31st Jan '2024

Recap :- Joint MLE for $N(\mu, \sigma^2)$

Continuing with Section 3.1 :

Revisiting Ex 2.2.1

$y_i \to$ # hits required to fracture	1	2	3	≥ 4	Total
$x_i \to$ # specimens	112	36	22	30	200
$\hat{E}(x_i) \to$ Estimated # specimens	100	50	25	25	200
$\lvert x_i - \hat{E}(x_i) \rvert$	12	14	3	5	0

Recall : $\theta = P(\text{surviving hit})$
We assumed, $y \sim Geom(1-\theta)$
We found, $\hat{\theta} = 0.5$

It seemed like our $\hat{E}(x_i)$'s were not matching up with our observations very well

Let's come up with a new model for finding the estimated # specimens.

↳ Let us assume that a proportion of the specimens are flawed and will always fracture on the 1st hit,

Proportion : $(1-\lambda)$ are flawed $\qquad \lambda \in (0,1)$

$\Rightarrow \lambda$ are good.

θ and λ are our two parameters now.

Let's reason the model

∴ Frequencies (X_1, X_2, X_3, X_4) are still Multinomial ☑

— 200 independent trials

— Constant probabilities

— 4 possible outcomes : 1 hits, 2 hits, 3 hits, ≥ 4

$$f(x_1, x_2, x_3, x_4) = \binom{200}{x_1, x_2, x_3, x_4} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4} \quad ; \quad \sum_{i=1}^{4} x_i = 200$$

$$\sum_{i=1}^{4} p_i = 1$$

Recall that :-

$$P_i = \mathbb{P}(Y = i) \quad \text{for } i = 1, 2, 3$$

$$P_4 = \mathbb{P}(Y \geq 4) = 1 - \sum_{i=1}^{3} p_i$$

$\boxed{\text{Now we want } P_i\text{'s as functions of } \lambda \text{ and } \theta}$

Let, $M = \begin{cases} 1 \ ; & \text{if plastic is flawless} \\ 0 \ ; & \text{otherwise} \end{cases}$

M is an indicator function for "good" pieces.

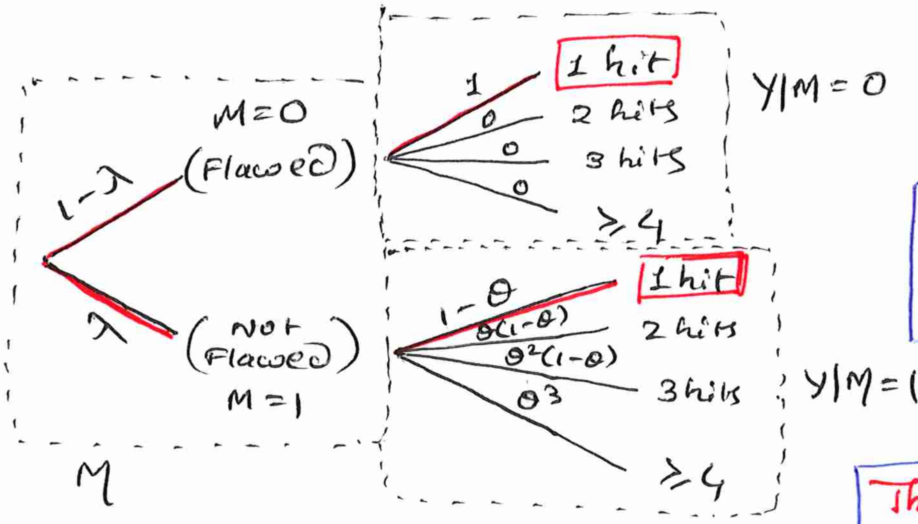
$$M \sim \text{Bernoulli}(\lambda), \quad \lambda = \mathbb{P}(\text{plastic is good})$$

What we had previously : $Y \sim \text{Geom}(1-\theta)$

Now we have : $Y \mid M = 1 \sim \text{Geom}(1-\theta)$

$Y \mid M = 0 = 1$

$Y \mid M = 0$ is degenerated at 1 hit



Conditional probability

$P(A \cap B) = P(A|B)\,P(B)$

The highlighted paths in the tree donote P_1

$P_1 = \mathbb{P}(\text{fracture on one hit})$

$= \mathbb{P}(Y=1)$ [Total Probability : $\mathbb{P}(A) = \sum_n \mathbb{P}(A \cap B_n)$]

$= \mathbb{P}(Y=1 \cap M=0) + P(Y=1 \cap M=1)$

$= \mathbb{P}(Y=1|M=0)\,\mathbb{P}(M=0) + P(Y=1|M=1)\,\mathbb{P}(M=1)$

$= 1(1-\lambda) + (1-\theta)\lambda$

$= 1 - \alpha\theta$ $\boxed{\therefore P_1 = 1-\alpha\theta}$

$P_2 = \mathbb{P}(Y=2)$

$= \mathbb{P}(Y=2|M=0)\cdot\mathbb{P}(M=0) + \mathbb{P}(Y=2|M=1)\,\mathbb{P}(M=1)$

$= \cancel{0 \cdot (1-\lambda)} + \theta(1-\theta)\lambda$

$= \alpha\theta(1-\theta)$ $\boxed{\therefore P_2 = \lambda\theta(1-\theta)}$

$\left.\begin{array}{l} P_3 = \lambda\theta^2(1-\theta) \\ P_4 = \alpha\theta^3 \end{array}\right\}$ Found similarly.

Sanity Check! $\sum_{i=1}^{4} P_i = 1$

Now lets write our joint likelihood function :

$$p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4} \quad ; \quad x_1 = 112 , x_2 = 36 , x_3 = 22 , x_4 = 30$$

$$L(\theta,\lambda ; x_1,x_2,x_3,x_4) = (1-\lambda\theta)^{112} \cdot (\lambda\theta(1-\theta))^{36} \cdot (\lambda\theta^2(1-\theta))^{22} \cdot (\lambda\theta^3)^{30} \quad ; \lambda \in (0,1), \theta \in (0,1)$$

$$= (1-\lambda\theta)^{112} \lambda^{36+22+30} \theta^{36+44+90} (1-\theta)^{36+22}$$

$$= (1-\lambda\theta)^{112} \lambda^{88} \theta^{170} (1-\theta)^{58} \quad ; \lambda \in (0,1), \theta \in (0,1)$$

joint Log-likelihood function

$$l(\theta,\lambda) = 112 \log(1-\lambda\theta) + 88 \log(\lambda) + 170 \log(\theta) + 58 \log(1-\theta) \quad ; \lambda \in (0,1), \theta \in (0,1)$$

Find $\hat{\lambda}$ and $\hat{\theta}$ ...

Try it yourself, we'll Continue on the next class!!!