Introduction to Fluid Dynamics

Problem Set #4 Solutions *11/21/2007*

1. Consider (again) "Plane Poiseuille Flow" as described in Kundu and Cohen 9.4 (and Fig. 9.4d).

A[20]. Integrate the kinetic energy equation (here given as kinetic energy per unit volume, in "Eulerian" form)

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)}_{1}=\underbrace{-\nabla\bullet\left[\mathbf{u}\bullet\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)\right]}_{2}\underbrace{-\nabla\bullet(\mathbf{u}p)}_{3}+\underbrace{\nabla\bullet\left[\nu\nabla\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)\right]}_{4}+\underbrace{p(\nabla\bullet\mathbf{u})}_{5}\underbrace{-\rho g w}_{6}\underbrace{-\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}_{7}$$

over a rectangular volume that goes from the bottom plate to the top plate, as done in class 11/14/2007 for Couette flow. Which of the 7 terms above contributes to the energy balance. How is this different from Couette flow?

%%% Use the notation of Kundu and Cohen, so the bottom plate is at $y=0$ and the top plate is at $y=2b$. Define the volume of integration, V, to have length L in the x-direction, and width B in the z-direction. Hence $V=2bLW$. The volume integration is quite straightforward, so I won't go into great detail.

Term 1 is zero because the flow is steady.

Term 2 is zero because the inflow of KE_V is balanced by the outflow of KE_V.

Term 3 is readily calculated. Note that the pressure is not a function of y. If we define the pressure to be p_1 at the left end of the volume, then is will be $p_1+L\frac{\partial p}{\partial x}$ at the right end of the volume. Thus:

$$-\int_V \nabla\bullet(\mathbf{u}p)\,dV=-\left(p_1+L\frac{\partial p}{\partial x}\right)\int_{A_2}u\,dA+p_1\int_{A_1}u\,dA=-L\frac{\partial p}{\partial x}W\int_0^{2b}u\,dy$$

And note that $\int_0^{2b}u\,dy=-\frac{2}{3}\frac{b^3}{\mu}\frac{\partial p}{\partial x}$, and $\overline{u}=\frac{1}{2b}\int_0^{2b}u\,dy=-\frac{1}{3}\frac{b^2}{\mu}\frac{\partial p}{\partial x}$. Thus

$$-\int_V \nabla\bullet(\mathbf{u}p)\,dV=-LW\frac{\partial p}{\partial x}\left(-\frac{2}{3}\frac{b^3}{\mu}\frac{\partial p}{\partial x}\right)=-V\overline{u}\frac{\partial p}{\partial x}$$

Term 4 is zero because the velocity is zero on the top and bottom walls.

Term 5 is zero because the flow is incompressible, and term 6 is zero because there is no vertical velocity.

Term 7, the volume integrated rate of dissipation, is readily calculated as

$$-\mu\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV = -\mu\int_V \left(\frac{\partial u}{\partial y}\right)^2 dV = -WL\mu\int_0^{2b}\left(\frac{\partial u}{\partial y}\right)^2 dy = V\overline{u}\frac{\partial p}{\partial x}$$

Thus the balance is between pressure work (which adds energy) and net dissipation, which removes it.

B[20]. Shift your frame of reference so that you are moving to the right with the average flow speed, $\overline{u}$ (that is the average of u over the distance between the plates). This is called a "Galilean Transform" and the equations of motion are unchanged by such a transformation. Note however that the velocity you observe, call it u' is given by $u' = u - \overline{u}$, where u is the x-velocity in the original frame of reference. Note also that now the solid boundaries are moving. Repeat your volume integral of the kinetic energy equation. Now which of the 7 terms are important? This is an example of how the "story" told by the energy equation may be different depending on the frame of reference.

%%% Here the logic is mostly identical, but now the pressure work term will be zero, because in the new frame of reference $\overline{u'} = 0$. But the integral arising from term 4 will give exactly the same value as we got from term 3 in part A. Thus the new balance will be between work done by viscous stress on the moving boundaries (which adds energy), and the net loss of energy to dissipation.