Exercise 3.1. Prove by mathematical induction that the following property P holds for all natural numbers.

$$P(n) \iff \Sigma_{i=1}^{n}(2i-1) = n^2$$

(The notation $\Sigma_{i=k}^{l} s_i$ abbreviates $s_k + s_{k+1} + s_{k+2} + ... + s_l$ when k, l are integers with $k < l$).

We proceed by mathematical induction on n.

Base Case

We choose $P(1)$ as the base case. $P(0)$ is vacuously true; i begins at 1, so $\Sigma_{i=1}^{0}(2i-1)$ is not defined.

$$\begin{aligned} P(1) &\iff \Sigma_{i=1}^{1}(2i-1) = 1^2 \\ &\iff (2(1)-1) = 1^2 \\ &\iff 1 = 1 \end{aligned}$$

Inductive Case

Assume the proposition P holds for an arbitrary natural number m. We need to show that the proposition $P(m+1)$ also holds.

$$\begin{aligned} P(m+1) &\iff \Sigma_{i=1}^{m+1}(2i-1) = (m+1)^2 \\ &\iff \Sigma_{i=1}^{m}(2i-1) + (2(m+1)-1) = (m+1)^2 \\ &\iff m^2 + 2m + 2 - 1 = (m+1)^2 \\ &\iff m^2 + 2m + 1 = m^2 + 2m + 1 \end{aligned}$$

Thus: $\forall m \in \mathbb{N}, P(m) \implies P(m+1)$. We've proven the base case and the inductive case, completing the proof. QED

Exercise 3.2 A string is a sequence of symbols. A string $a_1 a_2 ... a_n$ with n positions occupied by symbols is said to have length n. A string can be empty in which case it is said to have length 0. Two strings s and t can be concatenated to form the string st. Use mathematical induction to show there is no string u which satisfies $au = ub$ for two distinct symbols a and b.

We proceed by mathematical induction on the length of u.

Base Case

For the base case, we choose u to be the empty string, which has length 0 by definition. Therefore:

$$au = a, ub = b$$

Since a and b are two distinct strings by definition ($a \neq b$), we can apply the transitive property to derive the following:

$$au = a \neq b = ub \implies au \neq ub$$

Inductive Case

For the inductive case, we need to show that, given an arbitrary string u of arbitrary length n satisfying $au \neq ub$, $au' \neq u'b$ holds with an arbitrary string u' of length $n + 1$. This is the inductive hypothesis.

Assume we make a single character addition to u to produce u' with length $n + 1$. Then we have that $au \neq au'$ and $ub \neq u'b$, because u and u' have different lengths by definition and therefore cannot be concatenated to produce the original strings au and ub. Applying our inductive hypothesis and the transitive property, we can then derive the following:

$$au' \neq au \neq ub \neq u'b \implies au' \neq u'b$$

We've proven the base case and the inductive case, completing the proof. QED

Exercise 3.4 Prove by structural induction that the evaluation of arithmetic expressions always terminates, i.e., for all arithmetic expression a and states σ there is some m such that $\langle a, \sigma \rangle \Downarrow m$.

We proceed by structural induction on cases of arithmetic expressions **AExp**.

Numeric literals m. If a is a numeric literal m, the only rule for its evaluation is $\langle a, \sigma \rangle \Downarrow m$. Thus, we've shown that $\langle a, \sigma \rangle \Downarrow m$ exists for this case.

Addition $a_0 + a_1$. If a is an addition expression, then its evaluation matches the structure of the following rule:

$$\frac{\langle a_0, \sigma \rangle \Downarrow m_0 \qquad \langle a_1, \sigma \rangle \Downarrow m_1}{\langle a_0 + a_1, \sigma \rangle \Downarrow m}$$

By the induction hypothesis applied to a_0 and a_1 we obtain the numbers m_0 and m_1, which together sum to the number m.

Subtraction $a_0 - a_1$. If a is a subtraction expression, then its evaluation matches the structure of the following rule:

$$\frac{\langle a_0, \sigma \rangle \Downarrow m_0 \qquad \langle a_1, \sigma \rangle \Downarrow m_1}{\langle a_0 - a_1, \sigma \rangle \Downarrow m}$$

By the induction hypothesis applied to a_0 and a_1 we obtain the numbers m_0 and m_1, whose difference is the number m.

Multiplication $a_0 \times a_1$. If a is a multiplication expression, then its evaluation matches the structure of the following rule:

$$\frac{\langle a_0, \sigma \rangle \Downarrow m_0 \qquad \langle a_1, \sigma \rangle \Downarrow m_1}{\langle a_0 \times a_1, \sigma \rangle \Downarrow m}$$

By the induction hypothesis applied to a_0 and a_1 we obtain the numbers m_0 and m_1, whose product is the number m.

This completes all cases $a \in \mathbf{AExp}$ and proves that:

$$\forall a \in A . \exists m \mid \langle a, \sigma \rangle \Downarrow m$$