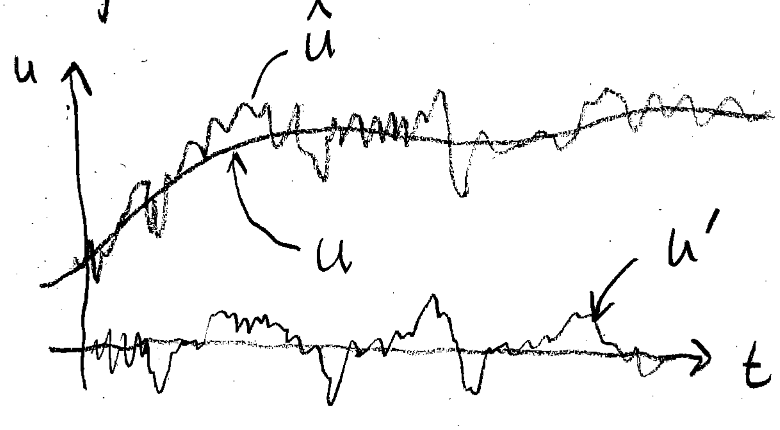
5.3

Effects of Turbulence: Ekman Layer

- The ocean + atm. are full of "turbulence," complex 3-D motions that take energy from large scale down to viscous dissipation (mm scales)
- Time scales less than $\frac{2\pi}{N}$, $\mathcal{O}(5 \text{ min.})$
- Space scales $\leq \mathcal{O}(100 \text{ m})$ atm + $\mathcal{O}(10 \text{ m})$ ocean (limited by stratification + Coriolis)
- The "velocity" we have used in this class is really the "Reynolds Averaged" velocity, meaning we have averaged out the turbulence:



$\hat{u} = u + u'$

$\hat{u}$ ← full velocity

u ← Reynolds Averaged velocity

u' ← turbulence

$$u \equiv \frac{1}{T}\int_t^{t+T} \hat{u} \, dt = \langle \hat{u} \rangle$$

use ⟨ ⟩ to denote the average

- Most terms in the equations are unchanged in form when we Reynolds average

e.g. $$\langle f\hat{v}\rangle = f\{\underbrace{\langle v\rangle}_{\equiv v} + \underbrace{\langle v'\rangle}_{0}\} = fv$$

Note: $\langle *'\rangle \equiv 0$

- But <u>non linear</u> terms are different, and some have persistent effects...

- most important is <u>vertical</u> eddy flux of horizontal momentum:

add $\hat{u}(\nabla\cdot\hat{\underline{u}}) = 0$

e.g. $$\frac{D\hat{u}}{Dt} = \hat{u}_t + \hat{\underline{u}}\cdot\nabla\hat{u} = \hat{u}_t + \nabla\cdot(\hat{u}\hat{\underline{u}})$$

so $$\left\langle\frac{D\hat{u}}{Dt}\right\rangle = \frac{\partial}{\partial t}\langle\hat{u}\rangle + \langle\hat{u}\hat{u}\rangle_x + \langle\hat{u}\hat{v}\rangle_y + \langle\hat{u}\hat{w}\rangle_z$$

And note $$\langle\hat{u}\hat{w}\rangle_z = \underbrace{\langle uw\rangle_z}_{(uw)_z} + \underbrace{\langle uw'\rangle_z}_{0} + \underbrace{\langle u'w\rangle_z}_{0} + \underbrace{\langle u'w'\rangle_z}_{\text{Not zero!}}$$

Doing this for all terms:

$$\left\langle \frac{D\hat{u}}{Dt} \right\rangle = u_t + \nabla\cdot(u\underset{\sim}{u}) + \langle u'u' \rangle_x + \langle u'v' \rangle_y + \langle u'w' \rangle_z$$

(1) (2) (3)

(3) >> (1) or (2) because:

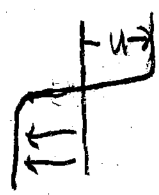
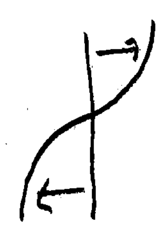
- u', v', w' all have same scale
- but $\frac{1}{H} >> \frac{1}{L}$ for our GFD-scale flows

- Term (3) is the divergence of vertical eddy flux of horizontal momentum / unit mass (also called "Reynolds stress" divergence
- We parameterize it as a "Fickean" diffusion:

with $\langle u'w' \rangle = -A \frac{\partial u}{\partial z}$ (downgradient)

A = "Eddy Viscosity" $\sim 10^{-2}\ m^2 s^{-1}$ Ocean. Boundary layer

$\sim 30\ m^2 s^{-1}$ atm. " "

Tends to decrease shear:  ⇒ 

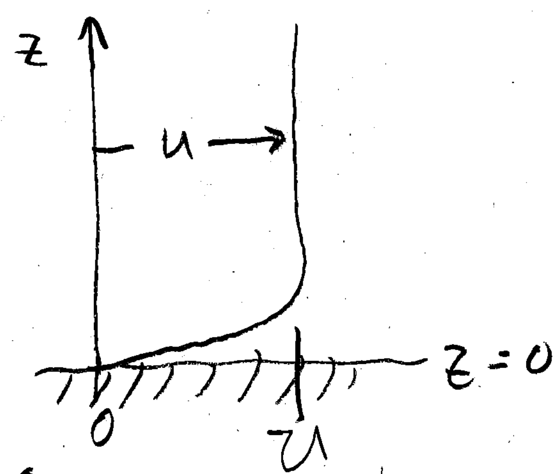
So the "Reynolds averaged" x, y mom are:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0} p_x + (A u_z)_z$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0} p_y + (A v_z)_z$$

For the "Ekman layer" problem, consider steady flow over a flat boundary, with "no-slip" b.c., and A = const. Large scale flow driven by pressure gradient: $f\mathcal{U} = -\frac{1}{\rho_0} p_y$

So our (linear) equations are



$\boxed{\text{x mom}}$ $-fv = A u_{zz}$

$\boxed{\text{y mom}}$ $fu = f\mathcal{U} + A v_{zz}$

b.c.'s (need 4) $u = v = 0$ at $z = 0$

$$\left.\begin{array}{l} u \to \mathcal{U} \\ v \to 0 \end{array}\right\} \text{ as } z \to \infty$$

Solution method: define $S = u + iv$ $\quad (iS = iu - v)$

+ form $\boxed{x \text{ mom}} + i \boxed{y \text{ mom}}$

$$\Rightarrow \quad ifS = ifU + AS_{zz} \quad \text{or} \quad \boxed{S_{zz} - \frac{if}{A} S = -\frac{if}{A} U}$$

- full solution = "particular" + "homogeneous" = $S^P + S^H$

- easy to guess $S^P = U$

- homogeneous $S^H_{zz} - \frac{if}{A} S^H = 0$

guess $S^H = S_0 \exp \alpha z$ $\quad$ (S_0 complex)

$$\Rightarrow \alpha^2 = \frac{if}{A} \quad \Rightarrow \alpha = \sqrt{i} \sqrt{\frac{f}{A}}$$

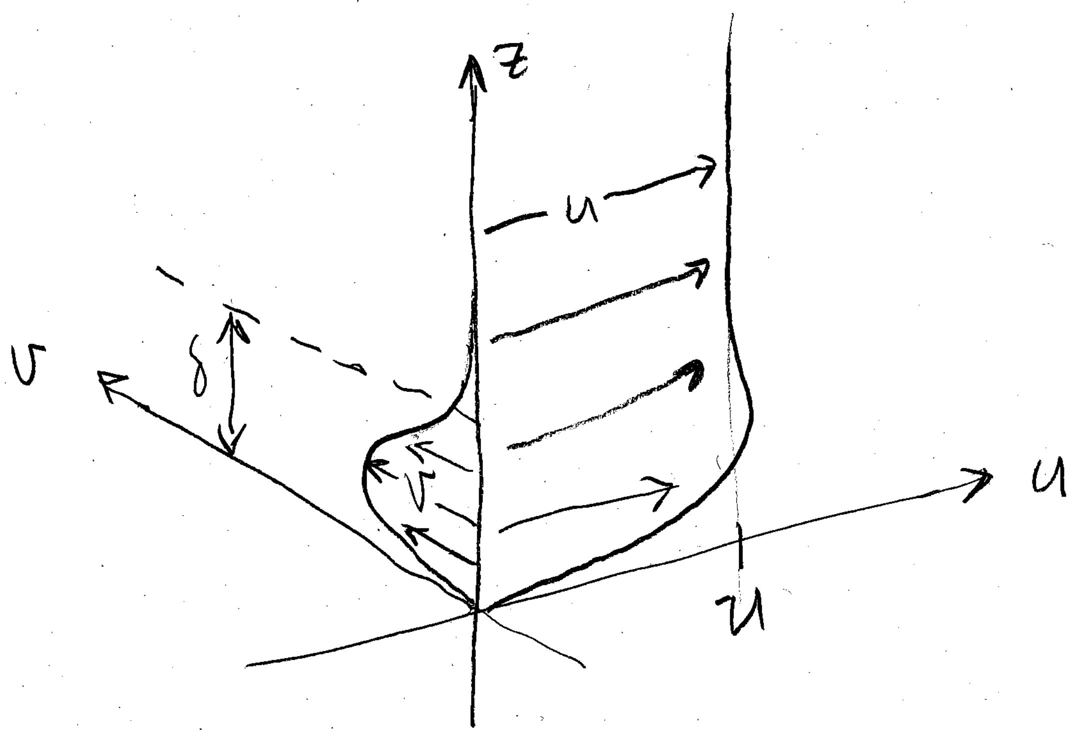
and $\sqrt{i} = \pm \frac{1}{\sqrt{2}} (1 + i)$ $\quad$ choose neg. root to get solutions that decay w/ z

$$\Rightarrow \quad \alpha = -\frac{(1+i)}{\delta} \quad \text{where } \delta = \sqrt{\frac{2A}{f}} = \text{"Ekman Layer Thickness"}$$

[see Appendix for solution details ...]

find $u = \mathcal{U}\left[1 - \exp\left(\frac{-z}{\delta}\right)\cos\left(\frac{z}{\delta}\right)\right]$

$$v = \mathcal{U}\exp\left(\frac{-z}{\delta}\right)\sin\left(\frac{z}{\delta}\right)$$



Bottom Boundary drag slows $\mathcal{U}$, and Coriolis pushes to the right, creating v

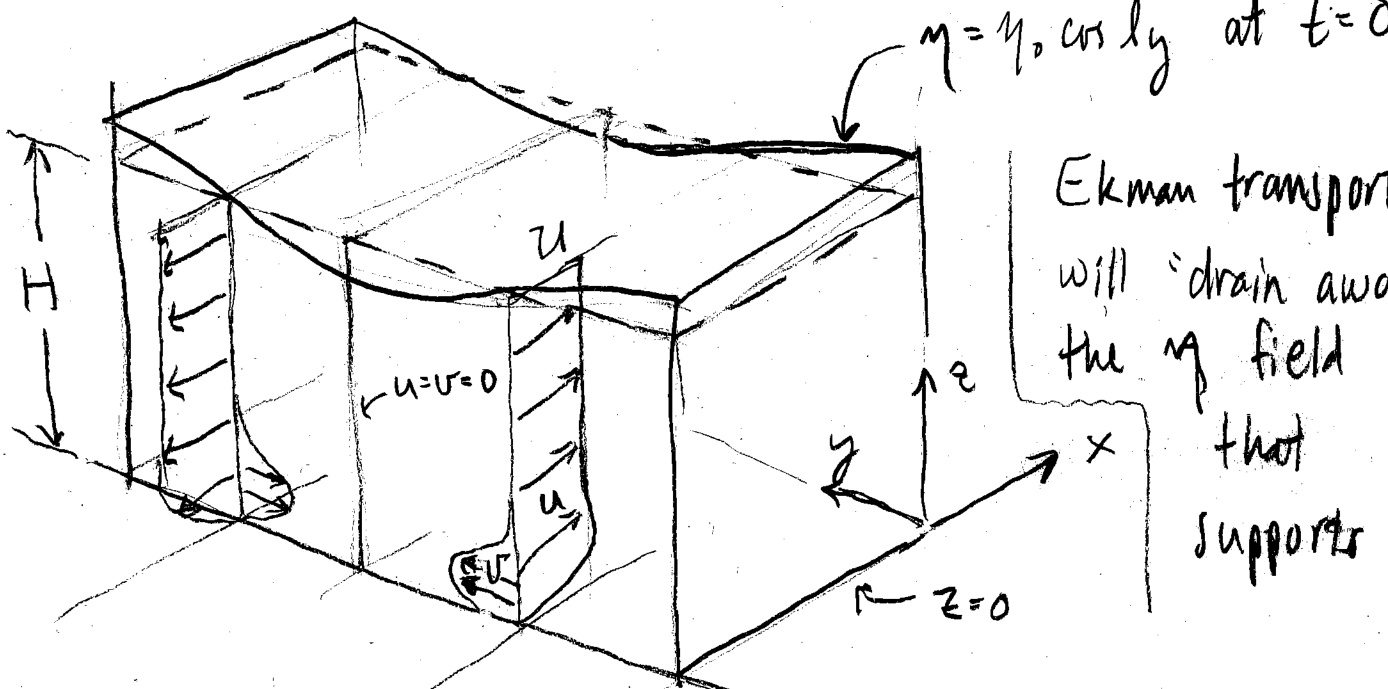
Note: $\boxed{y \text{ mom}}$ $\quad -fv = A u_{zz}$ $\qquad A u_z\big|_0 = \frac{\text{boundary stress}^x}{\rho_0}$

$$\Rightarrow \int_0^\infty v\,dz = -\frac{A}{f}\left(u_z\big|_\infty - u_z\big|_0\right) = \frac{A\mathcal{U}}{f\delta} = \frac{\text{stress}^x}{f\rho_0} = \text{"Ekman Transport"}$$

Note: δ = how far <u>diffusion</u> gets in a time f^{-1}

Ekman transport is down the large scale pressure gradient (recall $fU = \frac{-1}{\rho_0} p_y$) causing $\boxed{\text{Spin Down}}$

Consider a situation with $U = U(y)$



Ekman transport will "drain away" the η field that supports U.

$\boxed{\text{mass}}$ $\eta_t + \frac{\partial}{\partial y}\left[\int_0^{\eta} v \, dz\right] = 0$

As we did in the quasi-geostrophy lecture, we split v into two parts $v = v_g + v_a$, and since $\frac{\partial}{\partial x} = 0 \Rightarrow v_g = 0$

The ageostrophic v_a, also called the "secondary circulation" is due to two physical processes

- in the boundary layer friction is important
- in the interior time-dependence is important
- define: $v = \underset{0}{\cancel{v_g}} + v_a = v_a^E + v_a^I$

gives v_a^E (Ekman Layer)

$\boxed{\text{x mom}}$

$$u_t - fv = -\underset{0}{\cancel{g\eta_x}} + Au_{zz}$$

gives v_a^I (Interior)

from the Ekman Layer solution we know

$$\int_0^\eta v_a^E \, dz = \frac{A\mathcal{U}}{f\delta}$$

and then for the interior $v_a^I = \frac{1}{f}u_t$

$$\int_0^\eta v_a^I \cong \frac{H}{f}\mathcal{U}_t$$

then, since $\mathcal{U}$ is in ~geostrophic balance $\Rightarrow$ $\mathcal{U} = -\frac{g}{f}\eta_y$ (*)

so we may now write $\boxed{\text{mass}}$ as

$$\eta_t + \frac{A}{f\delta} u_y + \frac{H}{f} u_{yt} = 0 \quad , \quad \text{or using } (*)$$

$$\eta_t - \frac{Ag}{f^2\delta} \eta_{yy} - \frac{Hg}{f^2} \eta_{yyt} = 0$$

rearranging, and defining $a^2 = \frac{gH}{f^2}$ (Rossby Radius^2)

$$\Rightarrow \boxed{(\eta - a^2 \eta_{yy})_t - \frac{A}{\delta H} \eta_{yy} = 0}$$

and guess a solution of the form $\eta = \eta_0 \exp\left(-\frac{t}{\tau}\right) \cos ly$
(formally we would have used "separation of variables")

This yields the "spin down time" τ

$$\tau = \frac{(1 + a^2 l^2)}{a^2 l^2} \frac{2}{f} \frac{H}{\delta}$$

and note $\frac{1}{a^2 l^2} = \frac{\overline{APE}^y}{\overline{KE}^y}$

so there are two limits :

Large Lengthscale

$\frac{1}{l} \gg a$

APE dominate

$$\boxed{\tau \doteq \frac{1}{a^2 l^2} \frac{2}{f} \frac{H}{\delta}}$$

slow ⟵⟶ fast

Short Lengthscale

$\frac{1}{l} \ll a$

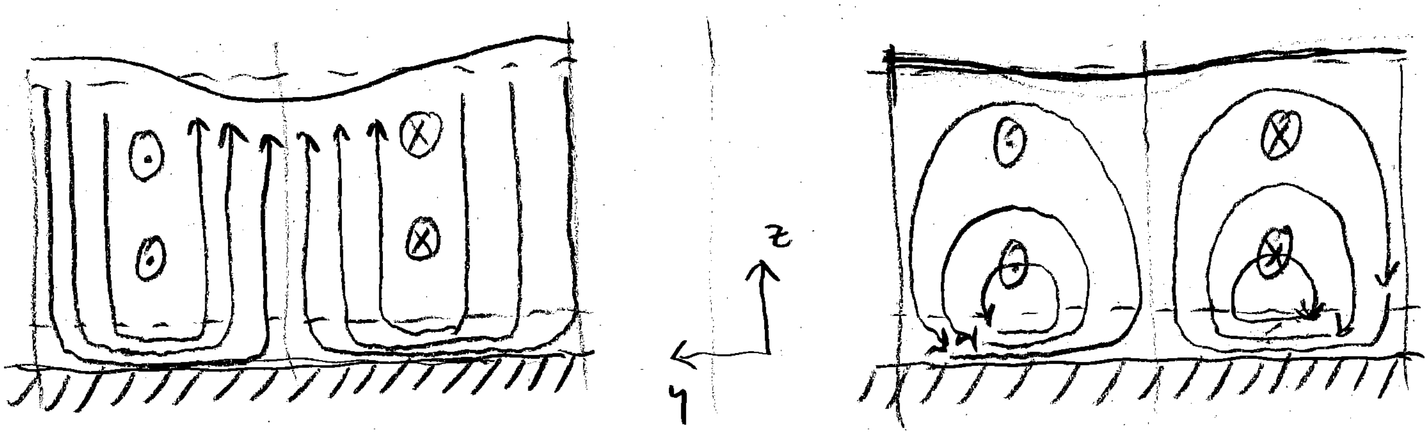
KE dominates

$$\boxed{\tau \doteq \frac{2}{f} \frac{H}{\delta}}$$

secondary circulation mainly lowers η

$(v_a^I \to 0)$

effectively a rigid lid and secondary circulation has $\int_0^\eta (v_a^E + v_a^I) = 0$



Some typical values (assume $\frac{1}{l} \ll a$: short Lengthscale case)
($f = 10^{-4}\,s^{-1}$)

	$\tau = \frac{2H}{f\delta}$	$\tau_{Diffusive} = \frac{H^2}{A}$ (from $u_t = A u_{zz}$)
Ocean $H = 10^3\,m$, $\delta = 10\,m$	$\rightarrow \tau = 20$ days	3 years
Atm. $H = 10^4\,m$, $\delta = 10^3\,m$	$\rightarrow \tau = 2$ days	1 month

$\Rightarrow$ Spin down is <u>fast</u> compared to frictional diffusion.

(but spin down is much slower for circulation with most of its energy "hidden" as APE – because the Ekman Layer is only driven by KE)

Appendix: Details of the Ekman Layer Solution

$$S = S^H + S^P = \overbrace{S_0}^{S_0^R + i S_0^I} \exp\left(\frac{-z}{\delta}\right)\underbrace{\left[\cos\left(\frac{z}{\delta}\right) - i\sin\left(\frac{z}{\delta}\right)\right]}_{\text{from: } \exp(-iz/\delta)} + \mathcal{U}$$

$$u = \operatorname{Re}\{S\} = S_0^R \exp\left(\frac{-z}{\delta}\right)\cos\left(\frac{z}{\delta}\right) + S_0^I \exp\left(\frac{-z}{\delta}\right)\sin\left(\frac{z}{\delta}\right) + \mathcal{U}$$

$$v = \operatorname{Im}\{S\} = -S_0^R \exp\left(\frac{-z}{\delta}\right)\sin\left(\frac{z}{\delta}\right) + S_0^I \exp\left(\frac{-z}{\delta}\right)\cos\left(\frac{z}{\delta}\right)$$

and at $z=0$:

$$\left.\begin{aligned} u &= S_0^R + \mathcal{U} = 0 \\ v &= S_0^I = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} S_0^R &= -\mathcal{U} \\ S_0^I &= 0 \end{aligned}$$

and this also satisfies $u \to \mathcal{U}$ as $z \to \infty$

leaving the full solution:

$$u = \mathcal{U}\left[1 - \exp\left(\frac{-z}{\delta}\right)\cos\left(\frac{z}{\delta}\right)\right]$$

$$v = \mathcal{U}\exp\left(\frac{-z}{\delta}\right)\sin\left(\frac{z}{\delta}\right)$$