Introduction to Fluid Dynamics
Solutions to Problem Set 1, 10/11/2007

1. [a] The pressure difference times the wing area has to hold up the mass, so

$$\Delta p = \frac{\text{weight of plane}}{\text{wing area}} = \frac{4\times 10^5 \text{ kg}\times 9.8 \text{ m s}^{-2}}{525 \text{ m}^2} = 7467 \text{ Pa}$$

[b] The fraction of atmospheric pressure is approximately

$$\text{Fraction} = \frac{\Delta p}{p_{ATM}} = \frac{7467 \text{ Pa}}{10^5 \text{ Pa}} \cong 7.4\%$$

2. The pressure at the bottom of either side tube (left or right) may be calculated by integrating the hydrostatic equation $p_z = -\rho g$, thus

$$\int_{z_{BOT}}^{z_{TOP}} \frac{\partial p}{\partial z} dz = -\int_{z_{BOT}}^{z_{TOP}} \rho g \, dz$$

Which gives

$$p_{BOT} = \rho g(z_{TOP} - z_{BOT}) + p_{TOP} = \rho g \Delta z + p_{TOP}$$

Now since the tubes on the left and right are the same height (as drawn) then the bottom pressure on either side must be the same, implying that there is no pressure force trying to accelerate the fluid in the lower tube in either direction. This reasoning is fine as long as p_{TOP} is the same on both sides. For the sake of argument, let's assume that p_{TOP} is bigger on the left than on the right. This would push the fluid in the top tube to the right (from high to low pressure) yielding the clockwise acceleration our optimistic inventor hopes to achieve. BUT, then our hydrostatic calculation indicates that there would also be a pressure difference along the length of the bottom tube, again with high pressure on the left. Thus this assumed pressure imbalance would push the fluid to the right in *both* the upper and lower tubes. Clearly these two competing forces will cancel out, and the fluid will not accelerate in either direction. Physically, the compression experienced by the tube on the right would quickly increase its pressure, eradicating any difference of p_{TOP}.

3. [a] Note that $T_1 = 293.15$ K. The pressure is given by that required to hold up the lid, so it is $p_1 = \left(10^4 \text{ kg}\times 9.8 \text{ m s}^{-2}\right)/\left(1 \text{ m}^2\right) = 0.98\times 10^5$ Pa. Then the density is given by the Universal Gas Law

$$\rho_1 = \frac{p_1}{RT_1} = 1.16 \text{ kg m}^{-3}$$

[b] The initial mass is just the volume times the density: $M = \rho_1 \times A \times (10 \text{ m}) = 11.6$ kg, and this is constant throughout the problem.

[c] The hydrostatic change is pressure (from $p_z = -\rho g$) is given by $\Delta p = (10\text{ m})\rho_1 g = 113.7\text{ Pa}$. This is just 0.1% of p_1 so we may safely neglect it.

Now we add 10^5 J of heat to the system. NOTE that the variables in the first law of thermodynamics were written in terms of "energy per unit mass" so for this change $Q = (10^5\text{ J})/(11.6\text{ kg}) = 8620.7\text{ J kg}^{-1}$.

[d] The change to the system is NOT adiabatic, because we added heat. The work done by the moving end may be assumed to be reversible (technically this requires that it be slow enough not to generate shock waves, and that it generate no turbulence – this will never be completely true, but in the situation shown it will be a very good approximation). The change was not isentropic, because of the heat added.

[e] What is the new state, after the heat has been added? First, we can say that the pressure is unchanged, because it is governed by the weight on top, so $p_2 = p_1$. Then recall the first law: $C_v dT = Q - p_1 dv$. The gas law is $v = (R/p_1)T$ (written in terms of the specific volume $v \equiv 1/\rho$) and so $dv = (R/p_1)dT$. From these we may show

$$dT = \frac{Q}{C_v - R} = \frac{Q}{C_p} = \frac{8620.7\text{ J kg}^{-1}}{1005\text{ J kg}^{-1}}\text{ K} = 8.58\text{ K}$$

Thus $T_2 = (293.15 + 8.58)\text{ K} = 301.7\text{ K}$. Then the new density is calculated from the gas law as $\rho_2 = p_2/(RT_2) = 1.13\text{ kg m}^{-3}$ (about a 3% decrease).

[f] The initial density can be related to the cylinder height as $\rho_1 = M/(AH)$, and after the change $\rho_2 = M/[A(H + \Delta z)]$. These may be rearranged to find $\Delta z = (\rho_1/\rho_2 - 1)H = 0.27\text{ m}$ (upwards).

[g] The work (per unit mass) done on the air in the cylinder by the moving end of the cylinder comes from the first law: Work done per unit mass $= C_v dT - Q$. Also note that $Q = C_p dT$ (from part [e] above). The change of internal energy of the system (again per unit mass) is given by $C_v dT$. It is easy to show that the ratio of these is given by

$$\frac{\text{Work done by end}}{\text{Change of internal energy}} = \frac{C_v - C_p}{C_v} = 1 - \gamma = -0.4$$

Note that this is negative, meaning that the work done by the end takes internal energy *out* of the system.

Now instead of adding heat we double the mass on top.

[h] The change is, of course, adiabatic, because no heat was added. The change is also isentropic and reversible (subject to the caveats above in [d]).

[i] The change in pressure is easy – we have doubled the mass on top so the pressure must also double: $p_2 = p_1 = 1.96 \times 10^5$ Pa. Since the change is isentropic we may use the expression $p/\rho^\gamma = \text{constant}$, and so $p_1/\rho_1{}^\gamma = p_2/\rho_2{}^\gamma$, and thus

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} = 2^{\frac{1}{1.4}} = 1.64$$

So $\rho_2 = 1.64\rho_1 = 1.9$ kg m^{-3}. Finally we use the gas law to get the new temperature

$$T_2 = \frac{p_2}{\rho_2 R} = 359.4 \text{ K}=66.25\ ^\circ\text{C}$$

[j] Using the same equation as in [f], but with the different value of ρ_2, we find $\Delta z = -3.9$ m (downwards this time, as one would expect).

[k] With no heat added the first law simplifies to $de = -pdv$, indicating that the ratio of work done by the moving end to the change of internal energy of the system is just 1 (i.e. they are the same).