1 Probability

The "probability", $P(x)$, of an event x occurring is a measurement of how "possible" x is using a scale of 0 to 1:

- An event x with a probability $P(x) = 0$ is impossible (x cannot happen).
- An event x with a probability $0 < P(x) < 1$ is "possible" (x may happen).
- An event x with a probability $P(x) = 1$ is certain (x must happen).

The closer $P(x)$ is to 1, the more likely it is that x will occurr.

The closer $P(x)$ is to 0, the less likely it is that x will occurr.

1.1 $P(tails)$

What's the probability of flipping a coin and getting tails? We intuitively know there is a 50% chance of this event occurring, but how do we demonstrate/prove/explain this? In other (more mathematical) words, if we let x be the event "flip a coin and get tails" then how do we measure $P(x) = 0.5$?

1.2 A theoretical model for $P(tails)$

When you flip a coin in the real world many events could happen:

- the coin could land with heads facing up (heads).
- the coin could land with tails facing up (tails).
- the coin could land on its side (yes, this can happen... Google it!)
- the coin could land in such a way that we can't tell which side is up (it falls between the floor boards or something like that)
- the coin could be stolen before it lands (by a thief)
- the coin could be disappeared before it lands (by a regime)
- the coin could be disintegrated before it lands (by a trusty blaster)
- some other unlikely event...

If we only consider the first two events (heads or tails) then we can construct a simple probability model that involves a set of two possible outcomes {heads, tails}. If we assume that each of these outcomes is equally likely, then we can determine that the probability of flipping tails should be exactly 0.5:

$$P(tails) = \frac{|\{tails\}|}{|\{heads, tails\}|} = \frac{1}{2} = 0.5$$

Notation: If A is a set, then $|A|$ is the number of elements in A.

1.3 Measuring $P(tails)$ experimentally

Of course we haven't actually measured the probability of flipping tails; we are relying on a simple model that assumes the coin cannot land on its side (or be stolen, disappeared, disintegrated, etc...) and that the chances of flipping heads or tails are the same.

We can validate our simple model by flipping a coin many, many times and recording the number of times we flipped tails. The measurement of the probability would be the number of times we flipped tails divided by the total number of flips. Simulating this in python gives:

```python
import random

def flip_coin(n):
    tails_count=0
    for i in range(n):
        tails_count+=random.randint(0,1) #0 is heads, 1 is tails
    print('P(tails) ≈', tails_count,'/',n,'=',tails_count/n)

flip_coin(10)
flip_coin(100)
flip_coin(1000)
flip_coin(1000000)
```

```
P(tails) ≈ 6 / 10 = 0.6
P(tails) ≈ 52 / 100 = 0.52
P(tails) ≈ 505 / 1000 = 0.505
P(tails) ≈ 499929 / 1000000 = 0.499929
```

So the more flips we do experimentally, the closer our measurement gets to the theoretical probability of 0.5.

1.4 A theoretical model for $P(x)$

Let S be the sample space consisting of all events that can occur.
Let E be a subset of S such that E only contains events in which x occurs.
If each event in S is equally likely, then the probability of x is:

$$P(x) = \frac{|E|}{|S|}$$

1.5 Measuring $P(x)$ experimentally

Randomly sample elements from the sample space S, over and over again, and record the number of times you get an element that is also in the event space E. The measurement of the probability $P(x)$ would be the number of times you got an element that was an element in E divided by the total number times you sampled an element from S.

1.6 Example

What's the probability of rolling a pair of 6-sided dice and getting a sum of 3?

The sample space consists of all 36 possible rolls:

$$\begin{aligned} S &= \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \\ &= \{(1,1),(1,2),(1,3),...,(6,4),(6,5),(6,6)\} \end{aligned}$$

The event space consists of the two ways of rolling a sum of 3:

$$E = \{(1,2),(2,1)\}$$

The probability of rolling a pair of 6-sided dice and getting a sum of 3 is:

$$\begin{aligned} P(\text{sum of } 3) &= \frac{|E|}{|S|} \\ &= \frac{|\{(1,2),(2,1)\}|}{|\{(1,1),(1,2),(1,3),...,(6,4),(6,5),(6,6)\}|} \\ &= \frac{2}{36} \\ &\approx 0.0556 \end{aligned}$$

We can measure $P(\text{sum of } 3)$ experimentally by simulating rolling a pair of 6-sided dice, over and over again, and checking each time if the sum is 3:

```
import random

def roll_pair(n):
    sum_count=0
    for i in range(n):
        d1=random.randint(1,6) #roll first dice
        d2=random.randint(1,6) #roll second dice
        if d1+d2==3:           #check if sum is 3
            sum_count+=1
    print('P(sum of 3) ≈', sum_count,'/',n,'=',sum_count/n)

roll_pair(10000000)
```

```
P(sum of 3) ≈ 555857 / 10000000 = 0.0555857
```

Exercises

1.1 What's the probability of rolling:

a) a 6-sided dice and getting a five?
b) a 6-sided dice and getting a two or a three?
c) a pair of 6-sided dice and getting a sum of five?
d) a pair of 6-sided dice and getting a sum of eleven or twelve?

1.2 A standard deck of 52 cards is shuffled and a single card is drawn. What is the probability that this card is:

a) the Ace of Spades?
b) not the Ace of Spades?
c) a ten?
d) a Joker?
e) a red suit?
f) a face card?
g) a prime number in a black suit?

1.3 A box of 12 laptops contains 6 Acers, 5 Lenovos, and 1 Mac. One laptop is selected at random from the box.

a) What is the probability that it is a Lenovo?
b) What is the probability that it is a Mac?
c) What is the probability that it is an Acer?
d) What is the probability that it is an HP?

1.4 A pair of 4-sided dice is rolled. Find the probability of rolling a sum of:

a) 2
b) 7
c) 5
d) 12

1.5 Let S_4 be the set of all strings of length 4. Find $|S_4|$ if the strings only contain:

a) uppercase and lowercase letters
b) lowercase letters and digits
c) letters in the word "internet"

1.6 Let S_3 be the set of all strings of length 3 made with letters in the word "likely". If an element $s \in S_3$ is selected at random, what's the probability that:

a) $s =$"lie"?
b) s starts with the letter "y"?
c) s does not contain the letter "e"?
d) s contains a single letter "k"?

1.7 Let $T = \{2x \mid 3 \leq x \leq 7, x \in \mathbb{Z}\}$. If an element $y \in T$ is selected at random, find the probability that:

a) $y = 7$
b) $y = 8$
c) $y \neq 10$
d) $y \geq 11$

1.8 Let A=$\{yoda, ben, luke, rey\}$ and D=$\{red, blue, yellow\}$. If an element x is randomly selected from D $\times$ A, find the probability that:

a) $x = (blue, ben)$
b) $x = (green, yoda)$

Answers (rounded to 4 decimal places)

1.1 a) 0.1667
b) 0.3333
c) 0.1111
d) 0.0833

1.2 a) 0.0192
b) 0.9808
c) 0.0769
d) 0.0000
e) 0.5000
f) 0.2308
g) 0.1538

1.3 a) 0.4167
b) 0.0833
c) 0.5000
d) 0.0000

1.4 a) 0.0625
b) 0.1250
c) 0.2500
d) 0.0000

1.5 a) 7311616
b) 1679616
c) 625

1.6 a) 0.0080
b) 0.2000
c) 0.5120
d) 0.3840

1.7 a) 0.0000
b) 0.2000
c) 0.8000
d) 0.4000

1.8 a) 0.0833
b) 0.0000

2 Conditional Probability

Determining the probability of a sequence of events involves conditional probability. For example if we have two events x and y, the probability of x occurring and then y occurring is given by the probability of x multiplied by the probability of y given that x has occurred:

$$P(x \text{ and } y) = P(x) \times P(y|x)$$

2.1 Example: A sequence of characteristics

What is the probability of selecting a card that is a Queen and a Spade?

We have that $P(\text{Queen}) = \frac{4}{52}$ because there are 4 Queens in a deck.

We also have that $P(\text{Spade} \mid \text{Queen}) = \frac{1}{4}$ because one of the 4 Queens is a Spade.

Then $P(\text{Queen and Spade}) = \frac{4}{52} \times \frac{1}{4} = \frac{1}{52}$.

Alternatively we could have noted that there is only one card in a deck that is the Queen of spades, so the probability must be $P(\text{Queen and Spade}) = \frac{1}{52}$.

2.2 Example: A set of characteristics

What is the probability of selecting a card that is a Queen or a Spade?

We have that $P(\text{Queen}) = \frac{4}{52}$ because there are 4 Queens in a deck.

We also have that $P(\text{Spade}) = \frac{13}{52}$ because there are 13 Spades in a deck.

For the last example we have that $P(\text{Queen and Spade}) = \frac{1}{52}$.

Then $P(\text{Queen or Spade}) = P(\text{Queen}) + P(\text{Spade}) - P(\text{Queen and Spade}) = \frac{4+13-1}{52} = \frac{16}{52}$

Alternatively we could have noted that there are sixteen cards in a deck that are Queens or spades, so the probability must be $P(\text{Queen or Spade}) = \frac{16}{52}$.

2.3 Example: at least one roll of 6

What's the probability of rolling a 6-sided dice three times and getting at least one roll of 6 in the three rolls?

We start by noting that the probability of rolling a 6 is $P(6) = \dfrac{1}{6}$, and the probability of not rolling a 6 is $P(\text{not } 6) = \dfrac{5}{6}$. Then we could compute the probability of all the possible cases of getting at least one roll of 6 in the three rolls:

- $P(\text{first roll is } 6) = \dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{5}{6} = \dfrac{25}{216}$
- $P(\text{second roll is } 6) = \dfrac{5}{6} \times \dfrac{1}{6} \times \dfrac{5}{6} = \dfrac{25}{216}$
- $P(\text{third roll is } 6) = \dfrac{5}{6} \times \dfrac{5}{6} \times \dfrac{1}{6} = \dfrac{25}{216}$
- $P(\text{first and second rolls are } 6) = \dfrac{1}{6} \times \dfrac{1}{6} \times \dfrac{5}{6} = \dfrac{5}{216}$
- $P(\text{first and third rolls are } 6) = \dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{1}{6} = \dfrac{5}{216}$
- $P(\text{second and third rolls are } 6) = \dfrac{5}{6} \times \dfrac{1}{6} \times \dfrac{1}{6} = \dfrac{5}{216}$
- $P(\text{first and second and third rolls are } 6) = \dfrac{1}{6} \times \dfrac{1}{6} \times \dfrac{1}{6} = \dfrac{1}{216}$

Adding up all the cases gives: $P(\text{at least one roll of } 6) = \dfrac{25+25+25+5+5+5+1}{216} = \dfrac{91}{216}$

2.4 Complement probability

Another (computationally simpler) method is to compute the complement probability of $P(\text{at least one roll of } 6)$, which is the probability of never rolling a 6 on all three rolls:

$$P(\text{never roll a } 6) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$$

Then we subtract this from 1 to obtain the probability of rolling at least one 6:

$$\begin{aligned} P(\text{at least one roll of } 6) &= 1 - P(\text{never roll a } 6) \\ &= 1 - \frac{125}{216} \\ &= \frac{91}{216} \end{aligned}$$

Exercises

2.1 Cards, marbles, and coins:

a) What's the probability of selecting a King and then another King if the cards are not replaced after selecting them?

b) A bag contains 3 green marbles and 4 red marbles. What's the probability of selecting a green marble and then selecting a red marble assuming the marbles are not replaced after selecting them?

c) What's the probability that when a coin is flipped three times, it lands tails, then heads, then tails?

2.2 What is the probability that when a coin is flipped six times in a row:

a) It lands heads every time?

b) It lands tails at least once?

c) It lands tails exactly twice?

2.3 What is the probability of selecting a queen and then a jack from a standard deck if:

a) the cards are not replaced after selecting them?

b) the cards are replaced (and shuffled) after selecting them?

2.4 What is the probability that a 6-sided dice never comes up even when it is rolled five times?

2.5 What is the probability that a randomly selected card from a standard deck of 52 cards is:

a) an ace or a king of spades

b) a club given that it is a black suit?

c) an ace or a heart?

2.6 A box of 12 laptops contains 6 Acers, 5 Lenovos, and 1 Mac. Two laptops are selected at random from the box without replacement.

a) What is the probability the first is a Mac and the second is an Acer?

b) What is the probability that both are a Lenovo?

c) What is the probability that an Acer and a Lenovo are selected (in any order)?

2.7 Suppose a math professor arrives early for class 24% of the time, leaves late 12% of the time, and arrives early and leaves late 7% of the time. What is the probability that for any given class the professor arrives early or leaves late (or both)?

2.8 A number is selected at random from $\{x \in \mathbb{Z}^+ \mid x \leq 20\}$. What is the probability that the number is:

a) a multiple of 3?

b) a multiple of 3, given than it's a multiple of 4?

2.9 A coin is flipped three times. What is the probability that the first flip was tails given that exactly two of the three flips were tails?

Answers (rounded to 4 decimal places)

2.1 a) 0.0045
b) 0.2857
c) 0.1250

2.2 a) 0.0156
b) 0.9844
c) 0.2344

2.3 a) 0.0060
b) 0.0059

2.4 0.0313

2.5 a) 0.0962
b) 0.5000
c) 0.3077

2.6 a) 0.0455
b) 0.1515
c) 0.4545

2.7 0.2900

2.8 a) 0.3000
b) 0.2000

2.9 0.6667