

2. (10 marks) Suppose there are three types of fish in a lake: Trout (T), Bass (B), and Carp (C). According to ecological theory, these fish species should occur in a certain lake with probabilities $p_T = \theta^2$, $p_B = 2\theta(1 - \theta)$, and $p_C = (1 - \theta)^2$ respectively, where θ is the unknown parameter denoting the proportion of each fish species in the lake, $0 \leq \theta \leq 1$.
- (a) (1 mark) Assume that in a random sample of size n from this lake, there are f_T Trout, f_B Bass, and f_C Carp. Write down the probability mass function for (f_T, f_B, f_C) assuming a multinomial distribution.
 - (b) (1 mark) Write the Log-likelihood function $\ell(\theta)$ as a function of θ based on the observed counts f_T, f_B , and f_C .
 - (c) (2 marks) Derive the expression for the Maximum Likelihood Estimate (MLE) of θ .
 - (d) (1 mark) Suppose a sample of 20 fish is taken from the lake, resulting in 6 Trout, 8 Bass, and 6 Carp. Compute the MLE of θ .
 - (e) (2 marks) Calculate the estimated expected frequencies of Trout, Bass, and Carp in the lake based on the MLE of θ and compare these with the observed frequencies. Comment on the results.
 - (f) (1 mark) Write down and simplify the Relative Likelihood function $R(\theta)$ under the conditions from part (e).
 - (g) (2 marks) Evaluate how plausible it is that the true value of θ is $\theta = 0.6$. Compare this to the plausibility of $\theta = 0.3$ based on the observed data.
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