22 FEB 2019

SECTION 4.6 – A NEAR-MINIMAX APPROXIMATION METHOD

GOAL: ASSUME f IS CONTINUOUS ON $[a,b]$. FIND $x_0, x_1, \ldots, x_n$ IN $[a,b]$ SO THAT THE POLYNOMIAL p OF DEGREE $\le n$ THAT INTERPOLATES f AT $x_0, x_1, \ldots, x_n$ IS ALMOST THE MINIMAX APPROXIMATION OF f OF DEGREE $\le n$ ON $[a,b]$

ie $\max\limits_{a \le x \le b} |f(x) - p(x)|$ IS ALMOST THE SMALLEST POSSIBLE

FOR NOW ASSUME $[a,b] = [-1,1]$

RECALL: THE ERROR IN INTERPOLATION IS GIVEN BY $f(x) = p(x) + \underbrace{\underbrace{(x-x_0)(x-x_1)\ldots(x-x_n)}_{\Psi_n(x)} \frac{f^{(n+1)}(c_x)}{(n+1)!}}_{\text{ERROR}}$

PLAN: CHOOSE $x_0, x_1, \ldots, x_n$ SO AS TO MINIMIZE $\max\limits_{-1 \le x \le 1} |\Psi_n(x)|$

$\Psi_n(x) = x^{n+1} + (\text{LOWER DEGREE TERMS})$

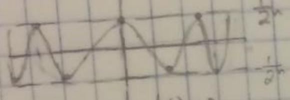
FROM SECTION 4.5, $\max |\Psi_n(x)|$ IS MINIMIZED IFF $\Psi_n(x) = \frac{1}{2^n} T_{n+1}(x)$ SO LET $x_0, x_1, \ldots, x_n$ BE THE ROOTS OF $T_{n+1}(x)$

$x_0, x_1, \ldots, x_n$ ARE THE ROOTS OF Ψ_n

THEN $(x-x_0)(x-x_1)\ldots(x-x_n)$ IS A CONSTANT MULTIPLE OF T_{n+1}

THE METHOD: INTERPOLATE f AT THE ROOTS $x_0, x_1, \ldots, x_n$ OF T_{n+1} WITH A POLYNOMIAL OF DEGREE $\le n$

THEN $\Psi_n(x) = (x-x_0)(x-x_1)\ldots(x-x_n) = \frac{1}{2^n} T_{n+1}(x)$



Ψ_n HAS $n+1$ ROOTS

$\therefore \Psi_n$ HAS n EXTREMA (MAX OR MIN) BETWEEN THE ROOTS

ALSO, Ψ_n HAS 2 EXTREMA AT THE END POINTS

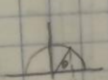
$\therefore \Psi_n$ HAS $n+2$ EXTREMA IN $[-1,1]$, OF EQUAL MAGNITUDE & OSCILLATING SIGN

$\therefore \Psi_n$ HAS THE EQUIOSCILLATION PROPERTY

$\therefore$ THE ERROR $f-p$ ALMOST HAS THE EQUIOSCILLATION PROPERTY

$\therefore p$ IS ALMOST THE MINIMAX APPROXIMATION OF DEGREE $\le n$ ON $[-1,1]$

FIND THE CHEBYSHEV POINTS $x_0, x_1, \ldots, x_n$



$x = \cos\theta$

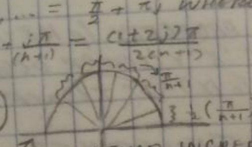
$0 \le \theta \le \pi$

$T_{n+1}(x) = \cos[(n+1)\underbrace{\arccos x}_{\theta}] = \cos[(n+1)\theta]$

$\cos[(n+1)\theta] = 0$ IF $(n+1)\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = \frac{\pi}{2} + \pi j$ WHERE $j \in \mathbb{Z}$

$\theta = \frac{\pi}{2(n+1)} + \frac{j\pi}{(n+1)}$, SO LET $\theta_j = \frac{\pi}{2(n+1)} + \frac{j\pi}{(n+1)} = \frac{(1+2j)\pi}{2(n+1)}$

LET $x_j = \cos(\theta_j) = \cos\left(\left(\frac{2j+1}{2n+2}\right)\pi\right)$



ALL OTHER INCREMENTS ARE $\frac{\pi}{n+1}$ EXCEPT END ONES ON SIDES ARE $\frac{1}{2}\left(\frac{\pi}{n+1}\right)$

$x_0 = \cos\theta_0$

$x_n = \cos\theta_n$

$x_0, x_1, \ldots, x_n$ ARE DISTINCT POINTS IN $[-1,1]$.

IF $j > n$ OR $j < 0$, x_j DUPLICATES ONE OF THESE. SO USE

$x_j = \cos\theta_j = \cos\left(\frac{(2j+1)}{(2n+2)}\pi\right)$ FOR $0 \le j \le n$

THIS IS EQUATION 4.102 ON p 174