BOOK 1: REASONING AND PROVING

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

1 In a well-shuffled 52-card deck, half the cards are red and half are black. If the number of red cards in the top half of the deck is added to the number of black cards in the bottom half of the deck, the sum is 30. How many red cards are in the top half? *(Copyright mathleague.com.)*

Many difficult problems that you come across can seem fairly straightforward, as frequently you have a good sense of where to start and what steps to take, even if each step is quite challenging. For some problems, though, the difficulty lies in not knowing at all what to do. In these cases it initially seems impossible to figure out how to get started.

To **Tinker** means to try possibilities and adjust your strategy based on what you've learned from your efforts. Tinkering is like trying to fit puzzle pieces together — you need to test out the pieces to see if they'll fit, but even if it doesn't work you now have a better sense of the shape of each piece.

The important thing is to *do something*, even if you aren't at all sure it will be helpful. Make a start. Don't be paralyzed by the problem. When you work on a math problem, as you tinker, you are gathering information — information that might lead to a solution or that might lead to a more interesting question.

Trying a specific example, even when the problem does not ask for one, is an example of **tinkering**. In the above problem, for instance, try just guessing the number of red cards that are in the top half of the deck, and see what you can figure out from there. Even though your first guess is likely to be wrong, that's perfectly OK. If you pay attention to the numbers you get for the amount of red and black cards in each half of the deck, you can adjust and make a much more educated guess the next time. That is, sometimes you will not notice a pattern until you generate some data for yourself.

tinker

Tinkering is a way of making abstract ideas and hard to approach problems concrete, so that you can really see what is going on.

The next few pages have a couple of problems for you to **Tinker** with. After you have tried a few different approaches, you can turn over the page and see if the solution provided was similar to yours, or if perhaps you came up with a better way!

Let's start with a bit of a mystery:

2

Each letter in the addition problem below represents a different number.

$$\begin{array}{r} AB \\ +\quad BC \\ \hline ADD \end{array}$$

What are the only possible values of A, B, C, and D? Try solving it before you turn over the page.

After perhaps tinkering with a number of different random values for A, B, C, and D, just to see what seems to be working and what isn't, you may have noticed that there is only one plausible value for A. What is it?

So now our problem has turned into:

$$\begin{array}{r} 1B \\ +\quad BC \\ \hline 1DD \end{array}$$

Could B be 7 or lower?

If you conclude that B is greater than 7, then it must be 8 or 9. So tinker a bit here. Replace B by 9 and see what happens.

The problem would then be:

$$\begin{array}{r} 19 \\ +\quad 9C \\ \hline 1DD \end{array}$$

Now what could D be? How about C? (Remember we were told all the letters represented *different* digits.)

At this point you might have seen that B must be 8. Now find C and D, and be sure to check that your solution works.

3

Here's a logic puzzle that might drive you a bit crazy at first, but you'll find the solution quite satisfying:

Al says: Bob is lying.
Bob says: Carl is lying.
Carl says: Al and Bob are both lying.

Who is lying? Who is telling the truth? There is only one possible answer.

If this seems impossible to solve, try just guessing right from the start who is telling the truth and who is lying, and see if your answer is consistent with what all three people are saying. If your initial answer doesn't work, then adjust your guess and try again.

The answer is on the back of this page.

There is no question that this problem can tie you up in logical knots! Let's try to approach it by assuming Al is telling the truth, and we'll see where that gets us. If this assumption works, great! And if it doesn't, hopefully we will have learned some valuable information about the problem.

If Al is telling the truth, that means that Bob is lying.

If Bob is lying, that means that he is lying about Carl lying, which means that Carl is telling the truth.

Now if Carl is telling the truth, that would mean that Al and Bob would both have to be lying. But we assumed at the start that Al was telling the truth! Al can't both be telling the truth and lying. Our assumption that Al was telling the truth causes us to contradict ourselves, so it must be wrong. Therefore, since Al can't be telling the truth, *Al must be lying.*

But does the problem really work out if Al is lying? Test it out and see.

So, who is lying, and who is telling the truth?

In case you were wondering if you could have started solving this problem by assuming, say, that Carl was telling the truth, and following the consequences from there, the answer is yes, absolutely. There are many ways to solve this problem, just as with many other math problems. The key is that all these different methods will eventually end up with the same answer.

The next few pages have a variety of "Tinker" problems for you to enjoy. Have fun!

4 Fill in the numbers 1,2,3,4,5 once each, to make the equation true:
$? \times ? - ? \times ? + ? = -13.$

5 If someone multiplied the first 211 primes together what would be the units digit of the product?

6 Add two numbers to the following set of numbers to make the mean 15 and the median 8.

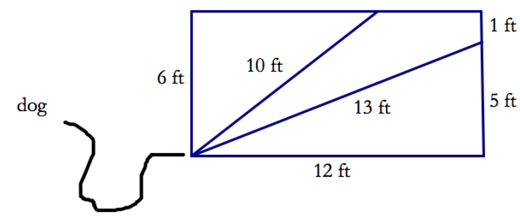
$\{3, 4, 17, 8, 39, 10, 2, 3, 71, 14\}$

7 The 3 missing numbers (all possibly different) in the equation below add up to what?

?45 – 3?7 = 55?

8 If one stacked 73 dice on top of each other on a table, what would be the sum of the numbers that are covered if the top die shows a 3?

9 A dog's leash is tied to a corner of a building. There are two tunnels going through the building that the dog can walk through. The lengths of each segment and tunnel are as marked. How long must the dog's leash be for him to be able to reach every part of the wall of the building?



10 In the film Die Hard With A Vengeance, the characters John McClane and Zeus Carver open a briefcase only to discover that in doing so they have armed a powerful bomb. It will explode in a matter of minutes unless they can disarm it. Inside the briefcase there is a scale. They have at their disposal two jugs — one holds exactly 5 liters and the other holds exactly 3 liters. To disarm the bomb, they have to fill the 5 liter jug with exactly four liters of water and place it on the scale. A few grams too much or too little will detonate the bomb. The water can be obtained from a nearby fountain.

How can they disarm the bomb?

HABITS

11 Make your own set of numbers that has:

a. a mean of 35, a mode of 30 and a median of 50.

b. a mean of 40, a median of 40 and a mode of 30.

12 On the portion of an 8x8 chessboard shown below, the X marks a knight. Knights can move in an "L" shape — 2 squares up or down and 1 left or right, or 2 left or right and 1 up or down. On the board below, the Y's mark where the knight can move.

		Y		Y			
	Y				Y		
			X				
	Y				Y		
		Y		Y			

a. If you can move as many times as you like, can the knight get to any square on the board?

b. What if the knight moves 3 squares in one direction and 1 in another (instead of 2 and 1)?

c. What about 1 and 1? Or 3 and 2?

d. What trends do you notice — under what conditions can the knight get to any square, and when can't it?

13 Andie never lies on Monday, Tuesday, Wednesday and Thursday. Leah always tells the truth on Monday, Friday, Saturday and Sunday. On the rest of the days, they may tell the truth, or they may lie. Both say they lied yesterday. What day is today?

14 You have 46 feet of fence, and you fence in a 120 ft^2 rectangular garden. What are the dimensions of your garden?

15 From a pile of a large number of pennies, nickels, and dimes, select 21 coins which have a total value of exactly \$1.00. In your selection you must also use at least one coin of each type. There are two answers to this problem — find both.

16 Peter said: "The day before yesterday I was 10, but I will turn 13 next year." Is this possible? Explain why or why not. *(Math Circles)*

17 THIS + IS = HARD; what must T stand for, if each letter stands for a different digit?

18 What is the units digit in 3^{101}?

19 Arrange the digits 1, 1, 2, 2, 3, 3 as a single six-digit number in which the 1's are separated by one digit, the 2's by two digits, and the 3's by three digits.

20 Alysha drives from Larchville to Oistin's Bay at 30 m.p.h. At what speed should she return so that her average speed (total distance traveled / total time taken) for the trip would be 60 m.p.h.?

21 Let $a + b + c + d + e = a \cdot b \cdot c \cdot d \cdot e$, where a, b, c, d and e are positive integers. Determine how many ordered quintuplets satisfy the equation.

22 What is the greatest number of points of intersection that can occur when 2 different circles and 2 different straight lines are drawn on the same piece of paper?
(from *Math Olympiads*, George Lenchner)

23 What is the smallest positive integer that is not prime that is also not divisible by any of the numbers up to 3? Up to 4? Up to 5? Up to 100?

1 The number 73 can be written as the sum of 73 consecutive integers. What is the product of these 73 integers? *(Copyright mathleague.com.)*

If you were reading a book about juggling for beginners, and it recommended starting out by trying to juggle 7 balls, you would rightly think that advice was crazy. Even someone who had no idea how to juggle could tell you that starting out with 3 (or even 2!) balls lets you get a better handle on what is going on. Before one tackles complicated problems, it makes sense to try easier, related ones first.

In mathematics, the way we cut a problem that is intimidatingly large or confusingly abstract down to size is to **Change or Simplify the Problem**. In the problem above, having to deal with 73 integers makes the problem initially seem extremely hard. How would one even approach finding the 73 consecutive integers that add to 73, and how on earth if you found them could you find their product in a reasonable amount of time?

Trying a Smaller Number is a very popular and useful way to change or simplify the problem. As with the juggling, it frequently allows you to get a much better understanding of what is going on. So in the problem above, why not see if you can simplify it by replacing all the 73's with 5's. Then, hopefully you can use the insight you have gained with the easier problem to solve the original problem as well. Give it a try!

Another variation on this theme occurs when a problem is stated in terms of variables like X or Z, such as:

2 At Bob's Breakfast Bungalow, the toast buffet involves X kinds of bread and Z kinds of jam. How many different bread-jam pairings are there, assuming that the customers use only one kind of jam each time they make toast?

When stated so abstractly ("X kinds of bread and Z kind of jam"), the problem can be hard to think about. **Changing Variables to Numbers** is another way to change or simplify the problem. Why not try specific numbers for X and Z to get a better sense of what is going on? It might make sense to try a few different pairs of numbers so that you can be sure that you have the right idea. Once you do, then, as with trying a smaller number, you can use what you have learned to go back and solve the original, more abstract (and likely more complicated) problem.

3. You add up X randomly chosen positive numbers. How many times larger is this sum than the average (mean) of those numbers? (Your answer may include an X in it.)

4. Some students stood evenly spaced in a circular formation. They counted off, starting at 1 and continuing by consecutive integers once around the circle, clockwise. How many students were there in the circle if the student furthest from student 19 was student 83? *(Copyright mathleague.com.)*

5. Giuseppe likes to count on the fingers of his left hand, but in a peculiar way. He starts by calling the thumb 1, the first finger 2, the middle finger 3, the ring finger 4, and the pinkie 5, and then he reverses direction, so the ring finger is 6, the middle finger is 7, the first finger is 8, the thumb is 9, and then he reverses again so that the first finger is 10, the middle finger is 11, and so on.

   One day his parents surprise him by saying that if he can tell them some time that day what finger the number 1,234,567 would be, he can have a new sports car. Giuseppe can only count so fast, so what should he do?

6. The following two problems are related.

   a. If Wenceslaus wrote a list of Z consecutive odd integers, by how much would the greatest number on his list exceed the smallest? (Your final answer may have a Z in it.)

   b. The sum of X consecutive odd integers is A. The sum of the next X consecutive odd integers is B. What does $B - A$ equal? (Your final answer will include an X in it.)

7. How many squares of any size does an 8 by 8 checkerboard have?

8. A blue train leaves the station at Happyville going East X miles per hour. A red train leaves the station at Nervoustown, $2(X + Y)$ miles away, at Y miles per hour and headed West towards Happyville. (Answers may include X and Y.)

   a. How long will it take the trains to crash?

   b. How far away will the red train be from Happyville when they crash? (Your answer may include an X and/or a Y in it)

   c. How far apart will the trains be one hour before they crash?

9 Yolanda tells Miguel that she can guess any integer he thinks of from 1 to 10 million in 25 yes-or-no questions or less. Miguel says that is ridiculous!

Who is right, and why?

10 Let $10^Z - 1$ be written fully out as a number (and thus with no exponents). Find the sum of the digits of this number. (You will have a Z in your answer.)

11 What's the minimum number of non-overlapping triangles into which you can divide a 2008-sided polygon? (Each triangle side must be a segment connecting two of the vertices of the polygon.)

12 Find two consecutive positive integers where the difference of their squares equals 3747.

13 Josephine writes out the numbers 1, 2, 3, and 4 in a circle. Starting at 1, she crosses out every second integer until just one number remains: 2 goes first, then 4, leaving 1 and 3. As she continues around the circle, 3 goes next, leaving 1 as the last number left.

Suppose Josephine writes out the numbers $1, 2, 3, 4, \ldots, n$ in a circle.

For what values of n will the number 1 be the last number left?

14 In the ordered sequence of positive integers: $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$, each positive integer n occurs in a block of n terms.

How many terms of this sequence are needed so that the sum of the reciprocals of the terms equals 1000?
(Copyright mathleague.com.)

HABITS

LESSON 1: DEFINING NEW SYMBOLS

Introduction

You have a pocket calculator that can only do one thing — when you type in two whole numbers, it takes the first number, adds the second number to it, adds the first number to the sum, then takes that whole answer and multiplies it by the second number. The number you see on the screen is its final answer after this series of steps.

1 If you type in 7 and 2, what will your calculator show?

2 What if you type in 2 then 7?

3 Your friend uses the calculator. You can't see the first number she types in, but the second number is 3. The answer that the calculator gives is 39. What was the first number?

Development

The calculator's rule (from the problems above) has a symbol to represent it: "$\triangle$". For example, "$7\triangle2$" refers to what you did in question #1.

4 What is $2\triangle10$?

5 What is $5\triangle y$, in terms of y? Simplify as much as possible (no parentheses in your answer).

6 What is $x\triangle8$?

7 What is $x\triangle y$, in terms of x and y? Write this as an equation.

If you invented the symbol "$\triangle$" for a math problem you wrote, and wanted to explain it in an equation rather than in words, you would say:

"Let $x \triangle y$ ______________________" (your answer to question 7).

This is called the algebraic rule for $\triangle$.

Here the word "Let" is used the same way as when a problem says "Let x be the number of boxes you purchase."

8 After trying out a few different whole numbers as inputs to the $\triangle$ rule, John makes the following claim: "To get an odd number for your answer from $\triangle$, you need to input odd numbers for a and b."

a. Does John's claim seem reasonable to you? If it doesn't, find a **counterexample** — a specific example which proves that his statement is not always true.

b. Is John's claim true?

These two questions are quite different — in many situations, there can be a variety of different predictions that seem reasonable, but only one of them may be actually true! The habit of seeking proof is not only about learning how to prove a statement to be true — it's also about learning to ask "Why would that be true?" when you are presented with a reasonable statement.

9 The symbol "&", applied to two whole numbers, means that you take the first number and then add the product of the numbers.

a. Find 4&7.

b. Find 7&4.

c. Write an algebraic rule for &.

d. Here is another of John's claims. "To get an odd number for your answer from $a \& b$, you need to input an odd number for a and an even number for b." Is his new claim true? If you think it is true, carefully explain why. If you believe it's not true, give a counterexample.

Practice

10 Just like any symbol regularly used in mathematics ($+$, $-$, $\cdot$, $\div$), symbols that we create can be used inside an equation. Let $m \sun n = 3n - m$. Calculate the following:

a. $5 \sun 2$

b. $6 \cdot (5 \sun 2)$

c. $(1 \sun 3.5) - (3.5 \sun 1)$

d. $2 \sun (3 \sun 5)$

e. $(2 \sun 3) \sun 5$

f. $(a \sun b) \sun c$

11 In problem 10, is $m \sun n = n \sun m$ true in general, for ANY input numbers m and n? Explain.

Problems

12

When you give © two numbers, it gives you the third number in the addition/ subtraction pattern. For example, ©$\{21, 23\} = 25$ (going up by 2's), and ©$\{95, 90\} = 85$ (going down by 5's).

a. Find ©$\{6, 11\}$ and ©$\{11, 6\}$.

b. Find ©$\{11,$ ©$\{6, 11\}\}$.

c. Write an equation for ©$\{m, n\}$.

13

When you give \$ two numbers, it gives you the third number in the multiplication/ division pattern. For example, $\$\{3, 15\} = 75$, and $\$\{48, 24\} = 12$.

a. Find $\$\{2, 3\}$, $\$\{12, 13\}$, and $\$\{102, 103\}$.

b. What is $\$\{2009, 1\}$?

c. If $\$\{x, 2\} = \$\{3, 1\}$, then what is x?

14

The symbol ¤ takes a single number, squares it, and then subtracts 4.

a. What is ¤(6)?

b. Can you ever get a negative answer for ¤(x)? Why or why not?

c. Find an x so that ¤(x) is divisible by 5.

15

Let the symbol ¥ mean: Add up the two numbers, then take that answer and subtract it from the product of the two numbers. What is 5¥8? 8¥5?

16

Look back at the problem above. Do you think the same thing would happen for any pair of numbers, if you used the same symbol? Explain your answer.

When switching the order of the input numbers never has an effect on the answer, the symbol you are working with is said to have the **commutative property**. For example, the symbol ¥ (from questions 15 and 16 above) had the commutative property, but the symbol © (from question 12) did not.

One important thing to note is that there's no such thing as "sometimes" having the commutative property. For example, ¥ has the commutative property because a¥b and b¥a are equal for ANY input numbers a and b, not just because it worked for 5 and 8.

Proving a statement false is as easy as finding one counterexample, but it is sometimes difficult to prove that a statement that appears to be true is indeed true. In the following problems (17-21), you will need to decide whether statements are true or false, and to also clearly support your position.

17 Fergie claims that each of the following symbols has the commutative property. Examine each of his claims.

a. $x \Downarrow y$ means add 1 to y, multiply that answer by x, and then subtract x.

b. Take two whole numbers x and y. To do $x\%y$, you divide x by 2, round down if it's not a whole number, and then multiply by y.

c. To calculate $x£y$, imagine that you walk x miles east and then y miles northeast. $x£y$ is how far away you end up from your starting point.

d. ¿ works by adding up the two numbers, multiplying that by the first number, and then adding the square of the second number.

e. ¢ takes two numbers. You reverse the first number (for instance, 513 becomes 315); one-digit numbers stay the same), then add the reversed number to the second number, and finally add up the digits of your answer.

18 Let $x \bigstar y = x^2 - y^2$. For example, $4 \bigstar 3 = 16 - 9 = 7$. True or false: $x \bigstar y$ always equals the sum of the two numbers — for example, $4 \bigstar 3 = 7$ which equals $4 + 3$. If it's true, justify your claim. If it's false, try to find out what kinds of numbers do make the claim work.

19 When you give $\heartsuit$ two numbers, it finds the sum of the two numbers, then multiplies the result by the first number. Finally it subtracts the square of the first number. Flinch claims that $\heartsuit$ is commutative. Is he correct?

20 Here's how you might prove Flinch's claim in problem 19 for *any* two starting numbers.

a. Let's call your first number m and your second number n. Write down and simplify as much as you can the expression for $m \heartsuit n$.

b. Write down and simplify as much as you can the expression for $n \heartsuit m$.

You should be able to convince anyone, using your work in problem 20 that Flinch's statement is always true, no matter which numbers we start with.

21 Is the rule below commutative?

The rule $\sim$ adds up the two numbers, doubles the answer, multiplies the answer by the first number, then adds the square of the second number and subtracts the square of the first number.

22 Let $x \clubsuit y = \frac{1}{1{,}000{,}000{,}000}\ x^y$.

a. Is there a value of y such that $10 \clubsuit y > 1$?

b. Is there a value of y such that $1.001 \clubsuit y > 1$?

23

The symbol "&", applied to *any* two numbers (not only whole numbers), means that you take the first number and then add the product of the numbers.

a. When you calculate $x\&y$, you get 120. What could x and y be? Give several different answers.

b. When you calculate $x\&y$, you get 120. Write an equation that expresses this fact, then solve for y in terms of x (meaning, write an equation $y =$.... with only x's in the equation).

24

The command "CircleArea" is a rule that finds the area of the circle with the given radius. For example, CircleArea$(3) = 9\pi$.

a. Find CircleArea(4).

b. Write the equation for CircleArea(x).

c. Can you find CircleArea(-4)? Why or why not?

25

It's January 1st, and you are counting the days until your birthday. Let m be the month (as a number between 1 and 12) and d the day (between 1 and 31) of your birthday, and pretend that there are exactly 31 days in each month of the year.

a. How many days are there until January 25? Until April 10?

b. For January 25th, $m = 1$ and $d = 25$, and for April 10th, $m = 4$ and $d = 10$. By looking at what you did in part a, explain how you can use the numbers m and d to count the days from January first to until any day of the year.

c. Let the symbol ♥ represent this count. Write an algebraic rule for calculating m♥d. Test your rule with an example.

26

Now, count how many days are from your birthday to New Year's Eve (December 31st). Represent this with the symbol ♈. Again, pretend there are 31 days in each month.

Write an equation for in terms of m and d, and use an example to show that your equation works. (You might want to try explaining it in words or in an example first.)

27

"max(a,b)" takes any two numbers and gives you the larger of the two. The symbol ∂ is defined by $\partial(a, b) =$ max(a,b) – min(a,b). Is ∂ commutative?

28 We say that the counting numbers (i.e., 1, 2, 3, ...) are "closed under addition" because any time you add two counting numbers, you get another counting number. Decide whether or not the counting numbers are closed under each of the following operations. In each case where the answer is no, try to find a group of numbers that *is* closed under that operation.

a. * (multiplication)

b. - (subtraction)

c. / (division)

d. The symbol ∂, from problem 27

29 The rule $\leftarrow$ adds twelve to a number and divides the sum by four. What number x can you input into $\leftarrow(x)$ to get an answer of 5? An answer of -3?

30 To do the rule $\forall$, add 5 to the first number and add 1 to the second number, then multiply those two answers.

a. What's $3\forall1$?

b. What's $x\forall y$?

c. Your friend tells you that she needs to find numbers x and y so that $x\forall y$ gets her an answer of A— a whole number that she does not reveal. In terms of A, tell her what to plug in for x and y. Make sure that your strategy would always get her the answer she wants.

d. What values of x and y give you an odd answer? Prove that your description is true and complete. (Make sure you know what it means to prove it's complete!)

e. What values of x and y would give you an answer of zero?

31 Create 3 different rules that give an answer of 21 when you plug in 2 and 8.

32 Let f be a rule that acts on a single number.

a. Create a rule for f so that $f(2) = 12$ and $f(11) = 75$.

b. Now, create a new rule g such that $g(5) = 26$ and $g(10) = 46$.

33

Let $x \triangle y = x^2 + 2xy$. ($x$ and y have to be integers.)

a. If you plug in 6 for x, find a number you could plug in for y to get an answer of zero.

b. Using part a as an example, describe a general strategy for choosing x and y to get an answer of zero, without making x zero. Explain why your strategy works.

c. Suppose you use the same number for x and y— call this number N. (So, you're doing $N \triangle N$.) What is your answer, in terms of N? Simplify as much as possible.

d. Suppose $x \triangle y = 20$. Solve for y in terms of x.

e. Describe a strategy to get any odd number that you want. Give an example, and also show why your strategy will always work (either give a thorough explanation, or use algebra to prove that it works).

34

Tinker to find a rule for $x \# y$ that gets the following answer: $5\#1 = 24$. Then try to write a rule that gives $5\#1 = 24$ and $4\#2 = 14$.

35

Create a rule α that works with two numbers, so that $\alpha(1,1) = 5$ and $\alpha(2,3) = 10$. Give your answer as an equation in terms of a and b.
$\alpha(a,b) =$ __________________________.

36

Let $\otimes\{a,b\}$ be the two digit number where the tens digit equals a and the units digit equals b. For example, $\otimes\{9,3\} = 93$.

a. In terms of f, what do you get when you do $\otimes\{5,f\}$? Write your answer as an equation. (Remember, writing $5f$ doesn't work, because it means $5 \cdot f$).

b. What do you get when you do $\otimes\{a,4\}$?

c. When you calculate $\otimes\{a,4\} - \otimes\{4,a\}$, what do you get in terms of a? Simplify as much as possible.

d. $\otimes\{6,x\} - \otimes\{x,3\} = 12$. Find x. Show your work algebraically.

37

Now, let's redefine $\otimes\{a,b\}$. Let $\otimes\{a,b\}$ be the three-digit number where the hundreds and units digits are a, and the tens digit is b.

a. In terms of b, what is $\otimes\{b,2\}$? (Again, $b2b$ won't work).

b. In terms of a and b, what do you get when you calculate $\otimes\{a,b\} - \otimes\{b,a\}$? Simplify as much as possible and check your answer with an example.

38

For any two positive numbers a and b, let $a \perp b$ be the perimeter of the rectangle with length a and width b.

a. Is $\perp$ commutative?

b. Is $\perp$ associative? In other words, does $(a \perp b) \perp c = a \perp (b \perp c)$?

Exploring in Depth

39 A positive whole number is called a "staircase number" if the digits of the number go up from left to right. For example, 1389 works but not 1549.

The rule $x \nabla y$ works by gluing x and y together. For example, $63 \nabla 998$ gives an answer of 63998. If x and y are both staircase numbers, and x is bigger than y, is it always true that $x \nabla y$ is bigger than $y \nabla x$? If you think it's always true, explain why. If not, explain what kind of input would make it true.

40 Suppose that, for any two positive numbers a and b, $a \blacksquare b$ represents the area (ignoring units) of the rectangle with length a and width b.

Is $\blacksquare$ associative?

41 In problem 23, you were asked to find some pairs of numbers x and y so that $x \& y = 120$. (Recall that $x \& y$ takes the first number and adds the product of the numbers).

a. Do you think you could find numbers x and y to get any number you wanted? Try a few possibilities and explain what you find.

b. Develop a rule for choosing x and y to get the number that you want, which we will call N. (Hint: Try starting out by picking a number for x, and then figuring out what y would have to be. You might have to try different numbers for x to find a solid strategy.)

42 Don't use a calculator for this problem.

a. Find $\frac{3}{4} + \frac{5}{6}$

b. Find $2 \div \frac{1}{2}$

c. Simplify $3x - 2(x+1)$

d. Write $.\overline{6}$ as a fraction.

e. Find $2\frac{1}{4} + 5\frac{7}{8}$

43 Let $x \square y = xy - x - y$. Develop a strategy to pick x and y so that you can get any number you want.

44 For a number x, $f(x)$ subtracts twice x from 100.

a. What is $f(10)$?

b. What is $f(-4)$?

c. Write an expression for $f(x)$.

d. Write an expression for $f(3x)$.

45 Let Let $x\Box y = xy - x - y$. Prove or disprove: if x and y are both larger than 2, then $x\Box y$ gives a positive answer. Make sure your explanation is thorough.

46 Write an equation for a rule $a\Diamond b$, so that the answer is odd only when both a and b are even.

47 Let $a \ \% \ b = a^2b - b^2a$. What would have to be true about a and b for $a \ \% \ b$ to be positive?

48 Let $a ☺ b = ab + b^2$. What would have to be true about a and b for $a ☺ b$ to be negative?

49 Let $x \ominus y = xy + y/2$, where x and y are whole numbers.

a. Describe what values of x and y would get you a negative answer.

b. Describe what values of x and y would get you an even whole number answer. (Careful! Your answer won't be just in terms of odds and evens. You'll have to tinker to see what type of numbers work. Think through it step by step and explain your reasoning).

c. What could x and y be to get 21? Give at least 3 different options.

d. Find a strategy to get any whole number you want, called N. You can explain your strategy in algebraic terms ("To get an answer of N let x equal ... and let y equal ...") or in words. Show an example to demonstrate that your strategy works.

50 For a number x that might not be a whole number, $\Phi\,(x)$ represents the integral part of x— for example $\Phi\,(6.51)$ is 6 and $\Phi\,(10)$ is 10. Using the Φ notation, write a rule @ that gives an answer of 0 if x is a whole number, and gives a non-zero answer if x is not a whole number. For example @(20) should equal 0 and @(20.3) should give an answer not equal to zero.

51 The rule $\lfloor b \rfloor$ is called the "greatest integer function" — it outputs the largest integer that is not above b. For example, $\lfloor 9.21 \rfloor = 9$, $\lfloor 12 \rfloor = 12$, $\lfloor .92 \rfloor = 0$, and so on.

Write an equation for the rule $\{b\}$, which rounds b down to the hundreds — for example $\{302\} = 300$, $\{599\} = 500$, and $\{2\} = 0$. For your equation to calculate $\{b\}$, you can use any standard operations, and you should use the operation $\lfloor b \rfloor$.

52 To understand the rule behind the symbol $\diamondsuit$, you need to draw a picture (graph paper will help). To draw the picture for $5 \diamondsuit 3$, pick a point to start and then draw a line stretching 5 units to the right. From there, draw a line stretching 3 units up. Then draw a line stretching 3 units to the right, and then a line stretching 5 units up. Finally, draw a line back to your starting point.

a. Let $5 \diamondsuit 3$ be the area of the shape you just drew. Calculate this number exactly.

b. Pick 2 new numbers (the 1st number should still be bigger). Draw the picture and calculate the answer that $\diamondsuit$ would give for your numbers.

c. Draw a diagram to help you find an algebraic rule for $x \diamondsuit y$ (again, you can assume that x is a bigger number than y). Check your rule by making sure it would give you the right answer for $5 \diamondsuit 3$ and for your numbers in part b.

d. Will your rule still work if the two numbers are the same, such as $4 \diamondsuit 4$? Explain why it will or will not always work.

e. Will your rule still work if the first number is smaller, such as $3 \diamondsuit 7$? Explain why it will or will not always work.

LESSON 2: COMBINATORICS INTRODUCTION

Development

John wishes to wear a nice pair of dress pants with a sport-coat when he goes out on his first date tonight. He has 5 different pairs of dress pants and 3 different sport-coats. John, wishing to make the best possible first impression, decides to try on each combination of a pair of dress pants and a sport-coat. After trying all of these combinations, he will decide which is the best. It will take him approximately five minutes to get dressed, look carefully at the jacket-pants combination, and then get undressed. If it is 5:30 p.m. now, and John wishes to be leaving his house by 7:00 p.m., will he make it? It's not hard to see that John should be interested in the number of jacket-pants combinations that he will be trying on. Determining this number is in the realm of an area of mathematics known as **combinatorics**.

1 How many jacket-pants combinations will John have to try?

2 Write down as many different ways of determining the answer to Problem 1 as you can think of. Compare these ways to the ways of other students in your class. Which method is most visual? Least visual?

John's situation is one that can be reasonably solved by listing all of the possible combinations. This approach, while sufficient, is not particularly useful with the task that Imelda Marcos would have had. She reputedly had approximately 3000 pairs of shoes, 2000 designer dresses, 500 bras, and 200 girdles.

3 How many shoes-dresses combinations did Imelda have?

4 How many shoes-dresses-bras combinations did she have?

5 Can you reasonably answer either Problem 3 or 4 by visualizing? What method(s) used to solve John's problem can be used with Imelda's?

6 Suppose you were trying to determine the number of combinations of n_1 things combined with n_2 things combined with n_3 things and so on up to n_k things. How many combinations are there?

The result you probably wrote in Problem 6 is known as the **Multiplication Principle of Counting**.

Practice

7 How many dress and girdle combinations did Imelda Marcos have?

8 You can make a sundae at *I Scream* by choosing one scoop of ice cream from 6 options (vanilla, chocolate, strawberry, banana, cherry, or coffee), one topping from 4 options (hamburger crumble, pickle pieces, bacon bits, or onion O's), and one sauce from 3 options (ketchup, mayonnaise, or balsamic). How many different sundaes do you get to choose from? Mmmm!

Going Further

John's and Imelda's problems led to our first significant idea in combinatorics. If this was all there were to combinatorics, however, the subject would not be of much interest to mathematicians, computer scientists, or businesses, nor would this lesson have more pages in it (as you can plainly see that it does). Let's look at some different situations from what we saw before.

9 The members of the Math Society wish to appoint a slate of officers that will be in charge of its meetings and activities this year. The slate will have 2 positions — President and Secretary. If there are 5 students in the Math Society, how many distinct slates can be formed? By distinct we mean to distinguish the slate consisting of Sheila (President) and Jack (Secretary) from the slate consisting of Jack (President) and Sheila (Secretary). You might think about visualizing in order to get a handle on how to count in this situation.

10 Suppose that the slate had 3 positions — President, Secretary, and Treasurer — and there were 6 students in the Math Society. How many distinct slates can be formed now?

11 Suppose that the Math Society expanded to 25 students and, therefore, decided that the governing body should have 4 positions — President, Vice-President, Secretary, and Treasurer. How many distinct slates can be formed now?

12 In how many different ways can Elsa arrange 5 books on the top shelf of her bookcase? She's going to arrange them side-by-side and standing up.

13 In how many different ways can Elsa arrange 20 books on the top shelf of her bookcase? Show your work.

Your response to determining the exact amount in Problem 13 might be to scream as you realize that you will have to write

$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Fortunately, Christian Kramp in 1808 showed some foresight and decided to co-opt the exclamation mark for just this situation. Given the above problem, Christian would have written 20!. ("20!" is read as "20 factorial.")

14 Write $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ using factorial notation. Write 6! without factorial notation.

15 Suppose we are going to create a 7-letter "word" by using the letters in the word "numbers". The "word" doesn't have to make sense.

a. How many different 7-letter words can we create if a letter cannot be used more than once in a word? Can you write your answer using factorial notation?

b. How many different 7-letter words can we create if a letter can be used more than once in a word? Can you write your answer using factorial notation?

c. What is the probability that a random 7-letter word created from the letters from "numbers", where each of the letters could be used more than once, also happens to be a 7-letter word that uses each of the letters in "numbers" exactly once?

Problem 15 Part a could have been written using different, but equivalent, language. A common way of expressing this problem is to ask, "How many different 7-letter words can we create if repetition is not allowed?" The phrase "repetition is not allowed" means in this case that a letter cannot be used more than once in a word. Problem 15 Part b, similarly, could have been written, "How many different 7-letter words can we create if repetition is allowed?" Paying attention to whether repetition is allowed is critical when trying to understand what you are trying to count. In other words, you need to develop the habit of **re-examine the problem** if you're going to avoid undercounting or overcounting.

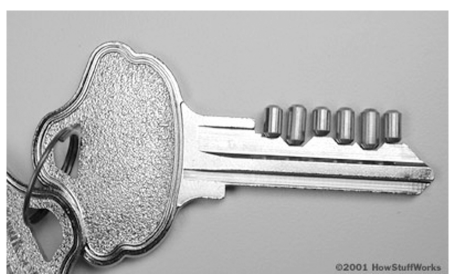
Practice

16 In how many ways can 6 people line up for a movie?

17 How many four-letter "words" can be made from the letters A, B, C, D, and E, if any arrangement is counted as a "word" and repetition is not allowed? Now consider using the 26-letter alphabet — how many 4-letter words are possible if, again, repetition is not allowed?

18 How many distinct license plates can be created in a state that has a 6-digit license plate (that is, plates have 6 digits and no letters) where digits can repeat? How many more license plates would there be if a 7th digit was added?

19 In the picture below, each cylinder is placed in a section of the key. The depth of the section is the "cut". If you have a key with 6 sections and each can contain one of four different cuts, how many different key configurations are possible?



20 Over the summer vacation, a student wants to read 14 books. In how many different orderings can the student arrange these books for reading?

21 In the upcoming school election, a completed ballot is one in which a student has voted for one person for president, one person for vice-president, one for treasurer, and one for secretary. If there are 5 people running for president, 4 for vice-president, 4 for treasurer, and 3 for secretary, how many distinct, complete ballots could be filled out?

Problems

In each problem you should think very carefully about the information that's been presented in order to determine what is relevant in solving the problem. Furthermore, you should consider visualizing — not necessarily with a tree diagram, but with slots or boxes — as a way of trying to organize the information. Lastly, if the numbers in the problem are large, don't be afraid to change them to smaller numbers and, therefore, make the problem a little easier to work with initially. By doing this, you are using or developing an important habit called **change or simplify the problem.**

22

Three-digit area codes on phones were introduced after it became clear that the restrictions on the seven-digit phone number limited the number of phones that could be in use. The original restrictions on these three-digit codes were as follows: the first digit can only be a digit from 2-9; the second digit must be 0 or 1; and the last can be any digit.

a. How many area codes were possible?

b. How many area codes began with the digit 4?

c. What is the probability that a random area code begins with the digit 4? What is the probability that it does not begin with a 4?

d. How many area codes ended with a 9?

e. What's the probability that a random area code forms an odd number?

23

In the United States, radio and television stations all have three- or four-letter "call signs" that begin with either K or W. Stations east of the Mississippi River start with W, and stations west of the Mississippi start with K.

a. In 1928 there were 677 radio stations (and 0 television stations) in the United States. If repetition of letters were allowed (so a station could be called WOW, for example), would it have been possible for each of these stations to have a three-letter call sign?

b. Today, using three- or four-letter call signs, how many TV or radio stations total could exist east of the Mississippi if repetition of letters were not allowed?

c. Of all the possible 3-letter call signs, what is the probability that if you randomly choose one it will have all of its letters in common with the station WJZ? For example: WJJ or WWZ.

d. Of all the possible 4-letter call signs, what is the probability that if you randomly choose one it will have exactly three of its letters in common with WTMD? (For example: KTMM.) How about four letters in common?

24

How many 3-digit numbers can be made from the digits 1 through 8 with no repetitions? What's the chance that one of these numbers will be divisible by 5?

25 A standard combination lock has a three number combination. Each number is in the range 0 to 49 and the order of the numbers matters — i.e., the combination 26-32-7 is different from 7-32-26. How many distinct combinations are there?

26 The rule IceCream takes the number of kinds of ice cream a store sells, n, and outputs the number of triple-scoop ice-cream cones they can possibly make. It counts in such a way that it matters which flavor is on the top or bottom, and multiple scoops with the same flavor are allowed. Write a formula for $IceCream(n) =$ ____________.

27 Suppose you were going to take a 20-question true-false exam. How many possible arrangements of responses are there? An arrangement in this case is a sequence of T's and F's that corresponds to how you answered the 20 questions.

28 To get onto computers, there are usually passwords required. Suppose your computer requires a 5-character code with the first three characters being digits (0-9) and the last two being letters from the alphabet.

a. How many codes are there?

b. If trying each password would take 5 seconds, how long would it take to try all of them?

c. If you were told that there weren't any vowels used in the password, how would this affect the amount of time it would take you to try all possible passwords? Also, what is the probability that a password chosen randomly does not have any vowels?

d. Would it take less time than in Part c if the 3-digit number were even and any of the letters of the alphabet could be used?

29 The symbol $\ominus$ takes any integer from 1 to 10 and gives back an integer from 2010 to 2014. How many different rules like this — using these same inputs, and giving as outputs any integer between 2010 and 2014 — could there be?

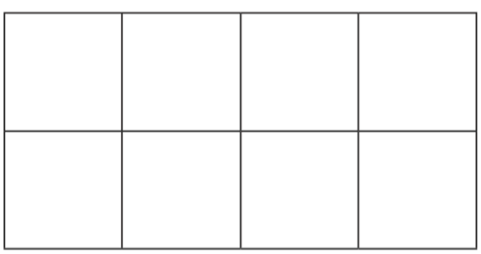
In Problems 30-33, you're going to need to think carefully about the given information and then decide how best to **prove** your answer. You should be prepared to give a step-by-step explanation that would convince any of your classmates. While each of the problems does involving counting, you'll need to decide whether the Multiplication Principle of Counting is to be used.

30 Show that in any group of five people, there are two people who have an identical number of friends within the group.

31 Daryl says that there are 210 possible 3-person committees that can be formed from 7 people. Alecia disagrees, saying that there are only around half that many. What do you think?

32 Do you have a better chance of getting a sum of 7 with two standard dice or with three?

33 Suppose that a class of 15 students is asked to shade one quarter of the area of the rectangle shown below. Only whole squares can be shaded, and the shaded squares cannot share a side. Can each student provide an answer that no one else in the class provides?



34 A bag contains all the rectangles with perimeter 36 whose dimensions are positive integers. If a rectangle is chosen at random from this bag, what are the chances that it will be a square?

35 Marbles are placed in a hat in a ratio of two red marbles to three blue marbles to five green marbles.

a. What is the probability that a marble you choose will be red or blue?

b. If there are 180 marbles in the hat, how many of the marbles are blue?

36 How many terms are possible when the following product is multiplied out: $(a+b)(c+d)(e+f)$?

37 You can think of the factorial symbol, !, as taking inputs and outputs, as is the case with many other symbols you've encountered.

a. For which values of n does $n!$ have odd outputs?

b. For which values of n does $n!$ have outputs that are multiples of six?

38 Rewrite as simply as possible.

a. $\frac{x!}{(x-1)!}$

b. $\frac{x! \cdot x!}{x^2}$

c. $\frac{x!}{(x+2)!} \cdot (x^2+3x+2)$

Exploring in Depth

39 Which of the following is $x(yzw)$ equal to? Answer all that are correct:

a. $(xy)(zw)$

b. $(xy)(xz)(xw)x$

c. $xy + xz + xw$

d. $xyzw$

e. $zxwy$

f. $y(zxw)$

40 Without a calculator determine which is bigger.

a. 2^{100} or 100^2

b. 2^{100} or $100!$

c. $100!$ or $50! \cdot 50!$

d. $100!/50!$ or $60!/30!$

41 9 girls and 8 boys are standing in a lunch line.

a. How many ways can they line up if all the girls are first, followed by the boys? Can you write this answer using factorials?

b. (Part a continued) What's the chance this will happen? Can you write this answer using factorials?

c. How many ways can they line up if they alternate gender? Can a boy be first in line? Can you write this answer in terms of factorials?

42 Don't use a calculator for this problem.

a. Simplify $(4x + 1) \cdot 2 - x$

b. Factor $x^2 - 19x + 84$

c. Factor $24x^2 - 48x$

d. Find $\frac{14}{5} + \frac{4}{15}$

e. We can use the symbol $r(x)$ to mean "the reciprocal of x." Find x if $r(x) = r(4) + r(6) + r(12)$.

43 Without determining what the explicit value is of 10!, how would you argue that $10! + 10! \neq 20!$? Is it ever the case that $k! + k! = (2k)!$?

44 The US Open Tennis Championship for women involves a single elimination tournament starting with 64 players (single elimination means that once you lose you're eliminated from the tournament).

a. How many tennis matches are played in the tournament?

b. Is it possible to run a single- elimination tournament with less than 64 players, if no player is allowed a "bye"? If so, what number of players can participate in such a tournament? If not, explain why.

c. Suppose the tournament was double-elimination — two losses and you're eliminated. How many matches are played in this tournament?

45 In Morse code, there are two symbols — a dot and a dash. Since there are 26 letters in the alphabet, there would have to be 26 different configurations of dots and dashes to use the Morse code successfully. Will any of the letters require a configuration with a total of five dots and/or dashes? Why?

LESSON 3: ARITHMETIC SEQUENCES AND SERIES

Introduction

1 Fill in the blanks to continue the pattern you see in this list:
$2,\ 7,\ 6,\ 11,\ 10,\ 15,\ \_\_,\ \_\_,\ \_\_,\ \_\_, \ldots$

In mathematics, a **sequence** is a list of numbers, usually with an identifiable pattern or a connection to real data — for example, the list of expected temperatures for the next ten days, or a sequence listing the profits that a business earned year by year.

The numbers in the sequence are called **terms**— for example, in the sequence above, 2 is the first term. Sometimes this pattern is very simple: $2, 4, 6, 8 \ldots$. Other times it might be harder to spot, like the problem above. Or, it might be an unusual type of pattern — $0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1 \ldots$.

2 Come up with the next two numbers in the sequence $1, 3, 6, 10, \ldots$, and a reason for why they should be the two numbers.

3 How about the next two numbers in the sequence
$2, 5, 10, 13, 26, 29, \ldots$?

4 Make up a sequence that you think will challenge your neighbor — then see if they can find the pattern.

5 A sequence starts with the numbers 5, 15, 35.

a. Describe a pattern that this sequence might be following, and give the next three terms according to the pattern.

b. Now describe a different pattern that the sequence might be following, and give the next three terms according to the new pattern.

The patterns you've seen so far have often been related to the differences between terms, such as the patterns in problems 1 and 2. When the pattern in a sequence is simply that the difference is always the same, the sequence is called an **arithmetic sequence**. For example, $7, 9, 11, 13 \ldots$ is one of these.

6 Some of the sequences below are arithmetic, and some are not. For each sequence, only a few terms are given. Identify whether or not the sequence appears to be arithmetic. Also for each sequence find a pattern and fill in the missing terms.

a. 3, 6, 12, 24, ___, ___, ...

b. 4, 11, 18, ___, ___, ...

c. 9, ___, 17, ___, 25, ...

d. 12, 13, 15, 16, 18, ___, ___,....

e. 47, 43, 39, ___, 31, ___, ___, ...

f. 13, 16, 22, 24, 28, 36, ___, ___, ___....

Development

In a sequence, it's not only the numbers that matter—the order of the numbers is important too. For instance, sequence "A" — 20, 10, 40, 30, 60, 50, ... — follows quite a different pattern from sequence "B" — 10, 20, 30, 40, 50, 60, ... — even though they contain all the same numbers.

In the first sequence, 30 is the 4th term, while in the second sequence, 30 is the 3rd term. To make this type of thing easier to write, we'll introduce some notation.

In the first sequence, we write "$A_4 = 30$" (pronounced "A sub four" or simply "A four") to mean that the sequence's 4th term is 30. Then the first sequence is called $\{A_n\}$ (because it is a set that consists of the terms A_1, A_2, A_3, etc) and the second sequence is called $\{B_n\}$.

7 Fill in the blanks according to the sequences below.

a. $A_1 =$ _____.

b. $B_4 =$ _____.

c. $B_{\_\_\_} = 30$.

d. $A_{\_\_\_} = 70$.

One last piece of notation you'll need to know is that, in an arithmetic sequence, the constant difference between terms is often called "d". In the sequence below, $d = 2$.

7, 9, 11, 13, . . .

8 In an arithmetic sequence called $\{T_n\}$, if $T_1 = 13$, and $d = 3$, then what does T_{10} equal? What does T_{100} equal?

9 In an arithmetic sequence, $T_4 = 8$ and $T_5 = 11$. Find the first three terms of the sequence.

10 An arithmetic sequence starts with $T_1 = 4$, $T_2 = 9$, ... Which term will equal 99?

11 What might the next two terms be in the sequence 4, 6, 9...? Find several different possible patterns.

12 Make up five sequences all beginning with 1, 2, but with a different third term. Look at what others in your class came up with, as seeing different patterns can give you ideas for making up some really interesting sequences.

13 Consider the sequence 5, 10, 6, 12, 8, 16, 12, 24, ...

a. Find the repeating pattern, and write the next 3 terms.

b. It looks like all of the numbers after the "5" are even — will the sequence ever produce another odd number? Why or why not?

14 Consider the three sequences below. Sequences $\{D_n\}$ and $\{E_n\}$ below have some missing information, but you can assume that they are arithmetic.

n	1	2	3	4
A_n	12	19	26	33

n	2	4	6	8	10
D_n	?	41	?	48	?

n	9	11	13	15	17
E_n	?	6	?	34	?

a. Which sequence, $\{D_n\}$ or $\{E_n\}$, has the same value of d as sequence $\{A_n\}$?

b. Plot the information from the three sequences on a graph (put n on the horizontal axis). Put all three on the same graph. How does what you see fit with your answer to part a?

15 In your thermometer, the mercury reaches a height of 46 mm when the temperature is 70 degrees Fahrenheit, and a height of 48 mm when the temperature is 73 degrees.

Think about these data as an arithmetic sequence where n is the temperature, and H_n is the height at that temperature. Note that "..." is used here to indicate that this table is not showing you values of n and H_n where n is less than 70 or greater than 75, but such values still exist.

a. Fill in the table below:

n	...	70	71	72	73	74	75	...
H_n	...	46			48			...

b. What is the value of d for this sequence?

c. What would the height be if the temperature were at 106 degrees Fahrenheit?

d. What would the height be if the temperature were at 65 degrees Fahrenheit?

16 In an arithmetic sequence, the *constant* difference is sometimes referred to as the rate of change of the sequence. Any idea why it might be so called?

17 If a sequence has $T_5 = 40$ but it has a rate of change of -6, find T_4, T_6, and T_{20}.

18 How would you find the value of d for the arithmetic sequence below? (Remember that "..." is used to indicate that there are other value(s) of n and T_n that are not listed in this table but certainly exist!) Also write an equation for T_b in terms of T_a, a, b and d.

n	...	a	...	b	...
T_n	...	T_a	...	T_b	...

Practice

19 Another thermometer has a different relationship between height and temperature. For this thermometer, the equation $H_n = 0.3n + 17.1$ expresses the height H_n (in mm) of the mercury in terms of the temperature n, in degrees Fahrenheit.

a. What's the height when the temperature is 50 degrees?

b. If you think of H_n as the nth term of a sequence, what's the first term?

c. What's the value of d for the sequence $\{H_n\}$?

d. What would be the temperature if the height of the mercury were 20.7 mm?

e. What would be the temperature if the height of the mercury were 3 mm?

20 An arithmetic sequence $\{T_n\}$ has $d = 5.5$, and $T_{12} = 48$.

a. Find T_4.

b. For what value of n would $T_n = 130.5$?

21 Each part below gives information about a sequence $\{A_n\}$. In each part, find the value of the unknown in the table for $\{B_n\}$ so that the two sequences have the same rate of change.

(Remember that "..." is used to indicate that there are other value(s) of n, A_n, and B_n that are not listed in a table but certainly exist. Thus in the first table of part a, n can be less than 6, can be 7 or 8, and can be greater than 9, even though those values of n aren't listed explicitly in the table.)

a.

n	...	6	...	9	...
A_n	...	40	...	82	...

n	...	4	5	...
B_n	...	a	48	...

b.

n	...	16	...	22	...
A_n	...	50	...	23	...

n	...	1	...	19	...
B_n	...	b	...	40	...

c.

n	...	9	...	31	...
A_n	...	40.5	...	71.3	...

n	...	4	...	14	...
B_n	...	4	...	c	...

d.

n	...	10	...	12	...
A_n	...	12	...	20	...

n	...	30	...	q	...
B_n	...	9	...	29	...

Problems

22 A sequence $\{A_n\}$ increases by 7 every 5 terms. Starting at the 2009th term, how far would you have to go to increase by a total of 2009?

23 In an arithmetic sequence...

a. The first term is 10, and the difference is d. In terms of d, what is the 20th term? How would you get the nth term?

b. The first term is T_1, and the difference is d. What is the 20th term? How would you get the nth term?

c. Write an equation for the nth term, T_n, in terms of T_1, n and d.

24 Suppose you have saved \$35 and you also earn \$15 for mowing a lawn.

a. How many lawns will you have to mow in order to have \$95 saved?

b. Suppose you wanted to save \$380. How many lawns do you have to mow, and how did you get your answer?

c. How about for \$395?

d. Let n be the number of lawns you mow, and M_n be the amount of money you have saved after mowing those lawns. Find M_1, M_{10}, and M_{100}.

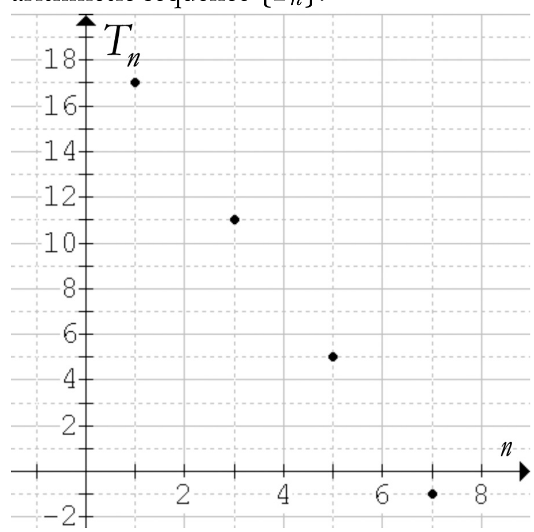
e. Write an equation for M_n in terms of n.

25 In an arithmetic sequence $\{C_n\}$, you know the following values: $C_1 = 12$, $C_3 = 20$, $C_5 = 28$.

a. What's the rate of change?

b. Does the equation "$C_n = 12 + 4n$" hold true for the sequence? If yes, explain why it works, and why it will work for every value of n. If no, explain why it fails, and fix it to make it work.

26 The graph below is a data plot for an arithmetic sequence $\{T_n\}$.



a. What is the rate of change of $\{T_n\}$?

b. Write an equation for the value of T_n in terms of n.

27 Using the information from sequence T_n below, do the following:

n	1	2	3	4	...
G_n	4	10	16	22	...

a. Write an equation for the nth term of the arithmetic sequence $\{G_n\}$ above, in terms of n ($G_n = \ldots$). Be sure to test your equation by plugging in a few values of n to make sure that it works.

b. Use your equation to determine what G_n will be when $n = 40$.

c. Use your equation to determine what value of n would make $G_n = 70$.

28 Suppose for an arithmetic sequence $\{T_n\}$, $T_1 = 378$ and $d = .001$.

a. Without using a calculator, roughly estimate what n would have to be so that T_n would be about 1 billion.

b. Now find the answer with a calculator and check that your estimate was a reasonable one.

29 Write an equation for an arithmetic sequence whose rate of change is 8, and whose fourth term is 60.

30 Find x if $2, 12, 3x + 10$ is an arithmetic sequence.

31 Find x if $x, 12, 2x$ is an arithmetic sequence.

32 When you give $\clubsuit(a, b)$ two consecutive terms, a and b, of an arithmetic sequence, it gives you the next number in the sequence. Write an equation for $\clubsuit(a, b)$.

33 Create three arithmetic sequences where $T_9 = 12$ and at least one is a decreasing sequence. Looking at your findings, is there any relationship between T_1 and the constant difference d? Explain.

34 Do the sequences with the following nth terms have any terms in common? $C_n = 18 + 3n$ and $D_n = 375 - 4n$.

35 For a sequence $\{C_n\}$, $C_{10} = 100$, and $d = -4$.

a. What is C_1?

b. How can this problem help explain why a negative times a negative is a positive?

36 Find x if the sequence is arithmetic: $7, x, 15$. Do the same for $3, x, 147$. What would be a plausible description of what x actually represents in each case? Would this always work when x is the 2nd of three numbers?

You get a **series** when you add up the terms of a sequence. For example if you add the first four terms of $3, 8, 13, 18, 23, 28, \ldots$, you get the series $3 + 8 + 13 + 18$. In the following problems, you'll develop clever ways to find the sum of a series without actually having to add every term individually — this way, if you have to add up a series that's 200 terms long, you won't have to type

200 numbers into your calculator.

37 Think of ways you could add up the series $1 + 2 + 3 + \ldots + 59 + 60$ without having to actually add sixty numbers.

38 Test out your strategies on the following series, then actually add them up number by number to check your answers:

a. $1 + 2 + 3 + \ldots + 10$

b. $1 + 2 + 3 + \ldots + 15$

39 How is the series $4 + 8 + 12 + \ldots + 236 + 240$ related to the problems above? Calculate the sum of this new series.

40 How about $5 + 8 + 11 + \ldots + 326 + 329$? Find one way of calculating the series based on your work above, and one way of calculating the series without using your work above. Describe your strategies (and make sure they both give you the same answer!).

41 Using any strategy, find a quick way to calculate:

a. $1 + 2 + 3 + 4 + 5 + \ldots + 299 + 300$

b. $3 + 4 + 5 + 6 + \ldots + 89 + 90 + 91$

c. $6 + 8 + 10 + \ldots + 140$

d. $2 + 9 + 16 + \ldots + 72$

42 Don't use a calculator for this problem.

a. Find $\frac{5}{8} + 3\frac{1}{10}$

b. Find $7 \div \frac{2}{3}$

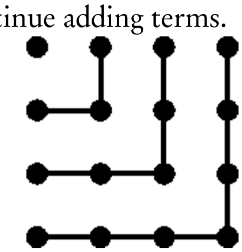
c. Factor $a^2 + 3a - 40$

d. Factor $x^2 - 9$

e. Recall that $|x|$, read "absolute value of x," means to make x positive, so $|3| = 3$ and $|-3| = 3$. Find $|1 - |3 - |1 - 5|||$.

43 Johnny said that when he added up the 100 terms of the arithmetic series 123 + 129 + 135 + ... + 717 and got about 13000. Mary added up the same series and got about 69000. Halle added them up and got about 42000. One of them is clearly closest to the right answer — figure out which one it must be in under 30 seconds.

44 Interpret the diagram below as a series (imagine the number of dots in each piece is the next "term", and add a few more pieces if necessary to see the pattern). Describe, in words, the terms being added, and describe the answers that you get as you continue adding terms.



Exploring in Depth

45 Imagine writing out the first twenty counting numbers: $1, 2, 3, \ldots 18, 19, 20$. Separate the sequence into odds and evens. Find the sum of the odd terms (if you have done problem 44, you'll know the answer right off). Now consider how the sum of the even numbers compare to the sum of the odd numbers; and then see if you can generate a formula for adding the numbers $1-20$; $1-100$; $1-2n$. Anything familiar about your findings?

46 You add up the first n terms of an arithmetic sequence: $T_1 + \ldots + T_n$.

a. Use one of your strategies to give an equation for the answer, in terms of the variables T_1, T_n, and n.

b. If d is the difference of terms in the sequence, rewrite your equation in terms of T_1, d, and n (without using T_n).

47 An arithmetic sequence has $T_1 = 7$. You know that d is any integer from 1 to 20. What's the probability that $T_n = 27$ for some n?

48 Compare the following two arithmetic sequences, $\{A_n\}$ and $\{B_n\}$:

n	1	2	3	...
A_n	40	44	48	...
B_n	2	8	14	...

a. Will sequence $\{B_n\}$ ever catch up with sequence $\{A_n\}$? How do you know?

b. Write an equation for $\{A_n\}$, and an equation for $\{B_n\}$. Check that your equations work.

c. Graph both of your equations on a calculator ("x" can stand for "n") and use your graph to determine whether or not $\{B_n\}$ ever catches up with $\{A_n\}$, and when.

d. Can you confirm that your answer in part c is right, just by using the equations and NOT the graph?

49 Consider the sequence $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots$.

a. Find a pattern and the next two terms.

b. Does the pattern consist of adding $\frac{2}{3}$ to each term?

c. Look at the ratios of each consecutive pair of terms — what do you find?

SUMMARY AND REVIEW

1 When $\because$ is given two numbers it multiplies the first number by 2 and adds to this outcome 3 times the second number.

a. Write an algebraic rule for $\because$.

b. If $a \because 9 = 130$, then what is a? Show your work or explain how you arrived at your answer.

2 The operator $\Uparrow$ takes two numbers and subtracts the product of them from two times the first number.

a. Write an algebraic rule for $x \Uparrow y$.

b. If $x \Uparrow 10 = 46$, then what is x? Use algebra to answer this question.

c. If $10 \Uparrow y = 46$, then what is y? Use algebra to answer this question.

3 Use algebra to determine the value of the unknown that makes the equation true.

a. $-5x + 7 = 29$

b. $11 - 4y = 6y$

c. $4(3y + 1) = 19$

4 A rule for doing calculations multiplies the first input by 4 then subtracts this quantity from 2 times the second input. This rule uses the symbol $\oplus$.

a. Calculate $4 \oplus 2$.

b. Calculate $-3 \oplus 1$.

c. Calculate $4 \oplus (1 \oplus -2)$.

d. Write an algebraic rule for $c \oplus d$.

e. Is this rule commutative? Explain.

f. When you calculate $x \oplus y$, you get 26. Write an equation that expresses this fact and solve it for x in terms of y. Now, find two different ordered pairs (x, y) that make this equation true.

5 Create a rule for $\odot$ if it is known that $4 \odot 3 = 7$ and $2 \odot 1 = 5$.

6 Suppose you know that $x \yen y = 5y + 2x$. If $x \yen y = 120$, explain how you can find a value for y that works if x is supposed to be 10. Now, explain how you can find a value for y given any value for x.

7

Solve each of the following equations for the unknown (letter) by using algebra.

a. $3(x-5)=96$

b. $2-2y=6+4y$

c. $\frac{b-3}{6}=4$

8

Writing "$2n$" is a generic way to represent an even number, assuming that n is an integer.

a. What types of numbers are represented by $3n$? Be specific.

b. What types of numbers are represented by $4n-1$?

c. What's a generic way of representing a number that is a multiple of 5 but is not a multiple of 10?

9

Let $c \odot d = c^2 + 3d$.

a. Calculate $2 \odot 4$.

b. Calculate $3 \odot (5 \odot 1)$.

c. If $c \odot d = 36$, give two different pairs of values of c and d that work. (For example, $c = 12$ and $d = 15$ would be one pair, except for the fact that $12 \odot 15$ doesn't actually equal 36!)

d. If $c \odot d = 36$, solve for d in terms of c. Then, explain how once could use this result to get thousands (or millions!) of pairs of values of c and d that make $c \odot d = 36$.

e. $4 \odot -1$ and $-1 \odot 4$ both equal 13. $5 \odot -2$ and $-2 \odot 5$ both equal 19. Can one then say that $\odot$ has the commutative property? Explain.

10

Without using a calculator evaluate each of the following.

a. $5!$

b. $\frac{5!}{4!}$

c. $\frac{100!}{99!}$

d. $\frac{100!}{98!}$

11 A high school coach must decide on the batting order for a softball team of 9 players.

a. The coach has how many different batting orders from which to choose?

b. How many different batting orders are possible if the pitcher bats last and the best hitter bats third?

12 In some states, license plates consist of 3 letters followed by 3 digits — for example, RRK 504. How many distinct (different from each other) license plates are possible?

13 A school has 400 students. Explain why at least two students must have the same birthday (month and day)?

14 A school has 677 students. Explain why at least two students must have the same pair of initials. Notice that middle initials are not being included.

15 A coin is tossed 8 times and the sequence of heads and tails is recorded. How many different sequences are possible?

16 Suppose you roll two, standard six-sided dice, and suppose one of these dice is green and the other is red. So, one possible roll is a red 1 and a green 3.

a. How many possible rolls are there?

b. What are your chances of rolling a sum of seven?

17 In a math class with 8 boys and 6 girls, the teacher randomly selects two students — one boy and one girl — to put homework problems on the whiteboard. How many different pairs of students can the teacher send to the board?

18 Without using a calculator evaluate the following.

a. $\frac{10!15!}{9!16!}$

b. $\frac{10!+9!}{9!}$

19 Bill has decided that he is going to sit six of his fifteen students in a row toward the front of the class. How many different ways can he do this?

20 (Continuation of Problem 19) Bill decides to arrange the other 9 students in a row behind the front row of six students. How many different ways can he arrange his 15 students in these two rows?

21 (Continuation of Problem 20) Bill decides that he's going to take his six favorite students and always place them in the front row. Is the number of ways that he can arrange his 15 students, now, the same as it was in Problem 19? If so, explain why. If not, provide an arrangement of students that would occur in one problem but not the other.

22 Suppose we want to form 9-letter words using the letters in the word "fisherman." Letters cannot be used more than once in each word.

a. How many 9-letter words are there?

b. How many 9-letter words are there if each word must begin and end with a vowel?

23 Suppose the letters of VERMONT are used to form "words." Letters cannot be repeated in the same word.

a. How many 7-letter words are there?

b. How many 6-letter words are there?

c. How many 5- or 6-letter words are there?

24 We're interested in determining the number of 3-digit numbers under certain conditions

a. How many are there that contain no 7's?

b. How many contain at least one 7?

25 We know that 3! is the solution to the following problem:

How many ways can you stack the three books — *Holes*, *Grendel*, and *Witches* — on a bookshelf?

Draw a tree diagram or table that clearly shows why the solution to the problem is $3 \cdot 2 \cdot 1$. Notice that just because your table or tree diagram has six entries in it doesn't mean that it clearly illustrates why the solution is $3 \cdot 2 \cdot 1$.

26 Suppose you roll three, standard, six-sided dice. So one possible roll is 1, 1, and 5.

a. How many possible rolls are there?

b. What's the probability of rolling a sum of 5?

27 There are 10 students in Bill's calculus class — 3 girls and 7 boys. Bill arranges 10 desks in a row. How many different ways can the students seat themselves at the 10 desks if

a. the girls sit at the first three desks on the left?

b. the girls sit at the third, fourth, and fifth desks?

28 Suppose students have either 2 initials or 3.

a. How many different students can have 2 initials without any two of the students having the same two initials? Answer the same question for 3 initials.

b. What is the smallest number of students the school can have to guarantee that at least two students must have the same initials? Briefly, but clearly, explain how you arrived at your answer.

29 Suppose you toss a coin 12 times, recording whether you get heads or tails.

a. How many possible sequences of heads and tails are there?

b. How many sequences have just one head?

30 You have 6 classes to do homework in but you only have enough time to complete three of the assignments. How many different ways can you complete the three homework assignments if the order in which you complete them matters? Assume that you don't go on to another assignment until you've completed the one you're working on at the moment.

31 You pull two socks out of a drawer containing 3 red socks and 4 green socks. What's the probability that both socks are red? It will help you think about the problem if you imagine pulling the socks out one at a time.

32 A soccer team has 16 players on the roster. Only 11 can play on the field at any given time.

a. How many different "teams" can the coach put on the field? (We are assuming here that even the same 11 players playing different positions is considered a different "team".)

b. If only 11 players showed up for next Saturday's game, now how many different teams can the coach put on the field?

c. The following Sunday, 11 players showed up, but the coach decided that one of them was the only one able to play goal keeper and another 3 of them could only play defense. Now how many teams can the coach field?

33 What is the chance that a random 4 digit number that is made only from the digits 1-8 does not have repeated digits?

34 License plates in Michigan always consist of three letters followed by three single digit (0-9) numbers. Assume repetition of letters is not allowed, but repetition of numbers is.

a. How many license plates are possible in Michigan?

b. How many license plates contain only letters taken from the word "Itchy"?

c. How many license plates contain only vowels and digits that are even prime numbers?

35 Suppose that you've flipped a coin four times and it landed heads each time. What's the probability that the next time you flip it, it will come up heads?

36 What's the probability that, if you toss a coin four times, you'll get heads exactly twice?

37 Suppose that there are 60 students in 9th grade, 60 in 10th, 58 in 11th, and 56 in 12th. Each grade elects a president, vice-president, and secretary. How many different student councils are there in each grade?

38 Use T_n notation to give a formula for each sequence: ($T_n = \ldots$)

a. $1, 4, 7, 10 \ldots$

b. the sequence you get by starting at 50 and going down by 4 each time.

c. $1, 1, 2, 3, 5, 8, 13 \ldots$

d. $1, 4, 9, 16, 25 \ldots$

e. $6, 18, 54, 162, 486 \ldots$

39 Using a method of your choice (except the method of typing lots of numbers into your calculator), find the sum of each series:

a. $3 + 8 + 13 + \ldots + 153$

b. The series with 45 terms, with $T_1 = 13$ and $d = 7$.

40 The third term in an arithmetic sequence is 20, and the 4th term is 28.

a. What is the 1st term?

b. What is the 739th term?

c. 996 is nth term in the sequence. What is n?

d. Prove whether 12754 is or is not a term in this sequence.

41 In an arithmetic sequence $c_8 = 20$ and $c_{23} = -70$.

a. Briefly, but clearly, explain how you can determine the value of c_{38} without determining the value of d, the constant difference.

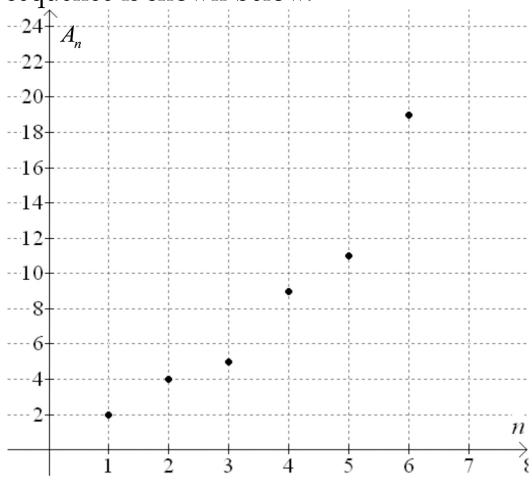
b. Determine the value of d.

c. Determine the value of c_2.

d. Write an equation for c_n.

e. Is there a term in the sequence c_n that has the value -432? If so, which term is it? If not, briefly explain why.

42 The graph of the first 6 terms of a sequence is shown below.



a. What is the value of A_3 ?

b. Is the sequence arithmetic? Explain.

c. What is the value of A_7 ? Briefly, but clearly, explain how you arrived at your answer.

43 In an arithmetic sequence, if $T_{50} = 1029$ and $d = 8$, find T_{20} and T_{142}.

44 In an arithmetic sequence, if $T_{45} = 79$ and $T_{33} = -77$, find T_4 and T_{50}.

45 If $d = 12$ and the 25th term of the arithmetic sequence is 148, what is n when $T_n = 784$?

46 What is d, if $T_{12} = -25$ and $T_{26} = -83$?

47 An arithmetic sequence goes as follows: $\ldots, 5, X, 11, \ldots, 236, \ldots$.
(Note that "..." means that there are other terms in the sequence that are not being explicitly listed here)

a. What is d, the common difference between terms?

b. It turns out that 11 is the 14th term of the sequence. What is the 3rd term?

c. 236 is the nth term in this sequence. What is n?

d. I'm thinking of a very large number, Z. Tell me how I can determine, in under a minute, if Z is a part of the sequence.

Park School Mathematics

This curriculum was written by members (past and present) of the upper school math department at the Park School of Baltimore. This was possible thanks to Park School's F. Parvin Sharpless Faculty and Curricular Advancement (FACA) program, which supports faculty every summer in major curricular projects. In addition to the support of FACA by the Nathan L. Cohen Faculty Enhancement fund and the Joseph Meyerhoff FACA fund, this project was also funded by grants from the E. E. Ford Foundation, the Benedict Foundation, Josh and Genine Fidler, and an anonymous donor.

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