CS325 Practice problem set 5

This problem set covers linear programming and reduction.

1. Consider the problem PDV 7.1, please reduce the linear problem to a different form of linear program in which all variables must non-negative, the constraints are all equations, and the objective is to be minimized.

$$\textit{minimize} - 5x - 3y$$
$$5x - 2y - s_1 = 0$$
$$x + y + s_2 = 7$$
$$x + s_3 = 5$$
$$x, y, s_1, s_2, s_3 \geq 0$$

   Note that to turn each inequality into a equality, we introduce one slack variable. If the original inequality is $\leq$, the slack is added to the leftside. If the original inequality is $\geq$, then the slack will be substracted from the left side. It is important to include the non-negative constraint for the slack variables.

2. Consider the following problem:

$$\max x_1 + x_2 \text{ s.t. } |x_1 - x_2| \leq 10$$

   Can you solve this problem with a linear program? If so, how? *This problem can be reduced to the following LP:*

$$\max x_1 + x_2 \textit{ s.t. } x_1 - x_2 \leq 10, x_1 - x_2 \geq -10$$

   This is a simply because $|x_1 - x_2| \leq 10$ is equivalent to $x_1 - x_2 \leq 10$ and $x_1 - x_2 \geq -10$.

3. Consider the following problem:

$$\min \max\{x_1, x_2, x_3\} \text{ s.t. } 3x_1 + 2x_2 - 5x_3 \leq 8$$

   Can you solve this problem with a linear program? If so, how? *This problem can be reduced to the following LP:*

$$\min y$$

   subject to

$$x_1 \leq y$$
$$x_2 \leq y$$
$$x_3 \leq y$$
$$3x_1 + 2x_2 - 5x_3 \leq 8$$

   To see this, we essentially introduced y to replace $\max\{x_1, x_2, x_3\}$. Now because y is the max of x_1, x_2, x_3, we have all three of them must be $\leq y$. And these three new inequalities are equivalent to $y = \max\{x_1, x_2, x_3\}$ given that we aim to minimize y, the minimal y that satisfy all three new inequalities is exactly $\max\{x_1, x_2, x_3\}$.

4. PDV 7.5 *[a.] Let F be the number of packages of Frisky Pup and H of Husky Hound.*

$$\max 7F + 6H - F - 2 * 1.5F - 2H - 2H - 1.4F - 0.6H = 1.6F + 1.4H$$

$$\textit{Subject to}$$

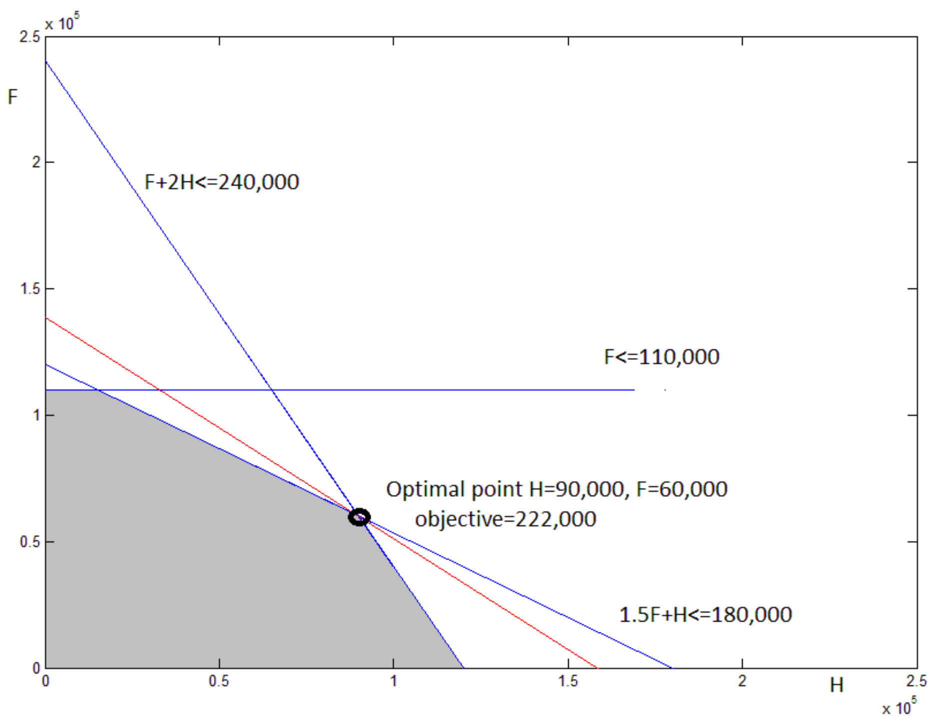
$$F + 2H \le 240,000$$

$$1.5F + H \le 180,000$$

$$F \le 110,000$$

$$F, H \ge 0$$

[b.] The shaded area in the following figure shows the feasible region. The red line shows equal-value line for the objective at the value of $222,000$*. Moving this line up and down will increase and decrease the objective value respectively. Moving this line further up will cause it to go out of the feasible region. Thus the optimal solution is at* $F = 60,000$ *and* $H = 90,000$ *with the objective valued at* $220,000$*.*



5. Please formulate the unbounded knapsack problem into a linear program. *The input to a knapsack problem includes a budget* B*, and a list of* n *items with their values and weights specified by* v_i *and* w_i*,* $i = 1, ..., n$*.*

To formulate late into a linear program, we introduce one variable x_i for each item i to represent the number of units that we take of item i. The objective is

$$\max_{x_i : i=1,\ldots,n} \sum_{i=1}^{n} v_i x_i$$

Subject to the constraint that:

$$\sum_{i=1}^{n} w_i x_i \leq B$$

and

$$x_i \geq 0 \text{ for } i = 1, ..., n$$

Note that knapsack typically assumes that x_i's are integers. This results in an integer linear program, which is much harder to solve.

6. Consider the following two problems.
   In the **hitting set** problem, we are given a family of sets $\{S_1, S_2, ..., S_n\}$ and a budget b, and we wish to find if there exists a set H of size $\leq b$ that intersects every S_i.

   In the **vertex cover** problem, we are given a graph $G = (V, E)$ and a budget b, and we wish to find if there exists a vertex cover S of size $\leq b$ such that S touches every edge in E.

   Show that the vertex cover problem can be reduced to the hitting set problem.
   To reduce the vertex cover problem to the hitting set problem, we assume there is a solver for the hitting set problem. Given an instance of a vertex cover problem, we need to construct an instance of the hitting set problem so that we can use the solver for hitting set to solve the vertex cover problem. The construction works as follows.

   Given a graph $G = (V, E)$ and a budget k, for each edge $e = (u, v) \in E$, we construct a set $S_e = \{u, v\}$. We can show that S is a vertex cover of size $\leq k$ for graph G, if and only if S is a hitting set of size $\leq k$ for the collection of S_e's.
   $\Rightarrow$ *:*
   If S is a vertex cover, for every edge $e = (u, v) \in E$, we must have $u \in S$ or $v \in S$ and $S_e = \{u, v\} \cap S \neq \phi$. Thus S is a hitting set of the collection of all S_e's.
   $\Leftarrow$ *:*
   If S is a hitting set, for every edge $e = (u, v) \in E$, we must have $S_e = \{u, v\} \cap S \neq \phi$, thus either $u \in S$ or $v \in S$. So S must be a vertex cover of G.