Introduction to Fluid Dynamics

Final Exam Solutions 12/13/2007

1[total=25]. Assuming that the pressure is hydrostatic (and thus determined by the horizontal gradient of the surface height) …

A[15]. …what is the functional form of the surface height field for a Rankine vortex in water with a free surface?

%%% By integrating the hydrostatic Z-MOM equation, $\frac{\partial p}{\partial z} = -\rho_0 g$, from some vertical position z up to the free surface at $z = \eta(r)$ where the pressure is atmospheric, one finds: $p = p_{ATM} + \rho_0 g(\eta - z)$. Taking the radial derivative of this, which proves useful below, gives: $\frac{\partial p}{\partial r} = \rho_0 g \frac{\partial \eta}{\partial r}$ (*). Then, defining the (known) azimuthal velocity at the edge of the core of solid body rotation ($r = R$) as $u_\theta(R) = U$, one may write the expressions for the velocity field of a Rankine vortex as:

$$u_\theta = \frac{U}{R} r \qquad \text{for } r \le R$$

$$u_\theta = UR\frac{1}{r} \qquad \text{for } r > R$$

The final piece of information you need is that the only non-zero terms in the radial momentum equation are $-\frac{u_\theta^{\,2}}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}$. Substituting in the expression (*) above gives the desired relationship between the azimuthal velocity (that we know) and the surface height (that we want to know). This is $\frac{\partial \eta}{\partial r} = \frac{1}{gr} u_\theta^{\,2}$, and so for the two regions

$$\frac{\partial \eta}{\partial r} = \frac{U^2}{gR^2} r \qquad \text{for } r \le R$$

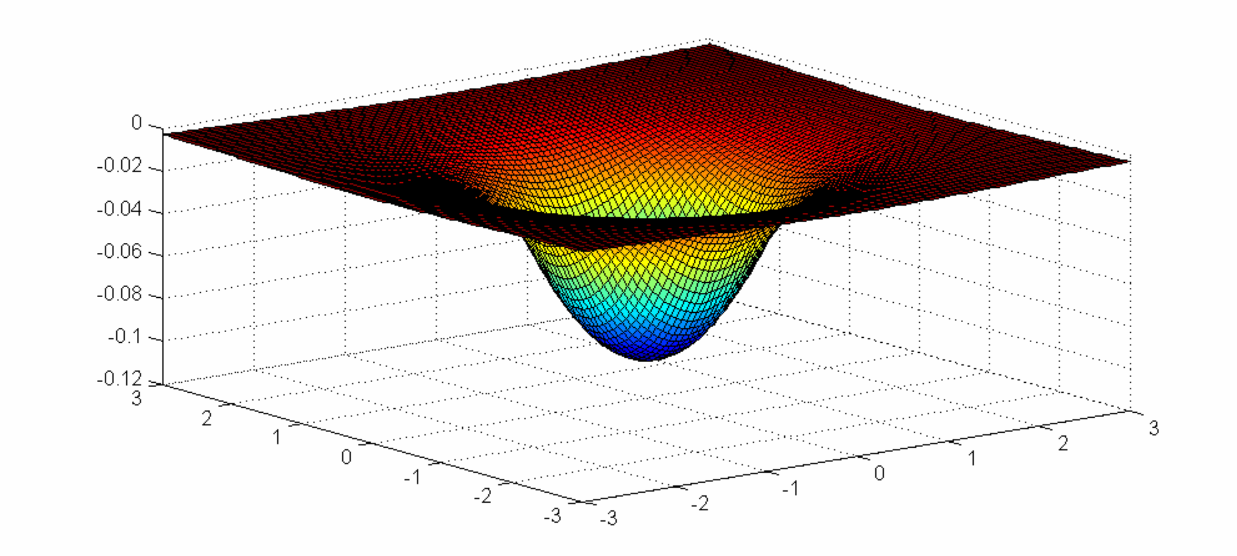
$$\frac{\partial \eta}{\partial r} = \frac{U^2 R^2}{g}\frac{1}{r^3} \qquad \text{for } r > R$$

These may be integrated in r. The constants of integration are found by (1) assuming that $\eta \to 0$ as $r \to \infty$, and (2) that the two solutions for η match at $r = R$. This gives the full solution:

$$\eta = -\frac{1}{2}\frac{U^2}{g}\left(2 - \frac{r^2}{R^2}\right) \qquad \text{for } r \le R$$

$$\eta = -\frac{1}{2}\frac{U^2}{g}\left(\frac{R^2}{r^2}\right) \qquad \text{for } r > R$$

This looks like a gentle "cone" that drops down toward the origin as r^{-2}, with a concave-upwards parabolic dish in the middle (in the region of solid body rotation).



This figure above is the surface height for a vortex with $R = 1$ m and $U = 1$ m s^{-1}. A strange property of the solution is that the change in surface height due to each part of the solution is the same.

B[10]. Assume that scales are similar to the "tidal headland eddy" formed by horizontal flow separation of tidal flow past a point in Puget Sound, as shown in the figure. Azimuthal velocity at the edge of the core of solid body rotation is about $u_\theta = 30$ cm s^{-1}, and the core has radius 500 m. What is the maximum deformation of the free surface height at the center of the eddy (make sure you specify the sign!), and what is its value at the edge of the core?

%%% Interestingly the value of R doesn't matter here! For the given value of U the surface height is given by:

$$\eta(r = 0) = -9 \text{ mm}$$

$$\eta(r = R) = -4.6 \text{ mm}$$

This is similar to the scales we have inferred for the surface height of the Three Tree Point headland eddy in Puget Sound.

2[25]. Sketch the distributions of velocity, pressure, and density in a sound wave. Discuss the energetics of a fluid parcel as sound waves pass through it. When (in relation to times of peak pressure) is kinetic energy added to the parcel, and when is internal energy added to it? Is the change of internal energy reversible?

%%% Assume that the velocity field for the sound wave is given by

$$u = u_0 \cos(kx - \omega t)$$

Then using the dynamical equations that we gave in class for sound waves

$$\text{X-MOM} \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\text{MASS} \quad \frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x}$$

It is straightforward to take derivatives and integrals of u to find expressions for the pressure and density, which are:

$$p = \frac{\rho_0 \omega}{k} u_0 \cos(kx - \omega t) + p_0 = p' + p_0$$

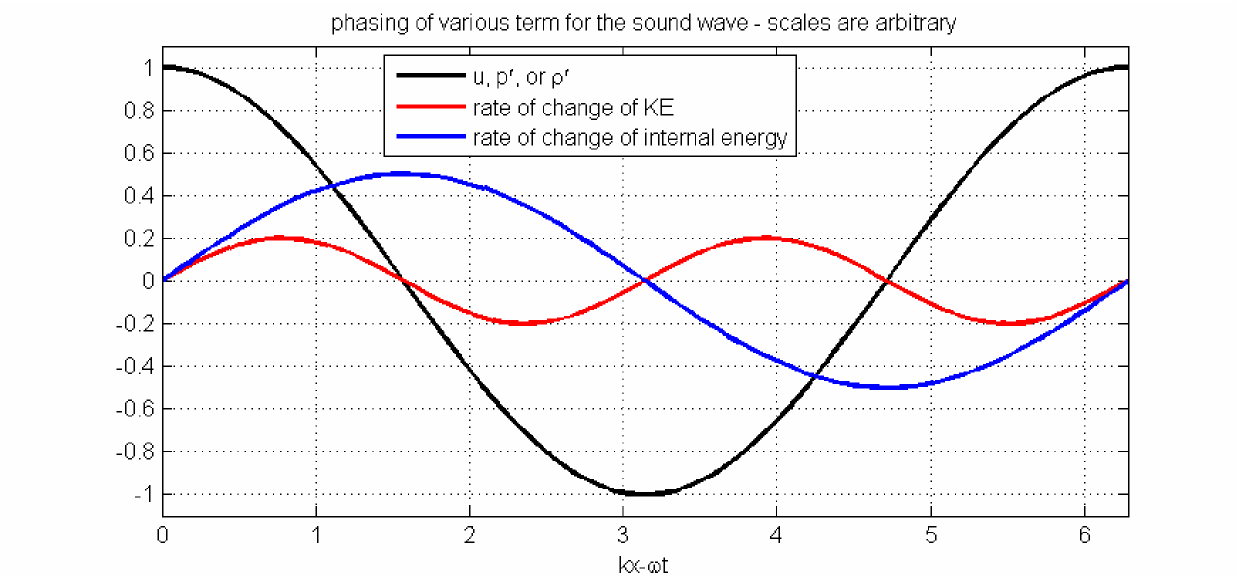
$$\rho = \rho_0 \frac{u_0}{c} \cos(kx - \omega t) + \rho_0 = \rho' + \rho_0$$

Note that we have taken the constants of integration to be the mean atmospheric pressure (p_0) and density (ρ_0). Both of these are assumed to be much larger than their respective perturbations associated with the sound wave (p' and ρ'). The statement about density perturbations being small is clearly consistent with the linearizing assumption that the sound wave velocity is much less than the sound speed $c = \omega/k$. So, in any case we may state from the above equations that the velocity, pressure and density perturbations are *all exactly in phase*, with maximum positive values of all three occurring at the same place (at a given time).

The phasing of the change in KE per unit mass for a fluid parcel as a wave passes by is governed by the linearized KE_M equation, which is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) = -\frac{1}{\rho_0} u \frac{\partial p'}{\partial x}$$

And then using our expressions for the velocity and pressure it is easy to show (using some trig identities) that either side of this is given by $\frac{1}{2}\omega u_0^{\,2} \sin\left[2(kx - \omega t)\right]$. This says that regions of maximum change in KE lie between peaks and nodes of velocity, as in the figure below:



The linearized equation for *internal* energy is given by

$$\frac{\partial e}{\partial t} = -\frac{p_0}{\rho_0}\frac{\partial u}{\partial x}$$

and thus it is easy to show that the RHS is given by $\frac{p_0 k u_0}{\rho_0}\sin(kx-\omega t)$. This is *out of phase* with the velocity, and corresponds with regions of maximum *rate of compression*. One may show that the magnitude of this term is much greater than the magnitude of the KE perturbation. Note also that regions of highest density (and pressure because they are in phase) will have a small positive temperature anomaly as well, for an ideal gas.

3[total=50]. Consider shallow water waves with bottom friction. In the "shallow water" limit (the wavelength, λ, is much greater than H, the undisturbed water depth) the velocity is predominantly horizontal ($u \gg w$) and is approximately independent of depth. Thus $u = u(x,t)$. The governing linearized equations are

$$\text{X-MOM} \qquad \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - Ru$$

$$\text{MASS} \qquad \frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

where $\eta(x,t)$ is the elevation of the fluid surface away from its rest position (i.e. it equals zero when the water surface is undisturbed). The term $-Ru$ is a simplified representation of the effect of bottom friction, and R is a constant with units of s^{-1}. Formally one would derive an equation like X-MOM above by taking a *vertical average* of the full x-momentum equation.

A[5]. Taking $R=0$ at first, combine X-MOM and MASS into a single "wave" equation for $\eta(x,t)$. This will have the form $\eta_{tt}-c^2\eta_{xx}=0$. This is a different way of getting to the shallow water limit of the surface gravity wave solution presented in class.

%%% By taking the x-derivative of X-MOM and the t-derivative of MASS, it is easy to eliminate the term u_{xt} between the two equations. This yields the desired "wave equation:"

$$\eta_{tt}-gH\eta_{xx}=0$$

which has wave speed $c=\sqrt{gH}$.

B[5]. Use the results of A to determine the expression for $\omega(k)$ (the dispersion relation). Hint: guess a solution of the form $\eta=\eta_0\,\mathrm{Re}\left\{\exp\left[i(kx-\omega t)\right]\right\}$. Are these waves dispersive? What is the wavelength if the wave period is 12 hours for water 10 m deep? This is like tides in a river/estuary channel.

%%% Plugging in the guess above yields the equation

$$\eta_0\,\mathrm{Re}\left\{(-i\omega)^2\exp\left[i(kx-\omega t)\right]-gH(ik)^2\exp\left[i(kx-\omega t)\right]\right\}=0$$

This will be true for the real part of { } (and for the imaginary part, as well, which represents sine solutions instead of cosines) if the coefficients inside sum to zero, i.e.: $(-i\omega)^2-gH(ik)^2=0$. This gives the dispersion relation:

$$\omega=k\sqrt{gH}$$

Note that I have taken the positive root. The choice is rather arbitrary – I prefer to always assume the frequency is positive (because time only goes in one direction) and then allow for the wavenumber to change sign which allows for waves traveling in different directions. The waves are non-dispersive because the phase speed $c=\omega/k=\sqrt{gH}$ is not a function of wavenumber. The wavelength for 12 hour period in 10 m deep water is easily found to be 428 km.

C[5]. For $u=u_0\cos(kx-\omega t)$, find an expression for u_0 in terms of c, η_0, and H, where $c\equiv\omega/k=(gH)^{1/2}$. What is u_0 for $\eta_0=1\text{ m}$ and $H=10\text{ m}$? Does this satisfy the linearization assumption $U/c\ll1$?

%%% Using X-MOM: $u_t=-g\eta_x$, and waves of the form $\eta=\eta_0\cos(kx-\omega t)$ and $u=u_0\cos(kx-\omega t)$, one finds $u_0=\eta_0 g\,k/\omega$. Using the known wave speed, and after some minor manipulation (involving multiplying by H/H) one arrives at the desired expression

$$u_0 = c\frac{\eta_0}{H}$$

This is worth memorizing, because the terms on the right are usually easy to know. Since the ratio $\eta_0/H = 0.1$ our linearizing assumption $u_0 \ll c$ is fairly well supported.

D[10]. Repeat A, but now assume non-zero R. The resulting equation will be different from the classical wave equation.

%%% The derivation is exactly as before, except that you will be left with a term involving Ru_x, which is annoying because you would like to have a PDE that is just in the single variable η. The trick is to use MASS again to change the term in u to one in η. This yields the new "wave-like" equation

$$\eta_{tt} + R\eta_t - gH\eta_{xx} = 0$$

E[10]. Repeat B with non-zero R. Hint: ω will be complex, and you may assume that the real part of the frequency (call this ω') is positive. Describe the behavior of the solution in words or sketches.

%%% Just repeating the process from B, the requirement that the coefficients inside $\{\ \}$ sum to zero gives the expression $\omega^2 + iR\omega - gHk^2 = 0$. This is a quadratic whose roots may be found in the usual way to find the frequency (which may be complex). This gives:

$$\omega = \sqrt{gHk^2 - \left(\frac{R}{2}\right)^2} - \frac{iR}{2} = \omega' - \frac{iR}{2}$$

where I have taken the positive value of the square root, on the grounds that this represents the real part of the frequency, which we will call ω'. The term inside the square root could itself be negative, for large enough R, but I will ignore this case.

The solution is thus

$$\begin{aligned}
\eta &= \eta_0 \operatorname{Re}\left\{\exp\left[i\left(kx - \omega' t + \frac{iR}{2}t\right)\right]\right\} \\
&= \eta_0 \operatorname{Re}\left\{e^{-Rt/2}\exp\left[i\left(kx - \omega' t\right)\right]\right\} \\
&= \eta_0 \operatorname{Re}\left\{e^{-Rt/2}\left[\cos\left(kx - \omega' t\right) + i\sin\left(kx - \omega' t\right)\right]\right\} \\
&= \eta_0 \cos\left(kx - \omega' t\right)e^{-Rt/2}
\end{aligned}$$

This is a progressive wave moving to the right at speed $c' = \omega'/k$, but decaying in amplitude exponentially in time, with e-folding time $2/R$. Note that the friction affects both the phase speed and the amplitude (for a given wavenumber). These waves are dispersive because the phase speed is a function of wavenumber:

$$c' = \sqrt{gH - \left(\frac{R}{2k}\right)^2}$$

F[10]. For waves like those in C, a reasonable value for R would be 10^{-3} s^{-1} (assuming a rough bottom). With this value of the bottom friction, how much is the phase speed $c' = \omega'/k$ different from $(gH)^{1/2}$? Assume that the wavelength is 100 km.

%%% For water 10 m deep, with the given values of R and k, the expression from E gives:

$$c' = 5.9 \text{ m s}^{-1}$$

which is about 60% of the undamped wave speed $c = \sqrt{gH} = 9.9 \text{ m s}^{-1}$.

G[5]. How much is the wave amplitude decreased by bottom friction in the time it takes the wave to travel one wavelength?

%%% The wave period is $T = \dfrac{2\pi}{\omega'} = 4.7 \text{ hours}$. Over this time the exponential decay is

$$e^{-RT/2} = 0.02\%$$

so these waves are almost completely gone after just one period. This case applies in many long tidal rivers, where the tidal wave only moves upstream with no reflection because it dissipates before it hits anything like a dam to reflect against.