Introduction to Fluid dynamics

Solutions for Problem Set #3, 11/1/2007

A. Note that the mathematical definition of the vertical average of the radial velocity over the gap is given by $\bar{u} \equiv \frac{1}{h}\int_0^h u\,dz$. Taking the vertical average of the mass conservation equation is straightforward, because h is not a function or r, and so the integral may freely pass through the radial derivative in the first term. Also, the vertical integral of $\frac{\partial w}{\partial z}$ just results in $w(z=h)-w(z=0)=\partial h/\partial t+0=h_t$. Thus we find

$$\frac{1}{h}\int_0^h\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}\right]dz=\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u})+\frac{h_t}{h}=0$$

Then, we may rearrange this to find an equation which may be integrated in the radial direction

$$\frac{\partial}{\partial r}(r\bar{u})=-r\frac{h_t}{h}$$

(noting that h and h_t are both independent of r, and hence may be treated as constants for this integration)

$$\int_0^r\frac{\partial}{\partial r'}(r'\bar{u})\,dr'=r\bar{u}-0=\int_0^r -r'\frac{h_t}{h}\,dr'=-\frac{1}{2}r^2\frac{h_t}{h}+0$$

And we have used a dummy variable r' because we retain r as a limit of integration. Thus we have

$$\bar{u}=-\frac{1}{2}\frac{rh_t}{h}$$

Note that h_t is negative in sign for a sinking disk, and so the radial velocity is positive (meaning outward from the center. Also, note that $\bar{u}$ is a function of r (explicitly in the equation above) and it is a function of time, through the temporal variation of h and h_t. It is not a function of z, of course.

B. The r-mom equation may be integrated directly twice in z (because we have assumed $\partial p/\partial r$ is not a function of z over the gap, yielding the general solution

$$u=\frac{1}{2}\frac{1}{\mu}\frac{\partial p}{\partial r}z^2+C_1z+C_2$$

Where C_1 and C_2 are constants of integration. These may be evaluated using the two boundary conditions: $u(z=0)=u(z=h)=0$ to find the desired solution

$$u=\frac{1}{2}\frac{1}{\mu}\frac{\partial p}{\partial r}\left(z^2-hz\right)$$

C. Taking the vertical average of the result from B, we find

$$\bar{u} = -\frac{1}{12}\frac{h^2}{\mu}\frac{\partial p}{\partial r}$$

D. Equating the results of A and C gives the following expression for the radial pressure variation

$$\frac{\partial p}{\partial r} = 6\frac{\mu h_t}{h^3} r$$

This may be integrated directly to find the pressure in the gap, to within a constant of integration, C:

$$p = \frac{3\mu h_t}{h^3} r^2 + C$$

E. The total (assumed) force balance on the disk consists of three vertical forces. The first is gravity: $F_g = -Mg$. The second is "buoyancy" which is the integral of hydrostatic pressure forces over the surface area of the disk. These are given by an upward force equal to the weight of water displaced by the disk, or $F_B = \rho_0 V g$. Note that this force already accounts for a constant pressure p_0 on the bottom face of the disk. So what we have to determine is the *additional* force on the bottom of the disk due to the viscous damping of the outward flow in the gap. We define the total pressure under the disk as $p = p_0 + p'(r,t)$, and since $p(r = R) = p_0$ we require $p'(r = R) = 0$. So our expression for the pressure from D may be written as

$$p' = \frac{3\mu h_t}{h^3} r^2 - p_0 + C$$

Using the condition $p'(r = R) = 0$ we may thus solve for C, finding

$$p' = \frac{3\mu h_t}{h^3}\left(r^2 - R^2\right)$$

and note that at $r = 0$ this is positive, because h_t is negative.

F. The upward force, F_p, due to p' is found by taking its integral over the plate area

$$F_p = \int_0^R 2\pi r p' dr = -\frac{3}{2}\frac{\pi\mu h_t R^4}{h^3}$$

Summing up the three forces on the disk and assuming they are in balance (i.e. the vertical acceleration of the disk is small compared to these other forces) gives $F_g + F_B + F_p = 0$. This may be solved (algebraically) to find the expression for the vertical velocity of the disk

$$h_t = -\frac{2}{3}\frac{h^3 g\left(M - \rho_0 V\right)}{\pi\mu R^4}$$

G. The expression above predicts that the disk will fall at $h_t = -1.8 \text{ m s}^{-1}$ for $h = 1 \text{ cm}$. This will decrease to $h_t = -1.8 \text{ mm s}^{-1}$ if the gap is 1 mm. When the gap is 0.1 mm the fall rate will be just $h_t = -1.8 \times 10^{-3} \text{ mm s}^{-1}$.

H. Using the code below to estimate the vertical acceleration at different heights, we may conclude that the acceleration term will be important for $h = 1 \text{ cm}$, but may be very safely neglected for the two smaller gaps. One could also estimate the sizes of some of the neglected terms in the r-mom equation, such as $u \partial u / \partial r$. (about 7 points per term considered, up to a maximum of 20).

```
% disk_accel.m  11/1/2007  Parker MacCready
% this gives simple estimates of the force due to acceleration of the disk
clear
% set constants
g = 9.8;    % [m s-1]
R = 1;      % [m]
rho0 = 1000;    % [kg m-3]
H = 0.05;       % [m]
V = pi*R^2*H;   % [m3]
mu = 1e-3;      % [gm m-1 s-1]
M = 1000;       % [kg]

disp(' ')
% set h (in a loop for different values)
for h = [0.01 0.001 0.0001]
    % calculate dhdt
    dhdt = -(2/3)*h^3*g*(M-rho0*V)/(pi*mu*R^4);
    % display result
    disp(['*********************************'])
    disp(['     h = ',num2str(h),' [m]']);
    disp(['  dhdt = ',num2str(dhdt),' [m s-1]']);
end

% calculate the error estimate in the vertical force balance
Fgb = g*(M-rho0*V);  % The downward force due to gravity and buoyancy [N]
% set h (in a loop for different values)
dt = 1e-10; % a small amount of time [s]
for h = [0.01 0.001 0.0001]
    % calculate dhdt
    dhdt = -(2/3)*h^3*g*(M-rho0*V)/(pi*mu*R^4);
    h2 = h + dhdt*dt;
    % calculate dh2dt
    dh2dt = -(2/3)*h2^3*g*(M-rho0*V)/(pi*mu*R^4);
    % calculate the acceleration and the size of the
    % resulting term in the vertical force balance
    acc = (dh2dt - dhdt)/dt;
    Facc = M*acc;
    % display result
    disp(['*********************************'])
    disp(['     h = ',num2str(h),' [m]']);
    disp(['  Facc = ',num2str(Facc),' [N]']);
    disp(['  Fgb = ',num2str(Fgb),' [N]']);
    disp(['  Facc/Fgb = ',num2str(Facc/Fgb),' [dimensionless]']);
end
```