BOOK 10: FUNCTIONS AND MODELS II

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

If you were told by a classmate that taking a cup of brackish lukewarm water three times a day for a month would heal your broken leg, you would probably laugh out loud. And even though your laughter might be a bit more subdued were the source of that statement the health segment of the evening news, I bet that you would nonetheless be quite skeptical. On many levels this would seem to you to be highly implausible, and so you might either absolutely dismiss it or yield to that nagging curiosity and check out some other sources.

On the other hand there are a host of other more reasonable sounding claims that you might be prepared to let slide. Examples might include the claim that an increase in oil prices pushes a decline in the Dow Industrial averages, or that there is a strong correlation between wealth and SAT scores. It doesn't seem unreasonable to hold a healthy skepticism for these as well.

In the context of mathematics, to **check for plausibility** is to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations.

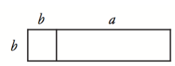
1

Your classmate proposed traveling 240 miles to the beach at an average speed of 80 mph, and the next day traveling home at 40 mph. He pointed out that since you have an average speed of 60 mph, the total 480 miles there and back takes 8 hours. Yet when you went to the beach, it didn't take 8 hours round trip, even though you two drove at precisely the speeds he had proposed. Why?

check for plausibility

It certainly seems at first glance that your classmate is correct, but just because a result seems superficially correct doesn't make it so. When there is an apparent contradiction in a problem, it often pays to make a quick check or two to see if what is being said is plausible. For example, how long does each half of the beach trip take?

2 Martin says, looking at the figure below, that the entire rectangle's area divided by the area of the square inside is equal to $\frac{a}{b} + 1$. Melissa thinks that such a peculiarly simple answer is unlikely to be correct. Martin insists he's right, and that in fact his formula would work for any values of a and b one might choose. How could Melissa check to see if Martin's formula is at least plausibly correct?



One particular way that we check the reasonableness of a solution to a problem is to **examine special and limiting cases**. In the problem above, what would those cases be? Well, one special case would surely be when $a = b$, because you could quickly determine the right answer and check if Martin's formula is correct there. What about if we examined when "a" was small and "b" was large—say $a = 1$ and $b = 100$, or even $a = 1$ and $b = 10000$? Does Martin's formula make sense in those cases as well? Can you come up with another limiting case, and see if his formula is plausible for that case as well?

Once you have checked a solution for all the limiting cases you can think of, if it still seems like a reasonable solution, you might think about how to prove it always works yourself from first principles. Can you come up with Martin's formula?

3 If one looks at $y = x^2$ on the calculator in "ZOOM SQR" mode (so that the scale in the x and y directions are the same and the graph looks most accurate), it appears that the graph is getting so steep, so quickly, that it will eventually become a vertical line. Does it? If so, estimate for what x value it becomes vertical; if not, explain why that can never happen.

4 While teaching in Brazil, Tony was approached by a fellow math teacher who said the following: "Hey Tony, do you know how to prove that all right triangles are similar? I was trying to show my students in class today and I couldn't quite do it." Could you have helped him out?

5 On a 3-D blueprint for an Olympic swimming pool, 1 ft. represents 16 actual feet. In order to determine how much water would be needed to fill the pool, Tim computes the volume from the blueprint, which is 4 ft^3, and then he multiplies by the scale factor of 16 to get 64 ft^3 of water to fill the pool. Tim is a little unsure if that is the correct amount, but it seems right. What do you think?

6 If Jorge told you that 3.16227765 was an exact solution to $x^2 = 10$, how could you determine without a calculator if he is correct, or slightly off?

7 Hero of Alexandria came up with a formula to determine the area of any triangle based solely on the lengths of its 3 sides. Below are 4 formulas, all of which purport to be Hero's formula. In all 4 formulas, a, b, and c are the lengths of the sides, and s is the semi-perimeter, which is equal to half the perimeter.

$$Area = (s-a)\,(s-b)\,(s-c)$$

$$Area = \sqrt{\frac{(s)(a)(b)(c)}{10}}$$

$$Area = \sqrt{(s)(s-a)(s-b)(s-c)}$$

$$Area = \sqrt{3(a+b+c)}$$

a. Which of these formulas do you think is the right one? Why?

b. Try seeing if the formulas give the kind of answers you would expect for various "common" triangles you have experience with.

c. Test to see if "extreme" triangles (ones with very large or small values of some of the sides) also give reasonable results.

d. What units do you typically measure area in? Does that also help you in deciding which formulas are most plausible?

Problem continued on the next page

e. Now pick the formula you are most confident is the right one. Does the fact that it has passed all your "tests" prove that it is correct? If so, explain why. If not, explain how you could become convinced that Hero was in fact correct.

8 Look at these 6 numbers: 1, 3, 6, 9, 11 and 12. Their mean is 7. Subtracting the mean from each of the numbers and adding those together gives us $-6 + -4 + -1 + 2 + 4 + 5$, which equals 0. Juniper is unimpressed, and says that you would always get 0, regardless of the 6 numbers you chose. Sassafras disagrees, and says it is highly dependent on choosing the right 6 numbers; for example, in this case, exactly 3 were above the mean and exactly 3 were below, and also there were no decimals to complicate matters. Who is right?

9 Zargo says that instead of doing lots of intricate calculations, he can find the area of a rhombus by just multiplying the diagonal lengths together. See if you can determine in a minute if his method is plausible.

10 If you draw a line from the vertex of any triangle to the midpoint of the opposite side (i.e. the median), will it be perpendicular to that side, or would it bisect the vertex angle from which it was drawn?

11 A pollster interviewed 100 families, and reported that the mean number of children was 2.037 and the median was 1.8. I do not believe either of these figures. Do you? Why?

12 Brian thinks he remembers that the area of a parallelogram is equal to the product of consecutive sides, but he isn't quite sure. You can't remember whether he's right either, but you know you can check his formula to see if it is plausible. Is it?

13 Show that the formula for the area of a triangle can be viewed as just a special case of the area of a trapezoid.

14 Which is bigger, $4^{\frac{1}{4}}$ or $10^{\frac{1}{10}}$? No calculators allowed! (Hint: try thinking about limiting cases.)

15 Hero's formula gives us a formula for the area of a triangle based only on the lengths of its 3 sides (see problem 7 in this lesson). No one has yet come up with a formula for the area of a quadrilateral based only on the lengths of its 4 sides. Why do you think that is?

16

a. Veneeta graphs $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 10$ on the same calculator screen using ZOOM STD. What does she notice about how their shapes compare to each other?

b. Grunchik changes the viewing window on his calculator so that it graphs x values from -4 to 4, and y values from -10 to 20. He then graphs the same three equations that Veneeta graphed. What does he notice about how their shapes compare to each other?

c. Veneeta and Grunchik compared their different answers, but aren't sure what to make of them. What do you think?

17

Prashad says that in a quadrilateral ABCD he has examined, AB + BC + CD + DA is equal to 1.8 times the diagonal AC. Evaluate whether what he says is possible or not.

18

a. When an object falls under gravity, its speed increases by a constant amount each second. Two stones are dropped at the same time from a cliff, but one of them is 10 feet higher up than the other at the time of dropping. As they fall, will the distance between them always be the same?

b. Later on, two stones are dropped at the same height from a cliff, but one stone is released one second before the other. As they fall, will the distance between them always be the same?

19

On the same screen as $y = x^2$, graph $y = x$ in ZOOM STANDARD. Then "ZOOM IN" once.

Note that the graph of $y = x$ is "above" the graph of $y = x^2$ for some values of x.

Hermione thinks that those values of x will be the only ones where $y = x$ is above $y = x^2$. What do you think?

check for plausibility

20 Bart is feeling a little sick. Having recently read about simpsonitis, a very rare and debilitating disease, and being somewhat hypochondriacal, he goes to see his physician, Dr. Kalvakian. The doctor checks him out and decides to administer a special blood test for detecting the disease.

This diagnostic test is 98% accurate (returns a positive result) for people who have simpsonitis and 95% accurate (returns a negative result) for people who do not have simpsonitis. Approximately 0.3% of people in the country actually have this disease.

Unfortunately, several days after taking the blood test, Bart receives a phone call from Dr. Kalvakian. The doctor tells him that he tested positive for the disease. Bart asks, "What's the chance that I actually have simpsonitis? I mean, you said that the test was not 100% accurate." Dr. Kalvakian replies, "Well, there's a 98% chance that you have the disease, Bart."

Bart initially has a cow, but then he decides to tell the brainy Lisa what Dr. Kalvakian said. What do you suppose Lisa told Bart in response?

21 Zollywog the crazy Geometry student has come up with a formula for the length of a median in a triangle! He claims that the length of the median m that bisects side a of a triangle (with other sides b and c, of course) is:

$$m = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}.$$

Could this possibly be true?

22 Felipe tells Bradley that he has just come up with a cool fact: a regular polygon of n sides, with distance r from the middle of the polygon to any one of the vertices, will always have an area less than $4r^2$. Bradley is skeptical of Felipe, since as n increases the area keeps getting bigger. Can you resolve their dispute?

23 Jillian says that, for any positive integer x, $\dfrac{420\,(x+1)!}{x}$ will always be an integer. Explain why she is correct.

HABITS

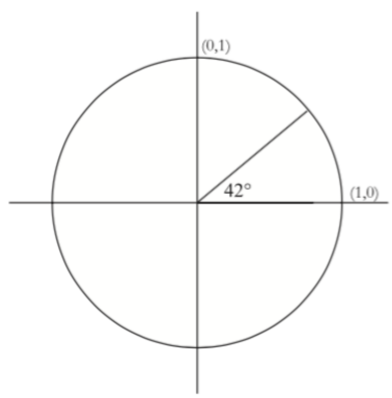
LESSON 1: GETTING COMFORTABLE WITH TRIG

Introduction

In an earlier lesson, you learned how the trigonometric functions sine, cosine, and tangent were defined for angles other than those between 0 and 90 degrees. Here is a quick refresher.

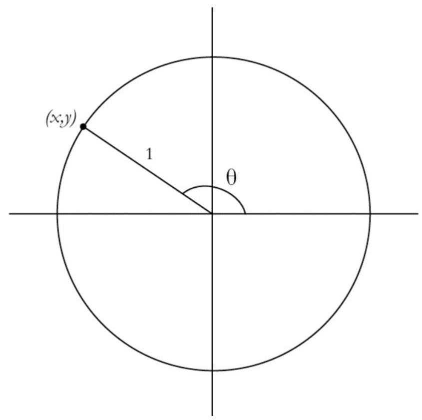
Imagine sitting at the edge of a merry-go-round with radius 1 meter. You've set up a coordinate system with (0,0) at the center of the merry-go-round, and the point (1,0) is due east from (0,0). When the wheel starts to turn counterclockwise, your position is (1,0).

1 If the merry-go-round has rotated 42° from its starting position, find the coordinates of your position.



2 If the merry-go-round has rotated 200° from its starting position, find the coordinates of your position.

This may be enough to remind you of the generalized definitions of sine and cosine. In the figure below, a "spoke" of a circle of radius 1 is drawn so that it makes a central angle of θ with the positive x-axis. The spoke intercepts the circle at the point (x, y). The **sine** of an angle θ is the y-coordinate of this point. The **cosine** of θ is the x-coordinate of this point.



3 Using the figure above, how would you define the tangent of θ?

4 In middle school, you learned that $\sin\theta$ is defined as $\dfrac{\text{length of the side opposite } \theta \text{ in a right triangle}}{\text{length of the hypotenuse}}$. Where is the right triangle in the figure above, and what are the opposite and hypotenuse?

5 How would you determine the sine, cosine, and tangent of an angle using a circle with a different radius than 1?

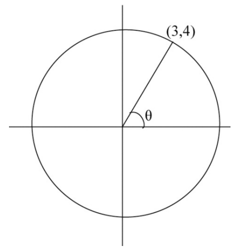
The circle with radius one is better known in mathematical circles as the "unit circle." You'll want to draw it — or a circle with a well-chosen radius — almost every time you solve a trig problem.

Development

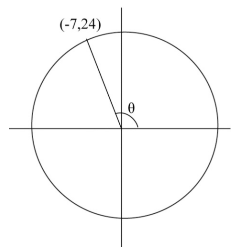
6 Cecilia is on a Ferris wheel and notices that, when she has rotated about 143 degrees from due east, she is 24 meters west and 18 meters above the center of the wheel. Use this information to approximate the sine, cosine, and tangent of 143 degrees. Then see how you did by asking your calculator for these values.

7 For each diagram, find the sine and cosine of the angle θ.

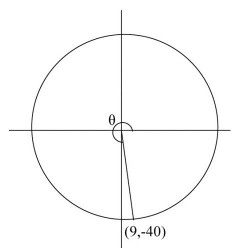
a.



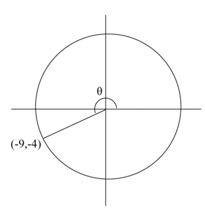
b.



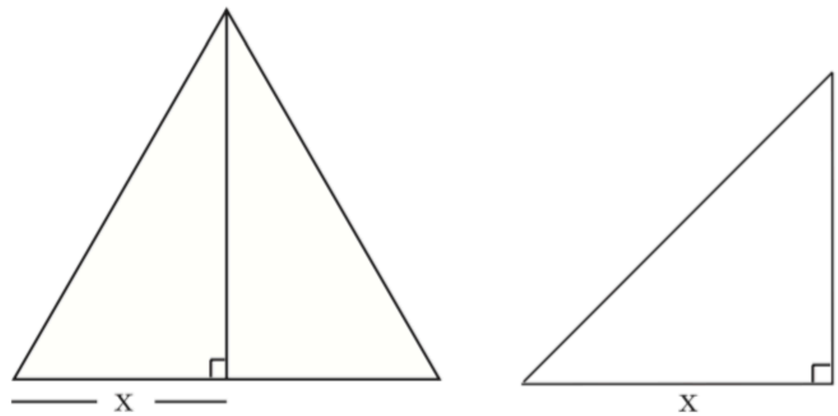
c.



d.



8 Copy the following 30-60-90 triangle and 45-45-90 triangle into your notebook. Then write in the remaining angles and find the remaining sides in terms of x.



Problem 2 may have reminded you that whether the sine or cosine of an angle is positive or negative depends on which **quadrant** the angle is in. This allows you to answer questions like the following.

9 Go back to your 30-60-90 and 45-45-90 triangles and think about positioning them on the unit circle.

a. Name all angles between 0 and 360 degrees that have a sine of $\frac{1}{2}$.

b. Name all angles between 0 and 360 degrees that have a cosine of $\frac{1}{2}$.

c. Name all angles between 0 and 360 degrees that have a tangent of 1.

d. Now explain how to find *all* the angles that have a sine of $\frac{1}{2}$.

10 Solve the equation $\sin\theta = \frac{1}{2}$. How many solutions are there? How many solutions are interestingly different from one another?

Of course, you might also have to solve equations that don't involve sines or cosines of angles in special triangles. For instance, if you had to solve $\sin\theta = .2531$, you would be forced to resort to the inverse sine function on your calculator.

11 Make sure your calculator is in degree mode, then take the inverse sine of .2531 to solve this equation.

12 Draw the "wheel" again, and draw a spoke at the angle that has sine .2531. Remembering that the sine of an angle is the height of a point above the x-axis, indicate the other point on the wheel that would have the same sine.

13 How would you calculate the size of this second angle with sine .2531? Do so.

14 Give *all* the solutions to the equation $\sin\theta = .2531$.

15 Now find *all* the solutions to the equation $\cos\theta = .2531$.

Practice

16 A wheel of radius 2 ft spins around the point $(0, 0)$. A ladybug sticker is initially at the point $(2, 0)$. Find the coordinates of the sticker once the wheel has spun

a. 115°

b. 220°

c. 300°

d. 660°

17 Calculate the exact value of each expression. Do not use a calculator.

a. $\sin 120^\circ$

b. $\cos(-60^\circ)$

c. $\cos 315^\circ$

d. $\tan 150^\circ$

e. $\sin 270^\circ$

f. $\cos 0^\circ$

18 Given that the sine of 138 degrees is about .6691,

a. Find all angles between 0 and 360 degrees that have a sine of .6691.

b. Describe all angles that have a sine of .6691, including those not between 0 and 360 degrees. Can you find a way to write your answer using symbols?

19 Find all solutions to the equation $\sin\theta = .6691$.

20 Find all solutions to the equation $3\sin\theta - 1 = 1.0074$.

21 Find all solutions between 0° and 360° to the equation $\tan\theta = \dfrac{-1}{\sqrt{3}}$.

22 Find all solutions between 0° and 360° to the equation $\cos\theta = -.8834$.

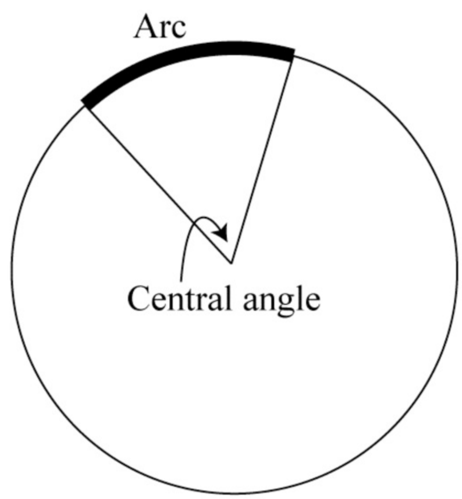
23 In which quadrants is the tangent negative? In which quadrants is the tangent positive?

Further Development

You're used to measuring temperature in degrees Fahrenheit, but you also know that much of the world measures temperature using degrees Celsius. Similarly, the world uses miles and kilometers, ounces and grams, gallons and liters.

When the Babylonians invented degree measure, they chose a 360-degree circle for two reasons. One was that, since there are 365 days in a year, a degree would be very close to one day's progress around the circular path they believed the sun made every year. Another reason is that 360 is divisible by lots of numbers, so we can describe a fourth, a sixth, an eighth, and a tenth of a circle as an integer number of degrees. (Try this with a 100 degree circle and you'll see it doesn't work as well.)

As time went on, though, mathematicians realized that the number 360 didn't really have anything to do with circles, and in fact some of their calculations would be easier if they used a different system. This system was inspired by the close relationship between a central angle of a circle and the arc it intercepts. It was also inspired by the importance of the unit circle.



Specifically, we'd like the measure of a central angle in the unit circle to be literally the same as the length of the arc it intercepts.

24 What is the circumference of the unit circle?

Since a 360 degree "central angle" intercepts the entire circle your answer to Problem 24 will be the equivalent of 360 degrees in our new system.

25 How much of a circle does a 180 degree central angle intercept? So, in this new system, how should we represent a 180 degree angle?

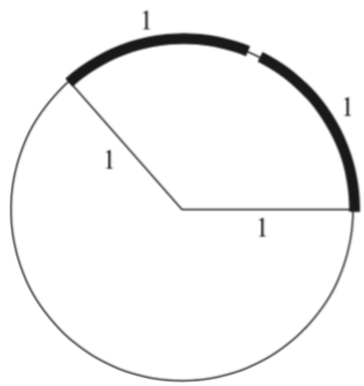
26 Under this new system, what should we call the measure of an angle that intercepts a quarter-circle?

27 What should we call the measure of an angle that intercepts a sixth of a circle?

When you measure angles this way, you are measuring in **radians**. (Try pressing the MODE button on your calculator and notice that there is a radian vs. degree option.) The next problems suggest a reason for the name.

28 How many times does the radius of a circle "fit" around the circumference? (Hint: remember that the circumference of a circle is $C = 2\pi r$.)

29 Say you have an angle that intercepts an arc on the unit circle equal to two of its radii. How many radians is that angle?



If you are about to solve an equation like $\sin\theta = \frac{\sqrt{3}}{2}$, you might not know whether to give your answer in degrees or radians. Your class should agree on which system to use.

Practice

30 If 60 degrees is equivalent to $\frac{\pi}{3}$ radians, how many radians are in

a. 120 degrees?

b. 240 degrees?

c. 30 degrees?

d. 15 degrees?

31 Use your answers to the previous problem to help convert each degree measure to radians.

a. 75 degrees

b. 150 degrees

c. 330 degrees

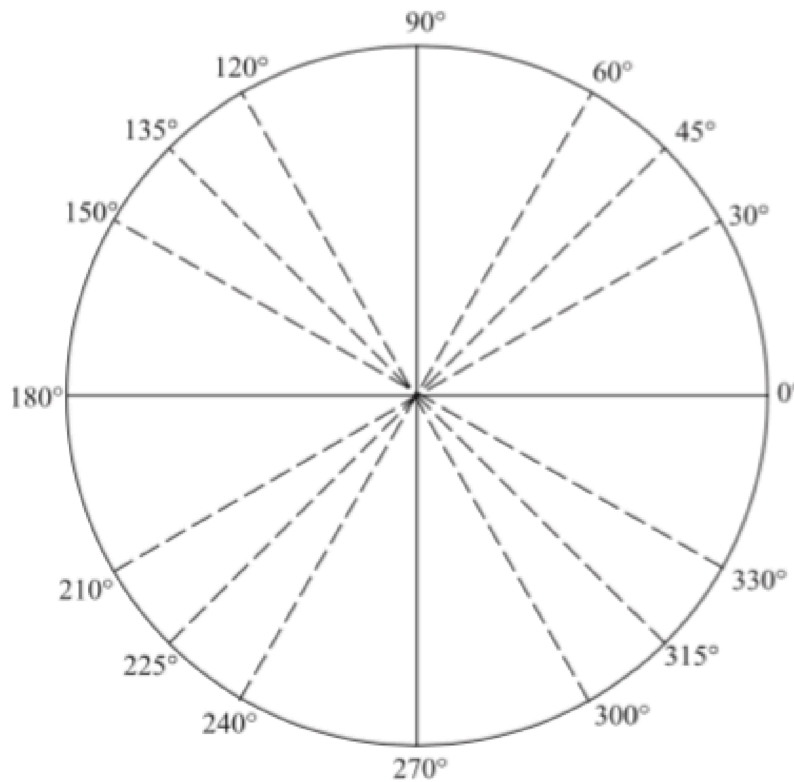
32 If 45 degrees is equivalent to $\frac{\pi}{4}$ radians, how many degrees are in

a. $\frac{3\pi}{4}$ radians?

b. $\frac{7\pi}{4}$ radians?

c. π radians?

33 Copy this unit circle into your notebook and write the radian measure of each angle marked.



34 Give the exact value of each expression.

a. $\sin \dfrac{\pi}{3}$

b. $\cos \dfrac{\pi}{3}$

c. $\sin \dfrac{2\pi}{3}$

d. $\cos \dfrac{2\pi}{3}$

e. $\tan \dfrac{7\pi}{4}$

f. $\cos \dfrac{5\pi}{4}$

Problems

35 Write each set of values in order of smallest to biggest. Do not use a calculator.

a. $\sin 60^\circ$, $\sin 70^\circ$, $\sin 80^\circ$

b. $\cos 60^\circ$, $\cos 70^\circ$, $\cos 80^\circ$

c. $\tan 60^\circ$, $\tan 70^\circ$, $\tan 80^\circ$

36 Write each set of values in order of magnitude (that is, you would write 2 before –10) and then say which are positive and which are negative.

a. $\sin 89^\circ$, $\cos 89^\circ$, $\tan 89^\circ$

b. $\sin 269^\circ$, $\cos 269^\circ$, $\tan 269^\circ$

c. $\sin 359^\circ$, $\cos 359^\circ$, $\tan 359^\circ$

37 Solve each equation for all values of θ. You may use your calculator, but keep in mind that it will not give you all the answers you need.

a. $\cos\theta = .4642$

b. $\frac{1}{3}\sin\theta = .2862$

c. $2\tan\theta - 4 = -3.5174$

d. $\sin(\theta - 15^\circ) = .7327$

38 Solve for all values of x. Give answers to Part a using degrees and answers to Part b using radians.

a. $\sqrt{2}\sin(x + 45^\circ) = 1$

b. $\sqrt{3}\tan(x + 4) = 3$

39 Solve for all values of θ.

a. $\tan^2\theta = 3$

b. $\left(\sqrt{3}\sin\theta + 1\right)(2\sin\theta - 1) = 0$

40 Does $\sin^{-1}x$ refer to an angle, or just a number that does not represent an angle? How about $\sin x$?

41 A wheel on a buggy has a radius of one foot. How many degrees has it spun counterclockwise if a chalk mark originally on the wheel's rightmost point is for the first time .8 feet off the ground?

42 Don't use a calculator for this problem.

a. Solve: $x^4 - 9x^2 = 0$

b. Solve: $\frac{3}{27}x^2 = 1$

c. Find the base-ten logs of a million, a billion, and a million times a billion.

d. Solve: $\frac{x+1}{x+3} = 14$

e. Simplify: $\frac{(x^2)^6}{x^3}$

43 A Ferris wheel has a radius of 132 feet. Its bottom seat is 6 feet off the ground. You are sitting in the seat at the wheel's rightmost point when the wheel begins to spin counterclockwise.

a. What is your height off the ground when the Ferris wheel has rotated $\frac{5\pi}{4}$ radians?

b. What is your height off the ground when the Ferris wheel has rotated $\frac{9\pi}{4}$ radians?

c. What is your height off the ground when the Ferris wheel has rotated $\frac{13\pi}{4}$ radians?

d. You have fallen asleep on the Ferris wheel. When you wake up, you realize that you are level with a tower that you know to be 100 feet off the ground. What are the possibilities for how many radians the wheel has rotated since the beginning of the ride?

44 How many degrees is one radian? How many radians is one degree?

45 Come up with a conversion formula between degrees and radians.

46 Another way of thinking of radian measure is as a ratio. In a circle, it is the ratio of the arc an angle intercepts to the radius of the circle.

a. Draw a circle of radius 5 inches, with an angle inside that intercepts an arc that is $\frac{1}{4}$ the length of the circle. Find the ratio of the arc length to the radius.

b. Repeat part a, but using a circle of radius 7 inches. Your angle will still intercept $\frac{1}{4}$ of the circle.

c. What is the radian measure of an angle that intercepts a quarter-circle?

47 In 10th grade, you found a formula for the length of an arc intercepted by an angle θ: arclength $= 2\pi r \cdot \frac{\theta}{360^\circ}$. Find a similar formula that works when the central angle is measured in radians instead of degrees.

48 In 10th grade, you also found a formula for the area of the sector the angle intercepts: sector area $= \pi r^2 \cdot \frac{\theta}{360^\circ}$. Find a similar formula that works when the central angle is measured in radians instead of degrees.

49 From what we've seen in this lesson, there are often two solutions to the equation $\cos\theta = A$ between 0° and 360°.

a. Are there any values of A for which there would only be one solution? How many values of A?

b. Are there any values of A for which there would be no solutions?

50 Suppose an angle has a sine of .4221.

a. How many angles have this sine?

b. Your calculator tells you that $\sin^{-1}(.4221) \approx 24.97^\circ$. Why do you suppose it only gives you one answer?

c. Try to figure out how the calculator "chooses" the answers it gives for the inverse sine button. What's the biggest answer it will give? What's the smallest?

51 Repeat part c of the previous problem for inverse cosine and inverse tangent. Why do you suppose the range of answers is not the same for the sine and cosine?

52 If $2\theta = 26^\circ + 360^\circ n$, where n stands for any integer,

a. Find all possible values for θ.

b. Did your possible values include $\theta = 553^\circ$? If not, revise your answer to part a.

53 If $3\theta = 150^\circ + 360^\circ n$, where n stands for any integer, find all possible values of θ.

54 Solve each equation.

a. $\sin 2\theta = .1647$

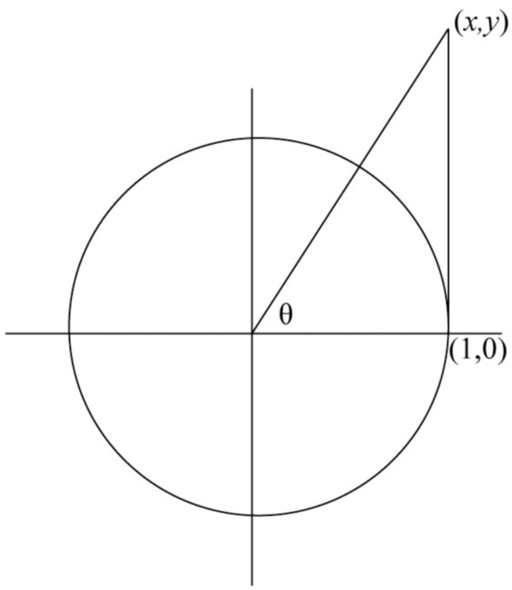
b. $\tan 2\theta + 5 = 42$

c. $2\sin 3\theta = \sqrt{3}$ (no calculator)

d. $\cos 3\theta = \dfrac{1}{\sqrt{2}}$ (no calculator)

Exploring in Depth

55 Find the coordinates of the point labeled (x, y) in terms of θ. Does the diagram suggest a certain word origin?



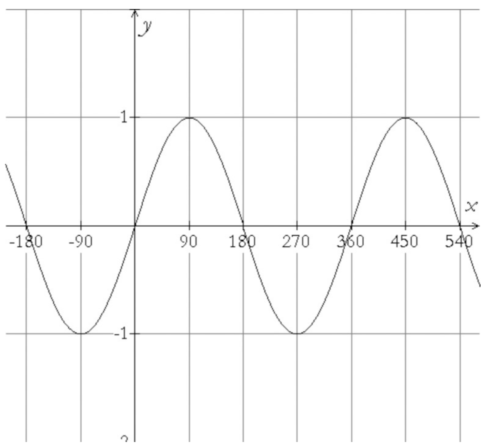
56 How many solutions does $\sin n\theta = \frac{\sqrt{3}}{2}$ have between 0° and 360°? Answer in terms of n.

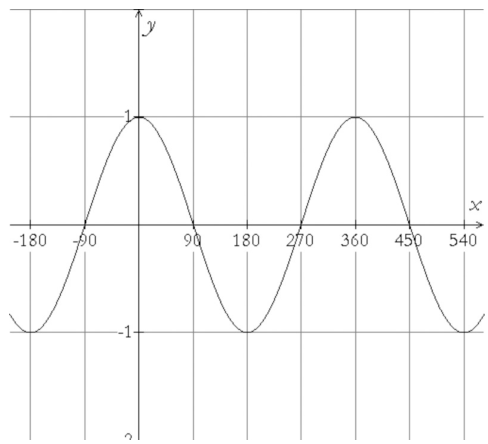
57 Refer back to Problem 46, in which you thought of the radian measure of an angle as a ratio. Explain why, unlike degrees, miles, or kilograms, radian measure doesn't require units.

LESSON 2: MODELING WITH TRIGONOMETRY

Introduction

The graph of the sine and cosine functions are very distinctive looking.





1 Which of the figures above is $y = \sin x$? Which is $y = \cos x$?

2 Using your knowledge of trigonometry, find coordinates for a few of the high and low points for each graph, and of the places they cross the x-axis.

Though you're used to using trigonometry to find missing angles and lengths in diagrams, the special shape of the graph of the sine and cosine functions make them very useful in modeling as well. Graphs that have a shape similar to that of a sine curve are called **sinusoidal** (even if they involve the cosine instead of the sine).

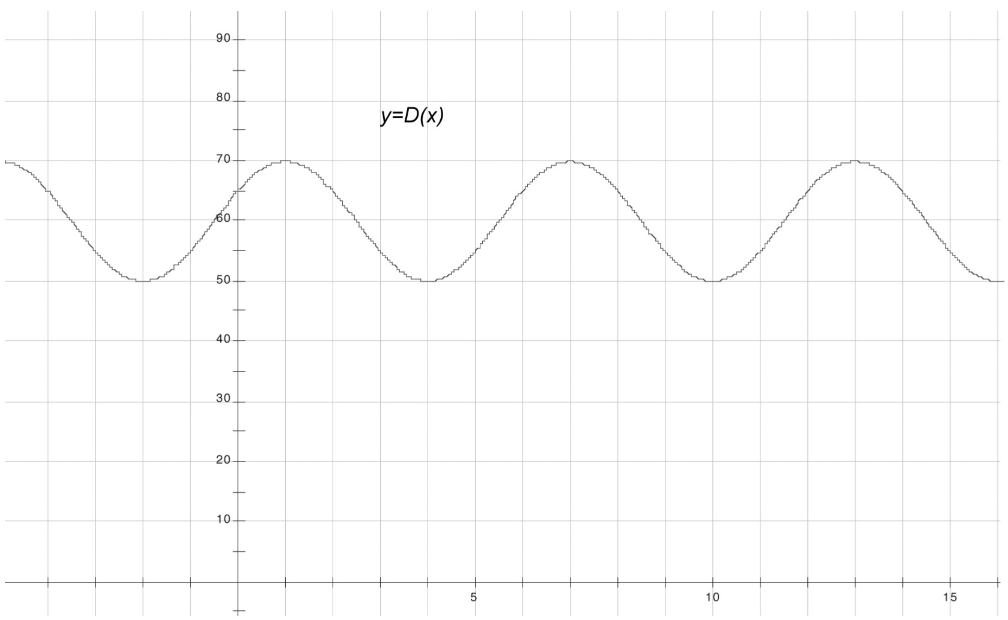
3 Come up with some situations in which a graph with the characteristics of a sine or cosine curve would provide a good model.

In general, sine and cosine curves are used to model situations that are **periodic** – that have some repeating pattern. (You might be familiar with this word from the periodic table, which is a chart of elements arranged so that, every time you begin a new row in the chart, the elements begin to repeat the same pattern of characteristics that was established in the previous row.) Sine and cosine functions are examples of periodic functions, and the distance you need to go on the x-axis before the values of the functions begin to repeat themselves is called the **period**.

4 What is the period of the sine function? How about the period of the cosine?

5 If you are using radians, what is the period of each function?

Not all situations that are periodic will have the same period. Nor will all be the usual sine or cosine graphs. For example, one situation that lends itself well to trigonometric modeling is the depth of a lake at a fixed location, measured over time as the tide comes in and out. Here is a graph of some data for a particular lake. $D(x)$ measures the depth of the water in inches at time x hours after midnight.



Though the shape of $D(x)$ is similar to that of a sine or cosine curve, clearly its equation is not $y = \sin x$ or $y = \cos x$. However, we can get the equation for this graph by doing various transformations of sine and cosine curves, much like we can take the graph of $y = x^2$ and use transformations to turn it into any parabola we choose. In order to figure out how to take a curve like $y = \sin x$ or $y = \cos x$ and turn it into the curve above, we'll **take things apart** by looking at each of the transformations separately.

Development

Let's first think about how we could change the period of a sine or cosine curve.

6 What type of transformation would changing the period be — a vertical stretch, vertical shift, horizontal stretch, or horizontal shift?

Suppose we wanted to cut the period of the sine function in half — make it 180° instead of 360°. In other words, the sine function starts to repeat itself after 360°, but we want to find a different function that repeats itself after 180°. And put a third way (!) a partial table of values for the sine function looks like this ...

x	30°	60°	90°	120°	150°	180°	210°
$\sin x$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

...but for our new function the table of values should look like this, taking on all the same values but twice as quickly.

x	15°	30°	45°	60°	75°	90°	105°
The new function's values	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

7 Come up with a function that looks like $y = \sin x$ but has half the period. If necessary, try a few formulas and test your conjectures.

8 Now try to find a formula for a function like the one above, except it has a period of 90°. Now find one with a period of 720°.

9 In terms of n, what is the period of the function $y = \sin nx$?

10 a. Write an equation for a function related to the sine function that has period P.

b. Does your idea work for cosine as well? Why or why not?

11 See if you can write an equation for a function that has the same period as the function in the tide example. You could pick either sine or cosine... for consistency, let's use cosine.

In addition to the change in period, the tide function differs from the sine and cosine functions in other ways. For one thing, it is vertically stretched as compared to those functions.

12 What is the height of the usual cosine function, $y = \cos x$, measured from the x-axis to its highest point?

13 How many times its normal height do you need to stretch a cosine curve in order to get the same height as the curve for the depth of water function, $D(x)$?

14 Conjecture an equation for a cosine-based function that would be vertically stretched by the right amount. If you've done it right and also changed the period, the graph of your function should be exactly the same shape as $D(x)$?.

The **amplitude** of a sine or cosine curve is defined as *half* the vertical distance from the top to the bottom of the curve. This was the quantity you found in Problem 12.

15 Why do you suppose the amplitude is defined as *half* the distance, rather than the entire distance from top to bottom?

16 What's the amplitude of $f(x) = \sin x$? What's the amplitude of $D(x)$?

17 Write an equation for a sinusoid curve that has amplitude A.

At this point, our graph has the right shape; we just need to translate it over to the right location. This is something you learned how to do when studying quadratics, though if you've forgotten, try experimenting with your calculator when answering the next few questions.

18 By how much do we need to vertically shift the graph?

19 Write an equation for a graph with the same period, amplitude, and vertical shift as $D(x)$ (again using cosine).

We could make this graph coincide perfectly with $D(x)$ by shifting it horizontally.

20 To remind yourself how to do a horizontal shift, first try writing equations that shift the graph of $y = x^2$ three units to the right, then three units to the left.

21 Now use the same technique to shift the graph of $y = \cos x$ by 90 degrees to the right, then 90 degrees to the left.

22 It's not obvious how to do a horizontal shift at the same time that you change the period of a graph. First make a sketch of $y = \cos 90x$. Then write an equation that you could use to shift this graph two units to the right. Check by using your calculator to graph your equation, and refine your equation if necessary.

23 Determine the required horizontal shift needed to turn your graph in Problem 19 into the graph of $D(x)$.

24 Now that you have all the tools, write an equation for $D(x)$.

Having this model allows us to make predictions. For example ...

25 Use your equation for $D(x)$ to predict the depth of the water at 10:15 pm.

26 Find all of the times today that the water will have a depth of 65 inches.

a. Algebraically — by solving an equation.

b. Graphically — by using your calculator.

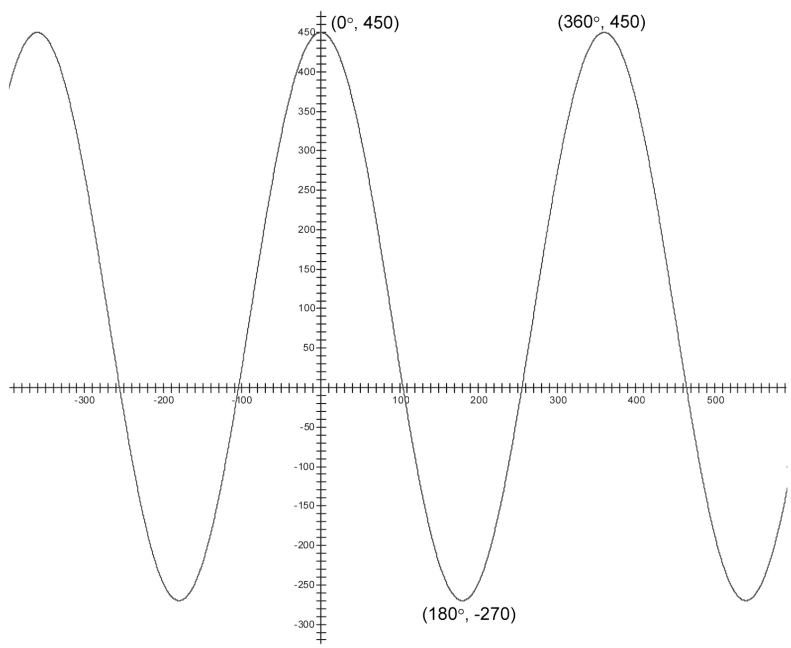
Practice

27 For each graph below,

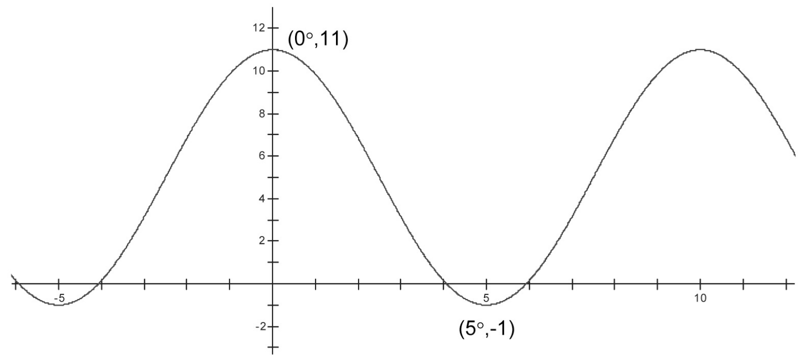
a. say what the period, amplitude, vertical shift, and horizontal shift is, assuming that the graph is a transformation of $y = \cos x$.

b. write an equation for the graph. Check it by graphing the equation on your calculator.

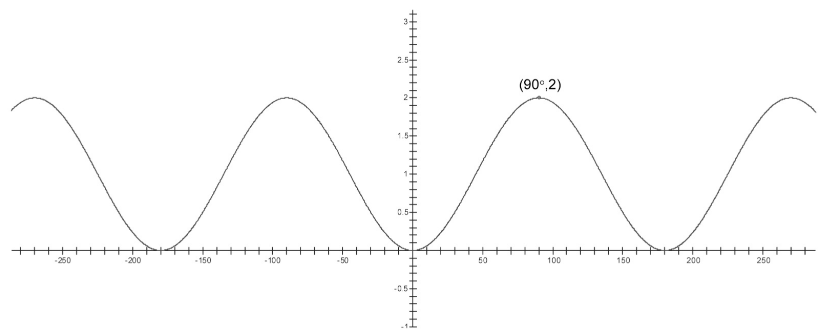
i.



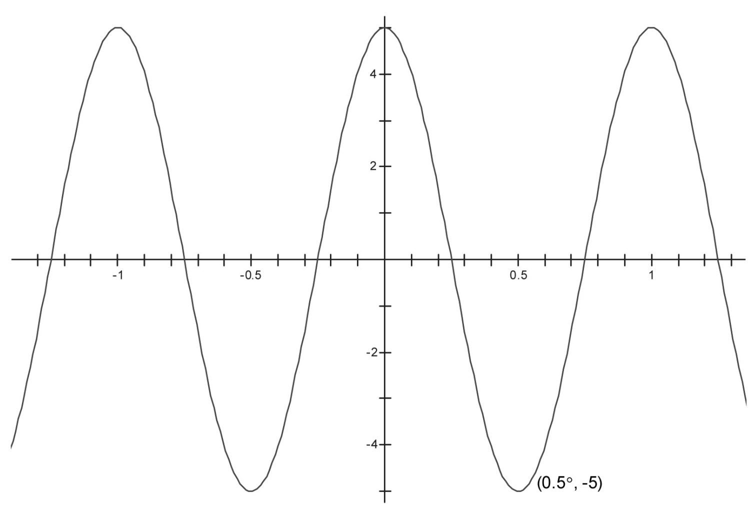
ii.



iii.



iv.



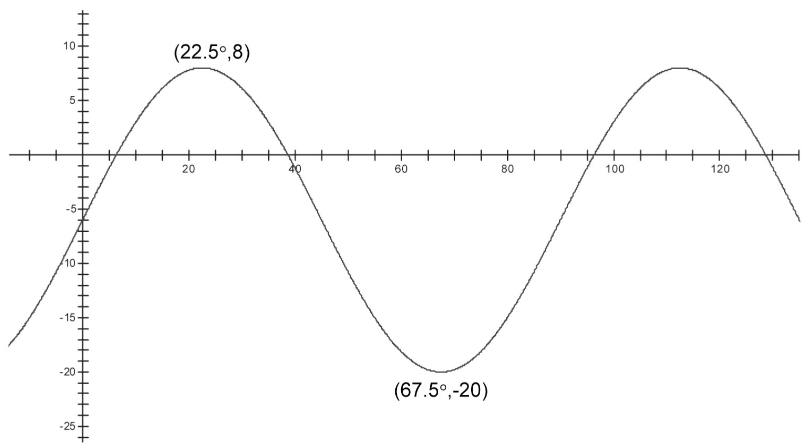
28

For each graph below,

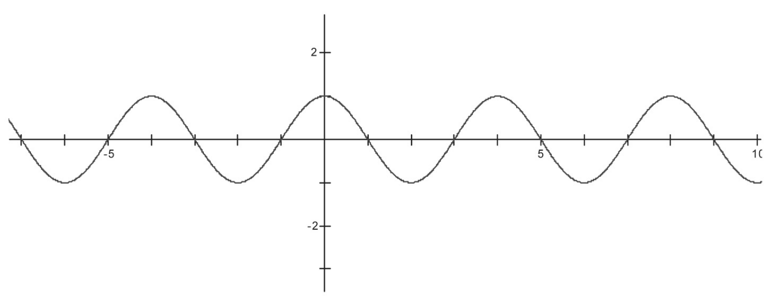
a. say what the period, amplitude, vertical shift, and horizontal shift is, assuming that the graph is a transformation of $y = \sin x$.

b. write an equation for the graph. Check it by graphing the equation on your calculator.

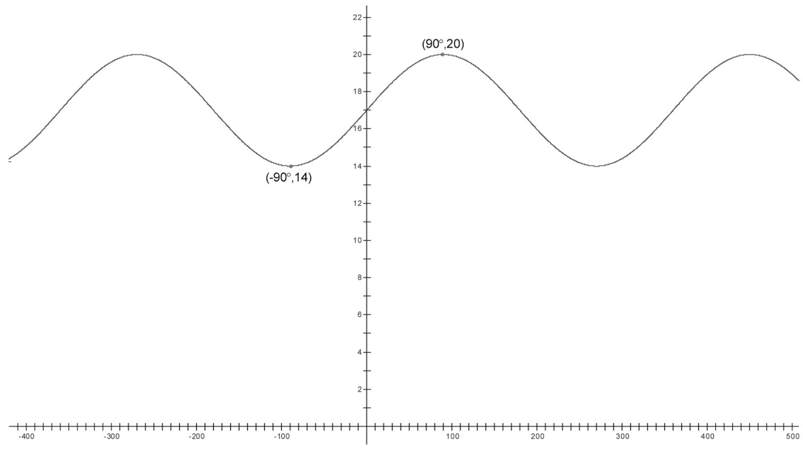
i.



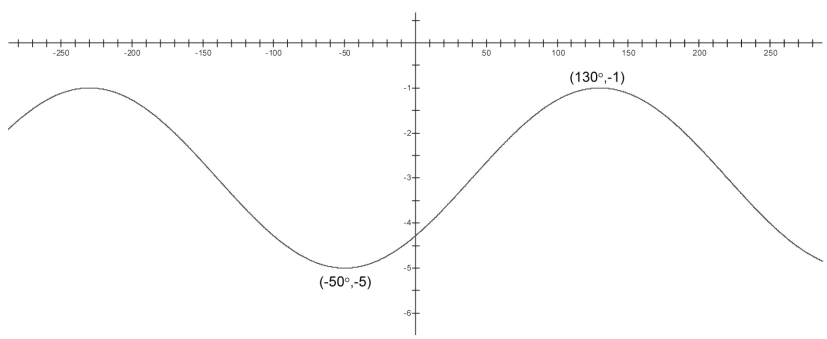
ii.



iii.



iv.



29 Find another equation for the function graphed in part i of the previous problem, this time using cosine instead of sine.

30 By identifying the period, amplitude, and shifts, draw a graph of each equation without the aid of your calculator (except to check).

a. $y = 2\sin x + 4$

b. $y = \frac{1}{2}\cos(2(x - 180))$

c. $y = \cos(3\,(x + 60))$

d. $y = \sin(3x + 60)$

31 How would you describe to someone the difference between what you do to change the period and the amplitude of a trigonometric graph?

32 Find an equation for a sinusoidal function that has a peak at the point $(2, 9)$, and has its first trough following that peak at $(7, -3)$.

33 Write a sinusoidal equation that has each of the following characteristics:

a. Period 36, amplitude .5, horizontal shift 5, vertical shift -7.

b. Period 6, amplitude 3, has a high point at $(7, 10)$.

34 Zinadine argues that in the function $D\,(x)$ the vertical shift should really be 69, because the high point of an untransformed cosine graph is 1 unit above the x-axis, but the high point of $D\,(x)$ is 70 units above the x-axis. Set him straight.

Problems

Note to the eager student: though most references to angles in this section use degree measure, feel free to solve the modeling problems using radians. In fact, your teacher may insist!

35 Write an equation for a function involving sine or cosine that has a period of 40 degrees.

36 Write an equation for a function involving sine or cosine that has a period of 91 degrees.

37 Write an equation for a function involving sine or cosine that has a period of 4π radians.

38 What would be the period — measured in radians — of the function $y = \sin \dfrac{2\pi}{3} x$?

39 Which of these equations shifts the graph of $y = \sin x$ or $y = \cos x$ six units to the right? How do you know?

a. $y = 4\cos(24\,(x-6)) - 7$

b. $y = 2\sin(3x-6) + 4$

40 Sketch the following transformations: first shift the graph of $y = \cos x$ 90 degrees to the right. Then horizontally compress the resulting graph towards the y-axis by a factor of two.

a. Write an equation for the resulting graph.

b. Can you "see" from the equation that there was a shift of 90 degrees?

c. Describe the transformations that must be done on $y = \sin x$ in order to get the graphs of $y = \sin(Ax - B)$ and $y = \sin(A\,(x - B))$. To what extent does order matter?

41 Graph the following equations:

a. $y = \sin(10x - 60)$

b. $y = \cos(4x + 128)$

42 Don't use a calculator for this problem.

a. Solve: $\dfrac{9}{x+5} = x - 5$

b. Evaluate: $\log 50 + \log 4 - \log 2$

c. Simplify $\sqrt{\dfrac{y^3}{x^2}}$

d. Solve the equation $x^3(x-1)(x-2)^2 = 0$. What is the degree of this equation?

e. Find k if $2^{2015} - 2^{2014} + 2^{2013} - 2^{2012} = k \cdot 2^{2012}$

43 You are at the Inner Harbor on June 22. At 11:30 am, low tide, you find that the depth of the water at the end of a pier is 0.5 ft. At 5 pm, high tide, the water is 1.1 ft deep.

a. Find an equation for depth as a function of time that has elapsed since 12 midnight at the beginning of June 22.

b. Use your equation to predict the depth of the water at 8 pm on June 23.

c. At what time does the first low tide occur on June 23?

d. What is the earliest time on June 24 that the water will be 0.85 ft deep?

44 In August, in Baltimore, the temperature can be modeled by the equation $y = -11\cos\left(\frac{\pi}{12}(x-2)\right) + 74$, where x stands for the number of hours past midnight on Aug. 1, and y stands for the temperature (in Fahrenheit) at time x.

a. Use algebra to find all times at which the equation predicts that the temperature will be *more than* 80°.

b. What does this equation imply that the maximum temperature will be? Explain your reasoning carefully. (And don't appeal to your graphing calculator)

45 A spring is attached to the ceiling of a room. You attach a weight to the bottom of the spring, then let go. Some time later, you start timing the oscillations of the spring. 0.4 seconds after you start the time, the spring stretches as far as it will go, to a distance of 24 inches above the floor. It then bounces back up, to a highest point of 50 inches above the floor, at 2 seconds, and continues to oscillate sinusoidally.

a. Sketch the graph of this function.

b. Find an equation for distance from the floor as a function of time.

c. What was the distance of the weight from the floor when you started the clock?

d. What is the distance of the weight from the floor after 4 seconds?

e. When does the weight first reach a height of 40 inches?

46 At time zero, someone puts a chalk mark at the top of your unicycle tire, and you immediately begin pedaling in such a way that the wheel makes two complete revolutions per second. The radius of your wheel is 1 foot.

a. Write an equation giving the height (measured from the ground) of the chalk mark as a function of time since you began pedaling. Be sure to show your work in finding the equation.

b. Use the equation to find the height of the chalk mark 2.3 seconds after you began pedaling.

c. Use the equation to find two times at which the chalk mark will be .8 feet above the ground. Then describe *all* the times at which the chalk mark will be .8 feet above the ground.

47 What might the graph of $y = -\sin x$ plausibly look like? Without using your calculator, check a few values in your head.

48 Try graphing $y = -\sin x$ on your calculator. Then try graphing $y = \sin(-x)$. Describe how these graphs differ from the graph of $y = \sin x$ using the language of transformations.

49 Use the results of the previous problem to predict what the graphs of $y = -\cos x$ and $y = \cos(-x)$ will look like. Then test your prediction by graphing them on your calculator. Describe how these graphs differ from the graph of $y = \cos x$ using the language of transformations.

50 Make a table of values for the function $y = \sin^{-1} x$. Then graph it, taking care to choose a scale appropriate for your values for x and y in the table. Does it appear to be visually related to the graph of $y = \sin x$ in any way?

51 (Hint: visualize!) Make up a sinusoidal function $f(x)$ that has exactly four solutions to the equation $f(x) = \frac{1}{2}$ between 0° and 360°. Then make one up that also has four solutions to $f(x) = \frac{1}{2}$ but also two solutions to $f(x) = 5$ between 0° and 360°.

52 Go back to the function $D(x)$, for which you should have found the equation $D(x) = 10\cos(60(x-1)) + 60$. Recall that x, the time, is measured in hours and $D(x)$, the depth of the water, is measured in inches.

a. Approximately how many feet per hour was the water level changing at 6 am?

b. How about at 3 am? Explain the sign of your answer.

c. At what times during the day would you expect the rate of change to be greatest? How about the smallest?

53 Let $f(x) = 2\sin x$ and $g(x) = 8\sin x$.

a. Is there an x-value at which the slope of g will be four times greater than the slope of f? Test your conjecture.

b. Why is it plausible that the slope of g would always be four times greater than the slope of f?

54 The website http://ptaff.ca/soleil/?lang=en_CA creates a "daylight hour graph" for major world cities.

a. Make one for Baltimore on today's date. Then, using x to represent days of the year and y to represent the number of hours of daylight on that day, write an equation to match the graph you see.

b. Imagine what the daylight hour graph for Vancouver would look like. Would you expect it to be different from the Baltimore graph in any of the following respects: period, amplitude, horizontal shift, vertical shift? Then create a graph for Vancouver and see if you are right.

55 Here is a table showing the number of daylight hours on the first of each month in the cities of Juneau and Fairbanks, Alaska. (Data from http://www.absak.com/library/average-annual-insolation-alaska)

City	Jan	Feb	Mar	Apr	May	June
Juneau	6:30	8:15	10:34	13:13	15:43	17:48
Fairbanks	4:00	6:55	10:07	13:35	17:01	20:33

	July	Aug	Sept	Oct	Nov	Dec
Juneau	18:10	16:30	14:01	11:30	8:55	6:53
Fairbanks	21:25	18:11	14:39	11:19	7:51	4:43

a. Write equations for the hours of daylight in both Juneau and Fairbanks. Use month of the year for x. Note that the numbers in the table are given in "hours:minutes" form, so you'll have to decide how to convert them to decimal notation.

b. According to the equation/graph you've made, what is the day of the year that receives the least daylight? Does this seem right? What data would you need in order to make a more accurate graph?

c. Graph the two equations on your calculator in order to estimate when Juneau and Fairbanks have the same number of hours of daylight. What time of year is this?

Exploring in Depth

56 Critics of the theory of global warming say that changes in the Earth's temperature are cyclic. The graph below shows the level of CO_2 in the atmosphere (which directly affects temperature).

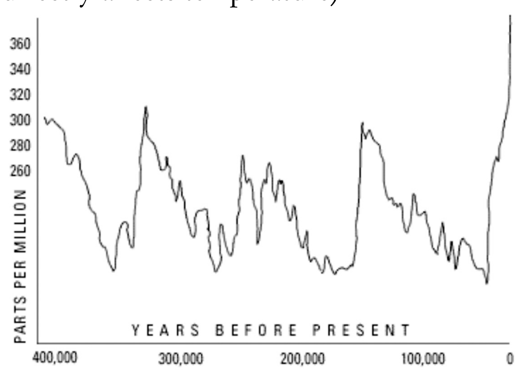


image from http://www.pbs.org/wgbh/warming/etc/graphs.html

a. Approximate some data points, and use these to come up with an equation giving the CO_2 level vs. year.

b. Use this equation to predict what the CO_2 levels should be in the early 21st century, and compare your answer with the graph.

57 (from Foerster's *Precaclulus*) The hum you hear on some radios when they are not tuned to a station is a sound wave of 60 cycles per second.

a. Is the 60 cycles per second the period, or is it the frequency? If it is the period, find the frequency. If it is the frequency, find the period.

b. The **wavelength** of a sound wave is defined as the distance the wave travels in a time equal to one period. If sound travels at 1100 ft/sec, find the wavelength of the 60-cycle-per-second hum.

c. The lowest musical note the human ear can hear is about 16 cycles per second. In order to play such a note, the pipe on an organ must be exactly half as long as the wavelength. What length of organ pipe would be needed to generate a 16-cycle-per-second note?

58 Make the following graph: put radian measure on the x-axis. On the y-axis, plot the value for the *slope* of $y = \sin x$. Plot enough points to get a general sense of the curve, then find an equation for it.

59 Repeat the previous question, this time plotting the slope of $y = \cos x$. Find an equation for this graph as well.

LESSON 3: VARIATION AND PROPORTION

1 You might recall from Physics class that Newton's 2nd Law of Motion tells us that the net force on an object divided by its mass equals its acceleration, i.e., $a = \frac{F_{net}}{m}$. Let's look at the case of a person pushing a bobsled and its passengers on the ice.

a. If the mass of the bobsled and passengers is 300 kg and the person pushes so that there is a net force of 900 Newtons on the bobsled, what is the acceleration of the bobsled (the units are in meters/sec^2)?

b. If the net force on the bobsled is tripled, does the acceleration triple?

c. If, instead, the net force remained at 900 N but the mass of the bobsled and passengers was tripled from part a, what would happen to the acceleration?

d. What would be the effect on the acceleration of tripling the net force and the mass from part a at the same time?

e. What would happen to the acceleration if the net force were the same as part a, but the mass of the bobsled and passengers was cut in half?

f. Finally, what would be the effect on the acceleration if one both doubled the net force from part a and halved the mass at the same time?

2 As you know, the circumference of a circle and the area of a circle of radius r can be found through the formulas $C = 2\pi r$ and $A = \pi r^2$.

a. If one circle has a radius of 1 meter and another has a radius of 3 meters, how many times bigger is the circumference of the second circle? How many times bigger is its area?

b. If one circle has a radius of 2.687 meters and another has a radius 3 times as big, how many times bigger is the circumference of the second circle? How many times bigger is its area?

c. Based on parts a and b, what do you think you can conclude in general? Prove your answer by considering two circles, one with radius X and the other with radius $3X$, and by doing some minor algebra.

d. The volume of a sphere of radius r can be found by using the formula $V = \frac{4}{3}\pi r^3$. If you have two spheres, where the larger sphere has a radius 5 times that of the smaller sphere, how many times bigger must the larger sphere's volume be? Prove that your answer is correct no matter what the radius of the smaller sphere is.

e. If one sphere is $\frac{1}{4}$ the radius of another, what fraction of the larger sphere is the volume of the smaller sphere? Again, prove your answer works for any two spheres that have this relationship.

3 The braking distance of a car is the minimum distance a car going at speed v can stop in (by using the brakes, of course). By using some fundamental physics, one can calculate the braking distance, assuming a car of average weight and a dry road, by the equation $d = .06v^2$, where v is in miles per hour and d is in feet.

a. If Jessup hits the brakes while driving in his jalopy at 20 mph, what is his braking distance? Alternatively, if Tess takes 384 ft. to stop, how fast was she going in her Miata?

b. Julie says that if she goes 40 mph she will stop in twice the distance than if she was going 20 mph. Is Julie correct? If so, show why. If not, how many times bigger is the 40 mph braking distance than the 20 mph braking distance?

c. Determine how fast Julie would *actually* have to go to stop in twice the distance.

d. How many times faster than 20 mph would she have to go to stop in 16 times as much distance as she would have stopped in at 20 mph?

e. How many times faster than 20 mph would she have to go to stop in M times as much distance as she would have stopped in at 20 mph?

4 Given a cube of side length x,

a. What is its volume?

b. What is its surface area?

c. If the length of the cube were to change to $5x$, then by what multiple would its volume increase?

d. If the length of the cube were to change to $5x$, by what multiple would its surface area increase? Why doesn't the "6" in the surface area formula affect your answer?

5 In the previous problem you examined the volume and surface area of a cube; you found that they were related to the side length x by a cubic (x^3) and a quadratic ($6x^2$). Is this true of more complicated shapes? Let's see.

a. Say that there is an irregularly shaped drawing that looks like, say, Big Bird.

Obtain a copy of this drawing from your teacher. How could you approximate its area in square centimeters to a reasonable degree of accuracy with a ruler?

b. Now if you doubled all the dimensions of this drawing (i.e. its height, its width, the distance between Big Bird's eyes — everything), how would its area change? Why? What if you scaled it up by a factor of 5 instead?

6 At the Macy's Day parade last year, there was an enormous balloon of Spongebob Squarepants that was 40 ft high. Toy copies of this balloon were being sold on the parade route and were 3 inches high.

a. Assuming the other dimensions were proportionally reduced, what is the scale factor of the copy in comparison to the original balloon?

b. Approximately how many of the toy copies would it take to fill the 40 ft. balloon?

The types of relationships between variables that you have been exploring in the previous problems are special cases of two broad categories: direct proportionality and inverse proportionality.

y is **directly proportional** to x when $y = kx$, where x and y are variables and k is a constant not equal to 0. Another way of saying this is that y **varies directly** with x. For example, in problem 2, where $C = 2\pi r$, we say that C is directly proportional to r, or that C varies directly with r.

y is **inversely proportional** to x when $y = \frac{k}{x}$, where x and y are variables and k is a constant not equal to 0. Another way of saying this is that y **varies inversely** (or **indirectly**) with x. For example, in problem 1c, where $a = \frac{900}{m}$, we say that a is inversely proportional to m, or that a varies inversely with m.

7 Answer the following questions by using the definitions of direct and inverse proportionality given above.

a. If $WZ = 100$, are W and Z directly or inversely proportional, or neither?

b. If $\frac{C}{T} = 64$, do C and T vary directly or indirectly, or neither?

c. If $P = Q + 20$, are P and Q directly or inversely proportional, or neither?

d. Suppose Z is directly proportional to S. As S increases, must Z also increase?

e. Suppose M varies inversely with R. As R increases, must M decrease?

Just as it is useful to say that "A varies directly with B", we can apply this language to a wider range of expressions as well. For example, in problem 4, where $A = 6x^2$, we can say that A varies directly with x^2, as A is equal to a constant times x^2. Alternatively, we could also have said in words that surface area is directly proportional to the square of the side length of the cube.

8 In the following questions, either write an equation (which may have an unknown constant k in it) based on the sentence given, or write a sentence based on the equation given.

a. The weight of a human is directly proportional to its volume.

b. The surface area of a sphere varies directly with the square of its radius.

c. From problem 3, $d = .06v^2$.

d. The intensity of light varies inversely with the square of the distance from the light source.

e. $G = \frac{5}{7w^3}$, where G are "galumphs" and w are "wagdoodles".

f. The height of an extra-terrestrial varies directly with the square root of the length of its yellow antennae.

Practice

9 If a large pizza is 1.5 times the diameter of a small pizza, how much more should it cost, assuming that price is based on area?

10 The width of a rectangle is 12 ft and the length is 24 ft. By what *multiple* does the area change if:

a. The width is doubled and the length is tripled?

b. The width is halved and the length is quadrupled?

c. The width is multiplied by some number A and the length is multiplied by some number B?

d. How would your answer to part a be affected if the original width and length of the rectangle were 100 ft and 500 ft?

11 You have a rectangular fish tank with dimensions 12 ft x 10 ft x 8 ft Which side, if you double it, will give you the biggest volume?

12 In the study of waves one discovers the relationship between the velocity, wavelength, and frequency of any wave: $v = \lambda f$. For example, if the wavelength is 4 meters and the frequency is 6 cycles/second, then the velocity would be 24 meters/second.

a. Say that you were looking at a different wave that had twice the wavelength and half the frequency. How many times bigger or smaller would the velocity be than in the example?

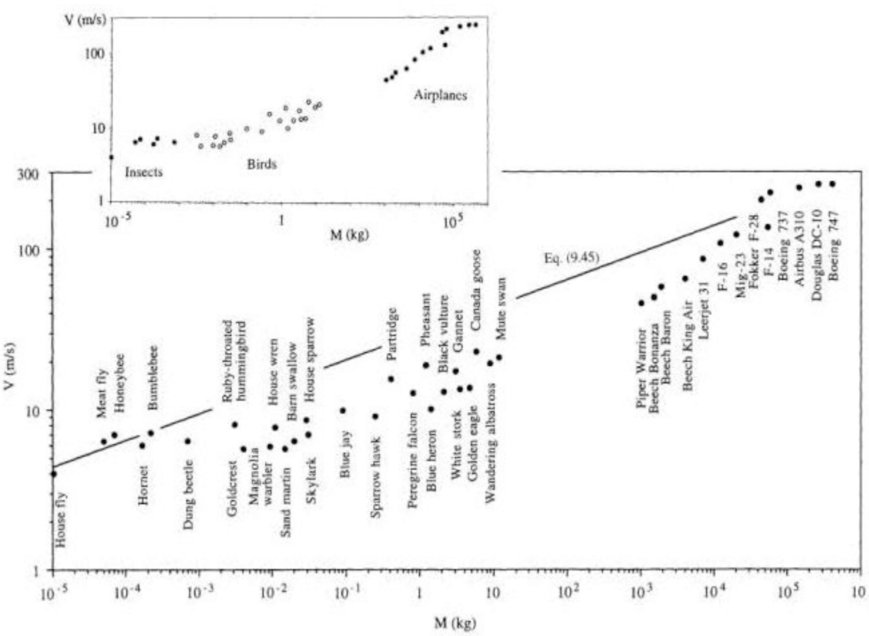
b. Say instead that you were looking at a different wave that had 2.5 times the wavelength and 4 times the frequency. How many times bigger or smaller would the velocity be than in the example?

c. Finally, say that you were looking at a different wave that had 6 times the velocity and half the wavelength of the initial example. How many times bigger or smaller would the frequency be?

13 The heat emitted per second (Q) due to radiation from an object is directly proportional to the fourth power of temperature T (in $^\circ K$) of the object. If the temperature of a quantity of metal triples, how much greater will its heat output be per second as a consequence?

14 Biologists have determined a good approximation of the relationship between the mass of a flying object and its optimum cruising velocity. "Optimum cruising velocity" is defined as the velocity at which an object travels the most distance for a given amount of energy it expends; the easiest analogy is with cars, where the most fuel efficient speed to drive turns out to be 55 mph.

The relationship is $M = \frac{V^6}{729000000}$, where V is the optimum cruising speed measured in meters/second and M is the mass of the object measured in kilograms. This relationship holds for birds, insects, and even planes!



Source: http://en.wikipedia.org/wiki/
File:Allometric_Law_of_Body_Mass_vs_Cruising_Speed_in_Constructal_Theory.JPEG

a. If one bird's V is double another's, how many times more massive is it? What if its V were triple another's?

b. If one bird's M is double another's, how many times faster is its optimum velocity?

15 a. If $R = PS$, are R and S in direct or inverse variation, or neither?

b. If $R = P^3S$, are P^3 and S in direct or inverse variation, or neither?

c. If $R^4 = PS$, are R^4 and P in direct or inverse variation, or neither?

16 In each question below, first find the constant of proportionality so that you can then answer the question being asked.

a. If y is directly proportional to x, and $y = 24$ when $x = 6$, what is y when $x = 13$?

b. If y varies inversely with x, and $y = 8$ when $x = 5$, what is y when $x = 80$?

c. If y varies directly with x^2, and $y = 63$ when $x = 3$, what is y when $x = 10$?

d. If y is inversely proportional to x^3, and $y = 12$ when $x = 2$, what is y when $x = 6$?

17 a. A cylinder has a base radius of 4 cm and a height of 10 cm. How will the volume of the cylinder be affected if you scale up the cylinder in all dimensions by a factor of 10? How will the surface area be affected?

b. A cylinder has a base radius of R cm and a height of H cm. How will the volume of the cylinder be affected if you scale up the cylinder in all dimensions by a factor of 10? How will the surface area be affected?

18 The area of a standard ghost in Mac-Pan is about 6.5 square units. What is the area of a ghost that has been scaled up by a factor of 2?

19 a. A beach ball has 8 times the volume of another beachball. How many times bigger is its circumference?

b. A beach ball has K times the volume of another beach ball. How many times bigger is its circumference?

20 Daddy bear, Mommy bear, and Preteen bear are all perfect copies of each other, except that each is .9 times the scale of the previous bear. How many more times than Preteen bear does Daddy bear weigh?

21 In the ideal gas equation from Chemistry, $PV = nRT$. R is an unchanging constant, but the other 4 quantities can change. Which of these pairs of quantities are in direct variation? Which pairs are in inverse variation?

22 Write an equation that predicts the number of rotations a wheel makes in a mile, given that you know the diameter of that wheel in inches (and note that 5280 feet = 1 mile).

23 You have an enormous vat of punch, 10 gallons (160 cups), which you have made for the Kiwanis-Elks Club-Rotary-March of Dimes annual dinner. You're expecting a big turnout. Write an equation relating the number of people who show up and the number of cups of punch each person can have.

Problems

24 You have two cans of soup. The large can is 3 times the scale of the small can. For each feature of the cans listed below, write how many times bigger it would get for the large can.

Feature	How many times bigger?
Example: Height of can	3 times
Amount of soup in can	
Amount of metal used to make the can	
How many people you can feed with the soup	
Number of calories if you eat all the soup	
Amount of ink needed to print label	
How many cans it takes to stack up to the ceiling	
Time it takes to open it with a can opener	
Number of peas in the soup	
How much heat rises off the can if it's hot	

25 For the following, make an educated guess as to whether the relationship between the given variables is approximately direct, inverse, or neither.

a. Average income in a town vs. average education level

b. Time it takes to read a book vs. how interesting the book is to you

c. Monthly sales of a video game vs. time the video game has been on the shelves

d. Time it takes to read a book vs. number of pages in the book

e. The number of hours of daylight vs. day of the year

f. Birth rate in a country vs. average income per family in that country

g. Pressure on a tire vs. the amount of air in a tire

h. Price of a new electronic device vs. number of people who want to buy one but are unable to due to limited availability

i. Number of diagonals in a polygon vs. number of vertices of the polygon.

26 It turns out that, to a very good degree of accuracy, the mass of an animal is inversely proportional to the fourth power of its resting heart rate. Given that a typical adult human has a resting heart rate of about 72 beats/minute and a mass of 160 pounds, what would you estimate the heart rate of a mouse (.055 pounds) and dinosaur (70000 pounds) to be?

27 You have a set of six Russian dolls. Each is a perfect copy of the others, except that each is only .8 times the scale of the previous one. If it took 3 oz of paint to paint the largest doll, how much paint was needed to paint the second-largest doll? How much paint was needed to paint the smallest doll?

28 The force on a car that is moving in a circle is described by the centripetal force equation: $F = \frac{mv^2}{r}$, where m is the mass of the car in kilograms, v is the velocity of the car in meters/second, r is the radius of the circle the car is moving in, and F is the force in Newtons. Suppose the force on a car is 12000 Newtons as it travels in a circle.

a. If the radius of the circle the car is traveling in doubles, what will the new force on the car be?

b. If the speed of the car doubles, what will the new force on the car be?

c. If the speed of the car doubles at the same time that the radius of the circle it is moving in doubles, what will the new force on the car be?

d. If the force on another car is 27000 Newtons as it travels in a circle, and the radius of the circle is suddenly cut to a ninth of the original radius, how much slower would the car have to travel for the force to stay unchanged?

29 The gravitational attraction of two objects is described by the equation $F = \frac{k}{d^2}$ (where d is the distance in meters between the centers of the objects, k is a constant based on the masses of the objects, and F is the force of attraction between them, measured in Newtons). For brevity, one often says that Gravity is an "inverse square" force, which just means that the force is inversely proportional to the distance squared.

a. If initially the force between a person and a planet is Q_0 and the distance between their centers is d_0, and then the person moves so that the distance between their centers is doubled, what is the new Force Q_1 between the objects, in terms of Q_0?

b. What if the distance were tripled instead of doubled? Now what would the force be in terms of Q_0?

c. Lastly, what if the distance were halved? What would the force be in terms of Q_0 then?

d. The force between a typical person and the Earth is about 980 Newtons. Given that the radius of the Earth is 6370 km and that the space shuttle orbits 390 km above the Earth's surface, determine what the force would be on the same typical person, if they were at the height above the Earth that the space shuttle is when it is in orbit.

30 According to Dr. Killjoy's research, how long a couple stays married can be predicted with absolute certainty: the length of the marriage is directly proportional to the number of times per day the couple holds hands and is inversely proportional to the square of the number of nasty looks per day they exchange. As an example, he says that a couple that holds each other's hands 6 times per day and exchanges 3 nasty looks will be married for 16 years.

If Dr. Killjoy is correct, how long will a couple remain married that holds hands twice a day but exchanges 4 nasty looks a day as well?

31 The so-called kinetic energy of a bullet is equal to $\frac{1}{2}mv^2$, where m is its mass in kg and v is its speed in meters/sec. The higher the kinetic energy of the bullet, the more its ability to cause damage. A typical rifle can shoot a .0042 kg bullet at 965 meters/sec.

a. If a second rifle can fire bullets that are twice as heavy but only at half the speed, how would the kinetic energy of the bullets of this rifle compare to those of the original rifle?

b. If a third rifle fires bullets that are half as heavy as the original rifle, but twice the speed, how does the kinetic energy of the bullets of the third rifle compare to those of the original rifle? To the second rifle?

c. If two rifles use bullets of the same mass, how many times faster must the muzzle velocity (i.e. speed at which the bullet leaves the gun) of one be than the other for it to be able to cause 20 times as much damage?

d. Two rifles are compared to see how far they can penetrate in to a target (this turns out to be directly proportional to the damage it can inflict). One has bullets with one-third the mass of the other, but its bullets still penetrate 4 times as far in to the target as the other's. How many times faster is the muzzle velocity of the rifle with the lighter bullets?

32 A beam that is supported horizontally at one end and free at the other can be deflected an amount D at the free end according to the equation $D = \frac{FL^3}{3K}$, where F is the force applied, L is the length of the beam, and K is a constant based on the shape and stiffness of the beam. Assume K does not change in all parts of this problem.

a. If the beam is currently being deflected an amount D_1, how much more force would you have to apply to triple the deflection?

b. If the beam was half the length it currently is, how much harder would one have to push to yield the same deflection D_1?

33 The number of alligators observed in a Martian swamp increases according to the equation $A = k \cdot 2^t$, where t is the number of days after the first alligators were observed, and k is a constant.

a. If there are 400 alligators in the swamp in the initial observation, how many alligators are there after 1 day? 2 days? 3 days? 6 days?

b. There are A_1 alligators at time $t = 13$. At what time will there be double this number of alligators?

c. If there are A_1 alligators after time t_1, how many alligators will there be after time $2t_1$? After time $3t_1$? After time nt_1?

34 The Richter scale indicates the intensity (I) of an earthquake by relating it to the amplitude (A) of waves at its epicenter in the following way: $I = \log_{10} A$. (Units are ignored in this problem for simplicity.)

a. What amplitude of waves would give an intensity of 1?

b. How many times bigger than in part a would the amplitude of the waves have to get for the intensity to be 2?

c. How many times bigger than in part a would the amplitude of the waves have to get for the intensity to be 6?

d. If the amplitude of the waves were double that of the waves in part a, what would the intensity be?

e. If the amplitude of waves at some time is A_1, and at some later time is $2A_1$, how did the intensity change between the two times?

35 The total cost C of producing N cars from scratch, where one has to first build the factory and the assembly line before producing a single car, follows the equation $C = 6000N + 12,000,000$. The factory can produce tens of thousands of cars in a single year. What affect does doubling the number of cars produced, from N_1 to $2N_1$, have on the total cost C? Justify your answer with a specific example or two.

36 An ideal gas obeys the equation $PV = nRT$. If the pressure is P_1 at some point in time, what will the new pressure be if, simultaneously, V is halved, T is tripled, R remains the same, and n is doubled?

37 The pull of a magnet turns out to be inversely proportional to the cube of an object's distance from the magnet; that is, $Pull = \frac{k}{d^3}$, where k is some constant.

Sarah, who is 24 feet from a magnet and holding a piece of iron, feels 512 times less Pull from the magnet than Isaac does, even though he is holding an identical piece of iron. If the magnet, Isaac and Sarah are all in a straight line in that order, how far apart are Sarah and Isaac?

38

"The Attack of the 50 Foot Woman!" was a campy "sci-fi" thriller from the 1950's, about a woman who, due to an alien encounter, grows to be a 50 ft giant, wreaking havoc on the metropolis she lives in. Could humans prosper at such heights if, say, our DNA gave our body permission to "keep growing"? Let's check it out.

a. Here are two important facts: 1) The weight of any animal is directly proportional to its volume, and 2) The strength of a bone is directly proportional to the cross-sectional area of the bone. Write equations that represent these relationships.

b. A 50 ft woman is 10 times taller than a 5 ft woman, indicating that one could roughly think of her as a 5 ft woman scaled up by a factor of 10 in all dimensions. How many times more would the 50 ft woman weigh than the 5 ft woman?

c. Since all dimensions are scaled by a factor of 10, how many times stronger would the 50ft. woman's bones be than the 5 ft woman's?

d. Bones in the leg break if they are under too much strain/pressure, that is, if there is too much weight above pressing down on the bones. Using your answers to parts b and c, determine how the pressure (i.e. $\frac{\text{Weight}}{\text{Cross}-\text{sectional area}}$ of the leg bones) on the 50ft woman's leg bones compares to the 5 ft woman's.

e. Thus explain why animals that are small would have difficulty if they grew too large, even given that their DNA permitted it. That is, why aren't there 50 ft women?

39

A spherical balloon becomes bigger and bigger as it is filled with more and more air, although it always retains its spherical shape. It starts out at $t = 0$ as a balloon with a volume of 10 cubic centimeters. With every second, 30 cubic centimeters more of air is pumped into the balloon.

a. Write an equation for the volume of the balloon as a function of time.

b. Write an equation for the volume of the balloon as a function of its radius.

c. Using parts a and b, write an equation relating time and radius of the balloon.

d. Now, using algebra, find radius as a function of time.

40

Examine the following crazy formula: $y = \frac{ab^2c^3\sqrt{d}}{ef^2g^3\sqrt{h}}$. Initially, y equals some unknown number based on current values of a, b, c, d, e, f, g, and h.

What happens to y if...

a. b is tripled?

b. g is halved?

c. d is quadrupled?

d. h is multiplied by 16?

e. c is doubled AND e is doubled?

f. d is multiplied by 9 AND g is tripled?

Exploring in Depth

41 The speed on a given planet with which an object must be propelled from its surface straight up so that it would never come down is called its escape velocity, determined by the equation $v_{esc} = \sqrt{\frac{2GM}{R}}$, where v_{esc} is the escape velocity in meters/second, G is a fixed constant, and M and R are the mass and radius of the planet in kilograms and meters.

a. The **escape velocity** on the surface of the Earth is 11,200 meters/second (you can see why we need rockets to achieve such velocities). If the Earth were twice as massive but the same size, what would the escape velocity then be? How many times more massive would the earth have to be for the escape velocity to be twice what it is now?

b. The moon has a radius that is .273 times the Earth's. It has a mass that is .0123 times the Earth's. What is the escape velocity on the surface of the moon?

42 Don't use a calculator for this problem.

a. Evaluate: $\log_3 45 + \log_3 2 - \log_3 10$

b. Solve for x: $x^2 = 10 - 3x$

c. Evaluate: $\frac{(2^3 3^2)^5}{4^7 3^8}$

d. Expand: $(2x - 3)^3$

e. Simplify: $\sqrt{6} \cdot \sqrt{35} \cdot \sqrt{21}$

43 Kleiber's Law for Animal Metabolic Rates is as follows: $q^4 = km^3$, where m is the mass of the animal in kilograms, k is a constant, and q is the metabolic rate.

a. If one animal is twice the mass of another, how many times bigger is its metabolic rate?

b. If one animal has 3 times the metabolic rate of another, how many times bigger is its mass?

c. If one animal has half the metabolic rate of another, how many times less massive is it?

44 The pressure exerted by an object A on another object B is directly proportional to A's mass and inversely proportional to the area of that part of A that rests upon B. (Remember, too, that the mass of an object is directly proportional to the cube of its height, as long as the other two dimensions change proportionally with height.)

Suppose A is a brick and C is another brick that is similar (in the mathematical sense) to A and made of the same material. Both bricks are resting on a table.

a. If C is two times as big (in each direction) as A, then how much pressure does C exert on the table in comparison to A?

b. If C's surface area is twice as big as that of A, then how much pressure does C exert on the table in comparison to A?

45 Physics tells us that an object falling in a vacuum will, because of gravity, get faster and faster — there's no limit on its top speed as it falls. In practice, though, the Earth's atmosphere causes **drag**. Because of this, a falling object (like, say a skydiver) will initially speed up, but eventually will reach its **terminal velocity** — its constant top speed, where the drag force perfectly cancels out gravity, so that there's no more acceleration.

According to Wikipedia, the square of the terminal velocity of an object is directly proportional to its mass and inversely proportional to its cross-sectional area.

Suppose you dropped two spherical balls, both made of identical material, from an airplane.

a. If the radius of the larger of the two balls is sixteen times the radius of the smaller, then which ball has a higher terminal velocity? How many times faster is it than the other one?

b. If, instead, the larger of the two balls weighs twice as much as the smaller, then which ball has the higher terminal velocity? How many times faster is it than the other one?

c. Now suppose that two balls, this time made of *different* materials, turn out to have the same terminal velocity. If the radius of the larger ball is twice the radius of the smaller, then which ball is made of a denser material? How do their densities compare? (Note: density = mass/volume.)

LESSON 4: MODELING WITH DATA

Introduction

1 The following data, taken from facebook.com, describe the approximate number of Facebook users at various times since the beginning of 2004.

t (months since January 2004)	P (millions of Facebook users)
12	1
24	5.5
36	12
46	50
56	100
61	150
62	175
64	200
67	250
69	300

Use your calculator to plot these data, with the Facebook population on the y-axis, and time in months on the x-axis. Then, try to predict how many people were on Facebook at the beginning of 2010.

Looking at the data, it's quite clear that the Facebook population has been growing over time, and it's also pretty easy to see that the *rate* at which it's growing is increasing, too. Even if we take those two observations into account, though, there seems to be quite a wide range of plausible predictions in problem 1. It's hard to know how to decide whether, say, 375 million users or 500 million users is the better prediction.

What if we could find an equation that appeared to fit the existing data very well? In that case, we would have reason to believe that the equation was a good mathematical model for Facebook population growth, and we could therefore make more specific, confident predictions.

If you've studied biology, you might have seen that exponential models are one of the simplest and most common types of models used for population growth. Even though Facebook isn't quite same as, say, bacteria in a Petri dish, it turns out that these data indeed appear to exhibit exponential growth. Let's find an exponential function that fits the data. Recall that, by definition, an exponential function has an equation in the form $y = a \cdot b^x$, where a and b are constants.

2 To find constants for our data, we'll need to substitute two points into the general exponential equation, and then solve the resulting pair of equations for a and b.

a. We have more points than we need: since there are only two constants to solve for, we actually only need two points. Which of the following pairs of points do you think we should use to get the best fit for the whole data set? Why?
$(12, 1)$ and $(24, 5.5)$
$(24, 5.5)$ and $(64, 200)$
$(36, 12)$ and $(62, 175)$

b. For each of the pairs of points in part a, find the equation of an exponential function that goes through the two points. Then, use your calculator to graph each of these three equations together with the actual data. Which one was the best fit? Which was worst? Why?

3 Using whichever equation from problem 2 fit the data best, predict the Facebook population at the beginning of 2010. Compare your predictions to the actual data, available at: http://www.facebook.com/press/info.php?timeline.

Development

Jenn Jeffings, ecologist extraordinaire, has taken some data, shown below, on chestnut oak trees in the woods around her house. She used the house as home base, and counted the number of chestnut oaks within certain distances. So, for example, the second row in the table indicates that Jenn counted 63 chestnut oaks that were each no more than 10 meters away from the house. Plot these data using your calculator.

Distance (meters)	Number of chestnut oaks
5	16
10	63
12	90
15	141
20	251
24	362

You can see clearly that, just as with the Facebook data, the y values are increasing at a faster and faster rate.

4 Are the data exponential? Either use an equation and some algebra, or else work directly with the numbers to answer this question.

It turns out that Jenn's tree data can be modeled accurately using a **power function**, which is any function that can be described with an equation of the form $y = a \cdot x^n$, where a and n are constants.

5 Strategically pick two points and use them to solve for the constants a and n. Check out the particular equation that results by graphing it on your calculator together with the data.

6 According to the model you made in problem 5, how many chestnut oaks are within 100 meters of Jenn's house?

The equation you came up with in problem 5 is just one example of a power function. Let's take a brief look at these functions in general.

7 Is the graph of $y = 5x^2$ exactly five times steeper than the graph of $y = x^2$? Is it exactly five times higher? Experiment with some values of a other than 5, and see if you can get any different effects. In each case, it will be helpful to compare your graph to the more basic $y = x^2$.

8 Graph $y = x^4$ and $y = x^5$ on your calculator. Compare them to $y = x^2$ and $y = x^3$. Now predict what $y = x^{94}$ and $y = x^{95}$ will look like. Check your prediction. What can you say in general about power functions where n is a positive integer?

9 Find the coordinates of a specific point that *must* be on the graph of $y = 3x^n$, regardless of the value of n. Then find one more.

Let's go back to the Facebook data we looked at in the introduction. In problem 2, you should have seen that the equation $P = 0.637 \cdot 1.094^t$ fit the data best. For consistency, let's use that equation for the following problems.

10 Suppose you were writing a *Postscript* article about Facebook's incredible exponential growth, and you wanted to include a prediction about when we would expect the Facebook population to reach 1 billion users. According to the equation, when will this happen?

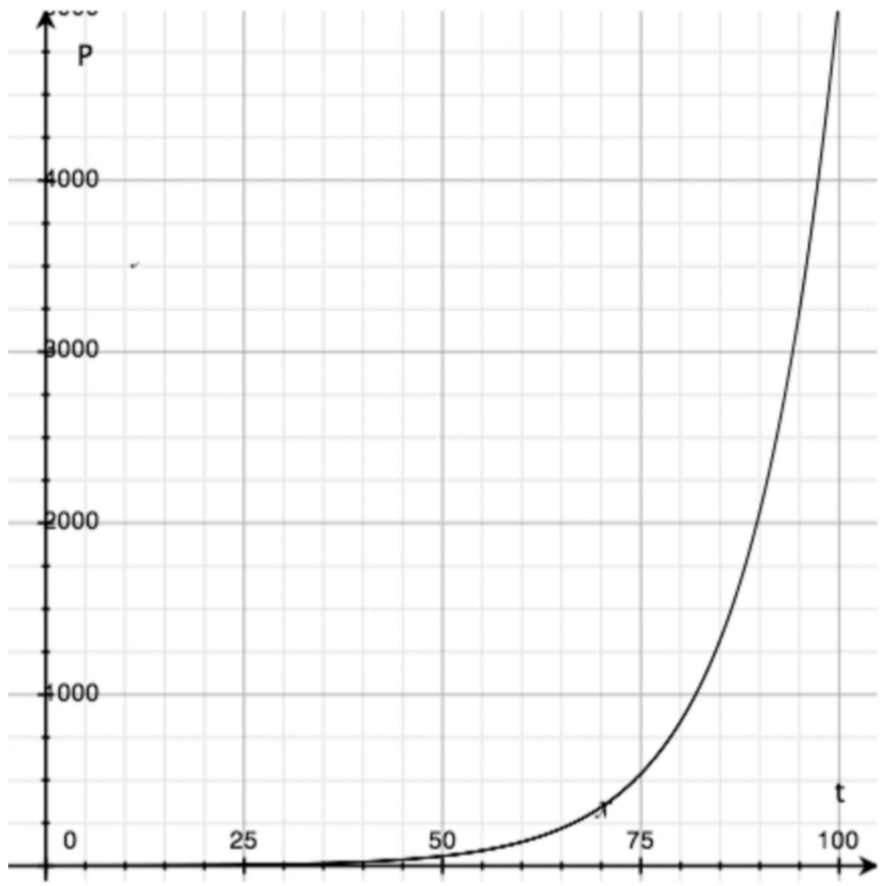
11 According to the equation, how long will it be after your answer to the previous problem until the population reaches 2 billion users? And how long after *that* until it hits 4 billion users?

12 Write an equation that expresses the time t, as a function of the population P. In other words, rewrite the equation $P = 0.637 \cdot 1.094^t$ in "$t =$ " form.

On the one hand, you could think of this equation as being the same as, or equivalent to, the original "$P =$ " equation. On the other hand, though, it's quite clear that in this equation, the "input" and "output" variables have been reversed. As you might recall from the lesson on algorithms, this means that the equation you wrote in problem 12 represents the **inverse** of the original function. (Actually, they are inverses of each other.)

13 Equations of the form $y = a\log_b(x) + c$ are called **logarithmic functions**. If you haven't already, use a bit of algebra to show that the equation you wrote in problem 12 is, indeed, a logarithmic function. Why does it make sense that the inverse of an exponential function would be a logarithmic function?

14 Below is a graph of the equation $P = 0.637 \cdot 1.094^t$.



a. Using the graph above, and without using your calculator, make a large (at least half-page) and accurate graph of your equation from problem 12. (Remember that, typically, we put the output on the vertical axis and the input on the horizontal axis.) What do you notice about the graph?

b. Plot your answers to problems 10 and 11 both on the graph above and also on your graph.

Practice

15

Fit a power function to each set of data.

a.

x	2	3	5	10	14
y	24	81	375	3000	8232

b.

x	4	9	11	17	22
y	52.8	139.7	177.7	299.6	408.2

c.

x	10	20	30	40	50
y	3.2	17.9	49.3	101.2	176.8

16

Fit a logarithmic function to the following data.

x	1	10	50	100	1000
y	4	5.5	6.5	7	8.5

17

Find the inverse of each of the following functions.

a. $H = 3t + 5$

b. $V = \dfrac{4}{3}\pi r^3$

c. $K = 65 + 30 \cdot 0.8^t$

d. $T = \log_2 n + 2$

Problems

18 In each of the following datasets, it appears that y increases at a faster and faster rate as x increases. Which of them are exponential?

x	1	3.3	7.5	15	27.8
y	0.75	8.17	42.19	168.75	579.63

x	1	10	21	34	42
y	2	63.2	192.5	396.5	544.4

x	1	3	5	7	9
y	3.14	28.26	78.5	153.86	254.34

19 When a rock is dropped from the top of a large cliff, it falls to the Earth in a very predictable way. Assuming air resistance is negligible (for the first few seconds at least, not a ridiculous assumption), here is a chart of time vs. distance fallen in the previous second. For example, at $t = 3$ seconds, the rock fell 80 feet in the previous second.

Time (secs)	Distance this second (feet)	Total Distance (feet)
1	16	
2	48	
3	80	
4	112	
5	144	
6	176	

Can you write an equation that relates total distance fallen (i.e. distance fallen from the top of the cliff) to time elapsed?

20 Suppose that you had the following data on the number of downloads D of Lady Gaga's latest single, as a function of time t in days since it came out.

t (days)	D (millions of downloads)
10	42
15	60
50	300

a. Fit an exponential function to the data provided, and use your equation to predict the number of downloads on the 100th day after release.

b. Now, fit a power function to the data, and use that equation to predict $D(100 \text{ days})$.

c. How much confidence do you have in each of these models? Explain.

21 Graph $y = (3/5)^x$ in the standard window on your calculator. It appears that this graph eventually touches the x-axis. Estimate at what value of x this happens.

22 In the previous problem, you probably saw that as x gets larger and larger, the graph of $y = (\frac{3}{5})^x$ and the x-axis become nearly indistinguishable. When this happens, we say that the graph of $y = (\frac{3}{5})^x$ and the x-axis are **asymptotic** to each other. We can also say that the x-axis is an **asymptote** of the graph of $y = (\frac{3}{5})^x$.

a. Is the graph of every equation of the form $y = b^x$ asymptotic to the x-axis? Explain.

b. The x-axis is a horizontal asymptote of $y = (\frac{3}{5})^x$. What *vertical* line is the graph of $y = (\frac{3}{5})^x$ asymptotic to?

23 It's hard to experience it without going into space, but it turns out that your weight actually depends on how far you are from the center of the earth. Here are some data showing how an individual's weight would vary:

Distance (miles)	Weight (pounds)
10000	18.9035532
20000	4.7258883
40000	1.181472075
80000	0.295368019

a. These data can actually be modeled by a power function. Using that fact, and the definition of a power function, find an equation for the data.

b. According to your equation, how would your weight change if you doubled your present distance away from the center of the earth? (Note: the radius of Earth is about 3969 miles... but does that even matter?)

c. How far up would you have to go in order for your weight to be 1% of what it is on the Earth's surface?

d. Graph the equation you found in part a. Does it have any asymptotes? If so, what do they mean in terms of the situation being modeled?

24 The equation you found in the previous problem is an example of an **inverse power function** — that is, a function of the form $y = ax^n$, where n is negative. Experiment with a few different values of a and n to get a sense for what inverse power functions can look like. Is it true that every inverse power function is asymptotic to both axes? Why?

25 The following data show the number of hours of sunlight in northern Maine based on the number of days since the winter solstice, which in 2009 was on December 21. (So, for example, the first row shows the number of hours of sunlight on December 21, the shortest day of the year.)

Sunlight hours	Days after winter solstice
8.5	0
8.6	7
8.7	14
8.9	21
9.2	28
9.5	35

(data from http://astro.unl.edu/classaction/animations/coordsmotion/daylighthoursexplorer.html)

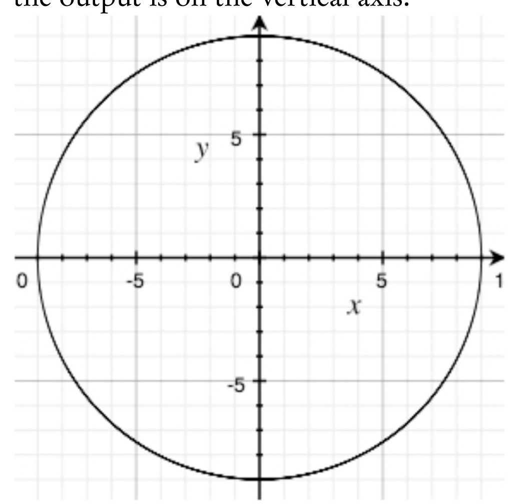
One night, at their vacation home in Maine, Brad Pitt and Angelina Jolie had a fight. Brad couldn't sleep that night, and so he was up when the sun rose at 7:04am. By the time it set at 4:22pm, they still hadn't made up. Using the data in the table, is it possible to say which day of the year Brangelina had their fight?

26 Often, when we are trying to model a phenomenon using mathematics, we're interested in using one variable, called the *input*, to predict the value of another variable, the *output*. This is generally only possible, however, if the input uniquely determines the output — or, in other words, when there is at most one output for any given input. When this is the case, we say that the relationship is a **function**. This was true, for example, in the Facebook population example at the beginning of the lesson, but it was *not* the case in problem 25. In each of the following, determine whether the given relationship is a function.

a. $\lambda = \dfrac{3 \cdot 10^8 \mathrm{m/s}}{f}$ (λ is the output.)

b. Your equation from problem 23.

c. The input is on the horizontal axis, and the output is on the vertical axis:



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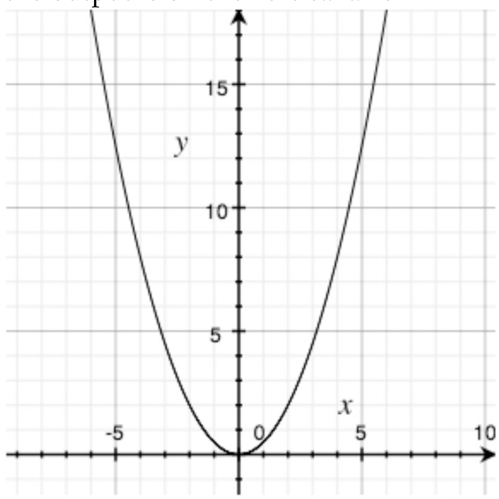
d. The input is on the horizontal axis, and the output is on the vertical axis:



e. Input: birth date. Output: age.

f. Input: month of the year (e.g., "January", "June", etc.).
Output: average temperature in Baltimore city.

27 How many vertical asymptotes does the graph of $y = 2x^5$ have?

28 Does $y = \log x$ have any vertical or horizontal asymptotes? Explain.

29 Google and others roughly measure the popularity of a given website based on the number of other pages on the Internet that link to the site. The following dataset shows how many blogs there are at different levels of popularity. As you might expect, there are significantly more unpopular blogs than there are popular ones.

Popularity (number of links from other sites)	Number of Blogs
5	50
10	40
16	30
35	20
132	10
530	4
4014	1

a. Find a power function that fits these data.

b. Based on your equation, how does the number of blogs with 1000 people linking to them compare with the number of blogs 500 people linking them?

c. Is the asymptotic nature of inverse power functions relevant to the situation being modeled? Explain.

d. According to this equation, how many links would the single most popular blog have?

30 Come up with a function whose graph is asymptotic to the line $y = 3$. Can you make one that is *also* asymptotic to the line $x = 5$?

31

In the chart below, there are four functions (f, g, h, and j) that are either “regular” power functions ($y = kx^n$, n positive) or inverse power functions ($y = kx^n$, n negative). For each function in the chart, determine which of these types of functions it is, and then find k and n.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-.0156	-.037	-.125	-1	---	1	.125	.037	.0156
$g(x)$	48	27	12	3	0	3	12	27	48
$h(x)$	5.2	3.9	2.6	1.3	0	-1.3	-2.6	-3.9	-5.2
$j(x)$	-.3125	-.5555	-1.25	-5	---	-5	-1.25	-.5555	-.3125

32

If you graphed the data and equations in the previous problem, you can clearly see four distinct shapes that a power function can have, depending on whether the power is even, odd, negative, or positive. The data below have yet another shape — and this data, too, can be modeled by a power function. Find an equation for the data. Does the graph of your equation have a horizontal asymptote?

x	1	8	27	36	43
y	5	10	15	16.5	17.5

33

The following data from the 2006 census shows the distribution of wealth in the United States. The second row, for example, says that there are 8,138,000 households having an income of $60-70 thousand dollars per year.

Income range	Thousands of households
10,000-20,000	14,447
60,000-70,000	8,138
110,000-120,000	2,920
160,000-170,000	992
210,000-220,000	336
240,000-250,000	245

(Data from: http://pubdb3.census.gov/macro/032006/hhinc/new06_000.htm.)

a. Based on what you know about the shapes of different functions, what types of functions *could* model these data? Does one type seem more promising?

b. Find an equation that accurately models the data above.

c. Using your equation, predict how many U.S. households have an income of $80,000.

d. Check out this visualization of the data above: http://www.visualizingeconomics.com/2006/11/05/2005-us-income-distribution/

The data you’ve seen in the previous problem, as well as in problem 29, are examples of what’s called the “80-20 rule,” or the “Pareto principle.” Vilfredo Pareto, an economist in the late 1800’s, made two oddly related observations: 80 percent of Italy’s wealth was owned by roughly 20 percent of the population, and 80 percent of the peas he harvested in his garden came from roughly 20 percent of the pea plants. (http://en.wikipedia.org/wiki/Pareto_principle.)

Since then, a surprising number of phenomena have been discovered to follow this general rule. Here's another example.

34 The data set below comes from a study of how often different words appear in a large sample of English text. The first column represents a certain level of rarity or commonness for a word, and the second column shows how many words had appeared with that frequency. So, for example, there were 586 words that appeared around 200 times, whereas there were only 73 words that appeared 600 times.

Approximate word frequency	Number of words with (approximately) this frequency	Example words with this frequency
200	586	"congress", "traffic"
300	236	"particularly", "restaurant"
400	147	"dinner", "dollar"
500	88	"question", "lives"
600	73	"sit", "neat"

(Data from: http://www.americannationalcorpus.org/frequency.html)

a. Is there evidence directly in the data above to suggest whether this situation would be modeled better with an exponential or a power function?

b. Find an equation that models the data above.

c. Use your equation to predict how many words would appear approximately 2010 times in the sample text.

35 You probably know that as you gain altitude, the temperature decreases. Here's some data that provides detail on this phenomenon for a particular location.

Height above sea level (m)	Temp (C)
1000	8.5
2000	2
3000	-4.5
4000	-11
5000	-17.5

(Data from: http://www.usatoday.com/weather/wstdatmo.htm)

a. If you wanted to fit an equation to these data, what type of function do you think you should try first?

b. Find an equation for the data, and use it to predict the temperature at an altitude of 7 kilometers above sea level.

36

At 10:00 am Tim left his hot cup of tea on the table in his classroom. The air conditioning in the room was on, so the temperature of the tea dropped fairly quickly, as indicated in the table below. The time is in minutes after 10:00 am and the temperature is in °F.

Time	Temperature
0	170.5
5	130.5
10	106.5
15	92.1
20	83.46
25	78.28
30	75.17
35	73.3
40	72.18

a. Plot these data on a coordinate axis system. To what temperature does the tea's temperature appear to be converging?

b. According to Newton's Law of Cooling, the temperature should be an exponential function of time. Try to fit an exponential equation to the data above. What happens? Why?

c. Find an equation that accurately models the given data. Based on your equation, predict the time at which Tim's tea will be within 1 degree of room temperature.

37

Sketch a graph of the equation you came up with in the previous problem. Now, estimate the slope of this curve at four different points.

a. What are the units of your slope values? What do these numbers mean about Tim's cup of tea?

b. Newton's Law of Cooling is actually stated thus: the rate of heat loss of an object is directly proportional to the difference between the object's temperature and room temperature. Do some calculations to verify this, using your answers to part a.

38

Suppose that the following data shows the height of a soccer ball as a function of time. As you probably learned in physics, these data should be parabolic, and therefore can be modeled using a quadratic equation.

Time(s)	Height(m)
0	0
0.5	11
1	19
1.5	25
2	28
2.5	29
3	27
3.5	23
4	17

a. Find an equation that accurately models the data.

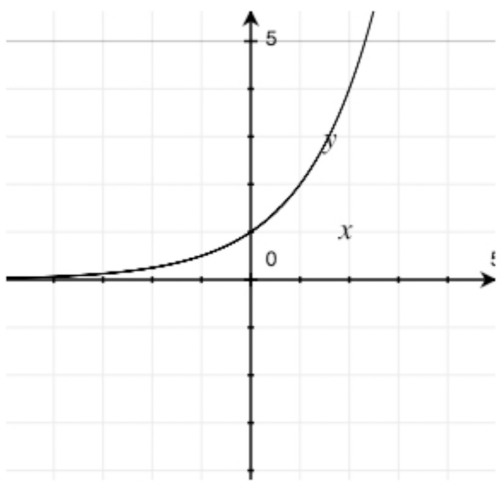
b. Mick Jagger, who was watching the game, took a picture on his cell phone when the ball was 22 meters above the field. How many seconds after the ball was kicked did Mick take the photo?

c. Is it possible to write a single equation that gives you t as a function of h?

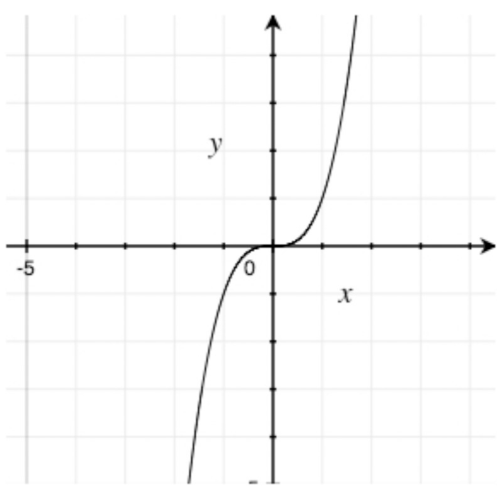
39 For each of the relationships in problem 26 that you decided was a function, determine whether or not it has an inverse function.

40 For each graph below, determine what type of function it is. If it's possible to be even more specific, then do so.

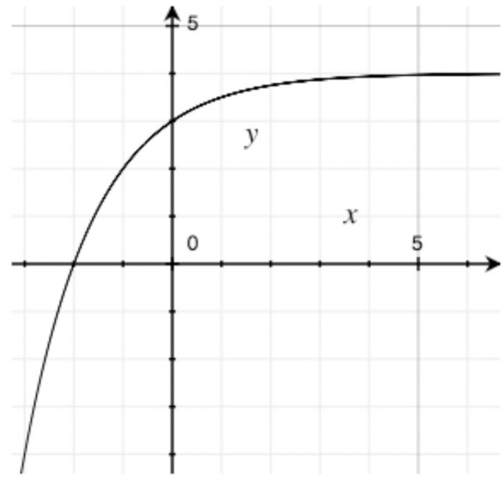
a.



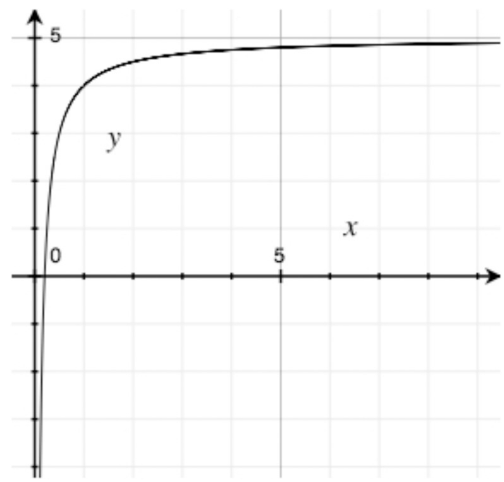
b.



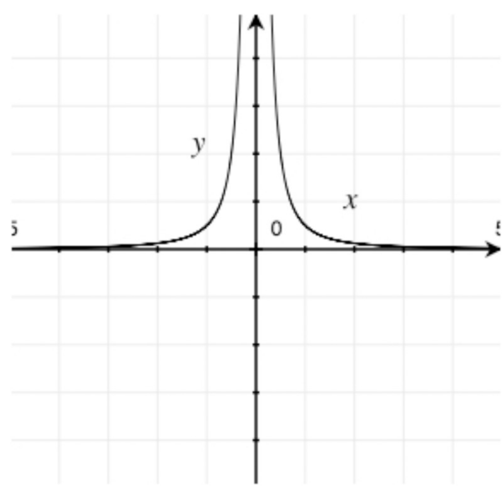
c.



d.



e.



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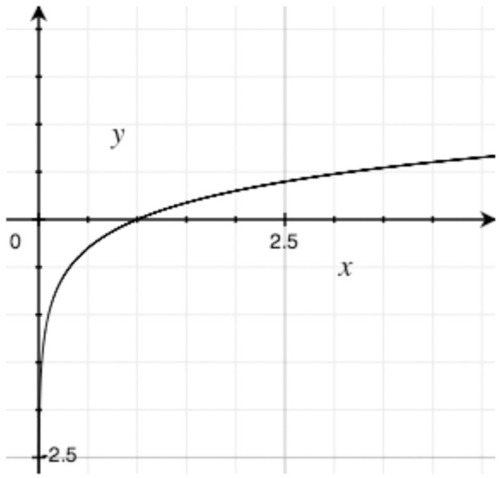
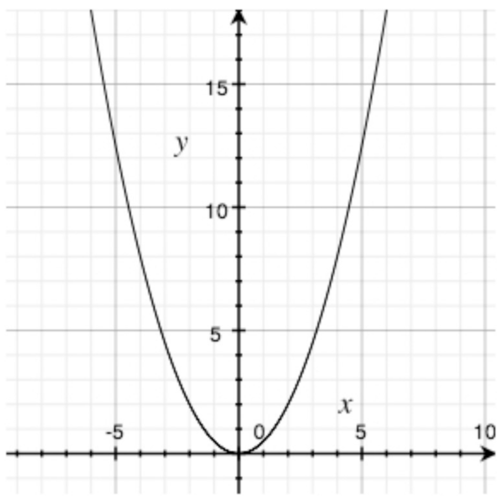
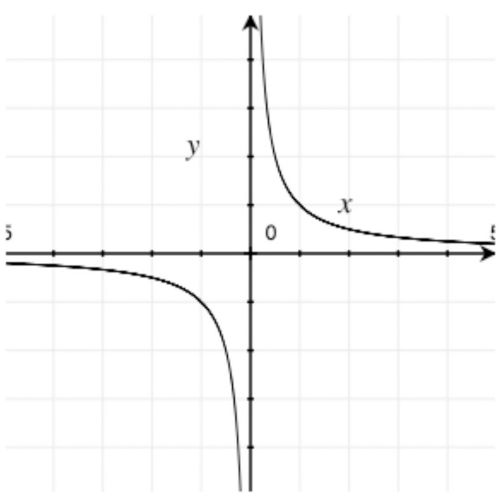
f.



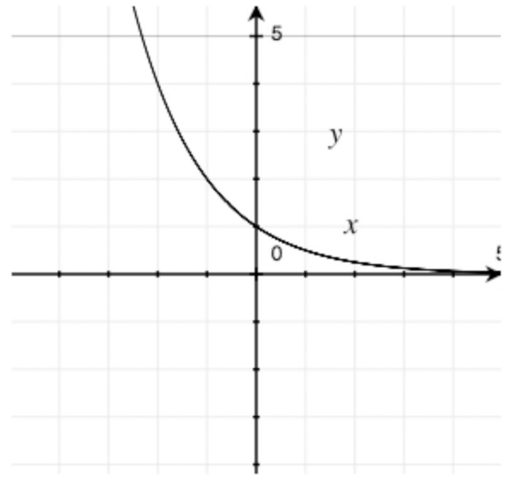
g.



h.



i.



41 The 1000 employees of the Acme Utensil Factory have been playing an absurdly long game of Blammo. They collected the following data on the number of surviving players on few different days.

Days	Survivors
0	1000
1	950
10	599
20	358
30	215
40	129

a. Based on what you know about the shapes of different functions, what types of functions *could* model these data?

b. Find an equation that models the data above.

c. RJ hypothesizes that the death rate in Blammo games is directly proportional to the population. Use your equation to estimate the Blammo death rate on days 1, 3, 10, and 12, and then test RJ's hypothesis.

42 Don't use a calculator for this problem.

a. Find x if $\log_2 x = 6$.

b. Solve for x: $(x-1)(x-3)^2(x+4) = 0$

c. Subtract: $\frac{3}{a} - \frac{7-b}{ab}$

d. Simplify: $\frac{\frac{x}{y}+1}{2-\frac{x}{y}}$

e. $\sqrt[3]{162}$ can be written in the form $a\sqrt[3]{b}$, where a and b are integers and b is as small as possible. What are a and b?

Exploring in Depth

43 In the beginning of this lesson, you saw that the inverse of an exponential function is a logarithmic function. Now, prove that the inverse of a power function is always a power function.

44 Prove that every equation of the form $y = a\log_b x + c$ can be written in the form $y = \log_B x + c$.

45 Look back at the equation you wrote for h as a function of t in problem 38. Using that equation...

a. Write an equation that tells you t as a function of the height h of the ball on its way up to its highest point.

b. Write an equation that tells you t as a function of the height of the ball on the way *down*.

46 Why isn't there only a single answer to the question "What angle has a sine of .4221"?

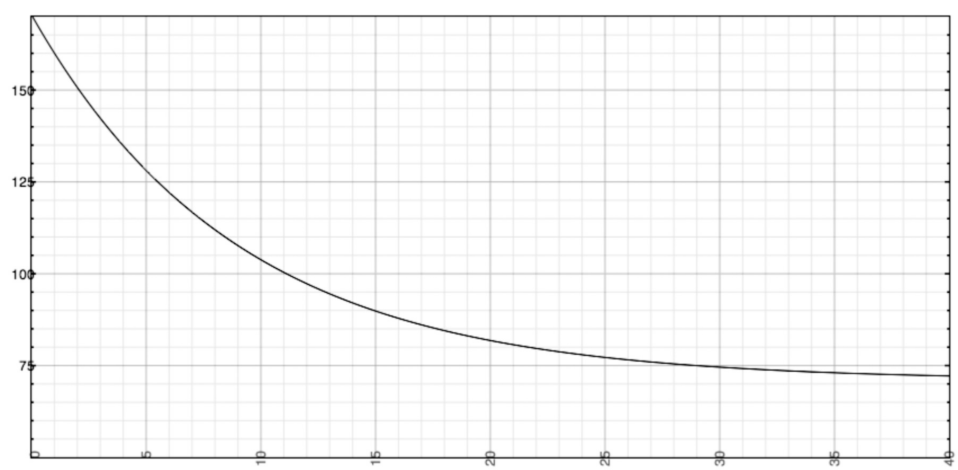
47 Your answer to the previous problem implies that the sine function does not have an inverse function. If that is the case, what the heck is your calculator doing when you do $\sin^{-1}$?

48 Is it possible to have an inverse function for just *part* of the sine function? What part?

LESSON 5: MAKING NEW FUNCTIONS

Introduction

In previous lessons, you may have used exponential, logarithmic, and power functions to model data. For instance, if you had evidence that the data could be modeled by an exponential function, then your task was to find values for a and b in the function type $f(x) = ab^x$. However, one problem in the Exponential Functions lesson involved a kind of exponential function for which it wouldn't work simply to find a and b. A 180° cup of coffee is left out to cool in a 70° room. Below is a graph showing the temperature of the coffee over time.

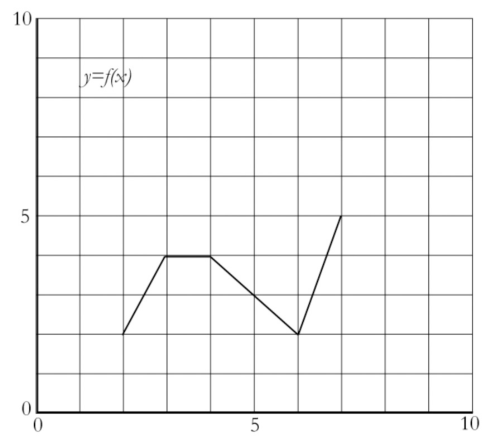


A function that does a decent job of modeling this data is $f(x) = 99.5(.895)^x + 71$. You can think of the function as a translation of 71 units up of the more familiar function $g(x) = 99.5(.895)^x$.

It's easy to imagine changing the vertical shift of g in order to produce lots of different functions, perhaps corresponding to cooling curves in rooms of different temperatures. More generally, in modeling data it's useful to be able to start with basic function types and use transformations in order to control the shape and position of those graphs. We will begin our study of this technique by examining more abstract types of functions.

Development

Below is a graph of a function f. Note that f is not one of the function types you have studied before. It does not have a nice formula. But its graph is still enough for you to answer questions about input and output.



A table of values for the function might look like this:

x	1	2	3	4	5	6	7	8
$f(x)$	---	2					5	---

1 Fill in the four missing values.

Note that we say "a table of values" "might" look like this because there are far more points on the graph of this function. If you wanted to, you could use $x = 3.5$ in the table, or even $x = \pi$. The dashes in the table are there because the function is not defined for certain inputs: specifically, for $x < 2$ and $x > 7$. Another way of saying this is that the **domain** of the function is $2 \leq x \leq 7$.

2 Using the same function f as in problem 1, draw the following graphs. If you need help getting started, remember that an expression like " $f(4) + 2$ " means "two more than the value of $f(4)$."

a. $y = f(x) + 2$

b. $y = \frac{1}{2}\, f(x)$

c. $y = -f(x)$

In Problem 2, you saw examples of a vertical translation, a vertical contraction, and a reflection over the x-axis. Horizontal transformations merit a little more attention.

3 Much as we can define the function $y = f(x) + 2$, we can define the function $y = f(x - 2)$.

a. Find the output of the function $y = f(x - 2)$ for the input $x = 5$.

b. Make a table of values with the headings x and $y = f(x - 2)$, and fill it in. Take care to think about which values of x it makes sense to use.

4 Using the values from your table in Problem 3, graph the function $y = f(x - 2)$. How was the graph of $y = f(x)$ transformed to make this new graph?

5 What is the domain of the function you graphed in Problem 4?

6 Conjecture a formula (like $y = f(x - 2)$) that translates the graph of $y = f(x)$ three units to the left. Then make a table and/or graph to see if you are right.

7 Now consider the function $y = f(2x)$. First find the output when $x = 2$. Then make a table of values with the headings x and $f(2x)$, and fill it in. Take care to think about which values of x it makes sense to use.

8 Using the values in your table from Problem 7, graph the function $y = f(2x)$. Describe the transformation of the graph of $y = f(x)$ this represents.

9 What is the domain of the function you graphed in Problem 8?

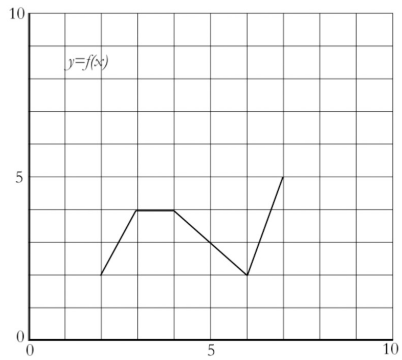
10 Conjecture a formula that horizontally stretches the graph of $y = f(x)$ so that each point is three times farther away from the y-axis.

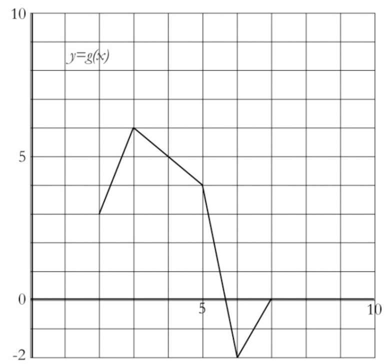
11 Use a similar process to graph $y = f(-x)$. What transformation of the graph of $y = f(x)$ does this represent?

12 Find a formula for $f(-x)$ if $f(x) = x^2 + 2x - 8$. Then sketch the graphs of $y = f(x)$ and $y = f(-x)$ on the same axes.

Using transformations is one way to build a new function from a known function. You can also combine two functions to get a third.

13 Let f be the function from the introduction (shown below, on the left), and g be the function defined by the graph below, on the right.





a. The function h is defined by the equation $h(x) = f(x) + g(x)$. For example, to find the value of $h(3)$, you would add together the values of $f(3)$ and $g(3)$. Find $h(3)$, then sketch a graph of h.

b. Now sketch a graph of $j(x) = f(x) - g(x)$.

The following exercise illustrates another way to combine functions, known as **composition** of functions. Composition is when you "chain" two functions together, by taking the output of one function and using it as the input to a second function. To see why this might be useful, consider the following situation.

14 Howie is chewing Super-Bubble gum and blows a super-bubble. The radius (in inches) of his bubble increases with time (in milliseconds) in a way given by the equation $r(t) = 2t^{0.25}$. What is the volume of Howie's bubble 5 milliseconds after he blows it?

In order to solve the previous problem, you had to do two things. First, you had to find the radius of the bubble at 5 milliseconds. Then you had to use the radius you found in order to find the volume of the bubble. In function language, if V is the familiar function $V(r) = \frac{4}{3}\pi r^3$ giving the volume of the bubble in terms of the radius, then you first found $r(5)$, and then found $V(r(5))$.

15

Using the functions r and V as in the previous problem,

a. find $V(r(3))$.

b. find a formula, in terms of t, for $V(r(t))$.

16

Let functions f and g be defined as in problem 13.

a. Find $f(g(3))$.

b. Find $g(f(3))$.

c. Find $f(g(5))$.

d. Sketch a graph of $g(f(x))$.

17

The tribble population grows exponentially, and the level of the squeaky noise they collectively make is directly proportional to the size of their population. Let the tribble functions be defined as follows:

P outputs the tribble population at time t.

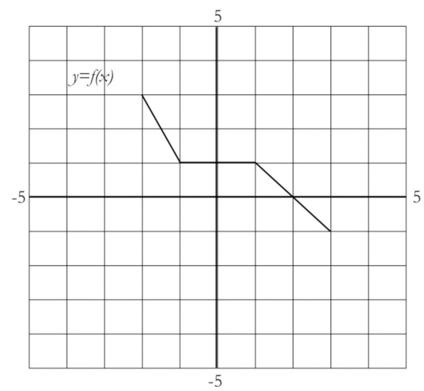
N outputs the noise level produced by a given number of tribbles.

D outputs the noise level given a time t.

Express one of these functions as a composition of the two others.

Practice

18 Let a function f be defined by the graph below. In your notebook, sketch a graph of each of the following equations.



a. $y = f(x-4)$

b. $y = f(x) - 2$

c. $y = f(-x)$

d. $y = -f(x)$

e. $y = f(\frac{1}{2}\, x)$

19 Without using your calculator, sketch a reasonable graph of $j(x) = \frac{1}{x}$. Identify the coordinates of three points on the graph. Then, without using your calculator, graph the following equations, including the images of the three points you identified. If the asymptotes move away from the axes, indicate them with dashed lines.

a. $y = j(x) + 1$

b. $y = j(x) - 0.5$

c. $y = 0.4j(x)$

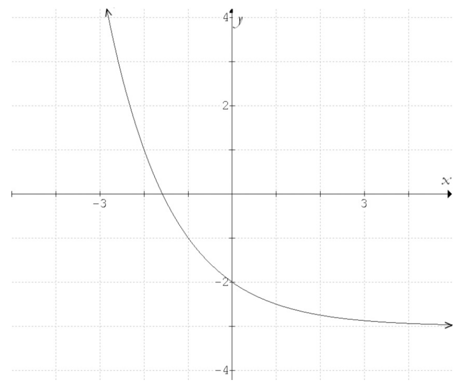
d. $y = -1.5j(x)$

e. $y = 2j(x-2)$

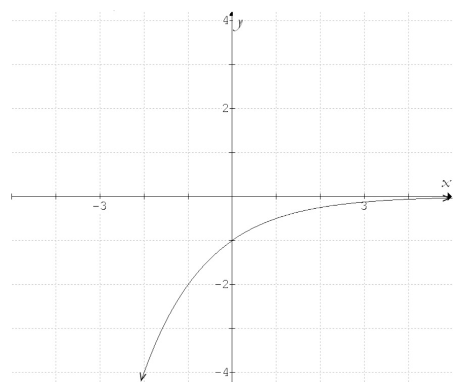
20 Write equations for each of the graphs you sketched in the previous problem.

21 Each graph below is a transformation of the graph of $y = 0.5^x$. For each graph, write an equation that models it.

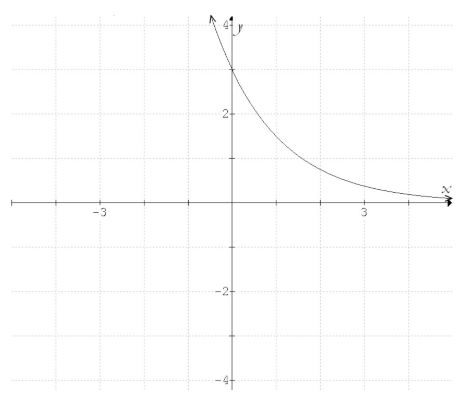
a.



b.



c.



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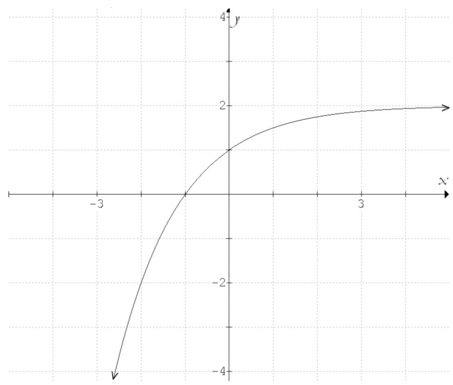
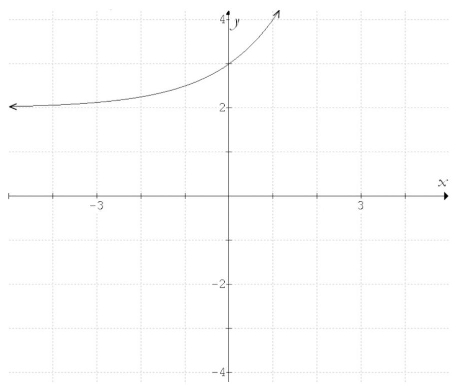
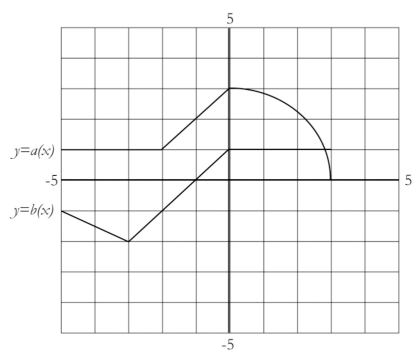
d.



e.



22 Let a and b be functions defined by their graphs below.



Sketch a graph of...

a. $y = a(x) + b(x)$

b. $y = a(x) - b(x)$

23 Let $f(x) = 3x$ and $g(x) = 2x + 1$. Sketch the graph of $h(x) = f(x) + g(x)$. What is its formula?

24

Let c and d be functions defined by their tables below.

x	0	1	2	3	4	5
c	5	-1	3	1	0	3
d	4	2	7	4	7	1

a. Find $c(d(3))$

b. Find $c(d(5))$

c. Find $d(c(5))$

d. If possible, find $c(d(2))$ and $d(c(2))$.

25

Let $f(x) = x^2$ and $g(x) = x^3$.

a. Write a formula for $f(g(x))$.

b. Compare your answer in part a to the formula for $f(x)g(x)$.

26

Give the domain...

a. of the square root function.

b. of the reciprocal function ($f(x) = \frac{1}{x}$)

Problems

27 Let $f(x) = x^2 + 2x - 8$.

a. Without using your calculator, graph $f(x)$ and $f(x-2)$ on the same set of axes.

b. By looking at your graph, find a formula for the function $g(x) = f(x-2)$.

c. Try the following alternative method for finding a formula for $f(x-2)$: Start with the fact that $f(x-2) = (x-2)^2 + 2(x-2) - 8$, then simplify. Why should this method work?

28 Find a formula for $f(2x)$ if $f(x) = x^2 + 2x - 8$. Then sketch the graph of $y = f(2x)$.

29 $y = f(x)$ is an exponential function that has a y-intercept of 2 and is asymptotic to $y = 0$. Find the y-intercept and asymptote of $y = -2f(x)$.

30 $(2, 5)$ is a point on the graph of the function $y = g(x)$. Identify the coordinates of the corresponding point on the graph of...

a. $h(x) = \frac{1}{2}\, g(-x)$.

b. $h(x) = -3g(x)$.

c. $h(x) = 6 - g(-x)$.

d. $h(x) = g(x-5) + 1$.

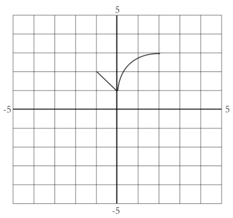
e. $h(x) = g(-2x)$.

f. $h(x) = c \cdot g(x) + b$.

g. $h(x) = -g(cx)$.

31

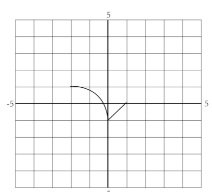
Let g be the function defined by the graph below.

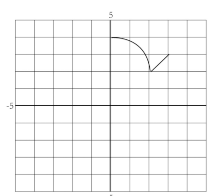


For the graphs in parts i through vi,

a. come up with a series of transformations that will turn the graph of g into the graph you see.

b. write an equation for the transformed function (for example, the first function shown below is $y = g(-x) - 2$).

i.


ii.


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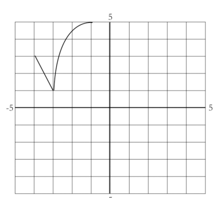

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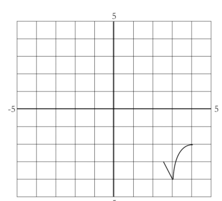

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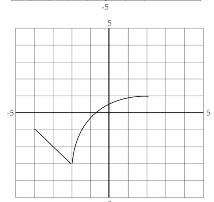
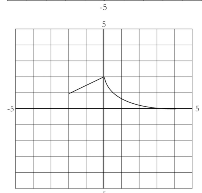
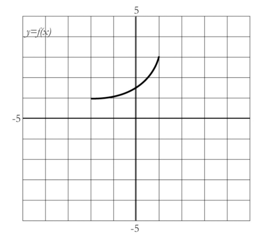


vi.


32

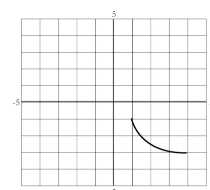
Let f be the function defined by the graph below.

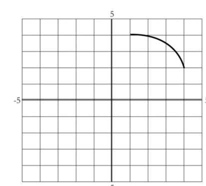


For each graph in parts i through vi,

a. come up with a series of transformations that will turn the graph of f into the graph you see.

b. write an equation for the transformed function.

i.


ii.


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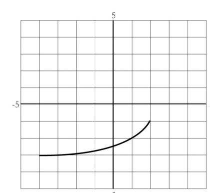

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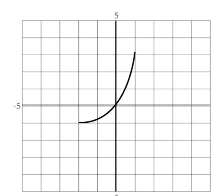
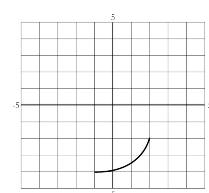
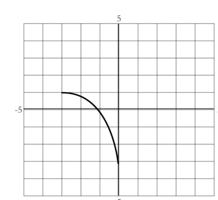


v.


vi.


33 (continuation of problem 17) When tribbles first appear on the Enterprise, there are four tribbles, collectively squeaking at a noise level of 6 decibels. Five hours after they appear on the Enterprise, there are 30 tribbles, squeaking at a noise level of 45 decibels.

a. Find a formula for the function giving population in terms of time, $P(t) = \ldots$. Assume that it is an untranslated exponential and therefore has the form $P(t) = ab^t$.

b. Find a formula for the function giving noise level in terms of population, $N(P) = \ldots$. (Recall that the noise level is directly proportional to the size of the tribble population.)

c. Now find a formula for $N(P(t)) = \ldots$. What does this formula tell you?

d. How loud will it be 24 hours after the arrival of the tribbles?

e. How long did it take until the noise level aboard the Enterprise reached an unbearable 150 decibels?

34 (Courtesy of *Functions Modeling Change*, Connally, et al) The number of pounds of fertilizer, $n = f(A)$, needed to fertilize a lawn is a function of the surface area A of the lawn, in m^2. Match each story (a-c) to one expression (i-iii).

a. I figured out how many pounds I needed and then bought 2 extra pounds just in case.

b. I bought enough fertilizer to fertilize my lawn and my neighbor's lawn, which just happens to be the size of mine.

c. I bought enough fertilizer to cover my lawn and my flower bed; the flower bed measures 2 square meters.

i. $2f(A)$
ii. $f(A+2)$
iii. $f(A)+2$

35 a. Sketch the graph of $y = |x|$. The point $(0, 0)$ is called the **vertex** of this graph.

b. Use transformation reasoning to predict what the graph of $y = |x - h| + k$ would look like, including the location of the vertex.

c. Investigate the effect of the constant a on the graph of $y = a\,|x - h| + k$. Does it affect the vertex?

36 Without using your calculator, sketch an accurate graph of $y = \sin(\frac{\pi}{2}\,x)$. Include at least two periods. On the same axes, sketch the graph of $y = x$. Now, by using what you know about function addition, sketch a graph of $y = x + \sin(\frac{\pi}{2}\,x)$.

37 Sally was asked to find $a(\frac{\pi}{4})$ in the function $a(x) = \sin(x-3)+2$, and found the answer 0.75. Without using your calculator in any way, decide if Sally's answer is plausible.

Earlier in this lesson, you learned that the **domain** of a function is the set of allowable inputs. Similarly, the **range** of a function is the set of outputs that a function might produce. For example, the range of the function $y = x^2$ is $y \geq 0$, since you can get any non-negative number you want by squaring some number, but you will never square a number and get an answer that is less than zero.

38 Give the range of each function.

a. Sally's function a from the previous problem

b. f(x) = $\frac{1}{x^2}$

c. $g(x) = e^x$

d. $h(x) = \log x$

e. $i(x) = 2^x + 3$

f. $j(x) = \frac{1}{x}$

39 Give the domain and range of each function. Visual thinking is encouraged. Thinking about transformations should obviate the need for your graphing calculator.

a. $f(x) = \frac{1}{x+2}$

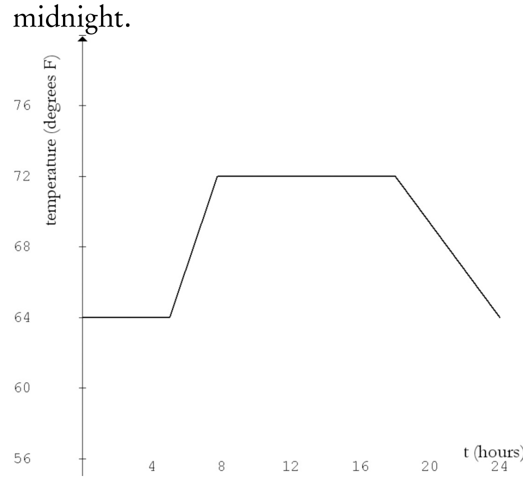
b. $g(x) = \frac{1}{(x+2)^2}$

c. $h(x) = \log(x+1)$

d. $i(x) = 2(x-3)^2 - 4$

e. $j(x) = -2\cos x - 4$

40 The function T graphed in the figure below gives the winter temperature in °F at a high school at time t hours after midnight.



a. Graph $c(t) = 142 - T(t)$.

b. Give the range of T and of c.

c. Explain why c might describe the cooling schedule of the school during the summer months.

41 An investigation into slope...

a. Find the slope of the line that heads in the same direction as the curve defined by $y = \log x$ at the point (10,1).

b. Make a prediction about the slope of the line heading in the same direction as the curve defined by $y = 3\log x$ at the point (10,3), and see if you are right.

42 Don't use a calculator for this problem.

a. Solve: $\frac{2}{x-1} = x + 3$

b. Evaluate: $\log_2 4^7$

c. By what factor does $\frac{w^5x^2}{y^3z^7}$ increase if w doubles and y is halved?

d. Solve for x: $\sqrt{4x} + x = 3$

e. Find all values of x so that

$$25^{-2} = \frac{5^{\frac{48}{x}}}{\left(5^{\frac{26}{x}}\right)\cdot\left(25^{\frac{17}{x}}\right)}$$

43 The greatest integer function takes an input and outputs the greatest integer that is less than or equal to that input. The greatest integer of x is written $\lfloor x \rfloor$. Here are some examples: $\lfloor 2.7 \rfloor = 2$, $\lfloor -4.5 \rfloor = -5$, and $\lfloor 5 \rfloor = 5$.

a. Sketch a graph of the greatest integer function. When you're done, ask your teacher about "open circle" notation.

b. Find the domain and range of the greatest integer function.

44 The greatest integer function is nicknamed the "floor function," which suggests the existence of a "ceiling function." The ceiling function takes an input and outputs the least integer greater than x. "Ceiling of x" is written $\lceil x \rceil$.

a. Sketch a graph of $y = \lceil x \rceil$.

b. Write a formula for $\lceil x \rceil$ in terms of $\lfloor x \rfloor$. Is one graph a translation of the other?

45 Sketch a graph of $y = \lfloor x \rfloor + x^2$ for $-2 \leq x \leq 2$.

46 Define the **signum function** as follows: $\text{sgn}(x) = 1$ if x is positive, -1 if x is negative, and 0 if $x = 0$. Sketch the graph of...

a. $y = \text{sgn}(x)$

b. $y = \text{sgn}(x) - 2$

c. $y = \text{sgn}(x) + 1$

d. $y = x^2 + \text{sgn}(x)$

47 The function © tells you how many siblings someone has, so ©(Malia Obama) = 1 and ©(Chelsea Clinton) = 0.

a. Find ©(you).

b. Give the domain and range of the function ©.

c. The function $\uparrow$ has the formula $\uparrow(x) = x^2 + 1$. Which makes sense: ©($\uparrow(x)$) or $\uparrow$(©(x))?

48

Below are tables for three functions, f, g, and h. These tables are complete tables — that is, the functions are not defined for any inputs other than the inputs given here.

x	-5	-2	1	2	3	7
$f(x)$	2	1	2	3	1	7

x	0	1	2	3	4	5
$g(x)$	-5	1	3	7	2	-2

x	-3	0	1	2	3	7
$h(x)$	0	1	2	3	4	5

a. Which compositions of functions are possible? That is, can you compute $f(g(x))$ for all values of x? How about $g(f(x))$? Find all pairs that work, paying attention to order.

b. Choose one of the compositions from part a and make a table of values for it.

c. Does $f(f(x))$ exist? How about $g(g(x))$ or $h(h(x))$?

49

An investigation into composing functions...

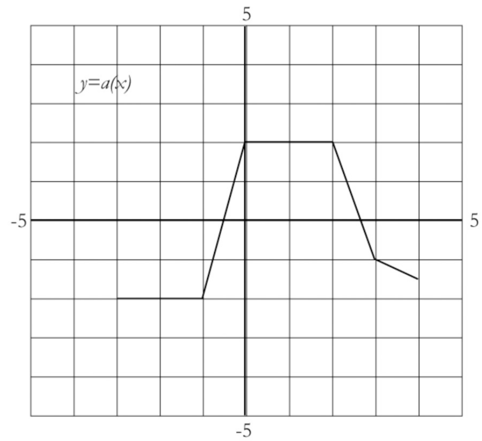
a. Make up tables for two functions, f and g, such that $h(x) = f(g(x))$ exists. Then make a table for the function h.

b. How do the domain and range of the function h relate to the domain and range of the functions f and g?

c. What (helpful) advice would you give to someone else in the class if they were trying to come up with two functions that could be composed?

50

Let the function a be defined by the graph below.

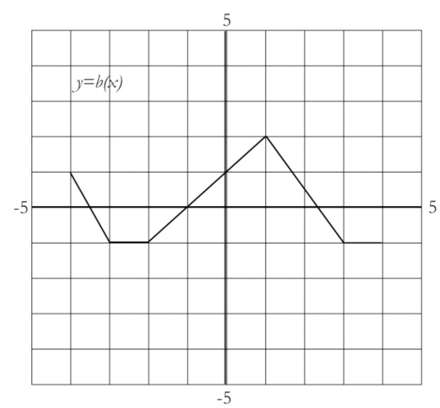


a. Sketch the graph of $y = |a(x)|$.

b. Sketch the graph of $y = a\,(|x|)$.

51

Let the function b be defined by the graph below.



a. Sketch the graph of $y = -\,|b(x)|$.

b. Sketch the graph of $y = b(|-x|)$.

52

a. Plot the graph of a function where, for all x, $f(-x) = f(x)$. Use the domain $-5 \leq x \leq 5$.

b. Plot the graph of another function that meets the criteria given in part a.

c. Make a conjecture about the graph of any function for which $f(-x) = f(x)$

53

a. Plot the graph of a function where, for all x, $f(-x) = -f(x)$. Use the domain $-5 \leq x \leq 5$.

b. Plot the graph of another function that meets the criteria given in part a.

c. Make a conjecture about the graph of any function for which $f(-x) = -f(x)$.

54

Are there any standard functions you know of that have the properties you conjectured in problems 52 and 53? What are they?

55

a. Plot the graph of function where, for all a and b, if $f(a) = b$, then $f(b) = a$. For example, if $f(2) = 4$, it must also be true that $f(4) = 2$. Use a domain of $-5 \leq x \leq 5$.

b. Plot the graph of another function that meets the criteria given in part a.

c. Make a conjecture about the graph of any function for which $f(a) = b$ implies $f(b) = a$. If necessary, make up more examples to test or refine your conjecture.

56

Let the function g be defined by the following chart.

x	-1	0	0.5	1	2	3	4
$g(x)$	-4	-3	-1	1	1.5	2	4

a. Plot the points on a graph.

b. From the information you have, make a table and graph of the inverse function for g. The standard notation for this function is "g^{-1}," read "g inverse."

c. Describe any patterns you notice.

d. The graph of g^{-1} is a transformation of the graph of g. Describe this transformation in precise language.

57

Find a formula for the inverse function of $f(x) = x^3$, and graph both f and f^{-1} on your calculator. Is the transformation you see the same one as in the previous problem? What evidence do you have for this?

58

The function f from problem 1 does not have an inverse. Why not? Does the function g from problem 13 have an inverse?

59 Which functions below DO have inverses?

a. $f(x) = x^2$

b. $g(x) = \sqrt{x}$

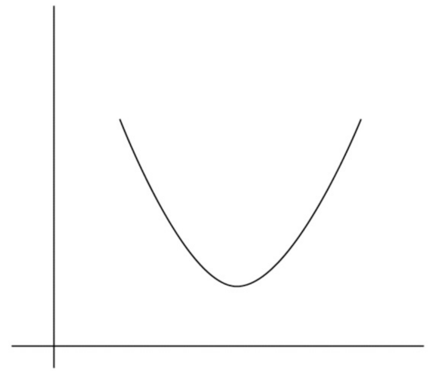
c. $h(x) = x^3$

d. $i(x) = 2^x$

e. $j(x) = x^6$

60 Which functions graphed below have inverses?

a.



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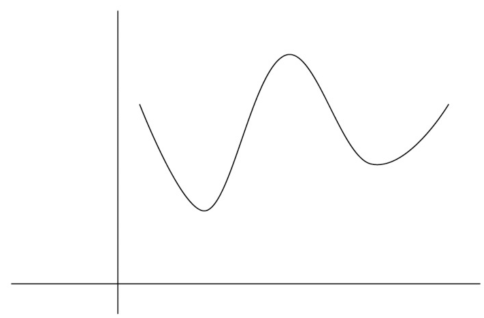

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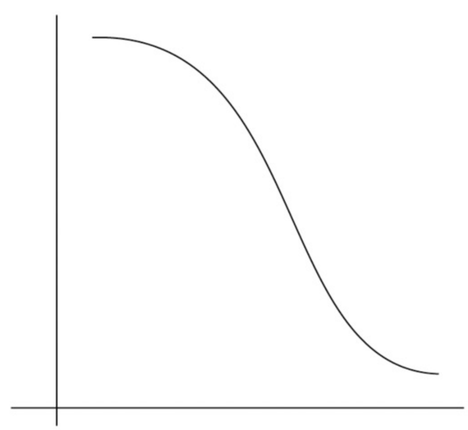

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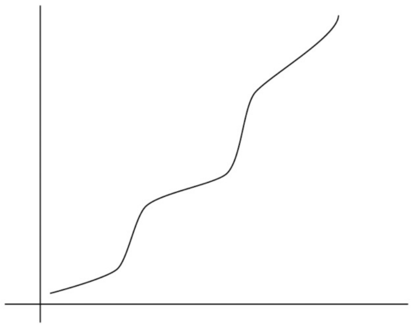
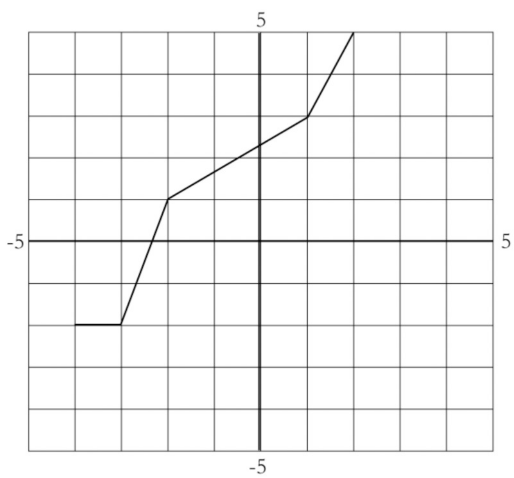
d.



61

The following questions refer to the function sketched below.



a. Find the domain and range of the function

b. Sketch the inverse of the function on the same axes

c. Find the domain and range of the inverse.

62

Find the domain and range of

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

63

The radius of a circle is initially 3 cm. It increases by two centimeters each hour.

a. Find a formula for the area of the circle in terms of time.

b. Sketch three graphs, labeling the axes appropriately:

i. $y = r(t)$ (radius of the circle vs. time)
ii. $y = A(r)$ (area of the circle vs. radius), and
iii. $y = A(t)$ (area of the circle vs. time).

c. When the circle has a radius of 7 cm, what is the rate of change of the area of the circle with respect to the radius? Make sure that your answer includes units.

d. At time t = 2 hours, what is the rate of change of the area of the circle with respect to time? Make sure that your answer includes units.

Exploring in Depth

64

Graph $f(x) = x^2$ and $g(x) = 2x$ on the same axes.

a. Make a rough sketch of what you think a graph of $y = f(x) + g(x)$ would look like. Then see if you are right by graphing the function $y = x^2 + 2x$.

b. Using the techniques of this lesson, write an equation for the graph you drew in part a. Is the equation equivalent to $y = x^2 + 2x$?

65 Graph the linear function $f(x) = 2x - 6$.

a. Reflect the line you've drawn over the line $y = x$.

b. Find an equation for the reflected line. Verify that it is the graph of f^{-1} .

66 If f and g are inverse functions, find $f(g(x))$.

67 Let $f(x) = x^2$, $g(x) = x - 2$, and $h(x) = 5x$. Write each of the following functions as a composition of f, g, and/or h.

a. $a(x) = 5(x - 2)$

b. $b(x) = 5x^2$

c. $c(x) = x^2 - 2$

d. $d(x) = (5x)^2 - 2$

e. $e(x) = 5(x - 2)^2$

f. $z(x) = (5x - 2)^2$

68 Find the domain and range of each function.

a. $a(x) = \sqrt{x - 5}$

b. $b(x) = \sqrt{x^2 + x - 6}$

c. $c(x) = \frac{x+5}{x-3}$

d. $d(x) = \frac{x^2}{(x+1)(x-2)}$

e. $e(x) = -x^2 - 7$

f. $f(x) = \frac{3x}{x}$

You have been using the "inverse sine" function for years, but thinking about it now, you can see that the sine function does not have an inverse! There are many ways to get an output of 1: when x is 90°, 270°, 450°, etc if you are working in degrees, or when x is $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, etc if you are working in radians. Thus, if someone says, "find $\sin^{-1}1$," you wouldn't know which of the values listed above to give in response.

The reason we can still talk about the inverse sine function is that we are thinking about only a limited set of inputs (angles) that you can plug in to the function $f(x) = \sin x$. Limiting the allowable inputs in this way is called a **domain restriction.**

69 Use your calculator to take the inverse sine of some angles. Notice the kinds of outputs it gives you. Can you figure out exactly which angles belong to the restricted domain of the sine function?

70 In your notebook, sketch a graph of the sine function, showing a few periods, and show in bold the part of the graph that is produced by using inputs from the restricted domain. Does the restriction seem like a logical choice? Would there be other ways to do it?

71 Do the $\cos^{-1}$ and $\tan^{-1}$ functions use the same domain restriction as $\sin^{-1}$? Play around with your calculator to see. If not, what are the other domain restrictions?

72 Let $f(x) = \sqrt{x}$ and $g(x) = x^2$. Do $f(g(x))$ and $g(f(x))$ have the same domain? First answer the question by looking at the algebra, then check your answer by having your calculator graph the unsimplified formula for each composition.

73 Let $f(x) = 2x$ and $g(x) = \sqrt{16 - x^2}$.

a. Find $f(g(x))$ and $g(f(x))$.

b. Give the domain and range of each of the functions in part a.

74 Let $f(x) = x^2$ and $g(x) = 4 - x$.

a. Find $f(g(x))$ and $g(f(x))$.

b. Give the domain and range of each of the functions in part a.

75 Trigonometric functions are not the only common functions that require a domain restriction in order to take their inverse. $f(x) = x^2$ does not have an inverse because for a given output, like 4, there are multiple inputs, -2 and 2, that produce that output. However, if you restrict the domain of f to allow only values of x greater than or equal to zero, then the function has an inverse. You can think of the square root function as the inverse of $f(x) = x^2$ on this restricted domain.

a. For each function below, suggest a domain restriction that would allow the function to have an inverse.

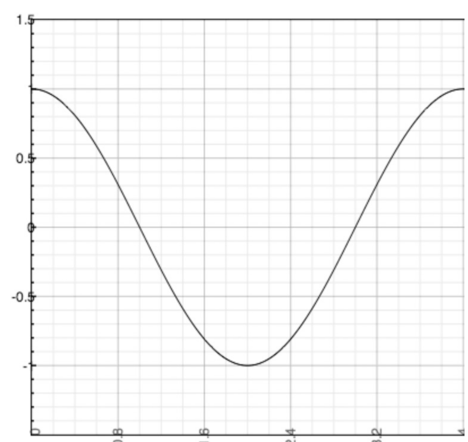
b. For functions i, ii, and iii, find an equation for the inverse of the function (with its domain suitably restricted).

i. $a(x) = (x-4)^2 + 3$

ii. $b(x) = \frac{1}{x^2}$

iii. $c(x) = \frac{1}{(x+3)^2}$

iv.



76 If $g(x) = \cos x$, find (using no calculator)...

a. $g(g^{-1}(\frac{1}{2}))$

b. $g^{-1}(g(45^\circ))$

c. $g^{-1}(g(37^\circ))$

77 Will it always be true that...

a. $\cos(\cos^{-1} a) = a$?

b. $\cos^{-1}(\cos a) = a$?

Park School Mathematics

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