Introduction to Fluid Dynamics

Midterm Exam Solutions 11/14/2007

1. Recall the steady, incompressible, 2D flow from Problem Set #2, defined by

$$(u,v) = D(x,-y)$$

where D is a constant with units s[-1]. The vertical velocity is zero, assume $\rho = \rho_0 = \text{const.}$, and that there is no gravity.

A[5]. What is the vorticity field for this flow?

The vorticity vector field has zero magnitude everywhere in space for this flow, because $\hat{k}\bullet\boldsymbol{\omega} = \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0$. This is just the vertical component of the vorticity vector. The other two components are also zero because all terms involve either w or $\partial/\partial z$, both of which are zero for this flow.

B[5]. What is the magnitude of the viscous term in the momentum equations?

The viscous terms are $\nu\nabla^2\mathbf{u} = \nu\left(\nabla^2 u, \nabla^2 v, \nabla^2 w\right)$. The third term is identically zero because there is no vertical velocity. The others are also zero, because the second derivative of the velocities is zero. For example:

$$\nabla^2 u = \frac{\partial^2(Dx)}{\partial x^2} + \frac{\partial^2(Dx)}{\partial y^2} + \frac{\partial^2(Dx)}{\partial z^2} = 0$$

Only the first term had a chance of being non-zero, but the linear increase of u does not change. If the viscosity is non-zero there IS viscous momentum (per unit mass) flux, given by

$$-\left(\nu u_x, \nu v_y\right) = -\nu(D,-D)$$

but this has the same flux into a point as out of a point (in either direction), so there is no accumulation of momentum due to viscosity.

C[5]. Justify the fact that the Bernoulli function is constant everywhere in this flow, and not just on a streamline (assume zero viscosity).

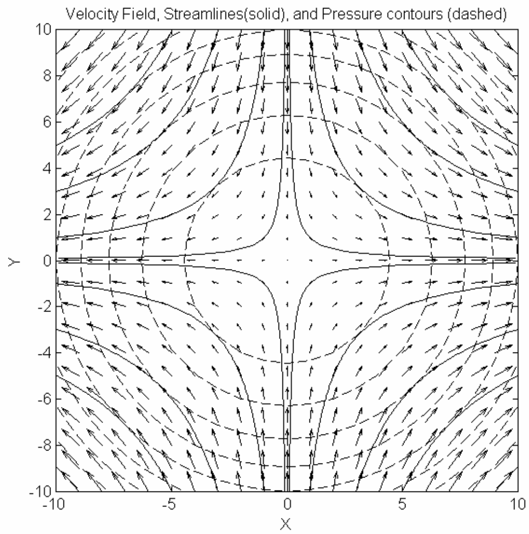
Since the flow is (i) steady, (ii) inviscid, and (iii) irrotational (zero-vorticity) it satisfies the conditions for having the Bernoulli function constant everywhere.

D[10]. Use Bernoulli to determine the pressure field $p(x,y)$ to within a constant. Sketch your answer along with some streamlines and velocity vectors.

since $\rho_0 B = \rho_0\left(u^2 + v^2\right)/2 + p = C = \text{const.}$, we may substitute in our values for the velocity to find

$$p(x,y) = C - \rho_0 D\left(x^2 + y^2\right)/2$$

Thus (I found this interesting and surprising) contours of constant pressure are just concentric circles centered at the origin. <u>The pressure is highest at the center</u>, and it drops off as the square of the radius (a parabolic hump). This looks like:



E[5]. Explain in words why the change in *magnitude* of the velocity along a streamline (and hence following a fluid parcel) makes sense in relation to the pressure field.
As a fluid parcel moves toward the origin from far away it is moving into higher pressure, so the pressure gradient along the streamline slows the parcel. Once it starts to move away from the origin is it moving down the pressure gradient, and so accelerates. Note that while the flow field is steady, the velocity *of a fluid parcel* does change over time.

F[5]. Explain in words why the change in *direction* of the velocity vector along a streamline (and hence following a fluid parcel) makes sense in relation to the pressure field.
For any parcel moving on a curved path (i.e. any path except those along the x or y-axes) there is always a <u>cross-path pressure gradient</u>, which will change the <u>direction</u> of the parcel velocity. The sense of change is that the parcel will turn toward lower pressure.

G[5]. Comment on the relative magnitudes of the cross-path and along-path pressure gradients at a point of maximum path curvature (anywhere on $y = \pm x$).

At this point there is zero along-path pressure gradient, but there is a finite cross-path gradient, so the pressure gradient is only cross-path. (To figure out if this is the point of maximum cross path gradient would take more work).

2. Consider a cup (a right circular cylinder with an open top) that is partially filled with water. The cup is floating in a large lake of density $\rho = \rho_0$, whereas the fluid in the cup is a bit denser, having density $\rho_{cup} = \rho_0 + \Delta\rho$.

A[10]. Derive an expression for the height of the water in the cup relative to that in the lake (HINT: the water surface level in the cup should be deeper than that of the lake, and the cup is tall enough above the water inside so that it does not sink).
There are many ways to go about this. Say the height of water in the cup is H, and the distance from the top of the water in the lake to the top of water in the cup is η (expected to be a negative number). Also assume that the area of the top or bottom of the cup is A. The buoyancy of the cup is given by the weight of the fluid it displaces, which is $\mathbf{F}_{BUOY} = \hat{k} g A (H - \eta)\rho_0$. The force of gravity on the water in the cup is $\mathbf{F}_{GRAV} = -\hat{k} g A H(\rho_0 + \Delta\rho)$. The sum of these two forces is zero (because the cup is not accelerating), so we may add them and solve for the surface height displacement

$$\eta = -H\frac{\Delta\rho}{\rho_0}$$

Note that we have to know the height of water in the cup, but we don't have to know the area A. Thus a skinnier, taller glass would sink deeper even with the same total volume of fluid in it. Also, doubling gravity would make no difference (alien boat builders take note!).

B[10]. If we drill a small hole in the bottom of the cup, which way will the pressure gradient cause water to flow through it? Justify your answer mathematically.
Integrating the hydrostatic equation outside the cup shows that the pressure in the outside water at the level of the base of the cup is equal to $p_{OUT} = p_{ATM} + \rho_0 g(H - \eta)$. The pressure inside the cup, at the bottom is similarly found to be $p_{IN} = p_{ATM} + (\rho_0 + \Delta\rho) g H$. Substituting in the value of η from the solution to 2A shows that these are identical. Thus if you drill a small hole in the bottom, pressure will not push the water in or out. (In real life however the situation would be gravitationally unstable, and the interface at the hole would start to deform, allowing drops of fluid to exchange in both directions.)

C[10]. If we drill a small hold in the side of the cup, which way will the pressure gradient cause water to flow through it? Justify your answer mathematically.

The full expression for pressure at any depth (below the fluid surface) in the cup is

$$p_{IN} = p_{ATM} + (\rho_0 + \Delta\rho) g (\eta - z)$$

where we have taken $z = 0$ at the lake surface. The expression outside the cup is

$$p_{OUT} = p_{ATM} + \rho_0 g(-z)$$

From these you can show that the pressure difference across the side of the cup, at any depth below the surface of fluid in the cup, is given by

$$p_{OUT} - p_{IN} = g\Delta\rho(H + z')$$

Where for convenience we have defined a new vertical coordinate $z' \equiv z - \eta$ which is the vertical position with zero at the surface of water in the cup. Note that z' is positive up, so it is a negative number in the cup, with magnitude always less than H. Thus the outside pressure is always greater than that inside the cup at any given depth, and this will cause water to flow <u>into</u> the cup through a small hole in the side.

3. Consider "Plane Poiseuille Flow" as described in Kundu and Cohen 9.4 (and Fig. 9.4d).

A[10]. Describe the momentum balance of a fluid parcel in words. Give your answer for a parcel at the center (midway between the two plates) and for one near one of the plates.

The only interesting momentum balance is x-mom. This only has two terms: the pressure gradient and the viscous stress divergence. For both parcels the pressure gradient pushes the parcel in the positive x-direction. Since the parcel does not accelerate there must be a force of equal magnitude pushing it in the negative x-direction. This force is the viscous diffusion of x-momentum in the y-direction. For the parcel at the center the viscous diffusion fluxes momentum away in positive and negative y-directions in equal measure. For a parcel near the bottom boundary the viscous momentum flux is downward (toward negative y), but the flux in from above the parcel is smaller in magnitude than the flux out below the parcel, so again there is a net loss of momentum due to viscosity.

B[10]. Why is the Bernoulli function not conserved along a streamline?

This flow does have vorticity, but because we are considering flow along a streamline this will not affect the integral which leads to the Bernoulli function. However the viscous term does not vanish on a streamline, and will lead to a decrease in B (sometime referred to as a "*loss of head*"). Mathematically we can still integrate along a streamline to find the change in B:

$$\left[\frac{1}{2}\mathbf{u}\bullet\mathbf{u} + \frac{p}{\rho} + gz\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \nu \frac{\partial^2 u}{\partial y^2} dx$$

For this flow $\partial^2 u/\partial y^2$ is constant. The only thing that can change in the Bernoulli function is the pressure, and so we see that the loss of pressure along a streamline may also be interpreted as a decrease of Bernoulli function due to viscosity.

C[10]. What terms are important in the mechanical energy balance equation (KE) for a fluid parcel? Give your answer just for a parcel at the center (midway between the two plates).

The full KE_V equation may be written in "Eulerian form" as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)=\nabla\bullet\left[-\mathbf{u}\bullet\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)-\mathbf{u}p+\nu\nabla\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)\right]+p(\nabla\bullet\mathbf{u})-\rho g w-\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}$$

For plane, steady Poiseuille flow this simplifies to

$$0=-u\frac{\partial p}{\partial x}+\nu\frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\rho\mathbf{u}\bullet\mathbf{u}\right)-\mu\left(\frac{\partial u}{\partial y}\right)^2$$

The physical meaning of the three terms are (i) pressure work divergence, (ii) convergence of viscous flux of KE_V, and (iii) rate of viscous dissipation. On the centerline only the first two terms are non-zero (because the shear $\frac{\partial u}{\partial y}$ is zero on that line). Thus we say that the net rate of pressure work on the parcel (which tries to increase its kinetic energy) is balanced by the viscous diffusion of kinetic energy away from the parcel, in the y-direction.