Introduction to Fluid Dynamics
Problem Set #6 Solutions
due 12/5/2007 at the start of class

Consider the potential flow problem of 2D flow around a cylinder.

A[5]. Derive the expression for the streamfunction, ψ (work in cylindrical polar coordinates). You can check your answer in Kundu and Cohen. It is useful to have the relations

$$u_R = \frac{\partial \varphi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

%%% This can be done from either expression, readily giving $\psi = U\left(r - \frac{a^2}{r} \right)\sin\theta$.

B[10]. Sketch (or plot) contours of ψ and φ, indicating the direction in which each grows larger.

%%% This is most easily done (for me) in MATLAB. This code:

```
% cylinder.m  12/3/2007  Parker MacCready
%
% this plots the streamfunction and velocity potential for potential flow
% around a cylinder

clear

% make axes
xymax = 2;
x = linspace(-xymax,xymax,100);
y = linspace(-xymax,xymax,100);
% note that x and y don't include 0
[X,Y] = meshgrid(x,y);

R = sqrt(X.^2 + Y.^2);
sin_th = Y./R;
cos_th = X./R;

U = 1;
a = 1;

phi = U*(R + a*a./R).*cos_th;
psi = U*(R - a*a./R).*sin_th;

figure
contour(X,Y,phi,[-3:.25:3],'-r');
hold on
```
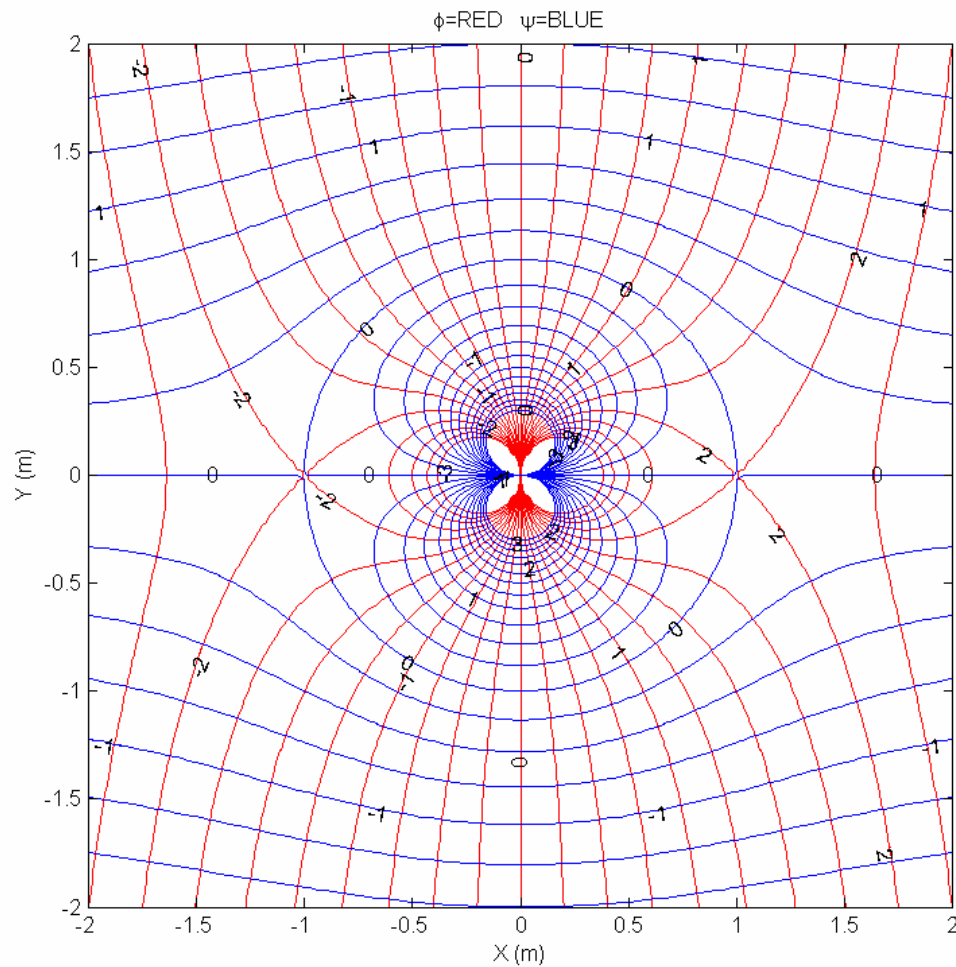
```
[cc,hh] = contour(X,Y,phi,[-3:1:3],'-r');
clabel(cc,hh);
contour(X,Y,psi,[-3:.25:3],'-b');
[cc,hh] = contour(X,Y,psi,[-3:1:3],'-b');
clabel(cc,hh);
xlabel('X (m)')
ylabel('Y (m)')
title('\phi=RED    \psi=BLUE')
axis equal
axis tight
```

Gives this figure

C[10]. Use Bernoulli to find the general expression for the pressure anomaly, $p' \equiv p - p_\infty$. Sketch this as a contour map for the region outside of the cylinder. Indicate regions of relatively high and low pressure.

%%% The expression is readily found to be

$$p - p_\infty = \frac{1}{2}\rho_0 U^2 - \frac{1}{2}\rho_0\left(u_R{}^2 + u_\theta{}^2\right) = \frac{1}{2}\rho_0 U^2\left[2\frac{a^2}{r^2}\left(\cos^2\theta - \sin^2\theta\right) - \frac{a^4}{r^4}\right]$$

using the same methods as in class.

A sketch (made using code similar to that above) is:
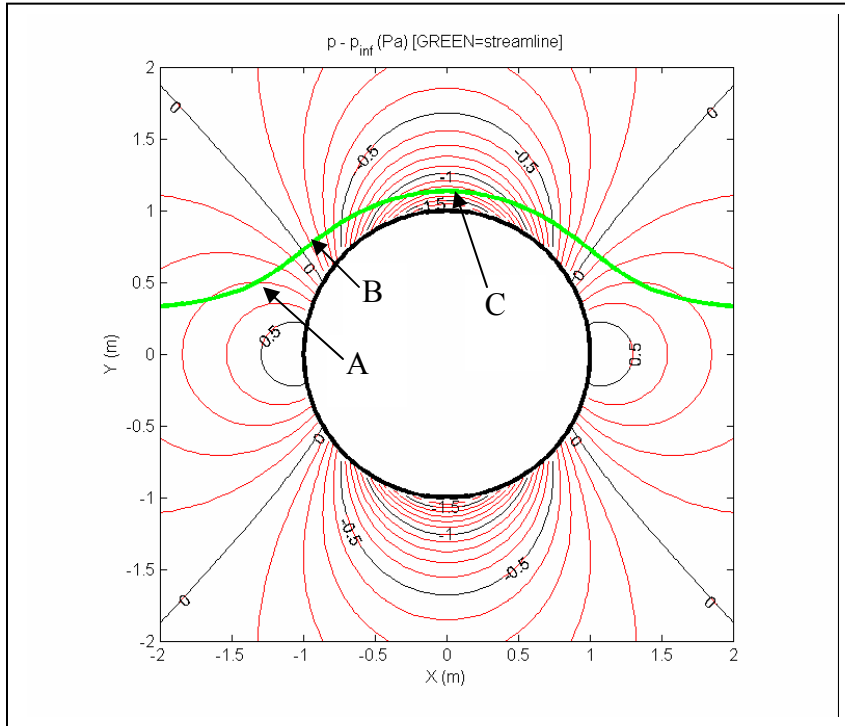
I have assumed a cylinder with 1 m radius, oncoming flow of $U = 1$ m s^{-1}, and $\rho_0 = 1.2$ kg m^{-3} (like air). Since atmospheric pressure at sea level is about 10^5 Pa the disturbances above are pretty small in comparison.

D[10]. Narrate how the change in parcel velocity (both magnitude and direction) along a streamline relates to the pressure gradient.

%%% Looking at the green streamline in the figure above, the velocity direction always curves when there is a pressure gradient *across* the streamline. Thus at position "A" it is curving to the left, toward lower pressure. At "B" there is zero curvature, and at "C" the curvature is to the right, again toward lower pressure. The parcel accelerates between A and C, as the pressure is decreasing all along that part of the path. Note of course that the flow is steady, even though there is non-zero Lagrangian acceleration.

E[10]. Argue (without resorting to detailed math) why the integrated x-momentum inside the cylinder should be zero. What does this tell you about the *momentum* of the cylinder in the case where we are in a frame of reference moving with the free-stream flow U (and so the ambient far field flow is stationary and the cylinder is moving in the negative x-direction)?

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%%% Since the streamlines for each half of the cylinder (upper and lower) are closed, that means there is the same net transport to the right between the widely spaced streamlines as there is to the left between the tightly-spaced streamlines near the origin. Thus the integral of u over the area of the circle must have equal positive and negative contributions, and so integrates to zero. If we were in a frame of reference moving with the free stream speed U, the cylinder would be moving to the left, and it would have momentum $-\rho_0 U \pi a^2$ (per unit length in the z-direction). This tells us that the momentum of such a feature is related to the size of the region of closed streamlines.