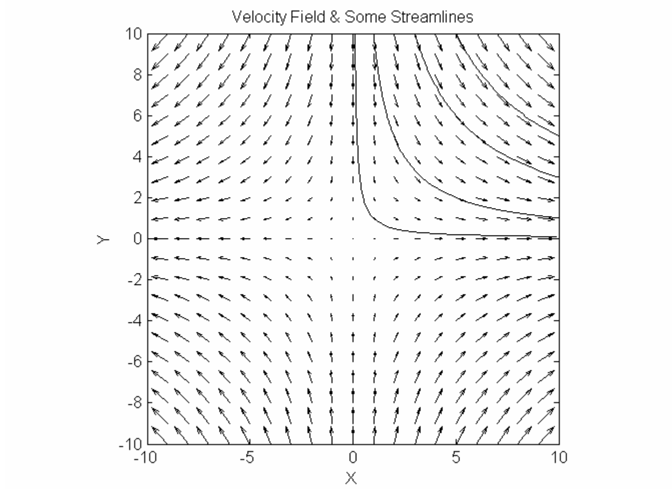
Introduction to Fluid dynamics

Solutions to Problem Set #2, 10/19/2007

1a. The pattern of (selected) streamlines and velocity vectors looks like:



The MATLAB code to make this plot is:

```
% ps2_1.m  10/19/2007  Parker MacCready
%
% this plots the streamlines and velocity vectors for the flow field in
% problem 1
clear
% create the axes
x = [-10:10]; y = [-10:10];
[X,Y] = meshgrid(x,y);
% create the velocity field
D = 1;
U = D*X; V = -D*Y;
% plot the velocity field
figure
quiver(X,Y,U,V,'-k');
hold on
% plot a few streamlines
ys = linspace(eps,10,100);
for C = [1 10 30 50];
    xs = C./ys;
    plot(xs,ys,'-k')
end
% add labels and fix scaling
axis equal
axis([-10 10 -10 10]);
xlabel('X'); ylabel('Y')
title('Velocity Field & Some Streamlines')
```

The field is non-divergent because $\nabla \bullet \mathbf{u} = u_x + v_y = D - D = 0$.

1b. Recall that the mathematical form for a streamline is governed by the expression $\dfrac{dx}{u} = \dfrac{dy}{v}$. Rearranging this expression, and substituting in our expressions for the velocity, we arrive at the first-order ODE $\frac{dy}{dx} = -\frac{y}{x}$, which has solution $y = C/x$, where C is a constant. Choosing different values of C places us on different streamlines, as shown in the figure.

1c. To do this part you have to remember that the velocity is defined as the local rate of change of the parcel position. We will call the parcel position $\left(x^L, y^L\right)$, where the "L" is for Lagrangian. So, for the x-position, the definition of velocity gives $u = \frac{dx^L}{dt}$ (*), and of course $u = Dx$. The trick is to realize that on the parcel path the x to use when evaluating the velocity has to be x^L. This means we can write the equation (*) as $\frac{dx^L}{dt} - Dx^L = 0$. The solution to this ODE, subject to the initial conditions stated in the problem set, is $x^L = x_0{}^L \exp(Dt)$. Similar reasoning gives $y^L = y_0{}^L \exp(-Dt)$.

1d. We just take the time derivatives of the results from 1c to find the velocity components following a fluid parcel: $\left(u^L, v^L\right) = \left[Dx_0{}^L \exp(Dt), -Dy_0{}^L \exp(-Dt)\right]$.

1e. The slope of the dye line is given by taking the ratio of the results from 1c: $y^L/x^L = \left(y_0{}^L/x_0{}^L\right)\exp(-2Dt) = \exp(-2Dt)$ (**). This makes use of the fact that for the dye line $y_0{}^L/x_0{}^L = 1$. Visual inspection of the figure above (1a) should make it clear that the distance between parcels on this dye line will change over time, because parcels have to remain on streamlines at the same time as they are on the straight line through the origin (**) whose slope is changing.

2a. The parcel velocities are given by

$$\frac{dx^L}{dt} = U, \ \frac{dy^L}{dt} = V\cos(\omega t)$$

These may be integrated directly in time (note that the velocity field is not a function of space) to give

$$x^L = x_0{}^L + Ut, \ \ y^L = y_0{}^L + \frac{V}{\omega}\sin(\omega t) \ \ (++)$$

And if the parcel started at the origin we would set $\left(x_0{}^L, y_0{}^L\right) = (0,0)$.

2b. Now, to construct a streakline we have to be concerned with the positions of a number of parcels that all had $\left(x_0{}^L, y_0{}^L\right) = (0,0)$ *but at different times*. For a given parcel to be at the origin at time t_*, its value of $x_0{}^L$ (to be used in (++)) must be given by

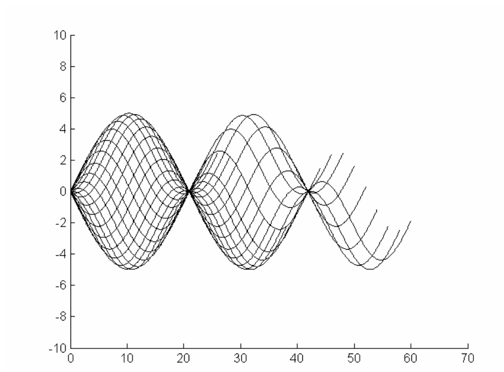
$x_{0*}{}^L = -Ut_*$. Similar reasoning for the y-position gives $y_{0*}{}^L = -\dfrac{V}{\omega}\sin(\omega t_*)$. Substituting these into (++) gives the positions of parcels on the streakline as functions of t (the actual time) and t_* (the time at which each parcel was released):

$$\left(x^L, y^L\right) = \left[U(t - t_*), \frac{V}{\omega}(\sin\omega t - \sin\omega t_*)\right]$$

A problem is that we don't know what t_* is, but we can rearrange the expression for x^L to get $t_* = t - x^L/U$. This may be substituted into the expression for y^L to show

$$\begin{aligned} y^L &= \frac{V}{\omega}\left[\sin(\omega t) - \sin\left(\omega t - \frac{\omega x^L}{U}\right)\right] \\ &= \frac{V}{\omega}\left[\sin(\omega t) - \sin(\omega t)\cos\left(\frac{\omega x^L}{U}\right) + \cos(\omega t)\sin\left(\frac{\omega x^L}{U}\right)\right] \end{aligned}$$

Note the one way to check this solution is to verify that when $x_0{}^L = 0$, it does indeed give $y_0{}^L = 0$. The above solution can then be plotted for the whole range of $x^L > 0$ (meaning that the dye injection has been going on for a longtime. The final solution is not steady. Try using this MATLAB code (contributed by your fellow student Stuart Evans) to visualize it:




```
% streakline.m  10/19/2007 Stuart Evans
%
% plots a solution to PS2 problem 2
clear
% Define u, v, w
u=20;
v=15;
w=6;
% uncomment following line to see long exposure version
close all;hold on;
% Step through time t.  For each t, calculate all tau, x, y.  Plot.
for t=1:0.1:3
    tau=linspace(0,t,500);
    x=u*(t-tau);
    y=v/w*(sin(w*t)-sin(w*tau));
    plot(x,y,'-k');
    xlim([0,70]);
    ylim([-10,10]);
    pause(0.1);
end
```