Introduction to Fluid Dynamics

Problem Set #5

Assigned 11/21/2007, Due 11/28/2007 at the start of class

Consider the "solar chimney," an invention designed to capture solar energy and turn it into electricity. Info on the real (as-yet-unbuilt) device is available at http://www.solarmissiontechnologies.com/. You will consider the fluid mechanics of a slightly simplified version of this device, paying particular attention to how it might be affected by vortex dynamics.

The solar chimney is built over flat land, and consists of a large flat disk and a long tube.

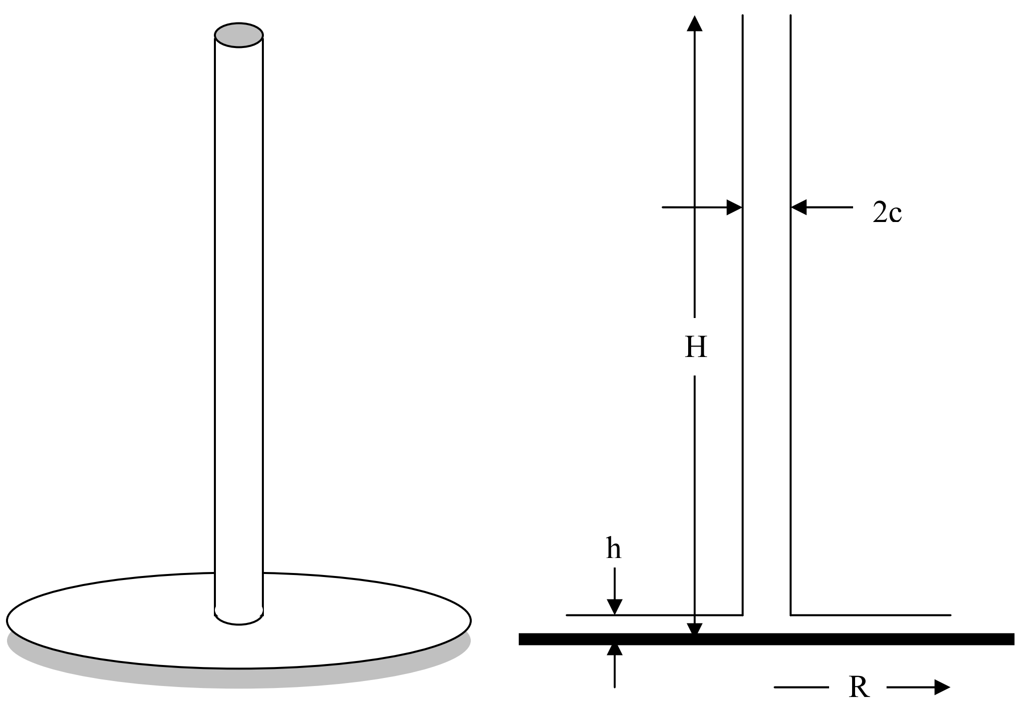

Figure 1. Sketch of the solar chimney, with some relevant dimensions

The disk has radius $R = 1 \text{ km}$, and is held a height $h = 10 \text{ m}$ above the ground. The tube radius is $c = 50 \text{ m}$, and it has a total height of $H = 1 \text{ km}$ above the ground (this is substantially taller than the world's tallest building!). Assume that the air in the tube is continually heated so that it always has a density difference $\Delta\rho = 0.1 \text{ kg m}^{-3}$ less than the outside air at the same height.

A[5]. What is the hydrostatic pressure difference at ground level between a point at the edge of the disk compared with a point at the center of the disk?
%%% This is simple because I gave you the density difference. It is just $\Delta p = g\Delta\rho H$.

B[10]. The air in the tube will want to rise, drawing air in radially from all directions under the disk. Assuming this process is steady and inviscid, and that air under the disk always has the same density ($\rho_0 = 1.2 \text{ kg m}^{-3}$), use Bernoulli to determine the increase in radial velocity of a fluid parcel as it moves from the edge of the disk ($r = R$) to the point where it gets to the edge of the tube ($r = c$).
%%% Here Bernoulli tells us that the exact expression for the *increase* in velocity is given by

$$\Delta u = \sqrt{\frac{2\Delta p}{\rho_0}\frac{1}{\left(1+\frac{2u_{R0}}{\Delta u}\right)}}$$

where u_{R0} is the radial velocity at $r = R$. So you can't solve this exactly without knowing u_{R0}. But by assuming $2u_{R0}/\Delta u \ll 1$ you arrive at the simpler expression

$$\Delta u = \sqrt{\frac{2\Delta p}{\rho_0}} \cong 40 \text{ m s}^{-1}$$

And note that this would imply $u_{R0} \cong \Delta u/20 \cong 2 \text{ m s}^{-1}$, so our error is only about 5%.

C[5]. Approximating the radial velocity at the edge of the disk as being zero, what is that volume flux through the tube?
%%% This is simply given by $Q = 2\pi c\,\Delta u\,h = 1.3\times 10^5 \text{ m}^3 \text{ s}^{-1}$.

D[10]. Now assume that the ambient winds have conspired to set up a small *circulation* in the air coming in under the edge of the disk (so there is a constant $u_\theta(r = R) \equiv u_{\theta 0}$, where u_θ is the fluid velocity in the azimuthal direction, as defined in Appendix B of Kundu and Cohen). Kelvin's circulation theorem says that a ring of fluid will keep its circulation as it moves. What is the resulting functional form for $u_\theta(r)$ between the edge of the tube and the edge of the disk?
%%% This is just given by

$$u_\theta = \frac{\Gamma}{2\pi}\frac{1}{r}$$

Where the circulation Γ must be big enough so that the condition at R is met. This requires that $\Gamma = 2\pi R u_{\theta 0}$.

E[5]. Show that even with this circulation the flow under the disk (which has both radial and azimuthal components) has zero vorticity. You need only consider the vertical component of vorticity.

%%% This is easy to show from

$$\zeta \equiv \hat{k}\bullet\boldsymbol{\omega} = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right) - \frac{1}{r}\frac{\partial u_R}{\partial \theta}$$

F[5]. Since the vorticity is zero, you may again use Bernoulli for the new flow field to find the maximum change in velocity that is supported by the pressure gradient, and of course it is the same as in B. But now the velocity when the fluid gets to the edge of the tube ($r = c$) has two components, and its magnitude is given by $|\mathbf{u}| = \left(u_R{}^2 + u_\theta{}^2\right)^{1/2}$. What is critical value of $u_{\theta 0}$ for which there will be zero net inflow (i.e. $u_R = 0$)?

%%% This will clearly require $u_\theta\left(r = c\right) \cong 40 \text{ m s}^{-1} = \frac{R}{c} u_{\theta 0} = 20 u_{\theta 0}$. And thus $u_{\theta 0} \cong 2 \text{ m s}^{-1}$.