Probability: If $A_1, A_2, \ldots, A_k$ are a collection of mutually exclusive and exhaustive events, then for any event B we have:

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots P(B \cap A_k)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)$$

Bayes' Theorem: If $A_1, A_2, \ldots, A_k$ are a collection of mutually exclusive and exhaustive events, then for any event B (where $P(B) \neq 0$) we have the following, for $1 \leq i \leq k$:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)}$$