대학원신입생세미나 과제1

통계학과 박현정 2015-20299

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Abstract

Probability Theory and Examples, Rick durret 교재의 $Theorem 1.6.8$을 정리.

1 Mesure theory

1.1 Expected Value

Theorem 1. *Suppose $X_N \to X a.s.$ Let g, h be contimuos functions with*

(i) $g \geq 0$ *and* $g(x) \to \infty$ *as* $|x| \to \infty$,

(ii) $|h(x)|/g(x) \to 0$ *as* $|x| \to \infty$ *and*

(iii) $Eg(X_n) \leq K < \infty$ *for all n.*

Then $Eh(X_n) \to Eh(X_n)$.

Proof. By subtracting a contrast from h, we can suppose without loss of generality that $h(0) = 0$. Pick M large so that $P(|X| = M)\, 0$ and $g(x) > 0$ when $|x| \geq M$. Given a random variable Y, let $\bar{Y} = Y 1_{(|y| \leq M)}$. Since $P(|X| = M) = 0, \bar{X_n} \to \bar{X}$ a.s. Since $h(\bar{X_n}$ is bounded and h is countinous, it follows from the bounded convergence theorem that

(a) $Eh(\bar{X_n}) \to Eh(\bar{X})$

(b) $|Eh(\bar{Y}) - Eh(Y)| \leq E|h(barY) - h(Y)| \leq E(|h(Y)|; |Y| > M) \leq_{\in M} Eg(Y)$ where $_{\in M} = \sup\{|h(x)|/g(x) : |x| \geq M\}$. To check the second inequality, not that when $|Y| \leq M, Y = Y$, and we have supposed $h(0) = 0$. The third inequality follows from the definition of $_{\in m}$. Taking $Y = X_n$ in (b) and using (iii), it follows that

(c) $|Eh(\bar{X}_n) \; - \; Eh(X_n)| \leq K_{\in M}$

To estimate $|Eh(\bar{X}_n) \; - \; Eh(X_n)|$, we observe that $g \geq 0$ and g is continuous, so Fatou's lemma implies

$$Eg(X) \leq \liminf_{n\to\infty} Eg(X_n) \leq K$$

Taking $Y \; = \; X$ in (b) gives

(d) $|Eh(\bar{X}) \; - \; Eh(X)| \leq K_{\in M}$

The triangle inequality implies

$$\begin{aligned} |Eh(X_n) \; - \; Eh(X)| &\leq |Eh(X_n) \; - \; Eh(X)| \\ &+ |Eh(\bar{X}) \; - \; Eh(\bar{X})| + |Eh(\bar{X}) \; - \; Eh(X)| \end{aligned}$$

Taking limits and using (a), (c), (d), we have

$$\limsup_{n\to\infty} |Eh(X_n) \; - \; Eh(X)| \leq 2K_{\in m}$$

which proves the desired result since $K < \infty$ and $_{\in M} \; \to \; 0$ as $M \to \infty$.

$\square$