26th Jan '2024

Continuing with Ex. 2.7.1.

b) We want to estimate the median family income, β.

So, $\mathbb{P}(X \leq \beta) = 0.5$

From the defination of median,

$$\mathbb{P}(X \leq \beta) = 0.5 \Rightarrow \int_1^\beta \theta x^{-(\theta+1)} dx = 0.5$$

$$\Rightarrow -x^{-\theta}\Big|_1^\beta = 0.5$$

$$\Rightarrow 1 - \beta^{-\theta} = 0.5$$

$$\Rightarrow \beta^{-\theta} = 0.5$$

$$\Rightarrow \beta^{\theta} = 2 \qquad \left[x^{-a} = y^{-b} \Rightarrow x^a = y^b\right]$$

$$\Rightarrow \beta = 2^{1/\theta} \quad ; \quad \theta > 0$$

Is this a 1-1 transformation? (Yes!)

Want $\hat{\beta}$; Can we use the invariance property?

↳ Does the inverse exist?

$$\theta = \frac{\ln(2)}{\ln(\beta)}$$

↳ Is the function $\beta = 2^{1/\theta}$ monotone?

$$\frac{d}{d\theta} 2^{1/\theta} = \ln(2)\, 2^{1/\theta} \frac{d}{d\theta}(1/\theta) \qquad \left[\frac{\partial}{\partial x} a^x = a^x \ln(a)\right]$$

$$\frac{d}{d\theta} 2^{1/\theta} = \frac{-\ln(2)\, 2^{1/\theta}}{\theta^2} < 0 \; ; \forall \theta > 0$$

Yes, β is monotone decreasing.

$\therefore$ We can use the invariance property to find

$$\hat{\beta} = 2^{1/\hat{\theta}}$$

So, $\hat{\beta} = 2^{1/0.522} \simeq \underline{3.773}$

Note, substituting, $\theta = \ln(2)/\ln(\beta)$ into $L(\theta)$

$$L(\theta) = L\left(\frac{\ln(2)}{\ln(\beta)}\right)$$

$$= \left(\frac{\ln(2)}{\ln(\beta)}\right)^{10} \prod_{i=1}^{10} x_i^{-(\ln(2)/\ln(\beta))}$$

$$= L_*(\beta)$$

So, we can find $\hat{\beta}$ by maximizing $L_*(\beta)$

[Exercise : Show that the maximizer of $L_*(\beta)$ is $\hat{\beta} = 2^{1/\hat{\theta}}$] $\rightarrow$ TREAT Question

We know MLE's are invariant under 1-1 transformations,

So, $\hat{\beta} = 2^{1/\hat{\theta}}$ and also, $R(\theta) = R_*(\beta)$,

under, $\theta = \ln(2)/\ln(\beta)$.

So, we can solve $R\left(\frac{\ln(2)}{\ln(\beta)}\right) = R_*(\beta)$

$R_*(\beta) - p = 0$ for a $100p\%$ L.I for β

By the definition of invariance property:

Since, 10% L.I for θ was $(0.24, 0.96)$, and β is monotone decreasing,

then 10% L.I for β is $\left(2^{1/0.96}, 2^{1/0.24}\right)$

$$\approx (2.058, 17.959)$$

This represents the set of plausible median income values based on Toronto sample data, relative to subsistence level, $x \geq 1$)

Comments –

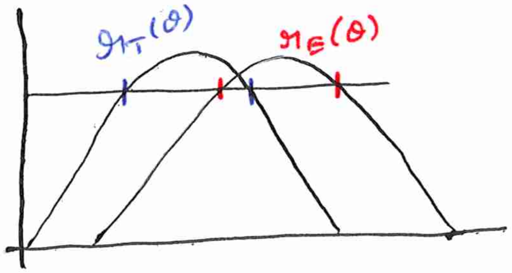
- Increased Sample size $\longrightarrow$ More peaked likelihood function.

  $\longrightarrow$ Narrower likelihood intervals, because our estimation is more precise.

- Can we pool independent samples? Assuming the same set up and distribution.

  $\hat{\theta}_T$ = Toronto Family income MLE

  $\hat{\theta}_E$ = England " " "



10% L.I | Compare $r(\theta)$ for both samples. If overlap present between 10% L.I's, then we can combine them and produce a common estimate of θ.

End of Chapter – 2