BOOK : REASONING AND PROVING III

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

1

At Arnaldo's small party, people shook hands with as many people they liked, from 0 up to everyone else at the party. Nayla has been paying close attention, and says that if you add up everyone's personal "handshake total", you will get 15. Can she be correct? Explain.

Often in life, when someone says something that seems surprising, counter-intuitive, or even reasonable, we will look not at the conviction with which the speaker is speaking, but instead for the evidence and reasoning needed to back up their claim. In mathematics, we do the same thing. It is a fun fact of mathematics that many things that initially appear to be false are in fact true, and as well many things that appear to be true are false. So the only way to know if a statement is correct is to use clear reasoning based on assumptions we all agree upon.

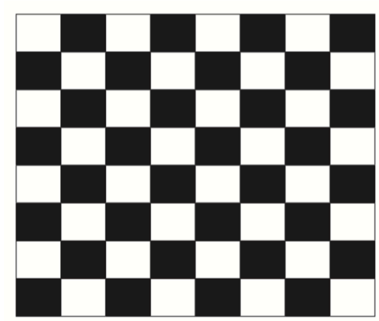
Proving is a way of demonstrating to yourself and others that something you think may be true (or false), must in fact be true (or false). Although there are many different ways of proving a statement, everyone agrees that the steps of a proof should be clear: a careful reader should be able to read each step of your proof and see how it connects and follows from the one before. That way, at the conclusion of your proof, they will agree with you!

In the "Arnaldo's party" problem above, at first it may seem a bit mystifying how to approach testing Nayla's claim. Probably the best thing to do at first is to Tinker by trying to see if you can get a "handshake total" for all party members to be equal or close to 15. If you can get to exactly 15, you're done; Nayla is correct, and the problem is over. On the other hand, if you have tried a number of different handshake "scenarios" and only gotten close to 15 in each case, nevertheless you will probably have noticed something interesting about handshake totals in general—something you might try to prove (and which will tell us if Nayla can be correct as well).

So, when presented with a claim, do three things. First, decide whether it seems reasonable – do you believe it at first glance? Next, try to find evidence that the claim is or is not true. Finally, for any claim that you believe to be true, try to prove that it's true by making a sequence of logical statements, each of which your peers could understand and agree to; and for any claim you believe to be false, provide an example that shows the claim is false—such an example is called a counterexample.

Here's a problem that many students have enjoyed:

2



A 64-square black and white checkerboard (see above) can clearly be completely covered by 32 dominos. If the top right and bottom left squares are removed, what is left is a 62-square board. How many dominoes will be required to cover the 62-square board?

3 [Continuation of problem 2] Did you check your answer to problem 2? A 64 square checkerboard is covered by 32 dominoes since every domino covers 2 squares, a white one and a black one. In the altered checkerboard, how many black squares are there? How many white squares? Would 31 dominoes do the job?

Here's another problem that might seem very tricky at first, but with a little persistence and some tinkering . . .

4 In a stock market game, Traci claims that she can purchase any stock above \$26 by using only \$5 and \$6 coupons. (For example, there is no way she could purchase \$13 of stock with these coupons, but she could purchase \$41 of stock by using one \$6 coupon and seven \$5 coupons.) Do you think that Traci can really pull this off?

As a token of our affection to you, the answer is given on the reverse.

While it may seem plausible that there is always some combination of $5 and $6 coupons that is equivalent to a specific amount of money, it at first seems unlikely that we may be able to prove that any amount over $26 can be produced by some combination of the two coupons. How would we know for sure that $329876 could be paid out?

Our best strategy is to begin by tinkering, and then see if there are any interesting patterns that emerge. We can then try to establish through careful reasoning that the patterns we see are no fluke, that they have to continue.

So let's start.

The first few dollar amounts one can figure out just by tinkering:

$27 stock can be paid by using three $5 coupons and two $6 coupons.

$28 stock can be paid by using two $5 coupons and three $6 coupons.

$29 stock can be paid by using one $5 coupon and four $6 coupons.

$30 stock can be paid by using no $5 coupons and five $6 coupons.

$31 stock can be paid by using five $5 coupons and one $6 coupon.

Wait a second. Isn't this sort of tedious? Doesn't it seem like this approach would take us forever, and even then we would still have an infinity of numbers left to check? And weren't we supposed to look for a pattern? There doesn't seem to be any pattern in the $5's and $6's at all.

Well, let's think about a $32 stock. Traci knows how to purchase a $27 stock—three $5 coupons and two $6 coupons. What coupon should she now add in order to purchase a $32 stock?

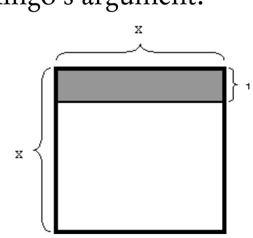
Now convince yourself that Traci's claim is valid.

5 Phillip says that every whole number has an even number of positive integer factors (for example, 6 is $1 \cdot 6$ and $2 \cdot 3$ so its factors are 1,6,2,3). Either explain why he is correct, or give a counterexample.

6 Think of any three consecutive integers and add them together. If you repeat this a few times you will notice something interesting about the sum. What is it?

7 If I own 12 blue socks, 16 red socks, and 18 green socks, and I reach in the dark in to my sock drawer, how many socks will I have to take out to guarantee that I have a pair that are the same color?

8 In the diagram below, the area of the gray rectangle is $x \cdot 1 = x$, while the area of the whole square is $x \cdot x = x^2$. Ringo says that since the gray rectangle fits inside of the square, this shows that x is smaller than x^2 for any positive number. What do you think of Ringo's argument?



9 In the "matches" game, 30 matches are laid on a table. The first player picks up anywhere from 1 to 6 matches, the second player also picks up anywhere from 1 to 6 matches, and the two then alternate picking up 1 to 6 matches until all the matches have been picked up. The player who picks up the last match loses. Can the first player always win with clever play?

10 If $\lfloor k \rfloor$ represents the smallest integer greater than or equal to k, we can agree that $\lfloor 2 \rfloor = 2$ and $\lfloor 5.3 \rfloor = 6$. Are either of the following true?

i. $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x + y \rfloor$

ii. $\lfloor x \rfloor \lfloor y \rfloor = \lfloor xy \rfloor$

11 The French mathematician, Pierre Fermat (1601-1665) claimed that no one could find positive integers n, m and r such that $n^x + m^x = r^x$ for any integer x greater than 2. But in an episode of *The Simpsons*, the equation $1782^{12} + 1841^{12} = 1922^{12}$ appears. What do you think?

12 Prove that if there are 6 people who can arbitrarily shake hands with one another, there is always a pair of persons who shake hands with the same number of people. Then prove this is true if there are n people.

13 When asked to square a certain integer, Joshua came up with an answer of 637812, Wendy came up with 637813, and Anthwaran with 637818. Without using a calculator, determine which of them could be right.

14 Can one make change for a 25-ruble bill, using exactly ten smaller bills of denominations of 1, 3, or 5 rubles? (The 1, 3, 5, and 25 ruble denominations used to be standard in Russian currency.)

15 Imagine you are at a school that has a row of 500 lockers, all shut. Initially a student goes along the row and opens every locker. Then a second student goes along and shuts every other locker beginning with locker number 2. A third student changes the state of every third locker beginning with locker number 3—that is, if a locker is open the student shuts it, and if a locker is closed the student opens it. A fourth student changes the state of every fourth locker beginning with locker number 4.

This continues until all 500 students have followed the pattern with the 500 lockers. At the end, which lockers will be open and which will be closed?

16 In a certain year there were exactly four Fridays and exactly four Mondays in January. Gisele says that, given only that information, she can determine what day of the week the 20th of January must fall on in that year. Is Gisele correct?

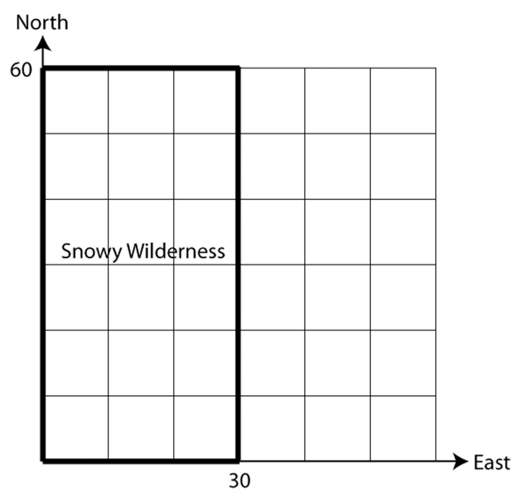
17 There are 12 million people in New York City. The number of hairs on a human head never exceeds 750,000. Must there be 2 people in New York who have the same number of hairs on their head?

HABITS

LESSON 1: ALGORITHMS

Introduction

An explorer is lost in the 30 by 60 mile rectangular section of snowy wilderness shown below. The search and rescue team knows that she started her journey at the southwest corner, and since there hasn't been any new snow for a while, they also know that they'll be able to see her tracks.



Obviously, the team wants to find the lost explorer as quickly as possible. There are different ideas about how to do this, though, so they decide to split into two groups. Group A will start at the southwest corner and simply follow the explorer's trail. Group B, though, has a different strategy. Here's what they do:

1. Group B starts out at $(0, 30)$, and walks directly east. Every time they cross the explorer's trail, they make a note of it. Suppose that by the time they reach the eastern border, they've crossed the explorer's trail 5 times. What can they conclude about where the explorer must be now? How about if they'd crossed the trail 8 times?

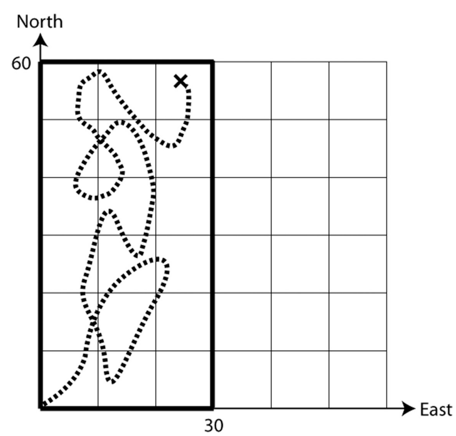
2 As you can see, Group B's west-to-east walk allows them to cut the remaining search area in half. Their idea is to keep repeating this until they've narrowed down the search area to a small horizontal strip. Suppose that on the first walk (along $y = 30$), they crossed the trail 3 times.

a. What horizontal line should they walk next?

b. Now, suppose that on the second walk, they cross the explorer's trail 6 times. Where could the explorer possibly be? Demonstrate your answer by drawing a picture of one such trail the explorer could have taken.

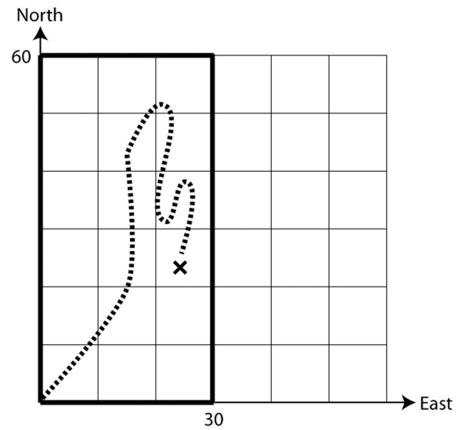
3 Is Group B's strategy better than Group A's? To help you answer this question, determine approximately how long each one would take to find the lost explorer in each of the following scenarios. (Assume that each group moves at the same speed, and that as soon as a group comes within a mile or so, they'll see the explorer's signal flares.)

a.



Problem continued on next page

b.



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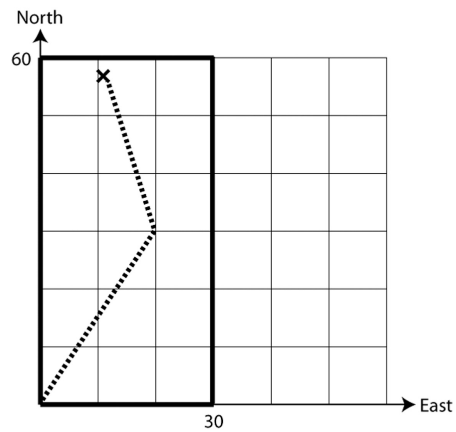


Development

The two strategies in the situation above are both examples of **algorithms**—though, of course, Group A's is a rather simple one. An **algorithm** is simply a clearly defined, step-by-step process for accomplishing a desired task. You can think of every algorithm as having three main parts: input, process and output. In the search and rescue situation, for example, the input would be the whole search area and the explorer's trail, and the output would be the explorer's location.

As you can hopefully see, thinking about and analyzing algorithms can be useful in situations where there are multiple possible approaches, and we want the "best" or most efficient one. But this kind of thinking isn't only useful in practical applications — it's also a source of some classic games and puzzles.

4 The Number Devil is thinking of an integer between 1 and 63 (inclusive), and your job is to guess it. If you guess wrong, the Number Devil will tell you (truthfully) whether your guess is too low or too high. Come up with a strategy for finding the Number Devil's number in as few guesses as possible.

Was there anything in common between your strategy in the previous problem and Group B's approach to finding the missing explorer? Group B's strategy — and, likely, yours — is an example of a **binary search** algorithm. (The Latin root *bini* means "two-by-two".) Here's another application of it.

5 Your English teacher has an alphabetized stack of 15 papers, but she's forgotten whose papers she collected. Now, she needs to check and see if she has Gabi's paper in the stack. She decides to use a binary search algorithm.

a. Simulate this with your group: have a group member make an alphabetized list of names without showing you. The list maker gets to decide whether or not "Gabi" is in the list. You, in the role of the teacher, can ask for any individual name in the list (e.g., "Who is the fifth person in the list?"). How many times do you have to ask before you can be sure whether or not Gabi's name is in the stack?

b. Make up an alphabetized list of 15 names that would force the teacher's binary search process to make the most guesses possible. This is called a "worst case" input for the algorithm. How many guesses does the algorithm take, in the worst case?

6 Now, you decide to play a little trick on the teacher. Carefully rearrange the ordering of names you made in part b of the previous problem so that, when your teacher uses binary search, she'll end up mistakenly concluding that Gabi's paper is not in the stack.

Based on what happened in problem 6, it's clear that there are certain situations in which the binary search algorithm isn't appropriate. This is always true: every algorithm makes some assumptions about its input.

7 What assumption(s) did the teacher make in using her binary search algorithm?

In addition to thinking carefully about the input, it's also helpful when working with algorithms to identify the **basic operations** your algorithm can use. The basic operations are always based on the situation you're dealing with: they are the actions you're able to do as a single "step" of your algorithm. In the Number Devil game, for example, your basic operation was to guess a number and find out if it was correct, too big, or too small. In the stack of papers example, the basic operation was to check the name on a single paper in the stack and compare it alphabetically with the name 'Gabi'. The basic operation in the following game is similar to these two, but slightly different.

8 Play the following game with a classmate:

Your partner should write five different numbers on slips of paper. Then shuffle the papers and put them face down on the table in a straight row.

You are the guesser. Your job is to figure out which paper has the largest number written on it. But you can never see the numbers themselves — you can only ask for comparisons. For example, you could point to the first slip and the fourth slip, and your classmate would have to tell you which one has the larger number.

Each comparison counts as one step, and your goal is to find the paper with the largest number with as few steps as possible.

a. Play the game and identify the paper with the largest number. How many steps did it take before you were certain about your answer?

b. Describe, in detail, an algorithm for finding the largest number in a list of numbers, using as few comparisons as possible.

9 Aaron has an algorithm that he claims can find the largest number every time using only three comparisons. Either figure out Aaron's algorithm, or explain why it's impossible.

Here's a somewhat different algorithmic puzzle that many consider a "modern classic." (In fact, the story goes that, before it got out onto the Internet, this was a question that Google interviewers asked of potential employees.)

You have two perfectly identical eggs. You need to figure out how high an egg can fall from a 100-story building before it breaks. You know nothing about the toughness of the eggs; they may be very fragile and break when dropped from the first floor. On the other hand, they may be super tough genetically altered eggs, so tough that that dropping them from the 100th floor doesn't even cause a scratch. The only thing you know for sure is that they both have exactly the same "toughness."

10 Joey takes his first egg, and drops it off of the fourth floor. It doesn't break. Then, he goes up and drops it off the tenth floor, and it shatters spectacularly on the sidewalk. Finally, he takes his second (and last) egg, and drops it off the seventh floor. It breaks. What can Joey conclude about the toughness of his eggs?

11 Come up with a strategy for figuring out your eggs' toughness. Your job is to guarantee the smallest number of egg-drops, in the worst case. You are allowed to break both eggs, but remember, two is all you get!

Practice

12 You have 16 identical-looking coins. Fifteen of them have exactly the same weight, and one is too heavy. Describe an algorithm for finding the heavy coin in as few steps as possible. The basic operation allowed is to use a simple balance to compare 2 piles of coins to see which pile is heavier.

13 In the algorithm you use for adding two very large numbers by hand, what are the basic operations? How about multiplying two numbers by hand? Dividing?

14 Describe the mental algorithm a street vendor might use to quickly make change for a $20 bill. Make sure your algorithm works by testing it out the following sale amounts: $1.50, $3.00, $11.45, $17.26.

15 Write an algorithm for finding someone's age based on their birth date and the current date. (For instance, Tom's birth date is 6/14/90 and today's date is .... so Tom is .... years old.)

16 Recall the Number Devil game from problem 4. Now let's say you're trying to guess a number between 1 and 255. If you were going to play using the binary search algorithm, what would be one worst-case number that the Number Devil could choose?

17 Consider the following strategy to the egg-drop problem:

Starting at floor 1, drop the first egg off of odd-numbered floors until it breaks. Then, go back down one floor and drop the second egg. If it breaks, that's the answer. If not, then the floor above is the answer.

Say you get to decide how tough the eggs are, and you want to force the strategy above to make the largest possible number of drops before it succeeds.

a. What is the worst-case input for this strategy? i.e., how tough should you make the eggs?

b. What's the worst-case number of drops?

Going Further

The algorithms we've worked with so far dealt with lists of information, whether they were lists of numbers, names, phone numbers, etc. Now, we're going to look at algorithms that transform a single item, such as a number, word, etc.

18 You and your friend are planning to share lockers, so as soon as you get your combinations, you plan to exchange them over the phone. Each combination has three numbers in it – for example, 11 – 2 – 41.

In case someone might be listening, though, you agree ahead of time to use the following algorithm to encrypt each number in the combination:

EncryptNumber (N):

If N is odd, then output $2 + N$.
Otherwise, output $2 \cdot N$.

Notice that the notation being used here is similar to the "function notation" that you've seen in some previous lessons. "EncryptNumber" is the name of the algorithm, and N is a name for the input to the algorithm.

a. Your combination is 12 – 7 – 11. What should you tell your friend?

b. Your friend tells you 80 – 28 – 29. What was her combination?

19 Your brother wants to use the same system with his friend, but he's confused about how it's possible to "decrypt" the numbers he receives. Give him detailed instructions for a "DecryptNumber" algorithm that he can use to understand what his friend tells him.

One convenient feature of using function notation is that we can use it to easily write down things about what an algorithm does for specific inputs. For example, we can write EncryptNumber $(4) = 8$, or DecryptNumber $(8) = 4$.

20 What is DecryptNumber (13)? How about DecryptNumber (24)? And DecryptNumber (18)?

21 Predict what would happen if you applied the EncryptNumber algorithm to each of your three answers in the previous problem. Then try it out to test your prediction.

When you use an algorithm to encode a message so that no strangers can understand it, it's called an "encryption" algorithm. This is essentially what a computer does when you enter a password or other personal information on a website, so that nobody besides the website owner can see it.

Another kind of encoding is called "compression". With compression, the idea is to shrink the size of a message down so it's faster to transport. This is why mp3's are such a big deal: they allow you to compress very large song files so that they're small enough to download quickly.

22 Emmett wants to compress all the old emails he's saved on his computer. Thinking of the shorthand he sometimes uses for text messages, he decides to go through each message and remove all the vowels and all the spaces. One problem with this approach is that it simply doesn't shrink the message by all that much. However, there's a much deeper problem. What is it? Can you suggest an alternative method for compressing Emmett's emails?

23 What are some characteristics that would make a good compression algorithm? What characteristics are absolutely necessary?

You saw with the compression and encryption algorithms above that they're only useful if they also have *de*compression and *de*cryption algorithms to go with them. These are two examples of a more general type of algorithm called an **inverse** algorithm. When you have two algorithms, and each one reverses the other, then the two algorithms are called **inverses** of each other. Using this language, we can say the major flaw in Emmett's compression algorithm was that it has no inverse algorithm. (Why not?)

24 You want to share your gym locker with a different friend from the one in problem 18. But she and your other friend don't get along, so you need a different code for this locker. Here's the one you use:

GymCode (N):

If N is odd, then output $2N$.
Otherwise, if N is a multiple of 4, output $N + 1$.
Otherwise, output $2N + 3$.

Write an inverse for this algorithm called GymDecode.

Now, let's say you wanted to write down all your passwords so you don't forget them. Since someone else might see the paper, you decide to encrypt all the passwords so that only you will know what they really are.

You've already got two encryption algorithms: EncryptNumber from problem 18 and GymCode from problem 24. So you can just combine the two to create the SuperSecretCode algorithm, as follows:

SuperSecretCode (N):
Output EncryptNumber (GymCode (N)).

Recall that GymCode (N) just means "the output of GymCode when you input N". Similarly, "EncryptNumber(GymCode (N))" means, "the output of EncryptNumber when you input GymCode (N)".

25 What is the value of SuperSecretCode (14)? (Hint: first figure out the value of GymCode (14), and then use substitution.)

26 Write the inverse algorithm, SuperSecretDecode.

Practice

27 I have three bags, two with marbles in them (bags A and B) and one without (bag C). First, I take half the marbles in bag A and add them to bag B. Then I recount the marbles in bag B, take out half, and put them in bag C. You know that when I'm done, the contents of the bags are as follows:

Bag A: 10 marbles
Bag B: 20 marbles
Bag C: 20 marbles

Find out how many marbles were in the bags originally.

28 Write an inverse algorithm to reverse the following process: Take a number and add 4 to it. Then multiply your answer by 3. Finally, subtract 1 from your answer.

29 Is there an inverse for an algorithm that sorts a list of numbers in ascending order? ("Ascending order" just means from least to greatest.)

30 An algorithm takes a number, multiplies it by 2, and then adds 5. If the output from the algorithm is X, what was its input, in terms of X?

31 Write an inverse algorithm to reverse the following process: Take an integer between 1 and 99. If it's a 2-digit number, multiply it by 5. If it's a 1-digit number, multiply it by 10 and then add 1. (Try the process on a few different numbers first.)

32 The ShiftCipher algorithm is a simple way of encrypting a word. Here's how it works:

Take a word, such as DOOR. Shift the first letter once – D becomes E. Then, since D is the 4th letter of the alphabet, shift all the other letters 4 times – OOR becomes SSV. If at any point you hit Z and need to go past it, then just wrap back around to A. So, ShiftCipher("DOOR") = "ESSV". (Another example: ShiftCipher("EYES") = "FDJX" – try it).

Write the DecryptShiftCypher algorithm.

Problems

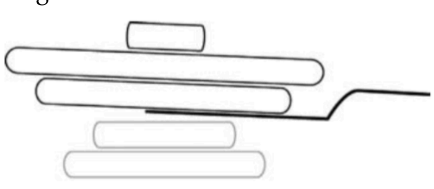
33 You always set your alarm clock to wake you up one hour before you have to leave. But if you have to leave before 7 am, you give yourself 15 extra minutes to get ready since you know you'll be groggy.

a. What's the input for the process described above? What's its output?

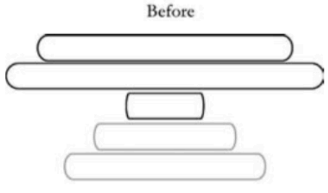
b. What time would you have to leave if the clock woke you up at 5:40am? 5:50am?

c. Your alarm wakes you up. Based on the time you see on the clock, how do you know when you'll have to leave?

34 You've got a stack of five pancakes, all of different sizes. Your only tool is a spatula, and the only basic operation allowed is: insert the spatula beneath any pancake in the stack and flip the whole section of the stack that's above the spatula onto the remaining stack below. Your job is to sort the stack so that the smallest pancake ends up on top and the pancakes increase in size as you go down the stack.



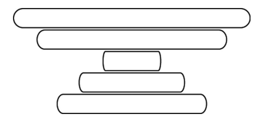
Before



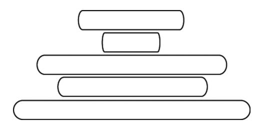
After

a. Sort each of the following stacks, using as few flips as possible.

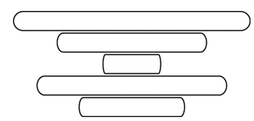
i.



ii.



iii.



b. Write a general algorithm for sorting any stack, and try it out on a few different initial arrangements of pancakes.

c. Make up a worst-case initial arrangement of five pancakes for your algorithm. How many flips does your algorithm take in the worst case?

35 You've lost your calculator, and your diabolical arch nemesis has decided to exploit this opportunity. He'll make calculations for you — for a price! Multiplication costs \$1, whereas exponentiation costs \$6. Unfortunately, you desperately need to compute 7^{10}, and you've only got \$5. What will you do?

36 You input any word, and the AlphaBlast algorithm outputs the same word but with all the 'a's replaced with 'z's. Is AlphaBlast reversible? Example: AlphaBlast("mathematics") = "mzthemztics". Test it carefully and then explain your answer. (An algorithm is called **reversible** if it's possible to create an inverse algorithm for it.)

37 For each of the following algorithms, assume that the input can be any number, and call it X. Read each description, and decide in each case whether the algorithm is reversible. If so, write the inverse.

a. The Foo algorithm either multiplies X by 2 (if it's a whole number) or doesn't change it (if it's not a whole number).

b. The Bar algorithm either multiplies X by 2 (if it's NOT a whole number) or doesn't change it (if it IS a whole number).

c. The FractionsAreFun algorithm takes X, adds 10, and divides that answer by 4, and outputs the result.

d. The NoReallyTheyAre algorithm divides X by 20 then adds 1 and outputs the result.

e. The Flip algorithm outputs the value of $100/X$, unless X is zero, in which case the output is just zero.

f. The Flop algorithm divides 20 by X, then adds 1 and outputs the result.

g. The Square algorithm simply outputs the value of X^2.

38 Here's a variation on the Number Devil game: the Number Devil is allowed to change its secret number after each time you guess. However, it can't change it in a way that makes any of its previous answers untrue. (So, if you guessed "5" the first time and the answer was "higher", it could not change its answer to "2", but it *could* change it to "7".)

a. Explain why binary search algorithm's worst-case guarantee is still the same.

b. Describe an algorithm the Number Devil could use to force a worst-case outcome every time, as long as you didn't get lucky and guess right the first time.

39 Here's a tougher variation of the game you played in problem 8. The setup is the same: your partner shuffles the cards and lines them up, and you can't look at them. But now, instead of just finding the largest one, your goal is to sort all the cards into descending order (i.e., line them up from largest to smallest). Your basic operations are (i) to ask your partner to compare two cards (just like last time), and (ii) to swap the positions of any two cards.

Write an algorithm for this game. That is, create an algorithm that will sort a list of 5 numbers. After writing the algorithm, go ahead and try it out with a group member. Does it work?

40 Look back at your pancake-flipping algorithm from problem 34. If you extended your algorithm to stacks of N pancakes, how many flips, maximum, could you guarantee it taking — that is, what's the worst it could do?

41 Here's a different version of the pancake problem. This time, you have three plates. Plate #1 has a stack of 5 pancakes, in order from the largest one on the bottom to the smallest on top. This time, though, you can only use the spatula to shift one pancake at a time to another plate. At no time can any larger pancake be on top of a smaller pancake. How many moves does it take to get the entire stack of pancakes from Plate #1 to Plate #3?

42 Don't use a calculator for this problem.

a. Simplify $\frac{20a^4b^7}{30a^3b^{-2}}$

b. Reduce the fraction $\frac{x^3+2x}{x^2+x}$

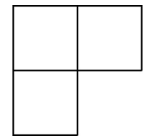
c. Solve for x: $x^{\frac{2}{3}} = 64$

d. Solve for x: $x^2 + 8x + 9 = 0$

e. Solve for x: $3x^2 + 7x + 3 = 0$

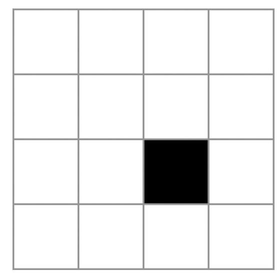
43 If you double the size of a list in which you're doing a binary search, will it (in the worst-case scenario) take you about twice as long to do the search?

44 The shape below is called a "tromino."

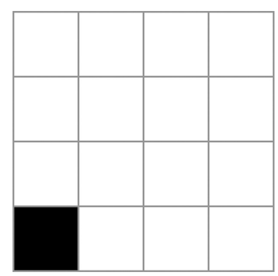


Each of the following "checkerboards" has had one square removed. For each board, find a way to cover *all* of the remaining squares with trominoes.

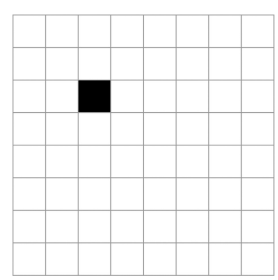
a.



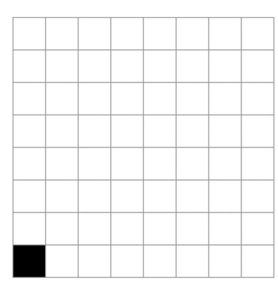
b.



c.



d.



45 You have only a compass and an unmarked straightedge. Come up with an algorithm for finding a point equidistant from three given points. In doing so, you will have proven that it is always possible to find such a point.

46 Alice is in the top right square of a giant chessboard, and the white rabbit she's chasing is in the bottom left square. Each turn, Alice gets to move one square up, down, left, or right. Then, after that, the white rabbit gets to move (up, down, left, or right). Alice catches the rabbit when she's able to move into the square he's sitting in. Assuming that both Alice and the white rabbit are strategically quite savvy, predict the outcome.

47 The Floor operation takes any number whatsoever as an input. $\mathrm{Floor}(x)$ simply takes x and returns the greatest integer that's less than or equal to x. In other words, the Floor function always "rounds down." Examples: $\mathrm{Floor}(2.001) = 2$, $\mathrm{Floor}(\pi) = 3$, and $\mathrm{Floor}(5.9999) = 5$.

The following algorithm, which uses Floor and some basic arithmetic operations, only takes positive integers for its input. What does it do?

$$\mathrm{Mystery}(n) = 5\left(\frac{n}{5} - \mathrm{Floor}\left(\frac{n}{5}\right)\right)$$

48 The RightShift and LeftShift operations work for any non-negative integers with five or fewer digits:

$\mathrm{RightShift}(N, d)$ chops off the rightmost d digits of N and outputs the result. Example: $\mathrm{RightShift}(3214, 2) = 32$.

$\mathrm{LeftShift}(N, d)$ moves each digit d places to the left, and fills in 0's in the empty spaces; then, it chops off all but the last five digits. Examples: $\mathrm{LeftShift}(75, 3) = 75000$, and $\mathrm{LeftShift}(3214, 2) = 21400$.

a. Starting with the number 6789, how can you use only the LeftShift and RightShift operations to end up with the number 7?

b. Generalize your work from part a to create the PickADigit algorithm, where $\mathrm{PickADigit}(N, d)$ outputs the dth digit of a positive integer N.

49 What does the following algorithm do, assuming that N is a positive, five-digit integer?

$$\begin{aligned}\mathrm{Ffleba(N)} = {} & \mathrm{LeftShift(N, 2)} \\ & + 10 \cdot \mathrm{RightShift(N, 4)} \\ & + \mathrm{RightShift(LeftShift(N, 1), 4)}\end{aligned}$$

50 You input a whole number to a certain algorithm. If it's even, the algorithm doesn't change the number. If it's odd, the algorithm turns the number backwards — for example, 57 becomes 75. Is this algorithm reversible?

51

Come up with a non-trivial numerical algorithm that reverses itself. In other words, come up with an algorithm Boing such that, for every N, $(\text{Boing}(\text{Boing}(N)) = N$.

52

Imagine your friend gave you a very long table of *all* the possible inputs to her algorithm and their corresponding outputs. Even if she didn't explain the process her algorithm used, describe using a single sentence how you could figure out whether or not the algorithm was reversible.

53

Your basic operations for the following are addition, subtraction, multiplication, division, and testing which of two numbers is the biggest.

a. Write an algorithm for figuring out whether or not a positive integer N is prime. Pick some three-digit numbers and try it out. (By the way, computer systems actually use algorithms like these to encrypt your private data.)

b. Suppose it takes you 3 seconds to use your calculator to do each of the basic operations above. How long would it take you to check whether or not 1,000,003 is prime? How about 1,000,005?

54

You have 4 identical-looking coins. Two of them are heavy and two of them are light (the two heavy ones weigh the same, and the two light ones weigh the same).

Call the coins A, B, C, and D. Your goal is to find out which two are heavy and which two are light. Your only instrument is a balance scale — you put some coins on the two sides of the scale, and it tells you which side is heavier (or that the two sides weigh the same).

Describe an algorithm for finding the light coins in as few steps as possible.

55

When you type a text message into a cell phone using multi-tap typing, you are essentially encoding English words into a long sequence of numbers (and pauses). For example, to type in the word "EIGHT", I type: 3 3 4 4 4 [pause] 4 [pause] 4 4 8. Use this picture of a cell phone's keypad to help you:

1	2 ABC	3 DEF
4 GHI	5 JKL	6 MNO
7 PQRS	8 TUV	9 WXYZ
	0 (space)	

a. I typed 7 7 7 7 , 7 3 3 , 3 3 2 2 2 4 4. (Here the commas represent pauses). What word did I type?

b. Write an algorithm for "decoding" a string of numbers and pauses into a word.

Exploring in Depth

56 You and a fellow explorer need to be able to communicate two numbers to each other — coordinates for your location, which will always be positive *whole* numbers, but can be small or large. The only means of communication you have is a carrier pigeon. Due to its tiny brain, the pigeon can only remember one *whole* number (though it can be as big as you want).

Somehow, you need to fit the two numbers together into one number, but you need to make sure that the process is reversible, so that your companion can figure out what numbers you're sending him.

a. Here's the first process you try. Say your two numbers are 67 and 302. Then you just put them together to make 67302, and send that number along with the pigeon.

b. Why is this non-reversible?

c. The next strategy you develop fails as well. What you tried to do was separate the numbers with "000". So, 67 and 302 gets written as 67000302.

d. This strategy will usually work, but not for every pair of numbers. When is this strategy non-reversible?

e. Create your own strategy, and show that it works — that it is reversible.

57 Look back at the sorting algorithm you wrote for problem 39.

a. Are there certain initial arrangements of the cards that make your algorithm finish in fewer steps?

b. Using your algorithm, what's the maximum number of comparisons you have to make, in the worst case, when you play the game with five cards?

c. If you used your sort algorithm from problem 39 to sort a list with N items in it, how many comparisons would it take? See if you can express your answer as a simple algebraic formula.

58 Five pirates come across 100 bars of gold. The pirates have a pecking order, with #5 and #1 indicating the top and bottom-ranked pirates, respectively. Each pirate wants to maximize his or her share of the gold bars. There are no coalitions or collusions between them. They are all very analytical — they can think things through!

Their process for splitting the loot is somewhat democratic. It begins with the top-ranked pirate making a proposal on how to divvy up the loot. (Note that the gold bars cannot be broken up, glued together, or otherwise changed; there are 100 bars of gold, period.) Each pirate gets one vote, up or down on the entire proposal. Remember, each pirate votes based on his or her own pocket and there are no side agreements of any sort. If the proposal gets 50% or more votes, it wins, and that's that. On the other hand, if it fails to muster 50%, then the pirate who made the proposal is thrown overboard, and the process continues with the next pirate down the hierarchy.

What is the maximum number of bars that the most powerful pirate can get and what allocation to each pirate will ensure him the 50% vote that he needs?

(from http://www.eogogics.com/talkgogics/ezine/tech-talk/pirates1)

59 Create an algorithm to find the median of a list of numbers. Your only basic operation allowed is to compare two numbers to see which is larger. Assume the list has an odd number of items in it.

60 Write an algorithm that finds the second largest number in a list of 16 numbers, using only comparison as the basic operation. It's possible to guarantee a correct answer in fewer than 29 comparisons — can your algorithm do this?

61 Sherlock and Watson create the following code to encrypt their messages:

First, turn each letter into a number (A is 1, B is 2, etc). Then for each number, multiply it by 7 and add 1. Finally, if any numbers have 3 digits, just ignore the hundreds digit — so 141 just gets written as 41, and so on.

a. Encrypt the word NO according to their code.

b. Sherlock sends Watson the message 36 34 22 8 13 36. Decode it.

c. Even though the code "erases" hundreds digits, it's still perfectly reversible — you can always decode any message and know exactly what was written. Explain why it's reversible.

62 You create an especially tricky locker code. Here's what you do to encode the 3-number combination: The new first number will be the sum of all three numbers in the original.

The new second number will be the sum of the first two numbers in the original.

The new third number will be the sum of the last two numbers in the original.

a. Encode 12 – 8 – 1. Do you see how you could find the 12, 8, and 1 from the coded version?

b. Decode 17 – 14 – 5.

c. You receive a coded combination: A – B – C (A, B, and C represent the numbers you receive). Write instructions or equations for decoding it.

63 This is a famous problem that's challenged many mathematicians. You have 12 coins that look alike. 11 are genuine and each have the same weight. 1 is counterfeit, and is *either* too heavy or too light.

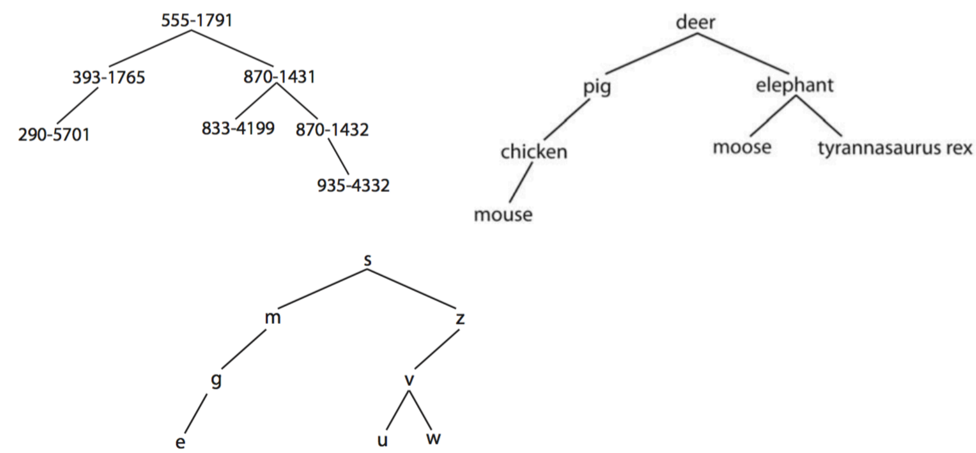
Write an algorithm for how to use a balance scale to find the fake coin, *and* determine whether it's too heavy or too light, in as few steps as possible. (It can be done with three weighings! See if you can figure out how.)

Hint: it will probably help a lot to make a chart to keep track of the different possibilities.

LESSON 2: LOGARITHMS

Introduction

The following tree diagrams, which are called **binary trees**, are all based on the same system for organizing information.



Each item (phone number, animal, letter) in the tree is called a "node", and the nodes immediately below a node are called its "children". As you can see, each node in the tree has at most two children. Furthermore, notice that as you start at the top and travel down the branches, every descendant down the right branch of a given node is somehow more than (or larger than, or after) that node, whereas every descendent to the left is somehow less.

1 Let's say we wanted to add the number "866-3162" to the phone number tree above. Even though it comes after "393-1765", it would be wrong to hang it to the right of that number, because that would still be down the left branch from "555-1791". Where should "866-3162" go? How about "squirrel" and "k" in their respective trees? There is only one correct answer for each of these.

2 Ask a group member to write down, without showing you, a list of 10 different numbers. Your group member will be reading the list to you one number at a time, and your job is to organize them into a binary search tree *as they're being read to you*. There's one rule: unlike in nature, these trees grow down. This means, for example, that the first number your group member reads will have to be the one at the very top of your tree.

a. Have your group member read the numbers to you, giving you time to add each number to the tree, without rearranging the ones you've added so far, before reading the next one out.

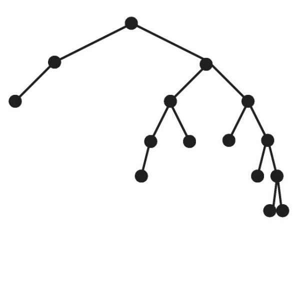
b. What would have happened if the numbers had been read to you in a different order? Try it and see.

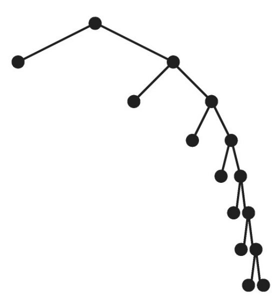
In these small examples, it's quite easy to see all the information at once, so they're not necessarily any better than just writing out a simple list. Imagine, though, that instead of the 7 phone numbers in the first diagram, you had a binary tree with hundreds of phone numbers in it. (By the way, this is essentially how a cell phone really does store phone numbers.)

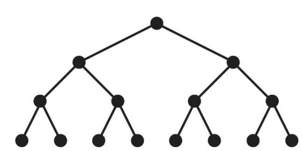
3 Alice the ant starts at the top node of this large phone number tree and wants to know if the exterminator's number, which she knows by heart, is in it. (She's hoping it's not!) Because Alice is so small, she can only read one node at a time, and she has to crawl down a branch to read another number.

a. Describe a simple algorithm she can use to find out where the exterminator's number would have to be, if it were in the tree.

b. Which of the following trees do you think Alice would rather search? Remember the old saying: when you're trying to find something, it's always in the last place you look!





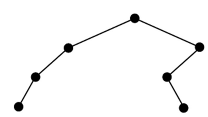


Development

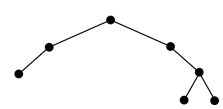
You saw from problem 2 that there is more than one possible "shape" that a binary tree can take, even with the same raw data.

4 For each of the following, come up with a sequence of items, which, if you used your approach from problem 2, would end up in a binary tree with the given shape. (The items can be anything you want: names, numbers, etc.)

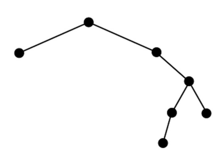
a.



b.



c.



The **depth of a node** in a tree is the number of branches it is away from the top. So, for example, "chicken" in the animal tree has a depth of 2. The **depth of the whole tree** is just the depth of its deepest node. (So the animal tree has a depth of 3, because "mouse", the deepest node, has a depth of 3.)

5 Rearrange the lists of items you used in problem 4 so that the resulting trees have the minimum possible depth.

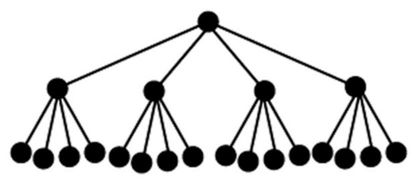
6 Do the same for the list your group used in problem 2.

The trees you made in problem 5 are examples of **balanced** binary trees, whereas the one you made in problem 6 is not balanced. To see why, think of the tree like a hanging mobile: if each node is perfectly balanced, then the whole thing is balanced.

7 Is it possible to arrange the numbers 1 through 7 into a balanced binary tree? How about 1 through 12? 1 through 15?

8 A **leaf** of a tree is a node that doesn't have any children. Notice that in a balanced binary tree, all the leaves have the same depth. If a balanced binary tree has 512 leaves, then what is the depth of the tree?

9 Of course, not all trees have to be binary. If you continued the regular pattern of the tree below, how many leaves would the next level of the tree have? How deep would it need to be in order to have 4096 leaves?



10 On some ancient computers, passwords had to consist of only the letters A through E. If I tell you that on these computers, there were 15625 passwords, and all of the passwords had to be the same length, can you say how many letters were required in a password?

In the last three problems, you were effectively looking for exponents. You might have found yourself asking, in rather awkward English, the question:

5 to the *what power* equals 15625?

Fortunately, there's a mathematical term for the number we're looking for here: it's called "the **logarithm**, base 5, of 15625". Symbolically, it's written "$\log_5 (15625)$".

So our awkward question is now both easier to pronounce:

What is the log, base 5, of 15625?

and easier to write:

$$\log_5(15625) = x.$$

11 Translate the equation above into an equation involving exponents.

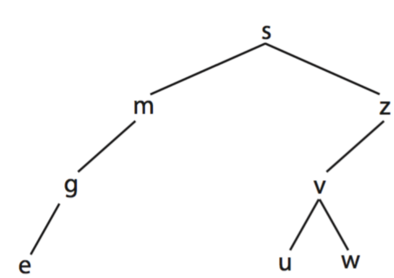
12 Go back and set up problems 8-10 with equations using log notation. (No need to solve them again—just set them up.)

13 Use symbols to create a formal definition for what it means to say that "y is the log, base b, of x". Your definition should have the form: "(*equation involving* log) if and only if (*equation involving exponents*)".

This is something you can refer to whenever you need to translate to and from "log" notation.

Practice

14 Consider the following binary tree.



a. Assuming this tree was built similarly to the one you built in problem 2, what order could the letters have been added in?

b. What ordering would you use to produce a tree with the greatest possible depth?

15

Determine the depth of each of the trees described below.

a. A balanced binary tree that has 1,048,576 leaves.

b. Every node that isn't a leaf has 4 children. There are 64 leaves.

c. Every non-leaf node has 6 children, and there are X leaves. (Answer in terms of X, using log notation.)

16

Unlike on this planet, primordial amoebas from Mars reproduce by splitting into three children. If life on Mars started with a single amoeba, but now there are 1,594,323 amoebas, then:

a. Estimate, without using a calculator, for how many amoeba-generations there has been life on Mars.

b. Now rewrite the question as a logarithm, and find an exact answer (calculator allowed this time).

17

Translate the following awkward bits of English into log notation. (Notice that some are complete sentences, and others are fragments.)

a. "What number could I raise 4 to, to get 64?"

b. "The power of 3 that gives you 81"

c. "4 to the *something-or-other* power equals 1024."

d. "The number to which 6 must be raised in order to produce 216"

e. "The number of times, starting with 1, that you have to double to get x"

18 Calculate the value of each of the following:

a. $\log_3(81)$

b. $\log_{13}(169)$

c. $\log_8(512)$

d. $\log_{11}(14,641)$

e. $\log_{100}(1,000,000)$

f. $\log_{1,000}(1,000,000)$

g. $\log_{10}(1,000,000)$

h. $\log_{15}(11,390,625)$

Going Further

19 Start with $x = 243$. Check with your calculator that $\log_3(x) = 5$. How much do you have to add to x to make $\log_3(x)$ go up by one? Up by two? Three?

20 What is the value of $\log_9(3)$?

Since the number 3 is not one of the powers of 9, it's not clear just what this question means, or how to find out what the value of $\log_9(3)$ would be. Remember, though, that it's also possible to have non-integer exponents. Let's see if we can use that to find an answer.

21 Start by rewriting $\log_9(3) = x$ in exponential form. Then, try tinkering with different ways to write this equation by applying some of your rules of exponents.

You might have noticed in the problem above that it was much simpler to solve once you'd rewritten "3" as "$9^{1/2}$". In the next problem, try using a similar strategy: tinker with different ways of rewriting things, using the definition of logarithms and your exponent rules. (You might want to take a minute with your group to remind yourselves of all the rules.)

22 Find the value of each of the following without using a calculator.

a. $\log_4(32)$

b. $\log_5\left(\frac{1}{25}\right)$

c. $\log_{10}(0.000000001)$

d. $\log_7(1)$

23 Why can you be sure that $\log_2(42)$ is not an integer?

24 Find every number from 1 to 1 million whose base-10 logarithm is an integer.

25 The "log" button on your calculator actually stands for "$\log_{10}$".

a. Before trying it out, can you estimate roughly what $\log_{10}(4000)$ will be? Check it on the calculator.

b. What are $\log_{10}(40)$, $\log_{10}(40,000)$, and $\log_{10}(4)$? What's the pattern?

26 Pick several three-digit numbers, and use your calculator to find the base-10 log of each one. Now do the same with several four-, five-, and six-digit numbers. What do you notice?

27 If $\log_{10}(2) = 0.301$, then what is the value of $\log_{10}(2 \cdot 10^a)$?

If you want to see why your answer to the previous problem is true, try using the definition of log and your exponent rules to tinker with different ways of rewriting things, like you did earlier in problems 20-22. This will also help with the next two problems.

28 If $\log_2(x) = 1.125$ and $\log_2(y) = 2.875$, then what is the value of $\log_2(x \cdot y)$?

29 If $\log_4(x) = 1.20$ and $\log_4(y) = 2.009$, then what is the value of $\log_4(x \cdot y)$?

Practice

30 Put the following in order from least to greatest:
$\log_2(47)$, 2, $\log_3(47)$, 3, $\log_2(35)$, 4, $\log_4(100)$, 5, $\log_3(240)$.

31 What are the nearest integers above and below $\log_2(24)$? Which of the two is the closest?

32 Find the value of each of the following.

a. $\log_{16}(4)$

b. $\log_3\left(\frac{1}{9}\right)$

c. $\log_4(.5)$

d. $\log_2\left(\sqrt{2}\right)$

33 Why does your calculator give you an error when you input "$\log(-100)$"?

34 Practice using the definition of logarithms and the rules of exponents to solve each of the following.

a. If $\log_4(x) = 1.6$ and $\log_4(y) = 3.2$, then what is $\log_4(xy)$?

b. If $\log_8(x) = 6$, then what is $\log_8(2x)$?

Problems

35 If $\log_5(x) = 5$ and $\log_5(y) = 7$, then what is the value of $\log_5\left(\frac{y}{x}\right)$?

36 If $\log_3(x) = 1.776$ and $\log_3(y) = 2.018$, then what is the value of $\log_3\left(\frac{y}{x}\right)$? (By the way: how old is the United States of America?)

37 Based on what you did in problems 28 and 29, complete the following generalization: "If $\log_b(x) = C$ and $\log_b(y) = D$, then $\log_b(xy) = ...$." Now prove it using algebra.

38 Revise the generalization you wrote in the previous problem into an identity that relates $\log_b(x \cdot y)$ to $\log_b(x)$ and $\log_b(y)$. (Remember that an **identity** is just an equation that is always true, for all values of the variables involved.) This identity is going to come in handy, so write it down somewhere where it will be easy to refer to it.

39 Given that $\log_2(5) = 2.322$ and $\log_2(3) = 1.585$, what is the value of $\log_2(15)$?

40 State and prove an identity that relates $\log_b\left(\frac{x}{y}\right)$ to $\log_b(x)$ and $\log_b(y)$.

41 If $\log_3(10) = 2.096$, then what is the value of $\log_3(30)$? How about $\log_3(100)$?

42 Don't use a calculator for this problem.

a. Simplify $16^{\frac{1}{2}} \cdot 2^3$.

b. Simplify $\frac{x^5y^{-3}}{x^{-1}y^2}$.

c. Solve for x: $4x^3 + 2x^2 = 0$.

d. Simplify $\frac{\sqrt{125}}{\sqrt{50}}$.

e. If $\frac{1}{x^2} - \frac{1}{x} = 1$, find $x^2 + x + 1$.

43 What is the value of $\log_2\left(4^{1005}\right)$?

44 For each of the following equations, use the log function of your calculator (and possibly some algebra) to approximate x to the nearest thousandth.

a. $10^x = 461.2$

b. $3 \cdot 10^x = 5$

c. $10^{x+2} = 50$

45 Mr. Golthramis was in a bad mood one day, and decided to take it out on his students by giving them this problem: "Write down, using no exponents, a number x such that $\log_{10}(x) > 1000$." The students, having recently learned the definition of log, were outraged, and refused to do the homework. Why?

46 Verify with your calculator that $\log_3 (12) = 2.2619$. Now, find the values of $\log_3(36)$, $\log_3 (4)$, and $\log_3(108)$.

47 According to the incomplete table below, $\log_5 (10) = 1.4307$. Is this (approximately) correct? Correct it if not, and then complete the rest of the table.

A	$\log_5 (A)$
2	
	0.5
4	
	1
8	
10	1.4307
16	
	2
	2.1535
50	

48 Suppose we visualize some number L as a tree diagram with L leaves, and another number M as a tree with M leaves. Using these as building blocks, how could you build a tree to visualize the quantity $L \cdot M$? Can you use this to illustrate the multiplication and division identities of logarithms you proved? (It might help to use specific numbers first: try $L = 27$ and $M = 9$.)

49 Consider a balanced binary tree with a depth of D.

a. True or false: the total number of nodes in the tree will be $1 + 2 + 2^2 + \ldots + 2^D$.

b. Find a simple formula for the total number of nodes in the tree. Answer in terms of D, and without using "...". (Hint: try several small examples, and look for a pattern.)

50 If a balanced binary tree has 1023 nodes total, then how many of the nodes are leaves?

51 Have one of your group members make up a binary tree and show it to you. Your job is to add nodes to the tree until you have doubled the total number of nodes, while keeping the tree as "shallow" as possible. Try this a few times with trees of different sizes and shapes. If the original tree has N nodes, then how many additional levels of depth do you need?

52 If you build a binary tree by adding the numbers 5, 2, 3, 7, 11, 17, and 13, in that order, the resulting tree would not be balanced. (Check this.) Instead of this, suppose you put these seven numbers into a hat and randomly chose one number at a time to add to the tree. What is the probability that your tree would be balanced?

53 Prove that each of the following are true:

a. $\log_b\left(x^2\right) = 2 \cdot \log_b(x)$

b. $\log_b(x^3) = 3 \cdot \log_b(x)$

c. $\log_b\left(x^4\right) = 4 \cdot \log_b(x)$

d. $\log_b\left(x^5\right) = 5 \cdot \log_b(x)$

54 Use the rule you just proved in the previous problem to solve each of the following for x, and give your answer to the nearest thousandth. (Hint: $A = C$ if and only if $\log_b A = \log_b C$.)

a. $1.02^x = 2$

b. $2^x = 1000000$

c. $1.5^{x-3} = 21$

55 What is $\log_{10}(\log_{10}(\text{one googol}))$? What is $\log_{10}(\log_{10}(\log_{10}(\text{one googolplex})))$?

(If you don't know what a "googol" or "googolplex" is, Google them.)

56 Professor Arlo G. Smith poses the following questions:

a. What is the ratio of 2^9 to 2^5? What is the ratio of 2^m to 2^n?

b. What is the ratio of $\log 2^9$ to $\log 2^5$? What is the ratio of $\log 2^m$ to $\log 2^n$?

c. What is the ratio of 100000000 to 1000000? What is the ratio of $\log 100000000$ to $\log 1000000$?

57 If:
$\log_b(X) = 3 \cdot \log_b(2) + 2 \cdot \log_b(3) + \log_b(5)$
then what is X?

58 Rewrite the expression $\log_5 7 + \log_5 15 - \log_5 3$ as a single logarithm in the form $\log_b a$.

59 Not all calculators have a built-in way to calculate $\log_2(47)$. However, using the definition and properties of logarithms, there is a way to figure this out with any scientific calculator. Here's a hint: re-write $\log_2(47) = x$ in exponential form, and then solve for x.

60 Can the $\log_{10}$ of an irrational number minus the $\log_{10}$ of a different irrational number ever equal a positive integer? Explain.

61 Solve each of the following equations, giving your answer to the nearest thousandth.

a. $1500 \cdot 1.13^t = 4000$

b. $3 \cdot 5^{x+5} = 8$

c. $\log_3 x - 2\log_3 7 = 1$

d. $5^{x-1} = 2^{3x-1}$

e. $7 \cdot x^3 = 1492$

62 A binary tree has 6000 nodes in it. What are its minimum and maximum possible depths?

63 Recall the Number Devil guessing game from the previous lesson, where the Number Devil picks a number between 1 and 15 and you have to guess it. When you guess, the Number Devil will tell you whether you're correct, too high, or too low. If you use binary search, how many guesses will you need in the worst case? What if the Number Devil chose a number between 1 and 1 million?

64 That pesky bad coin you saw in the previous lesson is back. (Maybe the Number Devil brought it along.) You have 27 identical-looking coins. The bad coin is too light, and the rest all weigh exactly the same.

a. Using a balance scale, how many weighings do you need to find the bad coin?

b. How many weighings would it take to find one bad coin among 2187 coins? How about 1 million coins?

65 If you look up logarithms on the Web, you're likely to see the following sentence, which is meant to be a helpful reminder when you're learning what logarithms are:

"A logarithm is an exponent."

What do you think people mean by this?

66 If $a_1, a_2, a_3, \ldots$ is a geometric sequence, then what kind of sequence is $\log_b(a_1), \log_b(a_2), log_b(a_3), \ldots$?

67 Recall that the Floor operation from the previous lesson takes a positive number x and chops off anything after the decimal point. Similarly, the Ceiling function essentially "rounds up": e.g., $\text{Ceiling}\,(3) = 3$, and $\text{Ceiling}\,(2.0001) = 3$. If N is a positive integer, what does $\text{Ceiling}\,(\log_{10}(N))$ tell you about N?

68 If you double a three-digit number 100 times, how many digits will the resulting number have?

69 Prove that squaring a number at most doubles the number of digits in the number.

70 Prove that $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$.

71 Generalize what you did in problem 59 into an identity about logarithms.

72 What is $\log_{10}(2) \cdot \log_2(10)$?
How about $\log_{10}(3) \cdot \log_3(10)$?
And $\log_7(3) \cdot \log_3(7)$?

73 In a **balanced ternary tree**, every non-leaf node has three children, and all the leaves have the same depth. How many nodes are in a balanced ternary tree with a depth of 10?

LESSON 3: COMBINATORICS: INDISTINGUISHABLE ORDERINGS

Introduction: Hey batta! Hey batta!

Jan bats .500 in softball — i.e., on average, for each 10 at-bats she gets five hits. We wish to determine the probability that she will get no hits if she comes to bat 5 times during a game. In order to calculate this probability we must assume that she stays a ".500 hitter" from one at-bat to the next, which is to say that success or failure at each at-bat is not affected by the previous at-bats. Given this assumption we can perform the following calculation:

Probability of no hits in 5 at-bats $= \frac{\text{number of ways of getting no hits in 5 at-bats}}{\text{number of possible outcomes in 5 at-bats}}$

$= \frac{1}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

$= 0.03125$

1 What is the probability that Jan will get exactly one hit in 5 at-bats?

2 What's the probability that Jan will get exactly 4 hits in 5 at-bats?

3 What's the probability that Jan will get exactly 2 hits?

The last question probably took some time to calculate since there are 10 different ways for Jan to get exactly 2 hits. For example, one string of at-bats might look like HNHNN, where H = "hit" and N = "no hit". Imagine trying to calculate the probability that Jan would get exactly 3 hits in 10 at-bats. Don't do this by listing all the strings of 10 at-bats with 3 hits. You will be cursing by the end of this process, if not earlier. Fortunately, there is a more elegant way of counting how many strings have exactly 3 hits in 10 at-bats. Let's look at one approach.

Development

This approach will require us, first, to review two basic counting principles that you learned last year. Suppose you're setting up a password for a new account.

4 If the password must be 3 letters followed by 3 digits, then how many possible passwords are there? The password is not case-sensitive so there's no distinction between "a" and "A".

5 If the password must be 3 letters followed by 3 digits, but no letter or digit can be repeated, then how many passwords are there? Again, the passwords are not case-sensitive.

6 If the password must be 10 digits, no letters, with no repetition, then how many passwords are there? There's a nice shorthand way of writing this answer. Do you remember it?

The problems above involve a basic principle of counting called the **Multiplication Principle of Counting**. You also needed to pay attention to whether repetition mattered. Lastly, Problem 6 alludes to some shorthand notation that comes in handy when the final count gets very large. Keep in mind that this is only notation and doesn't solve problems for you. As you progress through this lesson, you'll need to remember that the formulas you might derive are not all-purpose ones that solve counting problems. They tend to work in highly specialized situations. You will find more success if you read each problem carefully, think about what is being counted (visualize, even), and then apply basic principles like the Multiplication Principle of Counting. Now, let's see how to tackle the batting problem by looking at a similar problem.

7 The word *error* has 5 letters in it, two letters that appear once and one that appears three times. We're going to look at different 5-letter "words" that can be formed by rearranging these 5 letters. These words need not make any sense, so "rroer" is a word.

a. How many 5-letter words can be formed, assuming that you can tell the difference between the r's? In other words, think of the r's as being a red r, a blue r, and a green r (or as r_1, r_2, and r_3). By the way, each of the letters can only be used once per word.

b. Write down two 5-letter words that differ from each other only if the r's are different colors. How many other 5-letter words differ from these two only if the r's are different colors?

c. Think of another 5-letter word that is different from the ones in Part b. How many versions are there of this 5-letter word that are distinguishable from each other only if the r's are different colors? Would all of these 5-letter words be part of the count you made in Part a?

d. So, how many 5-letter words can be formed by rearranging the letters in *error*, if each letter can only be used once per word (and the r's are indistinguishable from each other)?

e. Lastly, you should be able to write your answer to Part d in the form $\frac{5!}{n!}$. What is n?

8 How many distinguishable 9-letter words can be formed from the letters in *freestone*, where each letter can only be used once per word?

9 How many distinguishable 9-letter words can be formed from the letters in *freewheel*, where each letter can only be used once per word?

10 How many distinguishable 9-letter words can be formed from the letters in *appraisal*, where each letter can only be used once per word?

11 Using the letters H, H, H, H, H, N, N, N, N, N, N, N only, how many 12-letter words can be formed? (Note: though the word "distinguishable" is not appearing you should assume you are just counting words that are different from each other.)

In the previous problems you were always counting letters. If you think about what you were doing, however, you should realize that the method of counting wasn't dependent on letters. Problem 11, for example, could have initially been a problem about how many ways can you step right five times and left seven times in a sequence of twelve steps.

12 Bill is flipping a coin 8 times, keeping track of the sequence of heads and tails. What is the probability that he'll get exactly 5 heads in total? Exactly 4 heads? 3 heads?

13 Alix's mathematics class wants to form a committee of 4 students to investigate whether AP classes should be taught in the school. If there are 16 students in the class then how many different committees can be formed?

The last problem may have seemed quite different from the previous ones, but there is a way of seeing it as no different. Imagine that you're working with all 16 students and that you have 16 Velcro labels. Twelve of these labels are a large N and four of them are a large Y. You line the 16 students up and then try to count how many different ways you can assign the labels to the students. In this sense, you are trying to determine how many different ways you can arrange 12 N's and 4 Y's in a sequence.

14 The answer to the last sentence (and to Problem 13) can be written in the form $\frac{a!}{b!\,c!}$, where a, b, and c have specific meanings in the context of the problem. What are a, b, and c and what are their meanings?

15 The fraction in Problem 14 can be simplified so that there is only one factorial expression in the denominator. Do this.

Look back at your answer to Problem 15 and remember that this fraction is an answer to Problem 13. Now, use your understanding of the relationship between this fraction and Alix's committee problem to answer the following question.

16 You're going to order a triple-scoop bowl of ice cream. If you plan on choosing three different types of ice cream from 8 possible choices, then how many different triple-scoop combinations are there?

In this lesson you have seen some problems where two different orderings of the same things are to be thought of as being different — e.g., the passwords 3857210964 and 6710892534. In other problems, you've had to discount these differences — e.g., the triple-scoop combination of chocolate-vanilla-pistachio does not differ from the combination pistachio-chocolate-vanilla. As you work through the remaining problems in these lessons, bear in mind these distinctions. Some problems will fall into one category and not the other, while some will fall into both. And, of course, some will fall into neither.

Practice

17 John and Mary are ordering hamburgers that come with 3 types of toppings: cheese (cheddar, blue, swiss), vegetable (lettuce, onion, tomato, mushroom), and sauce (mustard, ketchup). They will each order one of each type of topping.

a. John doesn't care about the order of his toppings, figuring that the whole mess ends up in the same place. How many different burgers does John think he can order?

b. Mary believes very strongly that the order in which the toppings are on her burger does subtly change the taste of the hamburger. How many different burgers does Mary think she can order?

18 There are 8 books on a shelf — five novels and three biographies. How many different ways can you line the books up on the shelf if

a. you naturally think of the books as being different from each other?

b. you don't distinguish the novels from each other or the biographies from each other, but you do distinguish novels as different from biographies?

19 You shuffle a standard 52-card deck several times.

a. How many different sequences of 5 cards are there at the top of the deck?

b. You deal a hand of 5 cards to yourself. How many possible 5-card hands are there?

c. Estimate the amount of time it would take to create all the possible 5-card hands of part b, assuming you can create 1 new hand every second. Would it take a day, a week, a month, a year, or a decade, or longer? What if you were trying to create all the possible 7 card hands? What about all the possible 52 card hands? Check your 3 estimates with a calculator and see if you were right in each case within a factor of 10 (also known as an "order of magnitude").

20 A true-false test has 25 questions on it. If you take the test by randomly deciding that 10 of the questions are true and the rest are false, then how many ways can you write down the answers to the test?

21 How many different ways can you arrange the letters in *Mississippi*?

22 Horatio has 3 tulips, 4 irises, and 3 daffodils growing in his yard. For each of ten days, he cuts one flower and brings it to his math teacher. How many ten-day sequences of flowers are there? (An example of one such sequence is iris-tulip-iris-iris-daffodil-tulip-daffodil-daffodil-tulip-iris.)

23 Using the seven digits 9, 9, 9, 3, 3, 3, and 3, once each, how many 7-digit numbers can be formed?

24 Each of the 240 students attending an Upper School dance has a ticket with a number for a door prize. If three different numbers are randomly selected, how many ways are there to award the prizes if the prizes are different from each other? if the prizes are identical?

25 In how many ways can you choose 3 letters from the word *problems*, if the order in which you choose the letters is important? is unimportant?

Problems

26 There are 20 cities in a certain country, and every pair of them is connected by an air route. How many air routes are there?

27 Remember Jan who bats .500 in softball. What's the probability that she will get exactly 4 hits in 10 at-bats? What's the probability that she will get no more than 4 hits in 10 at-bats?

28 How many "words" can you form from 5 A's and no more than 3 B's?

29 There are 6 books on a shelf. How many ways are there to arrange some (or all) of the books in a stack? The stack may consist of one book.

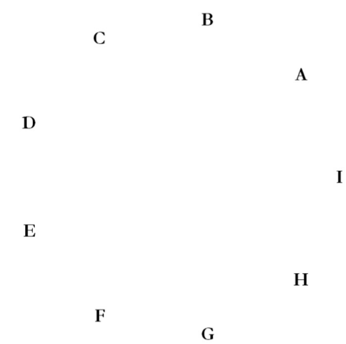
30 A teacher has a collection of 40 true-false questions. She wishes to create a test using 5 of these questions. How many different tests can she create if

a. the order of the questions doesn't matter?

b. the order of the questions does matter?

31 How many three-letter words can be formed from the letters A, B, C, D, E, F, G, H, and I, where each letter can only be used once per word?

32 The points A, B, C, D, E, F, G, H, and I are vertices of a regular polygon. If three of the points are chosen, and connected by line segments, a triangle is determined. How many triangles can be formed in this way? If four points were chosen to form a quadrilateral, then how many such quadrilaterals will there be?



33 You have six sticks of lengths 1, 2, 3, 4, 5, and 6 inches. How many non-congruent triangles can be formed by using three of these sticks as sides?

34 There are three rooms in a dormitory: one single, one double, and one for four students (called a "quad"). How many ways are there to house 7 students in these rooms?

35 *Creamy Heaven* serves 30 different ice cream flavors. A customer can order a single-, double-, or triple-scoop cone.

a. How many double-scoop cones are there if both scoops can be the same flavor and the order of the scoops doesn't matter?

b. How many possible cones are there if flavors can be repeated and the order of the scoops doesn't matter?

36 Three couples go to the movies and sit together in a row of seats. How many ways can they arrange themselves if each couple sits together?

37 Eight people are at Chen's house for a party (Chen is one of the eight). Chen has several options for sitting people for dinner.

a. If Chen gets a long table she can sit people on one side of the table so that they can watch a movie while they're eating. How many ways can she sit the eight people in this situation?

b. Chen can get a shorter, but rectangular table, and sit four people on one side and the other four on the opposite side. How many ways can she sit the eight in this situation? Assume that two arrangements that are reflections of each other over the long axis of the table are indistinguishable from each other.

c. Chen can sit the eight around a circular table. How many ways, now? Assume that two arrangements that are rotations of each other are indistinguishable from each other.

d. If Chen sits n people around a circular table, then how many ways can this be done if two arrangements that are rotations of each other are indistinguishable?

38 A bug jumps from lattice point to lattice point on a piece of graph paper according to the following pattern: from (a, b), the bug can jump only to $(a+1, b)$ or to $(a, b+1)$. How many different paths can the bug take if it wishes to start at $(0, 0)$ and end up at $(6, 4)$? Hint: tinker with this problem a bit so that you can get a feel for what the bug is doing.

39 Homer starts at the origin and performs a five-step random walk along a number line. Each second he steps either one unit to the left or one unit to the right.

a. Describe all the places that Homer can wind up at the end of this walk.

b. How likely is Homer to end up 5 steps to the right? 3 steps to the right?

c. If Homer were to perform this 5-step random walk 32000 times, what is your prediction of the average of all the final positions? What is your prediction of the average of all the distances from the origin to Homer's final position?

40 How many diagonals are there in a pentagon? a hexagon? a decagon? an n-gon? A **diagonal** is a straight line segment that connects any two non-consecutive vertices.

41 The rules of a checkers tournament indicate that each contestant must play every other contestant exactly once. How many games will be played if there are 12 participants? How many will be played if there are n participants?

42 Don't use a calculator for this problem.

a. Reduce the fraction $\frac{6a^2}{4a^2-6a}$.

b. Solve for x: $x^2 - 5x = 14$.

c. Solve for x: $x^2 - x - 1 = 0$.

d. Divide $\frac{2}{x} \div \frac{4}{x^2}$.

e. Write as one fraction: $\left(\frac{b}{2a}\right)^2 - \frac{c}{a}$.

43 How many ways can you split up 12 people into six pairs?

44 You have learned that any positive integer has a prime factorization of the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f \ldots$, where the "..." represents the rest of the primes, and a, b, c, d, e, f, and so on can each equal 0, 1, 2, 3, ... So for example, 360 is equal to $2^3 \cdot 3^2 \cdot 5^1$.

Remembering that every possible factor of 360 comes from one of the possible combinations of these prime factors (e.g. $20 = 2^2 \cdot 3^0 \cdot 5^1$ and $90 = 2^1 \cdot 3^2 \cdot 5^1$), and using the principles of counting that you learned studying combinatorics, determine the number of factors that 360 has. Note that 1 is considered a factor of 360 in this problem, i.e. $1 = 2^0 \cdot 3^0 \cdot 5^0$. (You might try answering for 84 instead of 360 if you want an easier example to start with.)

45 Generalizing Problem 44, find a way to determine the number of factors that any positive integer has. Try out your method with at least 3 other positive integers to check to see if it works.

46 Let $\binom{n}{r}$ stand for $\frac{n!}{r!(n-r)!}$, where n and r are nonnegative integers, and $r \leq n$.

a. Calculate $\binom{6}{2}$.

b. Calculate $\binom{7}{1}$.

c. Is $\binom{n}{r}$ ever not an integer? Explain.

d. Prove: $\frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 2\cdot 1} = \binom{n}{r}$.

e. Determine what should replace the question mark so that the following statement is true, then prove that the statement is true:
$\binom{n+1}{r} = \binom{?}{r-1} + \binom{?}{r}$

Exploring in Depth

47 How many different 5-person basketball teams can be chosen from a group of 10 people? How many ways can you split up 10 people into two different 5-person basketball teams?

48 The points A, B, C, D, E, and F are vertices of a regular hexagon. What is the probability that three randomly chosen vertices from this hexagon will form an equilateral triangle? What's the probability that four randomly chosen vertices from this hexagon will form a rectangle?

49 This is a continuation of Problem 38. In this problem you will see a connection between algebra and counting.

a. Expand $(r+u)^5$ and explain how this can help you determine how many paths there are from $(0,0)$ to $(2,3)$. Recall that when simplifying and collecting expressions such as rru, rur, and urr, we can write the final result as $3r^2u$.

b. What expression can you expand and simplify in order to answer Problem 38?

c. What is the coefficient of the x^3y^5 term in the expansion of $(x+y)^8$?

d. What is the coefficient of the x^5y^7 term in the expansion of $(2x+y)^{12}$?

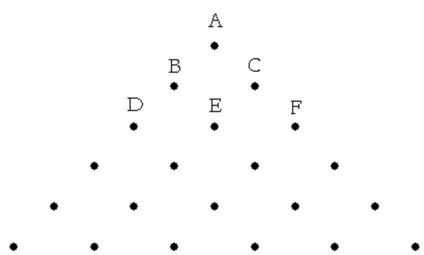
50 What is the sum of all the 3-digit numbers that can be formed using three different odd digits?

51 *Box Problems* You probably didn't realize that there are combinatorial problems that involve boxes, but there are.

a. Four boxes are numbered 1 through 4. How many ways are there to put 8 identical balls into these boxes (some of the boxes can be empty)? Hint: Imagine that you have 12 slots to work with.

b. Four boxes are numbered 1 through 4. How many ways are there to put 8 identical balls into these boxes so that none of them is empty?

52 Below is shown a triangle of dots. Suppose you start at dot A and move down through the triangle in the following manner: from any dot you can only move to the next dot that is down left or down right. So, if you are at dot A you can move to either dots B or C, but not directly to D or E. Below each dot in the triangle write the number of distinct paths from dot A to that dot.



53 There is a well-known triangle called Pascal's Triangle. What is this triangle and how does it relate to counting?

Park School Mathematics

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