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Monday, March 11, 2019

Final exam: Wednesday, March 20, 6:00 – 7:50 pm

Read. In Section 5.3 (Gaussian numerical integration), read pages 220, 223, Example 5.3.3 on pages 224–225.

Section 5.2 Error formulas

Consider trapezoidal and Simpson's rule approximations to $\int_a^b f(x)\,dx$

Trapezoidal rule (Theorem 5.2.1, page 204)

If f'' is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = (\text{Trap. rule approximation}) + \underbrace{\left(-\frac{h^2(b-a)}{12} f''(c_n)\right)}_{\text{error}}$$

for some c_n in $[a,b]$. Here, $h = \frac{b-a}{n}$

Simpson's rule (Theorem 5.2.5, p 208)

If $f^{(4)} = f''''$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = (\text{Simpson's rule approx}) + \underbrace{\left(-\frac{h^4(b-a)}{180} f^{(4)}(c_n)\right)}_{\text{error}}$$

Exercise (do not turn). Assume $f'' < 0$ on $[a,b]$. Use a graph to explain why the sign of the error for trapezoidal rule makes sense.

Suppose the stepsize h is halved

Trapezoidal rule: $h^2 \to \left(\frac{1}{2}h\right)^2 = \frac{1}{4}h^2$

The error may be multiplied by $\frac{1}{4}$

Note: (1) This assumes f'' is continuous on $[a,b]$

(2) $f''(c_n)$ could change if h is halved.

(3) For periodic f, the convergence can be faster than this. See Table 5.1 on page 194.

Simpson's rule. $h^4 \to \left(\frac{1}{2}h\right)^4 = \left(\frac{1}{2^4}\right)h^4 = \frac{1}{16}h^4$

This assumes $f^{(4)}$ is continuous on $[a,b]$.

Warning: See Example 5.2.7 and Table 5.4 on page 209; $\int_0^1 \sqrt{x}\,dx$

Asymptotic formula for the error in the trapezoidal rule

(from Atkinson, An Intro to Numerical Analysis, Section 5.5)

Theorem (Euler-Maclaurin formula)

Assume that f has continuous derivatives of order $\le 2m+2$ on $[a,b]$, and $h = \frac{b-a}{n}$

Let $E_n^T(f)$ = error in trapezoidal rule with stepsize h

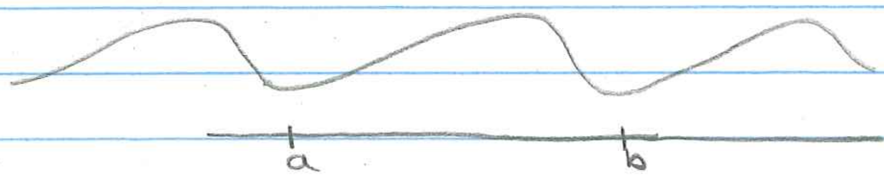
Then

$$E_n^T(f) = A_2 h^2 \left[f'(b) - f'(a) \right] + A_4 h^4 \left[f'''(b) - f'''(a) \right] + \dots + A_{2m} h^{2m} \left[f^{(2m-1)}(b) - f^{(2m-1)}(a) \right] + O(h^{2m+2})$$

where $A_2, A_4, A_6, \dots$ are constants with $A_j \to 0$ as $j \to \infty$

__Notation.__ An expression "$g(h) = O(h^p)$" means $\exists$ constant C such that $|g(h)| \le C|h|^p$ $\forall h$.

__Exercise__ (do not turn in). What happens if f is periodic with period $b-a$?



__Richardson extrapolation__

(Also discussed briefly in Section 5.2.3)

This leads to "Romberg integration", which is not in the text

Let $I = \int_a^b f(x)\,dx$.

Let $I_n^{(0)}$ = Trapezoidal rule approx with n subintervals and stepsize $h = \frac{b-a}{n}$

Let $I_{n/2}^{(0)}$ = " " $\frac{n}{2}$ " " " $2h$

Good to here

Assume that f has enough derivatives for the following to be valid.

$$2^2 \left[I = I_n^{(0)} + c_2 h^2 + c_4 h^4 + c_6 h^6 + \dots \right]$$

($c_2, c_4, c_6, \dots$ depend on f, a, b but not h.)

$$- \left[I = I_{n/2}^{(0)} + c_2 (2h)^2 + c_4 (2h)^4 + c_6 (2h)^6 + \dots \right]$$

(Same $c_2, c_4, c_6, \dots$)

$$(2^2 - 1) I = \left(2^2 I_n^{(0)} - I_{n/2}^{(0)} \right) + h^2 \cdot 0 + \text{constant} \cdot h^4 + \text{constant} \cdot h^6 + \dots$$

$$I = \underbrace{\left[\frac{4}{3} I_n^{(0)} - \frac{1}{3} I_{n/2}^{(0)} \right]}_{\text{Call this } I_n^{(1)}} + c_4^{(1)} h^4 + c_6^{(1)} h^6 + \dots$$

These depend only on f, a, b.

This involves stepsizes h and $2h$

Exercise (do not turn in). Show that $I_n^{(1)}$ is the Simpson rule approximation with stepsize h.

Keep going

$$2^4 \left[I = I_n^{(1)} + c_4^{(1)} h^4 + c_6^{(1)} h^6 + c_8^{(1)} h^8 + \dots \right]$$

$$- \left[I = I_{n/2}^{(1)} + c_4^{(1)} (2h)^4 + c_6^{(1)} (2h)^6 + \dots \right]$$

$$(2^4 - 1) I = \left(2^4 I_n^{(1)} - I_{n/2}^{(1)} \right) + h^4 \cdot 0 + (\text{constant}) \cdot h^6 + (\text{const}) h^8 + \dots$$

$$I = \underbrace{\left(\frac{16}{15} I_n^{(1)} - \frac{1}{15} I_{n/2}^{(1)} \right)}_{\text{Call this } I_n^{(2)}} + c_6^{(2)} h^6 + c_8^{(2)} h^8 + \dots$$

This involves stepsizes $h, 2h, 4h$