REASONING AND PROVING

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

1 In a well-shuffled 52-card deck, half the cards are red and half are black. If the number of red cards in the top half of the deck is added to the number of black cards in the bottom half of the deck, the sum is 30. How many red cards are in the top half? *(Copyright mathleague.com.)*

Many difficult problems that you come across can seem fairly straightforward, as frequently you have a good sense of where to start and what steps to take, even if each step is quite challenging. For some problems, though, the difficulty lies in not knowing at all what to do. In these cases it initially seems impossible to figure out how to get started.

To **Tinker** means to try possibilities and adjust your strategy based on what you've learned from your efforts. Tinkering is like trying to fit puzzle pieces together — you need to test out the pieces to see if they'll fit, but even if it doesn't work you now have a better sense of the shape of each piece.

The important thing is to *do something*, even if you aren't at all sure it will be helpful. Make a start. Don't be paralyzed by the problem. When you work on a math problem, as you tinker, you are gathering information — information that might lead to a solution or that might lead to a more interesting question.

Trying a specific example, even when the problem does not ask for one, is an example of **tinkering**. In the above problem, for instance, try just guessing the number of red cards that are in the top half of the deck, and see what you can figure out from there. Even though your first guess is likely to be wrong, that's perfectly OK. If you pay attention to the numbers you get for the amount of red and black cards in each half of the deck, you can adjust and make a much more educated guess the next time. That is, sometimes you will not notice a pattern until you generate some data for yourself.

tinker

Tinkering is a way of making abstract ideas and hard to approach problems concrete, so that you can really see what is going on.

The next few pages have a couple of problems for you to **Tinker** with. After you have tried a few different approaches, you can turn over the page and see if the solution provided was similar to yours, or if perhaps you came up with a better way!

Let's start with a bit of a mystery:

2 Each letter in the addition problem below represents a different number.

$$\begin{array}{r} AB \\ +\ \ BC \\ \hline ADD \end{array}$$

What are the only possible values of A, B, C, and D? Try solving it before you turn over the page.

After perhaps tinkering with a number of different random values for A, B, C, and D, just to see what seems to be working and what isn't, you may have noticed that there is only one plausible value for A. What is it?

So now our problem has turned into:

$$\begin{array}{r} 1B \\ +\quad BC \\ \hline 1DD \end{array}$$

Could B be 7 or lower?

If you conclude that B is greater than 7, then it must be 8 or 9. So tinker a bit here. Replace B by 9 and see what happens.

The problem would then be:

$$\begin{array}{r} 19 \\ +\quad 9C \\ \hline 1DD \end{array}$$

Now what could D be? How about C? (Remember we were told all the letters represented *different* digits.)

At this point you might have seen that B must be 8. Now find C and D, and be sure to check that your solution works.

3

Here's a logic puzzle that might drive you a bit crazy at first, but you'll find the solution quite satisfying:

Al says: Bob is lying.
Bob says: Carl is lying.
Carl says: Al and Bob are both lying.

Who is lying? Who is telling the truth? There is only one possible answer.

If this seems impossible to solve, try just guessing right from the start who is telling the truth and who is lying, and see if your answer is consistent with what all three people are saying. If your initial answer doesn't work, then adjust your guess and try again.

The answer is on the back of this page.

There is no question that this problem can tie you up in logical knots! Let's try to approach it by assuming Al is telling the truth, and we'll see where that gets us. If this assumption works, great! And if it doesn't, hopefully we will have learned some valuable information about the problem.

If Al is telling the truth, that means that Bob is lying.

If Bob is lying, that means that he is lying about Carl lying, which means that Carl is telling the truth.

Now if Carl is telling the truth, that would mean that Al and Bob would both have to be lying. But we assumed at the start that Al was telling the truth! Al can't both be telling the truth and lying. Our assumption that Al was telling the truth causes us to contradict ourselves, so it must be wrong. Therefore, since Al can't be telling the truth, *Al must be lying.*

But does the problem really work out if Al is lying? Test it out and see.

So, who is lying, and who is telling the truth?

In case you were wondering if you could have started solving this problem by assuming, say, that Carl was telling the truth, and following the consequences from there, the answer is yes, absolutely. There are many ways to solve this problem, just as with many other math problems. The key is that all these different methods will eventually end up with the same answer.

The next few pages have a variety of "Tinker" problems for you to enjoy. Have fun!

4 Fill in the numbers 1,2,3,4,5 once each, to make the equation true:
$? \times ? - ? \times ? + ? = -13.$

5 If someone multiplied the first 211 primes together what would be the units digit of the product?

6 Add two numbers to the following set of numbers to make the mean 15 and the median 8.

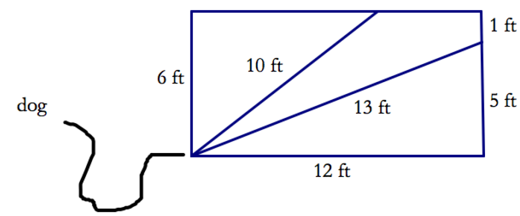
$\{3, 4, 17, 8, 39, 10, 2, 3, 71, 14\}$

7 The 3 missing numbers (all possibly different) in the equation below add up to what?

?45 – 3?7 = 55?

8 If one stacked 73 dice on top of each other on a table, what would be the sum of the numbers that are covered if the top die shows a 3?

9 A dog's leash is tied to a corner of a building. There are two tunnels going through the building that the dog can walk through. The lengths of each segment and tunnel are as marked. How long must the dog's leash be for him to be able to reach every part of the wall of the building?



10 In the film Die Hard With A Vengeance, the characters John McClane and Zeus Carver open a briefcase only to discover that in doing so they have armed a powerful bomb. It will explode in a matter of minutes unless they can disarm it. Inside the briefcase there is a scale. They have at their disposal two jugs — one holds exactly 5 liters and the other holds exactly 3 liters. To disarm the bomb, they have to fill the 5 liter jug with exactly four liters of water and place it on the scale. A few grams too much or too little will detonate the bomb. The water can be obtained from a nearby fountain.

How can they disarm the bomb?

11 Make your own set of numbers that has:

a. a mean of 35, a mode of 30 and a median of 50.

b. a mean of 40, a median of 40 and a mode of 30.

12 On the portion of an 8x8 chessboard shown below, the X marks a knight. Knights can move in an "L" shape — 2 squares up or down and 1 left or right, or 2 left or right and 1 up or down. On the board below, the Y's mark where the knight can move.

		Y		Y			
	Y				Y		
			X				
	Y				Y		
		Y		Y			

a. If you can move as many times as you like, can the knight get to any square on the board?

b. What if the knight moves 3 squares in one direction and 1 in another (instead of 2 and 1)?

c. What about 1 and 1? Or 3 and 2?

d. What trends do you notice — under what conditions can the knight get to any square, and when can't it?

13 Andie never lies on Monday, Tuesday, Wednesday and Thursday. Leah always tells the truth on Monday, Friday, Saturday and Sunday. On the rest of the days, they may tell the truth, or they may lie. Both say they lied yesterday. What day is today?

14 You have 46 feet of fence, and you fence in a 120 ft^2 rectangular garden. What are the dimensions of your garden?

15 From a pile of a large number of pennies, nickels, and dimes, select 21 coins which have a total value of exactly $1.00. In your selection you must also use at least one coin of each type. There are two answers to this problem — find both.

16 Peter said: "The day before yesterday I was 10, but I will turn 13 next year." Is this possible? Explain why or why not. *(Math Circles)*

17 THIS + IS = HARD; what must T stand for, if each letter stands for a different digit?

18 What is the units digit in 3^{101}?

19 Arrange the digits 1, 1, 2, 2, 3, 3 as a single six-digit number in which the 1's are separated by one digit, the 2's by two digits, and the 3's by three digits.

20 Alysha drives from Larchville to Oistin's Bay at 30 m.p.h. At what speed should she return so that her average speed (total distance traveled / total time taken) for the trip would be 60 m.p.h.?

21 Let $a + b + c + d + e = a \cdot b \cdot c \cdot d \cdot e$, where a, b, c, d and e are positive integers. Determine how many ordered quintuplets satisfy the equation.

22 What is the greatest number of points of intersection that can occur when 2 different circles and 2 different straight lines are drawn on the same piece of paper?
(from *Math Olympiads*, George Lenchner)

23 What is the smallest positive integer that is not prime that is also not divisible by any of the numbers up to 3? Up to 4? Up to 5? Up to 100?

1 The number 73 can be written as the sum of 73 consecutive integers. What is the product of these 73 integers? *(Copyright mathleague.com.)*

If you were reading a book about juggling for beginners, and it recommended starting out by trying to juggle 7 balls, you would rightly think that advice was crazy. Even someone who had no idea how to juggle could tell you that starting out with 3 (or even 2!) balls lets you get a better handle on what is going on. Before one tackles complicated problems, it makes sense to try easier, related ones first.

In mathematics, the way we cut a problem that is intimidatingly large or confusingly abstract down to size is to **Change or Simplify the Problem**. In the problem above, having to deal with 73 integers makes the problem initially seem extremely hard. How would one even approach finding the 73 consecutive integers that add to 73, and how on earth if you found them could you find their product in a reasonable amount of time?

Trying a Smaller Number is a very popular and useful way to change or simplify the problem. As with the juggling, it frequently allows you to get a much better understanding of what is going on. So in the problem above, why not see if you can simplify it by replacing all the 73's with 5's. Then, hopefully you can use the insight you have gained with the easier problem to solve the original problem as well. Give it a try!

Another variation on this theme occurs when a problem is stated in terms of variables like X or Z, such as:

2 At Bob's Breakfast Bungalow, the toast buffet involves X kinds of bread and Z kinds of jam. How many different bread-jam pairings are there, assuming that the customers use only one kind of jam each time they make toast?

When stated so abstractly ("X kinds of bread and Z kind of jam"), the problem can be hard to think about. **Changing Variables to Numbers** is another way to change or simplify the problem. Why not try specific numbers for X and Z to get a better sense of what is going on? It might make sense to try a few different pairs of numbers so that you can be sure that you have the right idea. Once you do, then, as with trying a smaller number, you can use what you have learned to go back and solve the original, more abstract (and likely more complicated) problem.

3 You add up X randomly chosen positive numbers. How many times larger is this sum than the average (mean) of those numbers? (Your answer may include an X in it.)

4 Some students stood evenly spaced in a circular formation. They counted off, starting at 1 and continuing by consecutive integers once around the circle, clockwise. How many students were there in the circle if the student furthest from student 19 was student 83? *(Copyright mathleague.com.)*

5 Giuseppe likes to count on the fingers of his left hand, but in a peculiar way. He starts by calling the thumb 1, the first finger 2, the middle finger 3, the ring finger 4, and the pinkie 5, and then he reverses direction, so the ring finger is 6, the middle finger is 7, the first finger is 8, the thumb is 9, and then he reverses again so that the first finger is 10, the middle finger is 11, and so on.

One day his parents surprise him by saying that if he can tell them some time that day what finger the number 1,234,567 would be, he can have a new sports car. Giuseppe can only count so fast, so what should he do?

6 The following two problems are related.

a. If Wenceslaus wrote a list of Z consecutive odd integers, by how much would the greatest number on his list exceed the smallest? (Your final answer may have a Z in it.)

b. The sum of X consecutive odd integers is A. The sum of the next X consecutive odd integers is B. What does $B - A$ equal? (Your final answer will include an X in it.)

7 How many squares of any size does an 8 by 8 checkerboard have?

8 A blue train leaves the station at Happyville going East X miles per hour. A red train leaves the station at Nervoustown, $2(X+Y)$ miles away, at Y miles per hour and headed West towards Happyville. (Answers may include X and Y.)

a. How long will it take the trains to crash?

b. How far away will the red train be from Happyville when they crash? (Your answer may include an X and/or a Y in it)

c. How far apart will the trains be one hour before they crash?

change or simplify the problem

9 Yolanda tells Miguel that she can guess any integer he thinks of from 1 to 10 million in 25 yes-or-no questions or less. Miguel says that is ridiculous!

Who is right, and why?

10 Let $10^Z - 1$ be written fully out as a number (and thus with no exponents). Find the sum of the digits of this number. (You will have a Z in your answer.)

11 What's the minimum number of non-overlapping triangles into which you can divide a 2008-sided polygon? (Each triangle side must be a segment connecting two of the vertices of the polygon.)

12 Find two consecutive positive integers where the difference of their squares equals 3747.

13 Josephine writes out the numbers 1, 2, 3, and 4 in a circle. Starting at 1, she crosses out every second integer until just one number remains: 2 goes first, then 4, leaving 1 and 3. As she continues around the circle, 3 goes next, leaving 1 as the last number left.

Suppose Josephine writes out the numbers $1, 2, 3, 4, \ldots, n$ in a circle.

For what values of n will the number 1 be the last number left?

14 In the ordered sequence of positive integers: $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$, each positive integer n occurs in a block of n terms.

How many terms of this sequence are needed so that the sum of the reciprocals of the terms equals 1000?
(Copyright mathleague.com.)

HABITS

LESSON 1: DEFINING NEW SYMBOLS

Introduction

You have a pocket calculator that can only do one thing — when you type in two whole numbers, it takes the first number, adds the second number to it, adds the first number to the sum, then takes that whole answer and multiplies it by the second number. The number you see on the screen is its final answer after this series of steps.

1 If you type in 7 and 2, what will your calculator show?

2 What if you type in 2 then 7?

3 Your friend uses the calculator. You can't see the first number she types in, but the second number is 3. The answer that the calculator gives is 39. What was the first number?

Development

The calculator's rule (from the problems above) has a symbol to represent it: "$\triangle$". For example, "$7\triangle 2$" refers to what you did in question #1.

4 What is $2\triangle 10$?

5 What is $5\triangle y$, in terms of y? Simplify as much as possible (no parentheses in your answer).

6 What is $x\triangle 8$?

7 What is $x\triangle y$, in terms of x and y? Write this as an equation.

If you invented the symbol "$\triangle$" for a math problem you wrote, and wanted to explain it in an equation rather than in words, you would say:

"Let $x \triangle y$ ______________________" (your answer to question 7).

This is called the algebraic rule for $\triangle$.

Here the word "Let" is used the same way as when a problem says "Let x be the number of boxes you purchase."

8 After trying out a few different whole numbers as inputs to the $\triangle$ rule, John makes the following claim: "To get an odd number for your answer from $\triangle$, you need to input odd numbers for a and b."

a. Does John's claim seem reasonable to you? If it doesn't, find a **counterexample** — a specific example which proves that his statement is not always true.

b. Is John's claim true?

These two questions are quite different — in many situations, there can be a variety of different predictions that seem reasonable, but only one of them may be actually true! The habit of seeking proof is not only about learning how to prove a statement to be true — it's also about learning to ask "Why would that be true?" when you are presented with a reasonable statement.

9 The symbol "&", applied to two whole numbers, means that you take the first number and then add the product of the numbers.

a. Find 4&7.

b. Find 7&4.

c. Write an algebraic rule for &.

d. Here is another of John's claims. "To get an odd number for your answer from a&b, you need to input an odd number for a and an even number for b." Is his new claim true? If you think it is true, carefully explain why. If you believe it's not true, give a counterexample.

Practice

10 Just like any symbol regularly used in mathematics ($+, -, \cdot, \div$), symbols that we create can be used inside an equation. Let $m \circledast n = 3n - m$. Calculate the following:

a. $5 \circledast 2$

b. $6 \cdot (5 \circledast 2)$

c. $(1 \circledast 3.5) - (3.5 \circledast 1)$

d. $2 \circledast (3 \circledast 5)$

e. $(2 \circledast 3) \circledast 5$

f. $(a \circledast b) \circledast c$

11 In problem 10, is $m \circledast n = n \circledast m$ true in general, for ANY input numbers m and n? Explain.

Problems

12 When you give © two numbers, it gives you the third number in the addition/ subtraction pattern. For example, $©\{21, 23\} = 25$ (going up by 2's), and $©\{95, 90\} = 85$ (going down by 5's).

a. Find $©\{6, 11\}$ and $©\{11, 6\}$.

b. Find $©\{11, ©\{6, 11\}\}$.

c. Write an equation for $©\{m, n\}$.

13 When you give \$ two numbers, it gives you the third number in the multiplication/ division pattern. For example, $\$\{3, 15\} = 75$, and $\$\{48, 24\} = 12$.

a. Find $\$\{2, 3\}$, $\$\{12, 13\}$, and $\$\{102, 103\}$.

b. What is $\$\{2009, 1\}$?

c. If $\$\{x, 2\} = \$\{3, 1\}$, then what is x?

14 The symbol ¤ takes a single number, squares it, and then subtracts 4.

a. What is ¤(6)?

b. Can you ever get a negative answer for ¤(x)? Why or why not?

c. Find an x so that ¤(x) is divisible by 5.

15 Let the symbol ¥ mean: Add up the two numbers, then take that answer and subtract it from the product of the two numbers. What is 5¥8? 8¥5?

16 Look back at the problem above. Do you think the same thing would happen for any pair of numbers, if you used the same symbol? Explain your answer.

When switching the order of the input numbers never has an effect on the answer, the symbol you are working with is said to have the **commutative property**. For example, the symbol ¥ (from questions 15 and 16 above) had the commutative property, but the symbol © (from question 12) did not.

One important thing to note is that there's no such thing as "sometimes" having the commutative property. For example, ¥ has the commutative property because a¥b and b¥a are equal for ANY input numbers a and b, not just because it worked for 5 and 8.

Proving a statement false is as easy as finding one counterexample, but it is sometimes difficult to prove that a statement that appears to be true is indeed true. In the following problems (17-21), you will need to decide whether statements are true or false, and to also clearly support your position.

17 Fergie claims that each of the following symbols has the commutative property. Examine each of his claims.

a. $x \Downarrow y$ means add 1 to y, multiply that answer by x, and then subtract x.

b. Take two whole numbers x and y. To do $x\%y$, you divide x by 2, round down if it's not a whole number, and then multiply by y.

c. To calculate $x£y$, imagine that you walk x miles east and then y miles northeast. $x£y$ is how far away you end up from your starting point.

d. ¿ works by adding up the two numbers, multiplying that by the first number, and then adding the square of the second number.

e. ¢ takes two numbers. You reverse the first number (for instance, 513 becomes 315); one-digit numbers stay the same), then add the reversed number to the second number, and finally add up the digits of your answer.

18 Let $x \bigstar y = x^2 - y^2$. For example, $4 \bigstar 3 = 16 - 9 = 7$. True or false: $x \bigstar y$ always equals the sum of the two numbers — for example, $4 \bigstar 3 = 7$ which equals $4 + 3$. If it's true, justify your claim. If it's false, try to find out what kinds of numbers do make the claim work.

19 When you give $\heartsuit$ two numbers, it finds the sum of the two numbers, then multiplies the result by the first number. Finally it subtracts the square of the first number. Flinch claims that $\heartsuit$ is commutative. Is he correct?

20 Here's how you might prove Flinch's claim in problem 19 for *any* two starting numbers.

a. Let's call your first number m and your second number n. Write down and simplify as much as you can the expression for $m \heartsuit n$.

b. Write down and simplify as much as you can the expression for $n \heartsuit m$.

You should be able to convince anyone, using your work in problem 20 that Flinch's statement is always true, no matter which numbers we start with.

21 Is the rule below commutative?

The rule $\sim$ adds up the two numbers, doubles the answer, multiplies the answer by the first number, then adds the square of the second number and subtracts the square of the first number.

22 Let $x \clubsuit y = \frac{1}{1{,}000{,}000{,}000}\ x^y$.

a. Is there a value of y such that $10 \clubsuit y > 1$?

b. Is there a value of y such that $1.001 \clubsuit y > 1$?

23 The symbol "&", applied to *any* two numbers (not only whole numbers), means that you take the first number and then add the product of the numbers.

a. When you calculate $x\&y$, you get 120. What could x and y be? Give several different answers.

b. When you calculate $x\&y$, you get 120. Write an equation that expresses this fact, then solve for y in terms of x (meaning, write an equation $y =$.... with only x's in the equation).

24 The command "CircleArea" is a rule that finds the area of the circle with the given radius. For example, $\text{CircleArea}(3) = 9\pi$.

a. Find CircleArea(4).

b. Write the equation for CircleArea(x).

c. Can you find CircleArea(-4)? Why or why not?

25 It's January 1st, and you are counting the days until your birthday. Let m be the month (as a number between 1 and 12) and d the day (between 1 and 31) of your birthday, and pretend that there are exactly 31 days in each month of the year.

a. How many days are there until January 25? Until April 10?

b. For January 25th, $m = 1$ and $d = 25$, and for April 10th, $m = 4$ and $d = 10$. By looking at what you did in part a, explain how you can use the numbers m and d to count the days from January first to until any day of the year.

c. Let the symbol ♥ represent this count. Write an algebraic rule for calculating m♥d. Test your rule with an example.

26 Now, count how many days are from your birthday to New Year's Eve (December 31st). Represent this with the symbol ♈. Again, pretend there are 31 days in each month.

Write an equation for in terms of m and d, and use an example to show that your equation works. (You might want to try explaining it in words or in an example first.)

27 "max(a,b)" takes any two numbers and gives you the larger of the two. The symbol ∂ is defined by $\partial(a, b) =$ max(a,b) – min(a,b). Is ∂ commutative?

28 We say that the counting numbers (i.e., 1, 2, 3, ...) are "closed under addition" because any time you add two counting numbers, you get another counting number. Decide whether or not the counting numbers are closed under each of the following operations. In each case where the answer is no, try to find a group of numbers that *is* closed under that operation.

a. * (multiplication)

b. - (subtraction)

c. / (division)

d. The symbol ∂, from problem 27

29 The rule $\leftarrow$ adds twelve to a number and divides the sum by four. What number x can you input into $\leftarrow(x)$ to get an answer of 5? An answer of -3?

30 To do the rule $\forall$, add 5 to the first number and add 1 to the second number, then multiply those two answers.

a. What's $3\forall1$?

b. What's $x\forall y$?

c. Your friend tells you that she needs to find numbers x and y so that $x\forall y$ gets her an answer of A— a whole number that she does not reveal. In terms of A, tell her what to plug in for x and y. Make sure that your strategy would always get her the answer she wants.

d. What values of x and y give you an odd answer? Prove that your description is true and complete. (Make sure you know what it means to prove it's complete!)

e. What values of x and y would give you an answer of zero?

31 Create 3 different rules that give an answer of 21 when you plug in 2 and 8.

32 Let f be a rule that acts on a single number.

a. Create a rule for f so that $f(2) = 12$ and $f(11) = 75$.

b. Now, create a new rule g such that $g(5) = 26$ and $g(10) = 46$.

33 Let $x \triangle y = x^2 + 2xy$. ($x$ and y have to be integers.)

a. If you plug in 6 for x, find a number you could plug in for y to get an answer of zero.

b. Using part a as an example, describe a general strategy for choosing x and y to get an answer of zero, without making x zero. Explain why your strategy works.

c. Suppose you use the same number for x and y— call this number N. (So, you're doing $N \triangle N$.) What is your answer, in terms of N? Simplify as much as possible.

d. Suppose $x \triangle y = 20$. Solve for y in terms of x.

e. Describe a strategy to get any odd number that you want. Give an example, and also show why your strategy will always work (either give a thorough explanation, or use algebra to prove that it works).

34 Tinker to find a rule for $x \# y$ that gets the following answer: $5 \# 1 = 24$. Then try to write a rule that gives $5 \# 1 = 24$ and $4 \# 2 = 14$.

35 Create a rule α that works with two numbers, so that $\alpha(1,1) = 5$ and $\alpha(2,3) = 10$. Give your answer as an equation in terms of a and b.
$\alpha(a,b) =$ __________________________.

36 Let $\otimes\{a,b\}$ be the two digit number where the tens digit equals a and the units digit equals b. For example, $\otimes\{9,3\} = 93$.

a. In terms of f, what do you get when you do $\otimes\{5,f\}$? Write your answer as an equation. (Remember, writing $5f$ doesn't work, because it means $5 \cdot f$).

b. What do you get when you do $\otimes\{a,4\}$?

c. When you calculate $\otimes\{a,4\} - \otimes\{4,a\}$, what do you get in terms of a? Simplify as much as possible.

d. $\otimes\{6,x\} - \otimes\{x,3\} = 12$. Find x. Show your work algebraically.

37 Now, let's redefine $\otimes\{a,b\}$. Let $\otimes\{a,b\}$ be the three-digit number where the hundreds and units digits are a, and the tens digit is b.

a. In terms of b, what is $\otimes\{b,2\}$? (Again, $b2b$ won't work).

b. In terms of a and b, what do you get when you calculate $\otimes\{a,b\} - \otimes\{b,a\}$? Simplify as much as possible and check your answer with an example.

38 For any two positive numbers a and b, let $a \perp b$ be the perimeter of the rectangle with length a and width b.

a. Is $\perp$ commutative?

b. Is $\perp$ associative? In other words, does $(a \perp b) \perp c = a \perp (b \perp c)$?

Exploring in Depth

39 A positive whole number is called a "staircase number" if the digits of the number go up from left to right. For example, 1389 works but not 1549.

The rule $x \nabla y$ works by gluing x and y together. For example, $63 \nabla 998$ gives an answer of 63998. If x and y are both staircase numbers, and x is bigger than y, is it always true that $x \nabla y$ is bigger than $y \nabla x$? If you think it's always true, explain why. If not, explain what kind of input would make it true.

40 Suppose that, for any two positive numbers a and b, $a \blacksquare b$ represents the area (ignoring units) of the rectangle with length a and width b.

Is $\blacksquare$ associative?

41 In problem 23, you were asked to find some pairs of numbers x and y so that $x \& y = 120$. (Recall that $x \& y$ takes the first number and adds the product of the numbers).

a. Do you think you could find numbers x and y to get any number you wanted? Try a few possibilities and explain what you find.

b. Develop a rule for choosing x and y to get the number that you want, which we will call N. (Hint: Try starting out by picking a number for x, and then figuring out what y would have to be. You might have to try different numbers for x to find a solid strategy.)

42 Don't use a calculator for this problem.

a. Find $\frac{3}{4} + \frac{5}{6}$

b. Find $2 \div \frac{1}{2}$

c. Simplify $3x - 2(x + 1)$

d. Write $.\overline{6}$ as a fraction.

e. Find $2\frac{1}{4} + 5\frac{7}{8}$

43 Let $x \Box y = xy - x - y$. Develop a strategy to pick x and y so that you can get any number you want.

44

For a number x, $f(x)$ subtracts twice x from 100.

a. What is $f(10)$?

b. What is $f(-4)$?

c. Write an expression for $f(x)$.

d. Write an expression for $f(3x)$.

45

Let Let $x \square y = xy - x - y$. Prove or disprove: if x and y are both larger than 2, then $x \square y$ gives a positive answer. Make sure your explanation is thorough.

46

Write an equation for a rule $a \lozenge b$, so that the answer is odd only when both a and b are even.

47

Let $a \;\%\; b = a^2b - b^2a$. What would have to be true about a and b for $a \;\%\; b$ to be positive?

48

Let $a ☺ b = ab + b^2$. What would have to be true about a and b for $a ☺ b$ to be negative?

49

Let $x \ominus y = xy + y/2$, where x and y are whole numbers.

a. Describe what values of x and y would get you a negative answer.

b. Describe what values of x and y would get you an even whole number answer. (Careful! Your answer won't be just in terms of odds and evens. You'll have to tinker to see what type of numbers work. Think through it step by step and explain your reasoning).

c. What could x and y be to get 21? Give at least 3 different options.

d. Find a strategy to get any whole number you want, called N. You can explain your strategy in algebraic terms ("To get an answer of N let x equal ... and let y equal ...") or in words. Show an example to demonstrate that your strategy works.

50

For a number x that might not be a whole number, $\Phi\,(x)$ represents the integral part of x— for example $\Phi\,(6.51)$ is 6 and $\Phi\,(10)$ is 10. Using the Φ notation, write a rule @ that gives an answer of 0 if x is a whole number, and gives a non-zero answer if x is not a whole number. For example @(20) should equal 0 and @(20.3) should give an answer not equal to zero.

51

The rule $\lfloor b \rfloor$ is called the "greatest integer function" — it outputs the largest integer that is not above b. For example, $\lfloor 9.21 \rfloor = 9$, $\lfloor 12 \rfloor = 12$, $\lfloor .92 \rfloor = 0$, and so on.

Write an equation for the rule $\{b\}$, which rounds b down to the hundreds — for example $\{302\} = 300$, $\{599\} = 500$, and $\{2\} = 0$. For your equation to calculate $\{b\}$, you can use any standard operations, and you should use the operation $\lfloor b \rfloor$.

52

To understand the rule behind the symbol $\diamondsuit$, you need to draw a picture (graph paper will help). To draw the picture for $5 \diamondsuit 3$, pick a point to start and then draw a line stretching 5 units to the right. From there, draw a line stretching 3 units up. Then draw a line stretching 3 units to the right, and then a line stretching 5 units up. Finally, draw a line back to your starting point.

a. Let $5 \diamondsuit 3$ be the area of the shape you just drew. Calculate this number exactly.

b. Pick 2 new numbers (the 1st number should still be bigger). Draw the picture and calculate the answer that $\diamondsuit$ would give for your numbers.

c. Draw a diagram to help you find an algebraic rule for $x \diamondsuit y$ (again, you can assume that x is a bigger number than y). Check your rule by making sure it would give you the right answer for $5 \diamondsuit 3$ and for your numbers in part b.

d. Will your rule still work if the two numbers are the same, such as $4 \diamondsuit 4$? Explain why it will or will not always work.

e. Will your rule still work if the first number is smaller, such as $3 \diamondsuit 7$? Explain why it will or will not always work.

LESSON 2: GRAPH THEORY

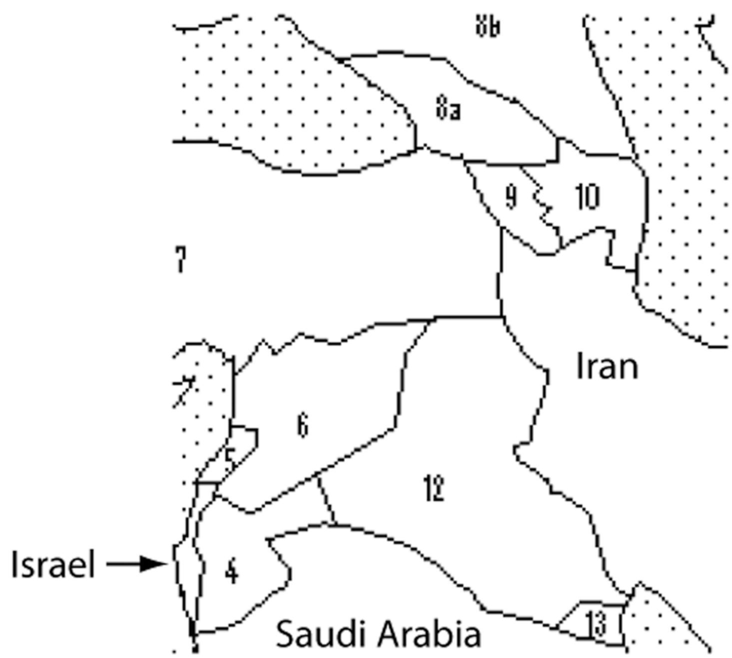
Introduction: Three Challenges

Felix will fly into Birmingham tomorrow to visit some family members. Today, he has his Alabama map spread out over a table in the local coffee shop to determine the route he will drive with his pay-per-mile rental car. He has a cousin in Birmingham, grandparents in Huntsville, an aunt in Cottondale, and a brother in Tuscaloosa. When Felix realized that he'll soon be served some of his grandma's chicken liver-sardine-cranberry casserole, he gagged on his cappuccino and accidentally spilled the whole cup on his map. The only usable part of the map remaining is the mileage chart between cities. Here is the mileage between the relevant cities:

	Birmingham	Huntsville	Cottondale	Tuscaloosa
Birmingham	X	102	50	56
Huntsville	102	X	148	143
Cottondale	50	148	X	9
Tuscaloosa	56	143	9	X

1 In what order should Felix visit his relatives to minimize the number of miles he drives with the rental car? Remember that he has to return to Birmingham to return the rental car.

Below is a map of part of the Middle East. You have different colors you can use to color in the different countries. If two countries are touching, then you should make them different colors so someone can easily tell they're different countries.



(For your reference and edification, the numbered countries are 4-Jordan, 5-Lebanon, 6-Syria, 7-Turkey, 8a-Georgia, 8b-Russia, 9-Armenia, 10-Azerbaijan, 12-Iraq, 13-Kuwait)

(image from http://catholic-resources.org/Courses/SCTR19-Spring2007-Worksheets.htm)

2 Color the map using as few colors as possible.
(If you don't have markers, just write the name of the color or use shading.)

Kim is figuring out the schedule for next year. It's tough, because out of thirteen electives, several students have expressed interest in taking more than one of them. She doesn't want to schedule two classes in the same block if there are students who want to take both of those classes.

In the grid below, an "x" indicates that at least one student has expressed an interest in taking both classes in that row and column of the grid.

	Poetry	Adv calc	Crim law	Foren.	Pract.	Stat	Prod.	Animal behavior	Civ. Lib.	Java	Shakespeare	Astro	Discrete
Poetry		x	x	x									
Adv Calc	x			x									
Crim Law	x			x	x	x							
Forensics	x	x	x		x						x		
Practicum			x	x		x	x	x			x		
Stat			x		x		x						
Production					x	x							
Animal					x				x	x	x		
Civ Lib								x		x		x	x
Java								x	x		x	x	
Shakespeare				x	x			x		x		x	
Astro									x	x	x		x
Discrete									x			x	

3

a. Assign each class a block (A through F) so that you never put two classes in the same block if there's anyone who wants to take both classes.

b. Now, suppose that all of these electives are to be scheduled on an ABC day. Can Kim create a successful, conflict-free schedule with only three blocks? If not, what is the fewest number of blocks that would accommodate these electives?

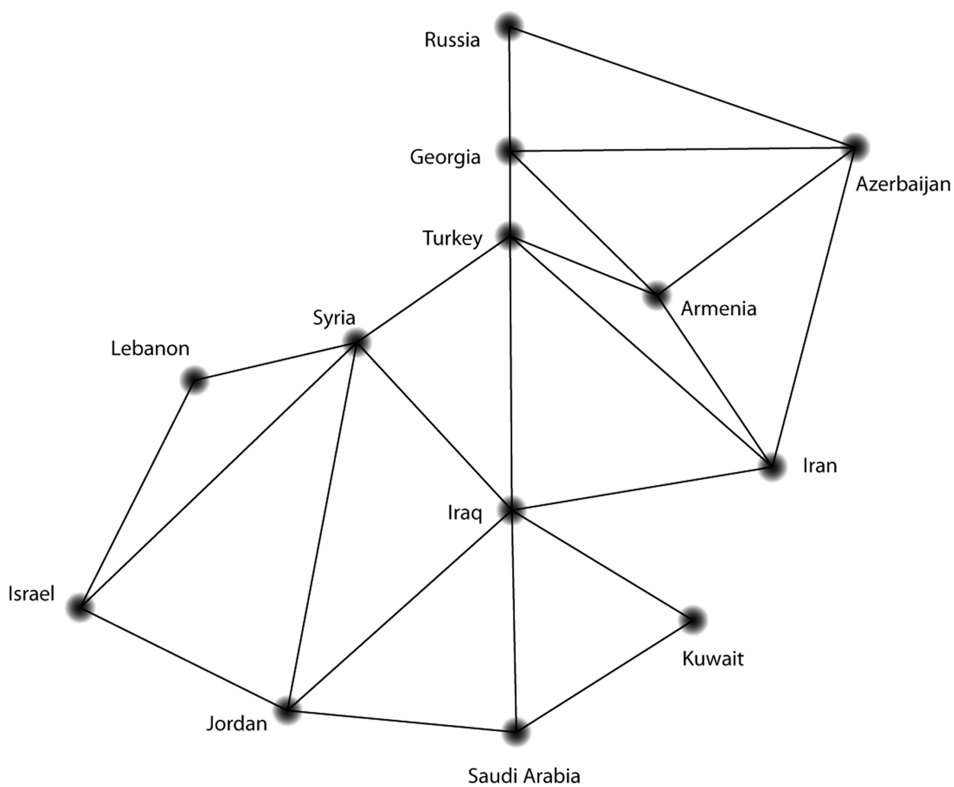
Development

To determine Felix's driving route, you probably recognized a way to represent the situation visually — whether you sketched out a map or drew a diagram.

The second and third problems, however, didn't have an obvious new representation to apply. Some people probably applied labels; others may have made lists. It turns out, though, if we use the method of the first problem with the second and third, we can learn much more from the data!

Most likely, the diagram you used in the first problem was actually a graph. A graph, in its most general sense, is a collection of dots (called "nodes") and lines connecting them (called "edges"). In your graph of Felix's problem, the nodes represented cities, and the edges related the nodes by showing the distances between.

We can represent the second situation with a graph, too, if we let each country be represented by a node. We can draw an edge between nodes when those countries border each other. Our graph comes out to look like this:

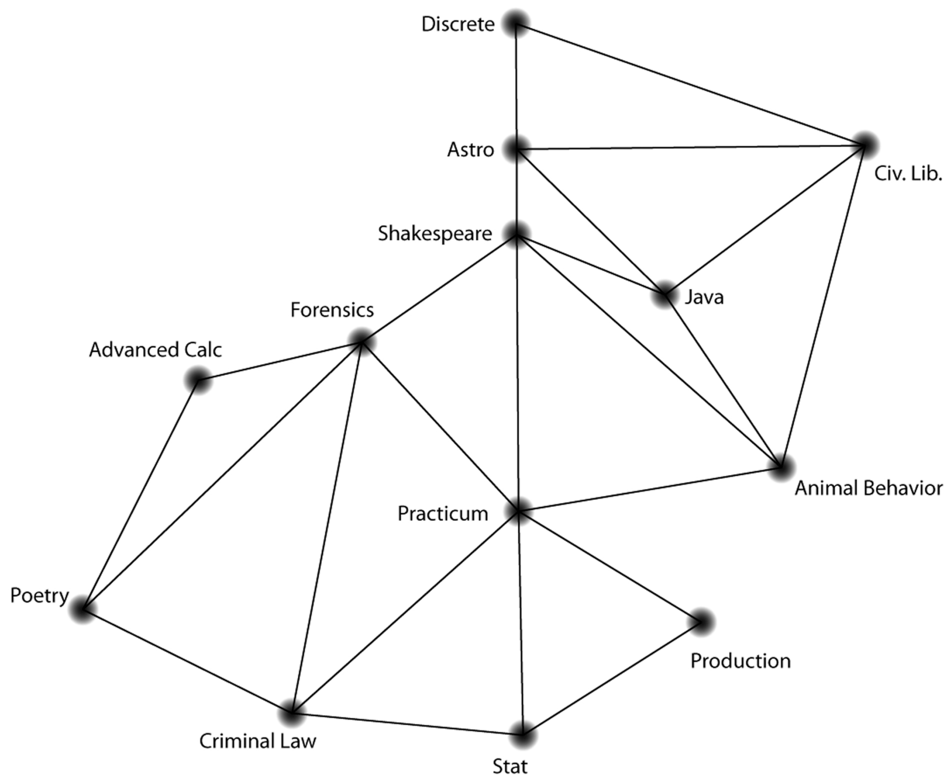


The problem now becomes to color each dot in such a way that any two dots connected by lines are always different colors.

4 What is the minimum number of colors you need now?

5 Did you think this version of the "coloring" problem was easier than in the introduction? Why or why not?

Similarly, we can represent the scheduling problem as a graph. The nodes can represent classes, and an edge can connect the nodes in the cases where some student wants to take both classes, as in the figure below.



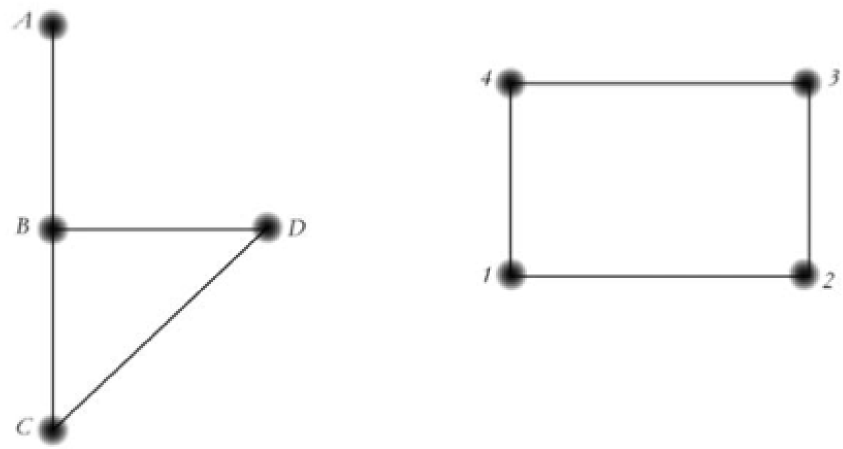
6 Solve the scheduling problem using this representation. What minimum number of blocks can you identify now?

7 Was this version of the schedule problem easier than in the introduction? Why or why not?

The tools you have been using so far in this lesson are the tools of **graph theory**. Already, graph theory has been über-useful in helping us represent relationships in a very streamlined way. What's more, we can now see that the map-coloring problem and the scheduling problem — initially very different — are actually remarkably similar.

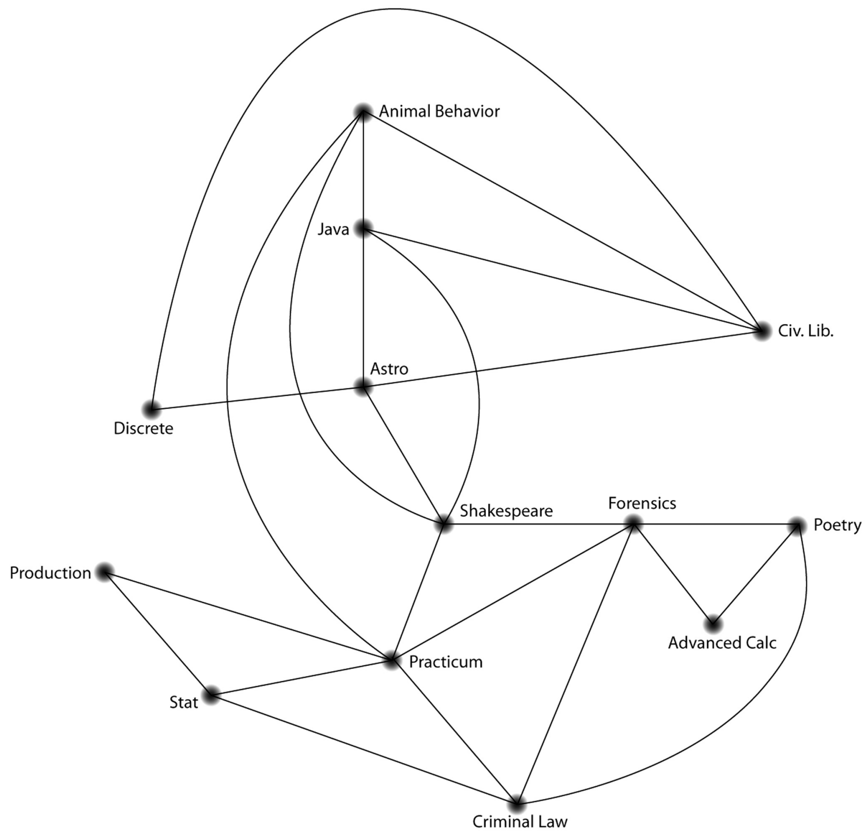
In fact, the identically structured graphs of the map-coloring problem and the scheduling problem are **isomorphic**. Two graphs are isomorphic when each node in one graph has a "partner" node in the other — a "partner" with the same connections. For example, Georgia in the first graph and Astronomy in the second correspond to one another. So do Lebanon and Advanced Calc. This correspondence comes from their identically structured connections, not their actual position on the page.

The following two graphs are not isomorphic.



An easy way to see that the graphs are not isomorphic is to see that node B has no "partner." In the left hand graph, B has three edges coming out of it. On the right, no node has three edges coming out of it, so B has no possible "partner." The number of edges connected to a node is a very important characteristic. The **degree** of a node (abbreviated deg(n)) is the number of edges coming out of it. So in the left-hand graph deg(A) = 1 and deg(B) = 3. What is deg(C)?

Isomorphisms can be trickier to spot than we've seen so far. For example, someone else with the task of drawing the scheduling graph might have come up with the following:



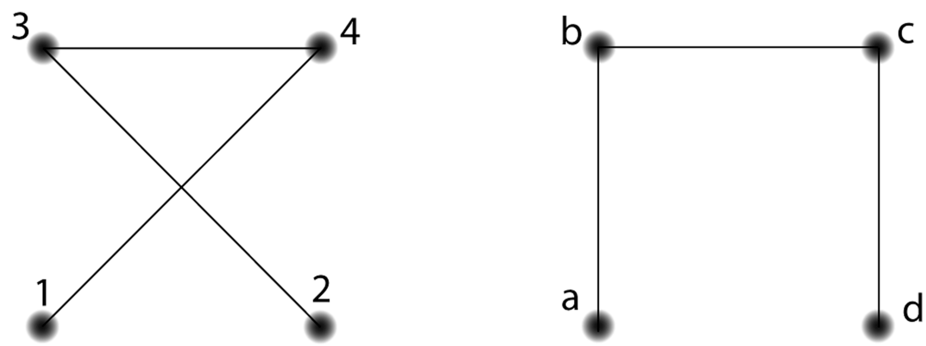
8 Check to see if this graph is still an accurate representation of the scheduling problem. Explain, in language a sixth-grader could understand, what you need to do to check if two graphs are isomorphic.

One final important concept that we will use is planarity. In the second version of the schedule graph, some edges crossed at places other than nodes. (For example, the edge connecting Animal Behavior to Shakespeare crosses the edge connecting Astro to Discrete.) In these cases, you can think of the edges passing under or over each other.

Graphs where you don't have to do this — where there is a way you can draw them without crossing the edges — are called **planar**.

9 Why do you think the word "planar" might be used?

Note that a graph might be drawn with the edges crossing, and yet be isomorphic to a graph that is clearly planar, as in the following two graphs (with pairing 1-d, 2-a, 3-b, 4-c):



In this case, we still say the first graph is planar because it can be drawn with no edges crossing and still represent the same information. Imagine picking up nodes 1 and 2, with their edges still attached, and "uncrossing" them to get something with the same shape as the second graph.

10 Draw a nonplanar graph.

11 Find some degrees of nodes in the Middle East graph.

a. What is deg(Saudi Arabia)?

b. What is deg(Syria)?

c. What country's node has the highest degree?

Practice

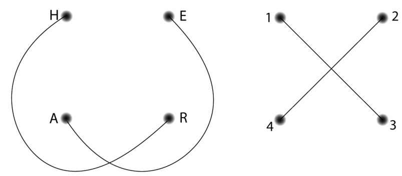
12 A shelter needs to find homes for nine cats. The table below indicates with an "x" which pairs of cats do not get along. Assuming that cats who do not get along cannot go with the same family, what is the minimum number of families needed to adopt all of these cats?

	A	B	C	D	E	F	G	H	I
A		x							
B	x		x		x				
C		x				x		x	
D					x		x	x	x
E		x		x		x	x		
F			x		x		x	x	
G				x	x	x		x	x
H			x	x		x	x		x
I				x			x	x	

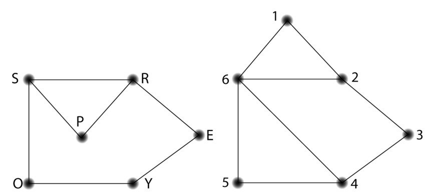
13 Which of these pairs of graphs are isomorphic? For graphs that are isomorphic, give the pairing of the nodes.

Now is a good time to think about visualizing the edges stretching and bending to see if you can make the shape on the left look like the shape on the right.

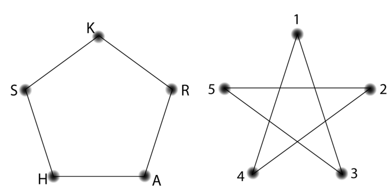
a.



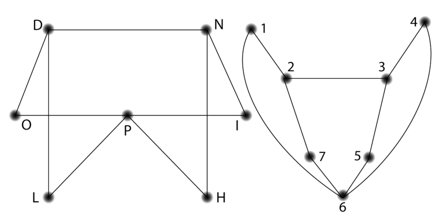
b.



c.



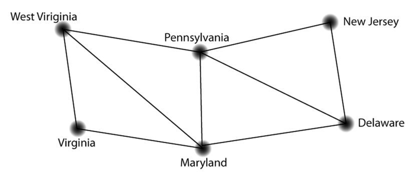
d.



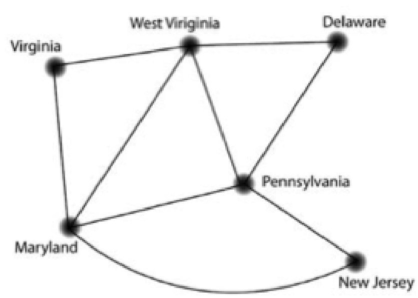
14 Below is a map of the region surrounding Maryland.
(image from https://anthropology.si.edu/writteninbone/comic/activity/Background_Tidewater_Chesapeake.htm)

Which of the next three graphs is an accurate representation of the borders shared by the states on the map?

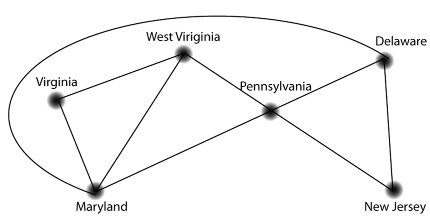
a.



b.



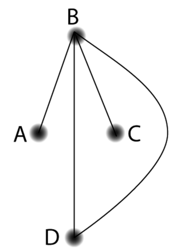
c.



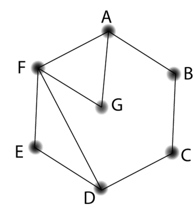
15

What's the degree of each node in each of these graphs?

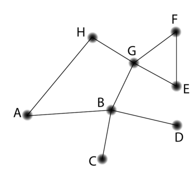
a.



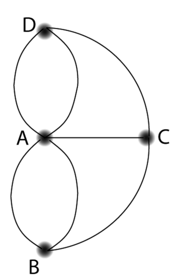
b.



c.

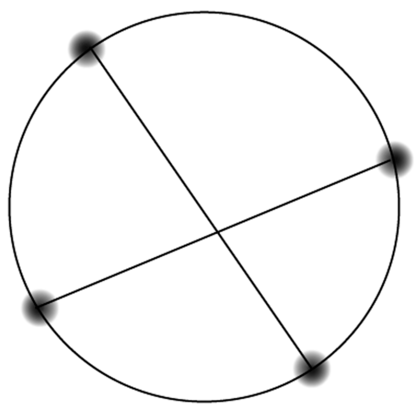


d.

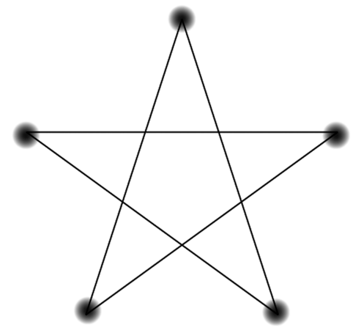


16 Which of these graphs are planar? For the ones that are planar, show how you can draw them with no edges crossing.

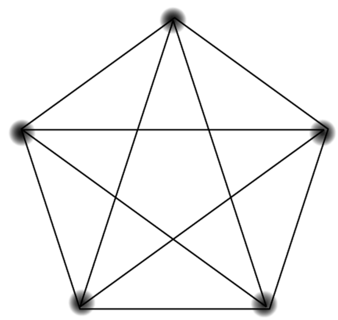
a.



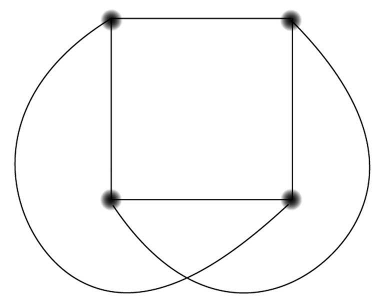
b.



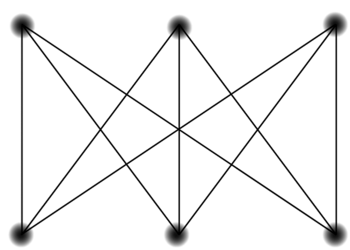
c.



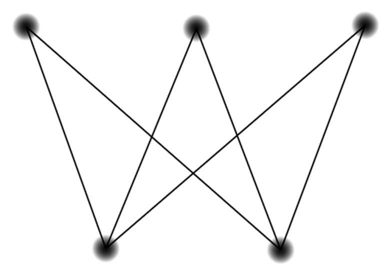
d.



e.



f.



17 A school has four exam periods. If one student is enrolled in two different courses, the exams for those courses need to be scheduled in different periods so that the student may take both exams. The following chart shows the pairs of courses having at least one student enrolled in both.

	Graphic Design	Philos-ophy	Architecture	Russian History	French	Statistics	Chemistry	20th Century Lit
Graphic Design		x		x		x		x
Philosophy	x			x		x		x
Architecture				x	x		x	
Russian History	x	x	x			x		x
French			x				x	
Statistics	x	x		x			x	x
Chemistry			x		x	x		
20th Century Lit	x	x		x		x		

a. Represent this situation as a graph.

b. Is it possible to create an exam schedule so that no student has to be in two places at once?

c. Find an exam schedule that uses four different exam periods.

Problems

18 (From Finkbeiner and Lindstrom) The Board of Directors of XYZ Corporation has 16 members and 6 committees of 5 persons each as indicated by the following chart.

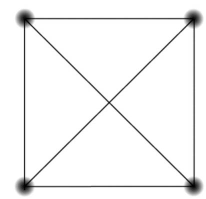
Committee	Members
1	A, D, G, K, N
2	A, B, G, L, M
3	H, J, L, P, R
4	C, E, F, M, Q
5	C, J, K, N, R
6	D, E, H, P, Q

How many distinct meeting times are needed to conduct the committee meetings of the board?

19 Draw a graph in which the **total degree** — that is, the sum of the degrees of each node — is odd.

20 The graph we used to represent bordering countries in problem 4 was planar. Create a map of your own countries for which the corresponding graph is nonplanar.

21 A graph is complete when every node is connected to every other node by an edge. Here's a complete graph with four nodes:



Draw:

a. A complete graph with five nodes.

b. A complete graph with six nodes.

c. A complete graph with three nodes.

22 How many edges will a complete graph with each of the following number of nodes have?

a. Five nodes

b. Six nodes

c. Three nodes

23 How would you calculate the number of edges of a complete graph with...

a. A hundred nodes?

b. n nodes?

24 Now that you know how the number of nodes and edges in a complete graph are related, investigate this relationship further.

a. Are the number of nodes and the number of edges in a complete graph related directly? That is, if you double the number of nodes, does the number of edges double, too?

b. Are the number of nodes and edges in a complete graph related linearly? Or does the number of edges go up “faster” or “slower” than in a linear relationship?

25 What’s the minimum number of nodes needed for a graph with 1000 edges, if at most one edge can connect two nodes, and no edge can connect a node to itself?

26 If a complete graph has 500 nodes, can you estimate the number of edges quickly? Is it more likely to be roughly 12500, 125000, or 1250000 edges?

27 In the complete graph with four nodes, pictured earlier, the graph is drawn so that some of the edges crossed. However, just because it is drawn that way in that particular diagram does not mean the graph is not planar.

a. Is the complete graph with four nodes planar?

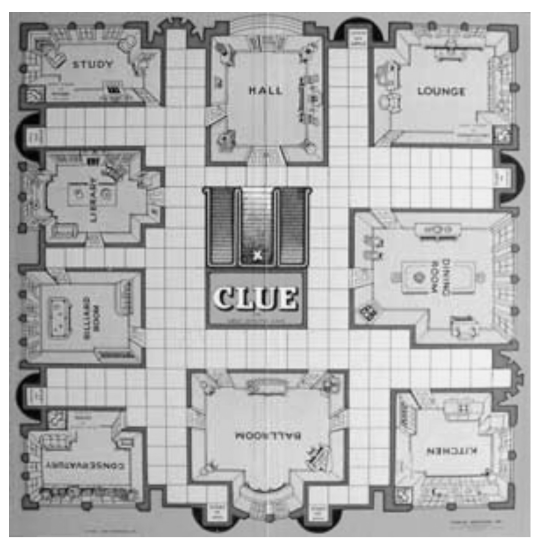
b. Is the complete graph with five nodes planar?

c. How about complete graphs with more than five nodes?

28 The symbol “Deg,” as in the degree of a node, represents a function. What kinds of objects does Deg take as input? What kinds of objects does it output? By the way, the set of allowable inputs to a function is called its domain, and the set of allowable outputs is called its range.

For the next seven problems, it will be useful to **change the representation** of the problem using the tools of graph theory. The first question to ask yourself is always, “what should the nodes represent, and what should the edges represent?”

29 In this “Clue” game board, draw a graph in which you show which rooms can be reached from the others by the roll of a 6 on a die. Include the secret passages that you can take no matter what you roll from the Lounge to the Conservatory and from the Kitchen to the Study. If you roll a number higher than the number of spaces you need, you can still go in the room. By the way, is the Clue graph planar?



30 A group of six sophomores is going down to the Lower School to read to a group of six first-graders. The teachers asked for a list of preferences, both from the first-graders and the sophomores, as to who would read to whom.

(First grader)	(would like to be read to by)
Algernon	Xenophon, Yvonne, Ulysses
Bartleby	Ulysses, Vlad, Yvonne, Zaphod
Curio	Vlad, Winnifred, Zaphod
Duncan	Winnifred, Zaphod, Xenophon
Evangeline	Ulysses, Xenophon
Francine	Ulysses ,Vlad, Winnifred, Zaphod

(Sophmore)	(would like to read to)
Ulysses	Algernon, Bartleby, Evangeline, Francine
Vlad	Bartleby, Curio, Francine
Winnifred	Bartleby, Curio, Duncan, Evangeline, Francine
Xenophon	Algernon, Evangeline
Yvonne	Algernon, Duncan, Francine
Zaphod	Bartleby, Curio, Francine

Is there a way to form partners that will make everybody happy?

31 (From Epp) A traveler in Europe wants to visit each of six cities shown on this map exactly once, starting and ending in Brussels.
The distance (in kilometers) between each pair of cities is given in the table. Suggest a sequence of cities she can visit that minimizes the distance traveled.

	Berlin	**Brussels**	**Dusseldorf**	**Luxembourg**	**Munich**
Brussels	783	--			
Dusseldorf	564	223	--		
Luxembourg	764	219	224	--	
Munich	585	771	613	517	--
Paris	1057	308	497	375	832

32 Wiki-Racing! The game of wiki-racing involves starting on a randomly generated Wikipedia page and taking Wikipedia links to navigate through until you reach a designated ending page. For example, two players on two different computers might agree to end on the page for "Fruit Bat." The first player to arrive at the Fruit Bat Wikipedia page using only the links on Wikipedia pages would win.

Below is a list of webpages in our miniature version of wiki-racing. Each webpage links to some, but not necessarily all, of the others.

Wikipedia Page	Links on that page
Toy Story	Walt Disney Pictures Pixar Animation Studios
Toy Story 3	Buzz Lightyear Toy Story
Pixar Animation Studios	Meet the Robinsons
Buzz Lightyear	Pixar Animation Studios Walt Disney Pictures Toy Story
Meet the Robinsons	Walt Disney Pictures Toy Story 3
Walt Disney Pictures	Pixar Animation Studios

a. How could you represent this situation using a graph?

b. What is the fewest number of clicks it will take to go from the Toy Story page to the Toy Story 3 page? From the Toy Story page to the Buzz Lightyear page?

33 Oh, no — your parents just announced that they're dropping by your apartment for a surprise visit. But your place is a mess. Fortunately, you've got some elves — as many as you need — to help you clean the place, but even an elf can't wash lights with darks. Here's what you and the elves have got to do, and the time it takes to do each thing:

- Wash your load of light-colored laundry
  (30 mins)
- Wash your load of dark-colored laundry
  (30 mins, and remember you've only got one washer)
- Dry your lights (45 mins, and you have to wash the lights first)
- Dry your darks (45 mins, you have to wash the darks first, and you've only got one dryer)
- Scrape off the dishes in your sink (10 mins)
- Load and run the dishwasher
  (60 mins, and you have to scrape first)
- Pick your stuff up off of the floor
  (20 minutes)
- Vacuum (10 minutes, and you have to pick up your stuff first)

a. Find a good graph-theory way to represent this problem. How long will it take you to put your apartment in order?

b. What is the minimum number of elves you need to do everything in that amount of time?

34 You have a 3×3 chessboard. The black knights are in the top two corners, and the white knights are in the bottom two corners. No other pieces are on the board. (Recall that the only moves knights are allowed to make is "2 up or down, 1 left or right," or "1 up or down, 2 left or right," but that they can "jump" pieces.)

a. How many moves will it take to switch the position of the knights, so that the black knights are now in the bottom two corners and the white knights are in the top two corners?

b. How many moves will it take just to switch the two rightmost knights?

35 Below is a map of the town of Königsberg, Prussia, as it existed in the 18th Century. A popular puzzle at the time was to cross all seven bridges without ever going back over a bridge you've crossed, and wind up at your starting point. Solve the puzzle yourself. (The mathematician Leonhard Euler invented graph theory by analyzing this problem.)

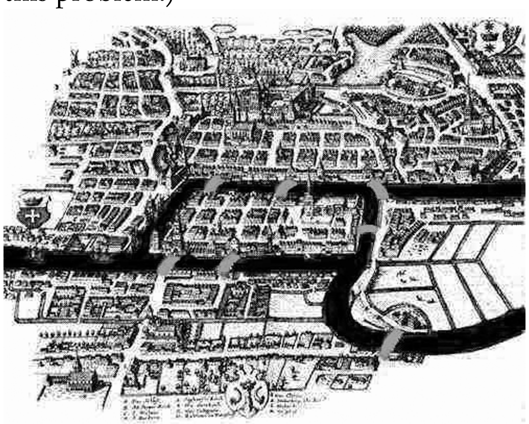
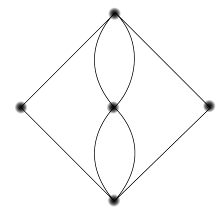


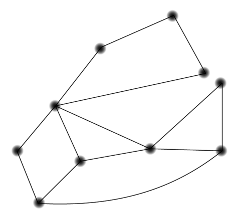
Image from solipsys.com.uk/new/KoenigsbergImages.html

36 For each figure, say whether you can trace it with your pencil without going over any segment twice, ending at your starting point.

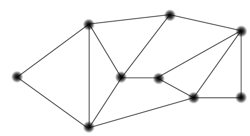
a.



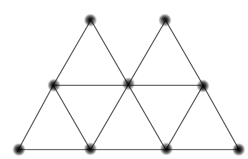
b.



c.



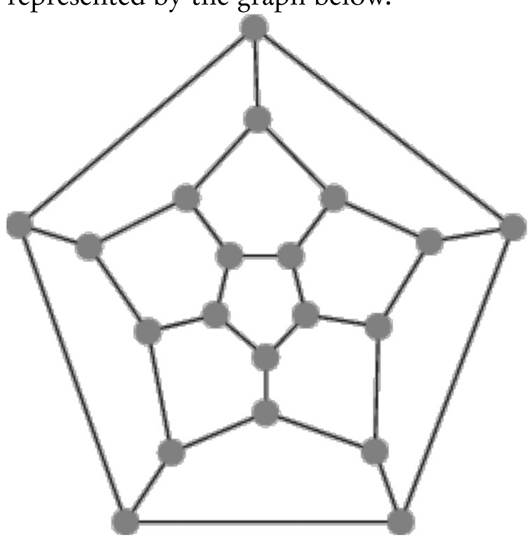
d.



37 A graph that can be traced in the manner of the previous problem is said to have an **Euler Circuit**, named after the mathematician who invented graph theory. Propose a method for determining if a given graph has an Euler circuit or not. (Are any of the concepts you've learned in this lesson relevant?) Make up new examples to test, if necessary.

38 Which complete graphs have Euler circuits?

39 The Irish mathematician William Rowan Hamilton invented a game in the 1850s called the Icosian Game. In it, you had to visit a list of cities connected by roads, as represented by the graph below.

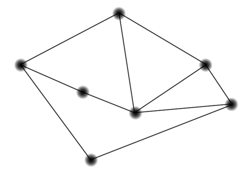


(image from mathworld.wolfram.com/IcosianGame.html)

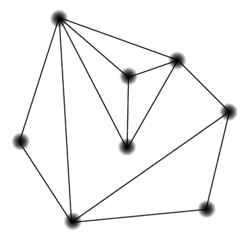
The challenge was to start in one city and follow the roads to visit all the cities, winding up at your starting city without ever having visited a city twice. Solve the Icosian Game.

40 If you read the previous problem carefully, you won't be surprised to learn that a circuit of the type you found — one where you need to visit each node exactly once and return to your starting node — is called a Hamiltonian circuit. Find Hamiltonian circuits for each graph below.

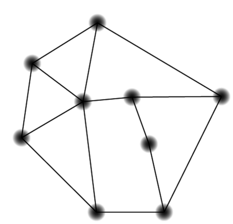
a.



b.



c.



41 Do you suppose that all graphs have Hamiltonian circuits?

42 Don't use a calculator for this problem.

a. Factor $x^2 + 8x + 16$

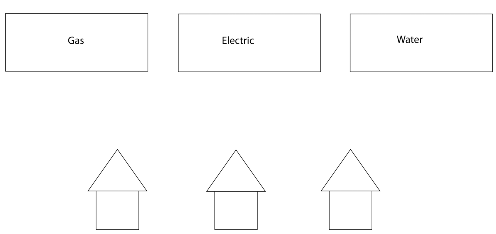
b. Factor $x^2 - 16$

c. Solve for x: $x^2 - 5x + 6 = 0$

d. Subtract $4\frac{7}{5} - 2\frac{1}{3}$

e. If $x^2 + y^2 = 30$ and $(x+y)^2 = 484$, what is xy ?

43 The diagram below shows a blueprint of three different utility companies and three houses. You want to add to the blueprint lines running from each utility company to each house, but the lines cannot cross one another. Suggest a way to do this or argue that it is impossible.



You may not have needed a graph to fully analyze the previous problem — but clearly, you could have drawn a graph to represent the situation. Graphs like the one in problem 43 that have two different groupings of nodes are called **bipartite**. In a bipartite graph, edges may only be drawn between nodes in two different groups (like house and utility); they never connect nodes in the same group. When you solved Problem 30, you also would have used a bipartite graph; however that graph was not a complete bipartite graph, whereas the graph in 43 is complete bipartite. Do you see why?

44 Draw a complete bipartite graph between two groups of two nodes. This graph is abbreviated $K_{2,2}$. Now draw $K_{2,4}$. Using this notation, what graph were you working with in Problem 43?

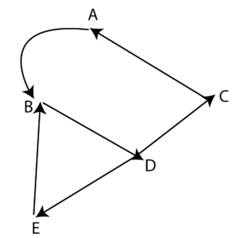
45 How many colors are needed to color the graph $K_{2,4}$? How about $K_{m,n}$?

46 Find a formula for the total degree of $K_{m,n}$.

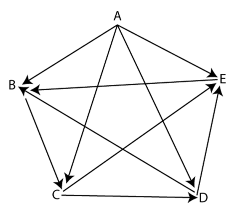
47 In the three-utilities puzzle (see problem 43), suppose now that lines are allowed to cross, and so the lines are actually run that way. If you accidentally cut one of the pipes without knowing which one it is, what's the probability that the gas in the leftmost house will go out? How about if there are m houses and n utilities, instead?

48 Any graph that could represent "who beat whom" in a situation where each team plays every other team once is called a tournament. Which of the graphs below are tournaments?

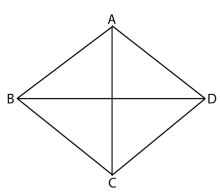
a.



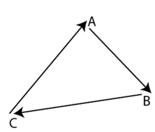
b.



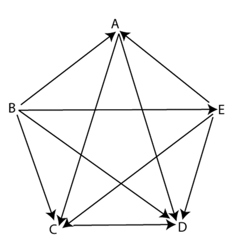
c.



d.



e.



49 *The Game of Sprouts*: Mark a few dots on a piece of paper. A move consists of two actions: connect two different dots with a single edge then place a dot at around the middle of this edge. You can only connect two dots if each has a degree of less than 3, and you may not repeat an edge or cross edges. The person who has no move is the loser.

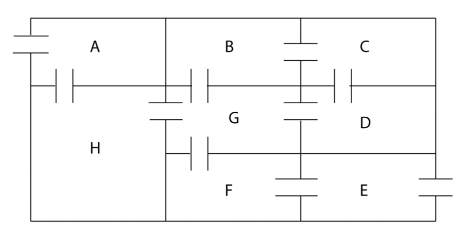
a. If you start with only two dots, then what is the maximum number of moves the game can last?

b. At the end of the two-dot sprouts game, what will be the total degree of the diagram?

c. At the end of the n-dot sprouts game, what will be the total degree of the diagram?

d. How many moves will the n-dot sprouts game take?

50 The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E?

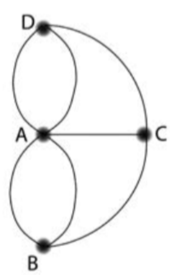


Exploring in Depth

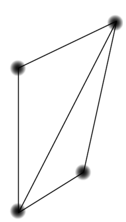
51 Earlier we learned that if two graphs are isomorphic, each node has a "partner" of the same degree. Is it also true that when nodes can be matched into partners having the same degree, the two graphs must be isomorphic?

52 An **Euler Path** is a way of tracing a graph so that you trace all the edges but wind up in a *different* place than where you started. Which of these graphs have Euler paths?

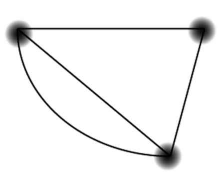
a.



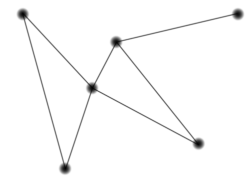
b.



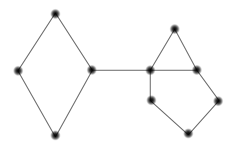
c.



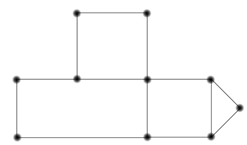
d.



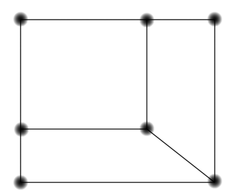
e.



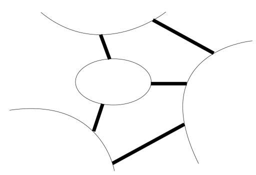
f.



g.



53 The following diagram represents land connected by bridges. Can you walk these bridges without retracing your steps? (Though not necessarily winding up in the same place you started.)



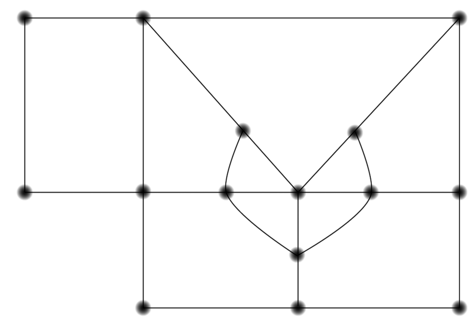
54 Propose a criterion by which you'd be able to tell if a graph has an Euler path. Does it matter which nodes are your starting and ending points? Remember to examine a similar problem by making up specific cases for yourself.

55 In problem 32 and the tournament problem (see problem 48), you drew a **directed graph** — a graph where the edges are one-way. The question of finding Euler circuits is more complicated on directed graphs. Do some experiments and propose criteria for whether a directed graph has an Euler circuit.

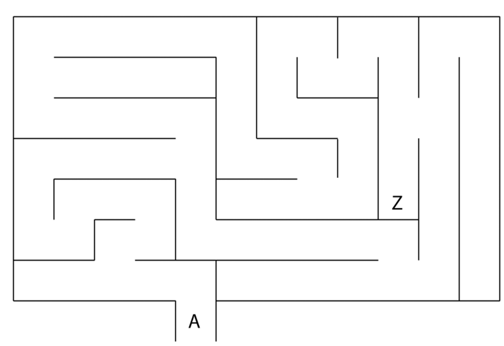
56 Does every tournament have a Hamiltonian path? What does it mean for the team rankings if there is more than one path through a tournament?

57 Come up with your own system to determine the winner of a tournament. It may help to create some of your own examples of tournaments.

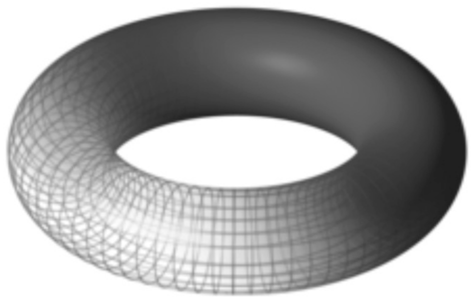
58 The following graph represents streets and corners; the portion of a street between two corners is a "block". The neighborhood association decides to place streetlights at some of the various corners so that the neighborhood never gets too dark at night. They want there to be enough streetlights so that every home is on a block that has a streetlight on at least one of its corners. Assume that there are homes between each pair of adjacent corners. Find the smallest number of street lights needed.



59 Use a graph to represent and solve the following maze.



60 The "doughnut" shape below is called a torus. You know you can't draw a complete graph with five nodes on a flat surface. But can you draw it on the surface of a torus?



61 If the blueprint for the houses and utilities (see problem 43) were printed on paper shaped like a torus, could you run the wiring with no overlaps?

LESSON 3: PRIMES PRODUCTS AND EXPONENTS

Development

As you learned last year, factoring a number means expressing it as the product of two or more other numbers. For example, 40 can be factored as $2 \cdot 20$. Of course, 40 can be factored in many other ways as well.

1 What are all the ways that 40 can be factored into two whole numbers?

Since 20 can be factored as $4 \cdot 5$, 40 can also be factored into 3 numbers as $2 \cdot 4 \cdot 5$; and since 4 can also be factored into $2 \cdot 2$, 40 can be factored into 4 numbers as $2 \cdot 2 \cdot 2 \cdot 5$. You might also think of saying it could be factored into 5 or 6 numbers as $1 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 5$, but since those aren't really different from $2 \cdot 2 \cdot 2 \cdot 5$, we will agree that 1 won't "count" as a factor in the following problems, even though technically 1 is a factor of every number. Thus, at this point, 40 can't be "broken down" any more than $2 \cdot 2 \cdot 2 \cdot 5$ (we are restricting ourselves to positive integers here, so we wouldn't consider turning 5 into $2.5 \cdot 2$ or into $-5 \cdot -1$).

2 Find two different ways to factor 72 into two positive integers. For each of your answers, continue factoring them until they can't be "broken down" any more. Finally, rearrange each of your final factorings so that the factors are in order from smallest to largest.

3 Factor 210 into the product of as many positive integers as possible. Compare your answer with a classmate's. Based on this question and on question 2, what generalization might you now conjecture?

A number has been factored as much as possible when each of its individual factors cannot be broken down any more. Numbers greater than 1 that cannot be broken down any more (i.e., cannot be factored into smaller positive integers) are called **primes**. We say that the **prime factorization** of 40 is $2 \cdot 2 \cdot 2 \cdot 5$, or $2^3 \cdot 5$ for short; we know it is a prime factorization because all of the factors (2, 2, 2, and 5) are primes. As suggested by questions 2 and 3, every positive integer greater than 1 has a unique prime factorization. As we will see, this uniqueness means that rewriting a number in its prime factored form will often be extremely useful, even though all we have done is written the same number in a different way.

Sometimes it can be tricky to know what the factors of a number are: that is, what divides into it evenly, leaving no remainder. For example, what numbers is 153 divisible by? It would clearly be helpful if one knew easy procedures (known as **divisibility tests**) that allowed one to see if smaller numbers divided arbitrary larger ones evenly. As a matter of fact, the divisibility tests for 2 and 5 are easy (what are they?), but it helps to know the tests for 3 and 11 as well (sadly, there is no easy test for 7).

If, when you add up the digits of a number, its sum is divisible by 3, then the number is divisible by 3 as well; on the other hand, if the sum is not divisible by 3, the number is not divisible by 3. So 14353 is not divisible by 3 (because $1 + 4 + 3 + 5 + 3 = 16$, which is not divisible by 3), while 55521 is. We will learn why the divisibility test for 3 is true later this chapter, but you might find it fun to ponder why it works on your own first!

Some of you may remember learning in 9th grade that a number is divisible by 11 only when the "alternating sum" is divisible by 11. So 4565 is divisible by 11 because is divisible by 11, as is 527494 because $+5 - 2 + 7 - 4 + 9 - 4 = 11$, but 6587 is not because $+6 - 5 + 8 - 7 = 2$ is not divisible by 11.

And now, a few problems to check your understanding of the above:

4 What is the largest number (other than 495 itself) that divides evenly into 495? Why do you know it's the largest number that can do so?

5 Is 4806 prime? How about 4807? Quickly name 5 numbers between 4800 and 4900 that are divisible by 11.

6 Determine the prime factorization of 2310.

7 Is 91 prime? How could you be sure of your answer? What about 221? Or 223?

8 Come up with a procedure that can determine, with reasonable efficiency, whether a number is prime or not.

Determining in general if a positive integer is prime (i.e., has no factors other than itself and 1) turns out to be an increasingly difficult task the larger the number becomes. In fact, one of the most powerful ways of encrypting data on the internet — the so-called "RSA algorithm" — is based on the difficulty of finding the prime factorization of very large (over 100 digits!) integers.

Prime factorization also allows us to determine the answers to questions about the relationships between two (or more) numbers. Remember that it can be helpful to write the final prime factorization using powers; for example, the prime factorization of 484 is $2^2 \cdot 11^2$.

9 What is the largest number that divides evenly into both 216 and 270? This number is called the **greatest common divisor** of the two numbers and is written as gcd(180, 156). (For example, gcd(24, 36) = 12, as 12 is the largest integer that "goes into" both 24 and 36.)

10 If your answer to question 9 involved testing out a lot of different numbers, try writing out the prime factorization of each original number (i.e. 216 and 270) first and see if you can use them to be more efficient in finding their gcd.

11 Find gcd(168, 448) efficiently. How can you be sure you have found the largest possible number that "works"?

12 Find $\gcd(2^{16} \cdot 3^4, 2^{13} \cdot 3^8)$, or put another way, gcd(5308416, 53747712). Looking at the original prime factorizations of each number, what do you notice about the prime factorization of your answer?

13 Create a procedure that allows one to efficiently determine the gcd of two positive integers. Check it with 3 examples that you make up, where at least one of the examples is similar to problem 12 (i.e., where the two numbers are large but have simple prime factorizations).

Crucially, then, a prime factorization is important because all the factors of a number derive from the number's prime factors. 24, for example, is $2^3 \cdot 3$, and all of its factors — 2, 3, 4, 6, 8, 12, and 24 — are made up of combinations of the 2's and 3's found in the prime factorization.

14

In the following exercises, learn how powerful prime factorization is by solving them without your calculator!

a. How can you check to see if $32 \cdot 35$ equals $28 \cdot 40$ without determining either product?

b. Juniper says that when she multiplies $6 \cdot 24 \cdot 55$ it is equal to $21 \cdot x$, where x is an integer. Explain if she is telling the truth or not.

c. Simplify $\frac{64 \cdot 25 \cdot 14 \cdot 33}{28 \cdot 44 \cdot 15}$ as much as possible.

d. What is the positive integer N for which $22^2 \times 55^2 = 10^2 \times N^2$? Copyright *www.mathleague.com.*

e. Is $6^4 \cdot 10^3$ greater than, less than, or equal to $8^2 \cdot 15^3 \cdot 6$?

The last few problems also are a reminder of the **laws of exponents** that you learned in middle school. Since 5^6 is just $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ and 5^4 is just $5 \cdot 5 \cdot 5 \cdot 5$, it makes sense that

$$\begin{aligned} 5^6 \cdot 5^4 &= (5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5) \\ &= 5^{10} \end{aligned}$$

Similarly, because $\frac{5^6}{5^4}$ is just $\frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 5}$, it makes sense that some pairs of 5's in the numerator and denominator would "cancel" (because $\frac{5}{5} = 1$) and thus

$$\frac{5^6}{5^4} = \frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 5}$$
$$= \frac{5 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot 5}{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}}$$
$$= 5^2.$$

Lastly, $(5^4)^3$ is just $(5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5)$, so $(5^4)^3 = 5^{12}$.

15

By using the reasoning of the previous paragraph, finish the following equations, where x is the base, and a and b are exponents:

$x^a \cdot x^b =$ $\qquad$ $\frac{x^a}{x^b} =$ $\qquad$ $(x^a)^b =$

16 For the second law in question 15, what are you assuming about a and b? Try different values of a and b to clarify your answer.

17 Simplify the following:

a. $x^{13} \cdot x^7$

b. $\frac{3^{28}}{3^{24}}$

c. $\left(x^5\right)^6$

d. $(\frac{x^8 \cdot x^9}{x^6})^2$

18 What do you think 3^0 should be equal to? Why? Come up with a specific argument to defend your view (and don't use your calculator!).

19 Let's explore the expression a bit more.

a. If $3^4 = \frac{3^p}{3^q}$, then what are possible pairs of values for p and q? Give at least 3 pairs.

b. Using similar reasoning as in part a, what would it appear 3^0 should be equal to? What about 497^0?

c. Which answer do you trust more, your answer to question 18, or your answer to part b above? Explain.

d. Finally, make a chart of the powers of 3, starting at 3^5 and going down to 3^1. Looking at this chart, what do you think 3^0 should be?

e. Based on part d and your own conclusions, can you revise your answer a little to question 16?

20 What do you think 6^{-3} should be equal to? Is your answer the same as $-(6^3)$? Again, come up with a specific argument to justify your answer.

21 Let's look a bit deeper into the idea of negative powers.

a. Using similar reasoning as in question 19a, what would it appear 6^{-3} should be equal to? (Write your answer as a reduced fraction.) Does your answer equal what you thought in question 20?

b. Make a chart of the powers of 6, starting at 6^5 and going down to 6^0. Following this idea, what do you think 6^{-1} would be? How about 6^{-2} and 6^{-3}? Again, write your answers as fractions, rather than using decimals.

c. Based on parts a and b and your own conclusions, can you revise your answer to question 16 even more than you did in question 19e?

22 What should x^6 be multiplied by to equal x^2? What should x^6 be divided by to equal x^2? Test that your answers work with a specific value of x, and then explain how the answers are related to each other.

23 Rewrite $\frac{x^{-4}}{x^7}$, $\frac{3^{-5}}{3^6}$, $\frac{x^8}{x^{-5}}$ $\frac{2^{-8}}{2^{-3}}$, $\frac{x^{-6}}{x^{-4}}$ and $\frac{x^{-7}}{x^{-11}}$ so that your final expressions have only positive exponents.

24 Simplify $\frac{x^8x^5}{x^3x^7}$, $\left(\frac{x^8x^5}{x^3x^7}\right)^4$, and $\left(\frac{x^8x^5}{x^3x^7}\right)^{-4}$. Then write each of your "simplified" answers with only positive exponents, if they aren't already.

Practice

25 Without using a calculator, find all the prime factors of 99792 that you can. Then, using a calculator, find the complete prime factorization of 99792.

26 Find gcd(1240, 4400). If you wish, you can express your answer not as a number, but as the prime factorization of that number.

27 Determine if $\frac{60480}{2268}$ is an integer without using a calculator!

28 What values of x and y satisfy $36 \times 5^x = 225 \times 4^y$?

29 Arrange from biggest to smallest: 3^{-3}, $(-3)^3$, 2^{-2}, $(-2)^2$, $(-3)^{-3}$, $(-2)^{-2}$.

30 Simplify $\frac{2^7 \cdot 2^5}{2^4 \cdot 2^6}$, $\left(\frac{2^7 \cdot 2^5}{2^4 \cdot 2^6}\right)^3$, and $\left(\frac{2^7 \cdot 2^5}{2^4 \cdot 2^6}\right)^{-4}$ so that each answer is extremely simple: 2 raised to an integer power. Don't use a calculator.

31 Can $3^4 \cdot 2^5$ be written in the simpler form a^b (where a and b are integers and $b \neq 1$)? If yes, check your answer by trying it out with specific numbers a and b just to be sure.

Going Further

32 Do the laws of exponents continue to hold when dealing with two variables? Let's check this out.

a. Does $(xy)^5 = x^5y^5$? Try testing this "rule" with specific numbers, and, if it seems to work, see if you can prove the rule using algebra.

b. Generalizing from part a, what would $(xy)^a$ be equal to?

c. Using similar reasoning, how could one rewrite $\left(\frac{x}{y}\right)^a$?

d. What would $(xy)^8(xy)^4$ equal? How about $(xy)^3(xy)^{-7}$?

e. Following the pattern in part d, how could you rewrite $(xy)^a(xy)^b$?

f. Can you rewrite $\frac{(xy)^a}{(xy)^b}$ as well?

g. How about $(x+y)^a$— does it simplify easily? How can you check your answer?

33 Now let's apply what we've learned in question 32.

a. Can $\frac{x^5y^7}{y^3x^2}$ be simplified? How about $\left(\frac{x^5y^7}{y^3x^2}\right)^3$? Finally, howzabout $\left(\frac{x^5y^7}{y^3x^2}\right)^{-6}$?

b. Simplify $\frac{2^3 \cdot 3^4}{2^5 \cdot 3}$ and $\left(\frac{2^3 \cdot 3^4}{2^5 \cdot 3}\right)^{-2}$ so that your answers are in the form 2^x3^y, where x and y are integers.

c. Simplify $\left(\frac{4x^{-3}y^7}{x^8y^3}\right)^{-2}$ and $\left(\frac{x^8y^3}{4x^{-3}y^7}\right)^2$. Are you surprised by the two answers?

d. Rewrite $\left(\frac{kx^ay^b}{x^cy^d}\right)^e$ so that your answer is in the form $k^Px^Qy^R$, where P, Q, and R are integers that you determine.

34 Here are a couple more doozies based on what you learned in question 33. Don't use a calculator!

a. What simple number is equivalent to $\left(3^2 \cdot 7^4\right)^4 \left(5^8 \cdot 7^{-5}\right)^3 \left(3 \cdot 5^4\right)^{-6}$?

b. What is $\left(\frac{16g^6}{9^6t^{-5}}\right)^{-2} \cdot \left(\frac{3^6t^{-3}}{2^2g^3}\right)^4$, simplified?

35 You know from the laws of exponents that $x^2x^5 = x^7$, or that $x^ax^b = x^{a+b}$. We can use this identity to help us think of square roots in a different way — as an exponent.

a. What does $(\sqrt{x})(\sqrt{x})$ equal?

b. So if $\sqrt{x} = x^N$, what must N be?

c. If $p \cdot p \cdot p = x$, what must p equal, in terms of x? Put another way, if we say that $p = x^N$, what does N equal?
(This number is called a "cube root" of x, i.e., since $2 \cdot 2 \cdot 2 = 8$, 2 is a cube root of 8. This can also be written as $2 = \sqrt[3]{8}$, where the $\sqrt[3]{}$ symbol indicates "cube root".)

d. If $p^5 = x$, what would p equal in terms of x? That is, if we say that $p = x^N$, what does N equal?

e. If $p = \left(\frac{1}{32}\right)^{\left(\frac{1}{5}\right)}$, what would p be? If $q = \sqrt[5]{\frac{1}{32}}$, what is q ?

f. In general, then, if $\sqrt[n]{x} = x^N$, how are N and n related? Give a specific example or two to clarify.

36 Using what you learned in question 35, simplify each expression as far as possible. No calculator necessary, although feel free to check!

a. $\left(16^{\frac{1}{2}}\right) + \left(16^{\frac{1}{4}}\right)$

b. $64^0 + 64^{\frac{1}{3}} + 64^{\frac{1}{2}} + 64^1$

c. $\left(\sqrt[3]{241}\right)\left(241^{\frac{1}{3}}\right)\left(\sqrt[3]{241}\right)$

d. $\left(81^{\frac{1}{4}}\right)^{-2}$

e. $125^{\frac{1}{3}} \cdot 36^{-\frac{1}{2}}$

37 You now know how to calculate 64^0, $64^{\frac{1}{3}}$, $64^{\frac{1}{2}}$, 64^1 and 64^2. Let's broaden our horizons even more!

a. Suppose a friend told you that he had just figured out how to calculate $64^{1.5}$. Based on what you already know about powers of 64, about how big a number do you think this is? Explain.

b. $64^{1.5}$ can also be written as 64^N, where N is a simple fraction. What is N?

c. Now further rewrite 64^N by using the property of exponents that tells us we can rewrite x^{12} as $\left(x^3\right)^4$.

d. Using your result from part c, you should now be able to compute a precise answer to $64^{\frac{3}{2}}$ $(= 64^{1.5})$. What is it, and does it agree with your answer in part a?

e. Similarly, what would $64^{\frac{2}{3}}$ $(= 64^{.\overline{666}})$ equal? How about $4^{2.5}$?

f. Finally, what is $7^{1.3}$? Explain what it *means* to raise a number to the 1.3 power. Once you've done that, do the same for $7^{1.29}$.

38 One can also solve equations quite easily when they contain powers. For example, $x^3 = 11$ can be solved by taking advantage of the laws of exponents and raising each side to the 1/3 power: $\left(x^3\right)^{\frac{1}{3}} = 11^{\frac{1}{3}}$, the idea being that because the exponents on the left side multiply, the exponent becomes "1", and so it simplifies to $x = 11^{\frac{1}{3}}$ (or $\sqrt[3]{11}$). Solve the following equations for x, keeping in mind the strategies in the paragraph above:

a. $x^5 = 32$

b. $x^{\frac{1}{5}} = 32$

c. $3x^3 = 81$

d. $\frac{16}{64}$ $x^5 = 8$

e. $x^{\frac{2}{3}} = 9$

f. $x^{2.71} = 126$

g. $5x^{3.87} = 1000$

Practice (No Calculators!!)

39 Simplify $\left(x^{-17}x^{8}\right)^{-3}$ in two different ways:

a. By simplifying inside the parentheses first, and then applying the outside exponent.

b. By applying the outside exponent first, and then simplifying.

c. Confirm that your answers in parts a and b are the same.

40 Simplify $\left(\frac{p^9q^{-5}}{q^7p^4}\right)^3$ and $\left(\frac{p^9q^{-5}}{q^7p^4}\right)^{-3}$. What do you notice?

41 Does $\left(\frac{w^6t^7}{t^{-8}}\right)^4$ equal $\left(\frac{w^3t^{20}}{w^{-5}}\right)^3$? Explain.

42 Don't use a calculator for this problem.

a. Divide $\frac{10}{3} \div \frac{5}{2}$.

b. Factor $4x^3 - 12x^2$.

c. Factor $x^2 - 25$.

d. Solve for x: $x(x+3) - 1 = -3$.

e. Solve for x, looking up the quadratic formula if necessary: $x^2 - 4x - 6 = 0$.

43 What is $\frac{(m^{-4}n^{12})^{-5}}{(n^{-6}m^3)}$ simplified?

44 Simplify $\left(\frac{12x^{-2}y^5}{8x^{-6}y^9}\right)^3$.

45 The volume of a cube is 2197 cm^3. Find the total surface area of the cube. (Calculators allowed.)

46 Write $5^5 \cdot 5^8 \cdot 8^5 \cdot 8^8$ in the form a^b.

47 What is $\sqrt{17^2}$? What is $\sqrt[3]{5^3}$?

48 Which is bigger: $\sqrt[3]{37^2}$ or $(\sqrt[3]{37})^2$? Why?

49 What does $8^{\left(\frac{4}{3}\right)} \cdot 4^{\left(\frac{3}{2}\right)}$ equal?

50 Simplify $\sqrt[3]{\sqrt[4]{x^{24}}}$.

51 Write $25^4 \cdot 125^{-2}$ as a power of 5.

52 Solve $2r^4 = 162$.

Problems (Still No Calculators!)

53 How many prime numbers under 10000 have digits that add up to 9?

54 If the sum of two prime numbers is 999, what is their product?
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55 Explain why the product of any three consecutive integers has to be divisible by 6.

56 The dimensions of a rectangular box (in cm) are all positive integers, and the volume of the box is 2002 cm^3. What is the least possible sum of the three dimensions?

57 (Calculators allowed.) What is the smallest positive integer that is NOT a factor of $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots \times 18 \times 19 \times 20$?

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58 Does $6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6$ equal 36^6? 6^{36}? 6^7? 36^{36}? Or something else in the form a^b? Explain.
(Appeared on AMC-12 competition, 1992)

59 Express $\frac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}}$ as a single, simple fraction. (Appeared on AMC-12 competition, 1987)

60 When 270 is divided by the odd number X, the answer is a prime number. What is X?
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61 What is the simplest expression for $\frac{2^{40}}{4^{20}}$?
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62 What is the simplified value of $\frac{4444^4}{2222^4}$?
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63 If you double x, by what factor does the expression $\frac{x^7 x^{-2}}{x^3}$ grow?

64 A composite number is a whole number that is greater than 1 and not prime — that is, it has factors other than itself and 1.

a. What is the smallest composite number that is not divisible by 2 or 3?

b. What is the smallest composite number that is not divisible by 2, 3, or 4?

c. What is the smallest composite number that is not divisible by 2, 3, 4, or 5?

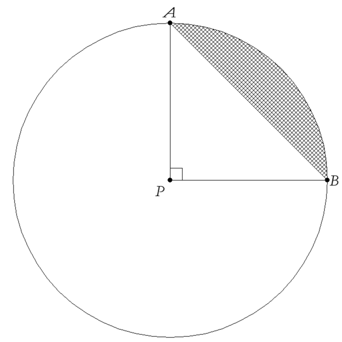
d. (Calculators allowed.) Now determine the smallest composite number that is not divisible by 2, 3, 4, 5, 6, 7, or any of the numbers up to and including 100.

65 Order these, from smallest to largest, without using a calculator: $1000^{\frac{1}{1000}}$, $\left(\frac{1}{1000}\right)^{1000}$, 1000^{-1000}, $\left(\frac{1}{1000}\right)^{\frac{1}{1000}}$, $\left(\frac{1}{1000}\right)^{-1000}$

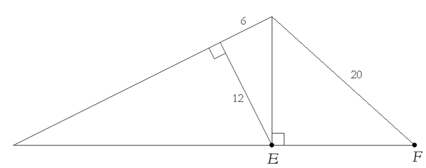
66 You are going to randomly choose three integers (repetition allowed) from 0 to 5, and call them k, m, and n. What's the probability that $2^k 3^m 5^n$ is NOT divisible by 5?

67 What is the only number x which satisfies $\sqrt{1992} = 1992\sqrt{x}$?

68 In circle P, the length of $\overline{AB}$ is $\sqrt{50}$. Find the area of the shaded region.



69 Find EF exactly.



70 Triangle ABC has a right angle at C. If sin B $= \frac{2}{3}$ what is tan B, expressed as a fraction?

71 Why must $\sqrt{17}$ be a non-terminating decimal? Put another way, why can't a terminating decimal multiplied by itself = 17?

72 What is the positive number x for which $x = \sqrt[3]{y}$ and $\sqrt{y} = 8$?
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73 If $x^{64} = 64$, what is the exact value of x^{32}?
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74 In simplest form, what is the numerical value of $\sqrt{1985}\ \sqrt[3]{1985}\ \sqrt[6]{1985}$?
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75 Which kind of number does not appear to have a 10th root? Which kind of number has an integer as its 10th root?

76 If $\sqrt[5]{x} = 4$, what is the value of $\sqrt{x}$?
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77 What does $\sqrt{7}\sqrt{3}\sqrt{2}\sqrt{5}\sqrt{10}\sqrt{21}$ equal? Think before calculating!!

78 Rewrite $\left(\frac{1}{4}\right)^{-\frac{1}{4}}$ as the root (i.e. square root, cube root, and so on—you choose !) of an integer that is NOT 4.

79 What is the value of x which satisfies $\sqrt[3]{x\sqrt{x}} = 4$?
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80 If $x \geq 0$, then $\sqrt{x\sqrt{x\sqrt{x}}}$ is equivalent to x^N. What is N?

81 The function s takes a number and outputs the sum of its “proper” divisors, meaning that 1 is considered to be a divisor but the number itself is not.

a. Find $s(8)$.

b. A perfect number is a number n for which $s(n) = n$. How many perfect numbers are between 2 and 10?

c. An abundant number is a number n for which $s(n) > n$. Find the first abundant number.

d. A pair of numbers is said to be amicable when the proper divisors of each number add up to the other number.

e. Verify that 220 and 284 are an amicable pair.

f. Write the definition of amicable numbers using the $s(n)$ notation.

Exploring in Depth (No Calculators Still!)

82 Which is bigger, 2^{3000} or 3^{2000}? Prove your answer.
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83 Write $\frac{15^{30}}{45^{15}}$ in the form a^b, if possible. If it is not possible, explain why not.
(Appeared on AMC-12 competition, 1993)

84 If $3^x = 5$, what is the value of $3^{(2x+3)}$?
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85 What is the integer n for which $5^n + 5^n + 5^n + 5^n + 5^n = 5^{25}$?
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86 Take any 6-digit number that repeats 3 digits twice in the same order, like 596596$. It will always be divisible by 7, 11, and 13. Why? (Calculators allowed.)

87 How many distinct pairs of positive integers (m, n) satisfy $m^n = 2^{20}$?
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88 How many positive integers less than 50 have an odd number of positive integer divisors?
(Appeared on AHSME 41 competition)

89 If $x > y > 0$, then express $\frac{x^y y^x}{y^y x^x}$ using only a single exponent.
(Appeared on AMC-12 competition, 1992)

90 Earlier, you learned how to calculate a number such as $8^{\frac{2}{3}}$.

a. In fact, there are two different plausible ways of evaluating a number like $8^{\frac{2}{3}}$: as $\left(8^{\frac{1}{3}}\right)^2$ or as $\left(8^2\right)^{\frac{1}{3}}$. Are these two numbers in fact equal? Explain why or why not.

b. Try calculating both $27^{\frac{4}{3}}$ and $32^{\frac{6}{5}}$ in each of the ways described in part a. Which is easier, and why?

91 Multiplying and dividing numbers and/or expressions that have roots and/or exponents in them becomes considerably easier if all the roots are converted to exponents. Also often helpful in harder problems is to try to give all the exponents the same base. For example, the easiest way to simplify $2^5 \cdot 8^4$ is to think of "8" as "2^3":

$$\begin{aligned} 2^5 \cdot 8^4 &= 2^5 \cdot \left(2^3\right)^4 \\ &= 2^5 \cdot 2^{12} \\ &= 2^{17}. \end{aligned}$$

Simplify the following:

a. $4^3 \cdot 64^5$

b. $2^{-4} \cdot 32^2$

c. $\frac{3^5 \cdot 18^4}{16}$

d. $7^{\frac{5}{6}} \cdot 49^3$

e. $\frac{81^{\frac{3}{8}}}{27^{\frac{1}{2}}}$

92 (Calculators allowed.) In addition to the greatest common divisor, another interesting numerical concept is called the least common multiple.

a. What is the smallest positive integer that is a multiple of 30 and also a multiple of 42? This number is called the "least common multiple" and is written as lcm(30, 42). (For example, lcm(6, 8) = 24, as 24 is the smallest number that is both a multiple of 6 and a multiple of 8.)

b. If your answer to part a involved testing a large amount of numbers, try using the prime factorizations of the numbers to help you see how to find the lcm efficiently.

c. Find lcm(84, 126) efficiently. How can you be sure you have found the smallest possible number that works?

d. Find lcm($2^{16} \cdot 3^4$, $2^{13} \cdot 3^8$). Looking at the original prime factorizations of each number, what do you notice about the prime factorization of your answer?

e. Create a procedure that allows one to efficiently determine the lcm of two numbers. Once again, check with 3 examples you devise, with at least one of the examples similar to part d.

93

Take any two positive integers x and y. (Calculators allowed.)

a. Find their gcd and lcm.

b. Multiply the gcd and lcm together.

c. Now multiply the original numbers x and y together. What do you notice?

d. Try parts a through c with 5 different pairs of positive integers x and y.

e. What is going on here? Explain carefully in terms of the prime factorizations of x and y. (Hint: Go back to questions 13 and 92e and look at how you were able to find the gcd and the lcm efficiently, and compare.)

94

Sometime around 5th grade, you were taught to add fractions $\frac{a}{b}$ and $\frac{c}{d}$ by finding a common denominator. Assume b and d are positive.

a. Explain whether you should be finding the gcd or the lcm of b and d if you want to find their smallest possible common denominator.

b. By the way, why do you need a "common" denominator to add fractions anyway? Why not just add them using different denominators?

95

If $x = \sqrt{2000}$ and $y = \sqrt{2001}$, what is the simplified numerical value of $(x + y)^2 + (x - y)^2$?
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96

What on earth could $4^{\sqrt{2}}$ mean? Could you determine approximately how big it would have to be?

97

Let's learn about how simplest radical form directly relates to prime factorization. (Calculators allowed.)

a. Put $\sqrt{99}$ and $\sqrt{884}$ into simplest radical form.

b. Explain, for an arbitrary integer n, how you would go about putting $\sqrt{n}$ into simplest radical form.

c. Now try simplifying $\sqrt{907}$. What makes you confident that you are definitely in simplest radical form?

d. Explain why you don't have to test any numbers above 31 when checking to see if 907 is prime.

e. Now try simplifying $\sqrt{529}$, $\sqrt{551}$, and $\sqrt{557}$.

f. Given what you learned in parts c through e, can your answer in part b be improved or clarified? Are you confident your method will efficiently put $\sqrt{n}$ into simplest radical form? Explain.

g. Try your method on $\sqrt{343}$, $\sqrt{379}$, $\sqrt{403}$, $\sqrt{765}$, $\sqrt{1517}$, $\sqrt{1373}$, and $\sqrt{1763}$.

98 Let's take a look at a proof the Greek mathematician Euclid came up with over 2000 years ago to show that there are an infinite number of primes. Euclid did this by assuming that there were a finite number of primes, and then showed that assuming that is the case turns out to be self-contradictory. Follow along to see his "moves"!

a. If there were a finite number of primes, we could call the largest one "P". Euclid then asked us to consider "Q", the number that has a prime factorization that includes each of the primes exactly once. Explain clearly how one could calculate "Q".

b. Euclid then made his biggest "move". Remembering that all the primes divide evenly into "Q", Euclid asks us to consider the number $Q + 1$. Which of the primes do you think would go evenly into $Q + 1$ as well? Why?

c. What can you conclude about $Q + 1$, now that you have established that it has no prime factors?

d. Why does your conclusion contradict what you assumed at the start of the problem?

LESSON 4: NUMBER GAMES AND ALGEBRA

And Now For Something Completely Different— A Mindreading Textbook!

Don't believe it? Try this!

1 Pick a number, any number.

a. Add 4 to it.

b. Square your new number.

c. Subtract 16 from your answer.

d. Divide what you have by the original number you picked.

e. Finally, subtract the original number.

f. This textbook knows what you got as your answer for Part e, and in fact it predicted ahead of time what you were going to get. Oh, you don't believe it? Check out Part g.

g. Why should I tell you? You think I'm a liar. Oh, alright, I'll tell you, but only if you promise to be impressed. See part h for enlightenment.

h. I'll tell you, not that you deserve it. Your answer to Part e is...(drum roll please)...eight. Now then, kiddo, never doubt a talking textbook again.

Development

You may have seen a trick like the one above before and wondered how it worked. Let's look at a few of them and see if we can figure out how they "tick".

2 Have a friend pick a number without telling it to you, and then, keeping their calculations to themselves, ask your friend to:

a. Add 8 to it.

b. Now multiply by 3.

c. Subtract 11.

d. Multiply by 2.

e. Add 4.

f. Finally, have them divide by 6 and tell you the answer.

g. From their answer to Part f, you can now figure out their original number and impress the socks off of them. How? Try out your solution with a classmate and see if it works.

3 Let's figure out what happened in Problem 2 by using algebra. Let's call the original number your friend was thinking of "x".

a. What number would they have after adding 8 to their original number?

b. And after multiplying by 3? Remember to use parentheses!

c. How about after subtracting 11? Simplify your answer.

d. After multiplying by 2 and adding 4 and simplifying, what expression do you have?

e. Now divide your answer in Part d by 6, and simplify your answer again. Remember that with fractions, $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

f. So, given your answer in Part e, how can you determine your friend's original number?

4 Here's a trickier one. Think of the age of anyone you know.

a. Multiply it by 10.

b. Add 1.

c. Multiply by 2.

d. Add 21.

e. Multiply by 5.

f. Your answer to Part e will end in the two digits "15". Remove those last two digits to get a new number.

g. Subtract 1 from the new number, and that is the age you were thinking of. Why? (Hint if you're having trouble: What you did in Part f can also be thought of as a subtraction followed by a division—how so?)

5 Here's a more difficult variation. Pick a positive number.

a. Three less than that number, multiplied by two more than that number is...

b. Now take your answer from Part a and add the original number.

c. Add 6 more.

d. Take the square root. Ta da! Why does this work?

Figuring out Problem 5 required that you understand how to multiply two expressions together like $(x-3)\,(x+2)$—you use the distributive law and then combine "like" terms. Here, then, we get

$x^2 + 2x - 3x - 6 = x^2 - x - 6.$

Recall that like terms are those that have the same power of x.

6

Take two 2-digit numbers, Betty and Crocker.

a. Divide Betty by Crocker. (i.e. $\frac{\text{Betty}}{\text{Crocker}}$)

b. Raise your answer to the 4th power, and then square the result. Call this "Vanilla".

c. Now, instead, divide Crocker by Betty.

d. Cube what you got, and then cube it again. Call this "Chocolate".

e. Multiply "Vanilla" times "Chocolate".

f. Multiply your answer in Part e by Betty. What do you get?

g. Instead, divide your answer in Part e by Crocker, and then take the reciprocal. Now what do you get?

h. Explain the mystery of the universe's existence, or at least Parts f and g.

7

Let's go back to Problem 1, the introductory problem.

a. Calling the original number x, represent what "happens" to the number algebraically in each step.

b. To understand why the trick works, we have to simplify the complicated expression obtained in Part a. The first two steps produces $(x+4)^2$; how can this expression be simplified? Check with an individual value of x to see if your simplification is plausible. Remember, $(x+4)^2 = (x+4)\,(x+4)$.

c. After the 3rd and 4th steps, you should have $\frac{\left(x^2+8x\right)}{x}$. This can be simplified by factoring the numerator. Do so.

d. Just as $\frac{6\cdot 13}{6}$ can be simplified by realizing that $\frac{6}{6}$ $= 1$ and that we can thus cancel out the 6's, leaving just 13, the same is true with x's and y's. For example, $\frac{y(x-5)}{y}$ is just $x-5$, and $\frac{w^5(x+w)}{w^5(x+7)}$ is just $\frac{(x+w)}{(x+7)}$. Now further simplify your answer to Part c.

e. After the next and final step, you should now understand the "trick"! To check your understanding, construct a similar problem where the first two steps are to subtract 5 from the number and then squaring the result, and the final step yields the original number. (No square rooting allowed!!)

Even when you are dealing with two or more variables, the following basic principles still hold and can be quite handy:

Combining like terms when adding or subtracting (meaning terms that have the same powers of the same variables)—

I. $$2xy - 5xy = -3xy$$

II. $$7w^2v^4 - 5w^2v^4 = 2w^2v^4$$

Distributing—

I. $$3xy\left(2xy - 5xy^2 + 4x^2y\right) = 6x^2y^2 - 15x^2y^3 + 12x^3y^2$$

II. $$\begin{aligned}(7x + 4w)\left(5x^2 - 11xw\right) &= 35x^3 - 77x^2w + 20x^2w - 44xw^2 \\ &= 35x^3 - 57x^2w - 44xw^2\end{aligned}$$

Simplifying (by Recognizing "1") when dividing—

I. $$\begin{aligned}\frac{30x^6y^9}{15x^6y^5} &= \frac{2 \cdot 15x^6y^5y^4}{15x^6y^5} \\ &= \frac{2}{1} \cdot \frac{15x^6y^5}{15x^6y^5} \cdot \frac{y^4}{1} \\ &= \frac{2y^4}{1} \\ &= 2y^4\end{aligned}$$

Finding a Common denominator and combining/separating—

I. $$\begin{aligned}\frac{xy^2}{3} - \frac{xy^2}{7} &= \frac{7xy^2}{21} - \frac{3xy^2}{21} \\ &= \frac{4xy^2}{21}\end{aligned}$$

II. $$\begin{aligned}\frac{3}{t} + \frac{4t}{r} &= \frac{3r}{tr} + \frac{4t^2}{tr} \\ &= \frac{3r + 4t^2}{tr}\end{aligned}$$

III. $$\begin{aligned}\frac{6x^3y - 22x^4y^2}{2x} &= \frac{6x^3y}{2x} - \frac{22x^4y^2}{2x} \\ &= 3x^2y - 11x^3y^2\end{aligned}$$

Practice

8 Ask a friend to think of any integer and add the next highest integer to it. Tell him to add 13 to that result, and then divide by 2. Finally, he subtracts the original number he thought of. What will his final answer be? Why?

9 Have a friend select two secret numbers of their choosing—call them Rosencrantz and Guildenstern.

a. Have them triple Rosencrantz, and then add 15 to their result.

b. Then have them multiply Guildenstern by 6, and add 21 to their answer.

c. Have them compute (Answer for Part a) - (Answer for Part b).

d. Tell them to divide their answer for Part c by 3. Let's call this number "Hamlet".

e. Ask them to add 2 to Hamlet, then to subtract Rosencrantz, and finally to tell you the number they've calculated.

f. Tell them you now know what Guildenstern is—because you do! Just divide the number they tell you in Part e by -2, and that is equal to Guildenstern. Why does this trick work?

g. If your friend told you "Hamlet" as well, you could also figure out what Rosencrantz is. Why? Explain.

10 Here's one that involves powers. Again, pick a number.

a. Double it, and then raise your answer to the 3rd power. Call what you get "Jack".

b. Now instead, multiply the original number by 6, and then square the result. Call what you get "Jill".

c. Compute $\frac{\text{Jack}}{\text{Jill}}$, and then multiply that by 4.5.

d. What in the world is going on?

11 Use the distributive property to help rewrite the expression so that it is as compact as possible, often called “collecting like terms”:

a. $-3(a+4b)-6(2b-a)$

b. $(2x-y)(x+3y-7)$

c. $\frac{2}{5}(4m-3n)+\frac{3}{10}(7m+n)$

d. $(j-k)^2-(j+k)^2$

12 Distribute and collect like terms.

a. $2x(-3x^2-4)-2(x^3-4x+7)$

b. $(4xy^2+2x-3y)(5y^2-7x^2+6)$

c. $\frac{(-6w-4)(-8w)}{2}$

d. $-12(3a-2b)-4\left(7a-\frac{3b}{2}\right)$

e. $3\sqrt{x}(\sqrt{x}+4)-7x(x-5)-12\sqrt{x}$

f. $(2k-3)^2$

13 Add or subtract by finding a common denominator, then collect like terms:

a. $\frac{7x}{8}-\frac{8x}{7}$

b. $\frac{3x+2}{y}-\frac{4y-6}{x}$

c. $\frac{3w^2-7u}{4u}+\frac{6u+9w}{5w}$

14

Separate the following fractions into separate terms, then simplify each term:

a. $\frac{6x-18y}{3}$

b. $\frac{5hg^3-10h^2g^5}{5hg^2}$

c. $\frac{8x^2-12xy+20y^2}{-4xy}$

d. $\frac{14y^{-3}-26y^4z^{\frac{3}{2}}+10z^{-\frac{5}{2}}}{4y^3z^{\frac{1}{2}}}$

15

Simplify each of the following by recognizing "1":

a. $\frac{30x^7y^4}{18x^6y^4}$

b. $\frac{48p^{24}q^8}{36q^{24}p^8}$

c. $\frac{51x^{-6}y^3z^0}{17x^{-7}y^5z^{-2}}$

d. $\frac{42\sqrt{z}y^3p^{\frac{5}{2}}}{14p^{\frac{1}{2}}y^4\sqrt{z}}$

16

If $\frac{x+5}{x+1} = 1 + \frac{M}{x+1}$, what number must M be?
(Appeared on Algebra 1 Math Contest, 1995-6, #4)

17

Are the following pairs of expressions always, sometimes, or never equal? If the answer is "sometimes", also find all the values of x when they are equal.

a. $3 - 4x$ and $4(6-2x)+7(x-3)$

b. $2x^2+6x-36$ and $2(x-3)(x+6)$

c. $2(2x^2-18)$ and $\frac{(2x-6)(2x+6)}{(x+17)}$

d. $4-2x-3(5-4x)$ and $-4(3-x)-2(x-3)+7$

18 For each of the following equations, say whether it is an identity, meaning that it is true for all values of the variable(s). Check by assigning numbers to the variables in the equation and seeing if your answer still seems correct!

a. $(x+y)(w)=(xw)(yw)$

b. $a^{-b}=-\left(a^{b}\right)$

c. $4x+3x=7x^{2}$

d. $b^{r}b^{s}=b^{rs}$

e. $(a-b)^{2}=-2ab+a^{2}+b^{2}$

f. $a^{4}b^{3}=(ab)^{12}$

g. $2+3x=5x$

h. $a^{-2b}=\frac{1}{a^{2b}}$

i. $j^{p}j^{q}=j^{(p+q)}$

j. $\frac{5+a}{5x}=\frac{1+a}{x}$

k. $(j-k)^{2}>-2$

l. $\left(\frac{5m-n}{n}\right)-\left(\frac{12n+3m}{2m}\right)=\left(\frac{10m^{2}-5mn-12n^{2}}{2mn}\right)$

19 If $\left(4^{-1}-3^{-1}\right)^{-1}$ were to be rewritten as a single number, what would it be? (No calculators on this one!)

20

Simplify the following (and show the steps you took):

a. $\frac{54x^8y^{-3}}{27x^2y^{-6}}$

b. $\left(\frac{76x^5w^{-7}}{19w^{-4}x^8}\right)^{-4}$

c. $\left(\frac{p^{-3}}{12q^{-2}}\right)^{-1}$

d. $\left(\frac{120x^{-8}y^7}{y^{-2}x^{-4}w^3 \cdot 15}\right)^{-2}$

e. $\left(\frac{x^{3n}y^6}{y^2x^n}\right)^{-2}$

Problems

21 Given that $(x+2)(x+b) = x^2 + cx + 6$ for all values of x, what must c be? (Appeared on AHSME 1988 competition, #5)

22 If $Q = 2(3\pi - 7)$, how many times bigger than Q is $8(12\pi - 28)$?

23 A tree broke at a point $\frac{1}{4}$ the distance up the trunk, and when it fell the top of the tree was 60 feet from its base (so a triangle is formed — the upright $\frac{1}{4}$ of a tree, the diagonal rest of the tree, and the ground). How tall was the tree?

24 If $x = y^2$, what is the value of $y^{1996} - x^{998} + 499$? Copyright *www.mathleague.com.*

25 f is a function that squares a number, then adds 16. g is a function that adds 4 to a number, then squares the result. Anthony says that f and g are really the same function. Is he right?

26 Prove that the square of an even number is always even. Then prove that the square of an odd number is always odd. (Hint: Every even number is a multiple of 2; every odd number is one more than a multiple of 2.)

27 Prove that the square of an odd number minus the square of another odd number is always a multiple of 4.

28 Take two numbers: the first should be a multiple of 12, and the second 3 more than the first. Prove that the second number squared minus the first number squared will never be divisible by 6.

29 If $\frac{1}{x-1} - \frac{1}{x+1} = \frac{C}{x^2-1}$, what number must C be?

30 A box whose dimensions are 4 ft, 4 ft and 4 ft. is packed with cylindrical cans that are 2 ft high with a diameter of 6 in. When the box is fully packed with cans, how much space is wasted in the box? Prove that your answer is the same as $16(4 - \pi)$ ft^3.

31 If $a + b = 0$, but $a \neq 0$, what is the value of $\frac{a^{2007}}{b^{2007}}$? Of $\frac{a^{2008}}{b^{2008}}$? Copyright *www.mathleague.com.*

32 The length (in cm) of the diagonal connecting opposite corners of a cube is the same as the volume of the cube (in cm^3). What is the surface area of the cube?

33 Three adjacent faces of a rectangular box have areas 20, 30, and 40 cm^2. What is the exact volume of the box?

34 Make up a problem like Problem 2 and try it out on your classmate. Make sure to prove that it works before doing it!

35 This trick involves two numbers — your age, and the amount of change in your pocket (expressed in pennies — a number from 1 to 99).

a. Double your age.

b. Add 5.

c. Multiply by 50.

d. Add the amount of change in your pocket (and if you don't have change, just think of a number between 1 and 99).

e. Subtract the number of days in a non-leap year.

f. Add 115.

g. Divide by 100.

h. Look to the left and to the right of the decimal point. Cute, no? How does this one work?

36 Here's a trick for you to finish. Take two positive numbers, x and y.

a. Multiply their sum by their difference.

b. Now multiply your answer by the sum of their squares.

c. Add y raised to the 4th power.

d. Take the square root of your result.

e. How could someone always figure out what "x" was if they had the answer to Part d?

37 Pick two numbers.

a. Multiply the first by three more than the second — call it "Sonny".

b. Multiply the first by three less than the second — call it "Cher".

c. Subtract Cher from Sonny, and then divide by 6. What do you notice?

d. Now add Cher to Sonny, and then divide by double the first number. What have you found?

e. Explain the mysteries of Sonny and Cher. (It might help to think of the two numbers you picked at the start as x and y.)

38

John from Cincinnati claims that he has a snappy way to calculate $37^2 - 33^2$ in his head — just multiply the sum of the two numbers by their difference (i.e. $70 \cdot 4 = 280$).

a. Does John's method actually work? Try it on a few pairs of numbers.

b. If two numbers that you pick are x and y, John's method claims that there is an easier way to calculate $x^2 - y^2$. In terms of x and y, what is John saying is the easier way?

c. Prove that John's method always works, or that it doesn't always work.

d. Calculate $104 \cdot 96$ in your head! (Hint: Work backwards!)

e. Distribute $(2w - 12z)(2w + 12z)$ and $(x^3 + y^7)(x^3 - y^7)$; what do you notice?

39

Jake made up the following number trick, which he was sure would dazzle all his friends. "Pick a number without telling me. Multiply it by 100, and subtract that from 2500. Now add the square of the original number, and tell me the answer." Jake knew that all he had to do was call the final result Y, use his calculator to compute $50 - \sqrt{Y}$, and he'd have their original number back.

a. Confirm that Jake's method works for a few numbers. Then explain why it works.

b. For a while, Jake's trick worked wonderfully. But the other day, his friend Rachel said that Jake had failed to read her mind! Her number had been 99, but Jake had guessed 1. How could he have been so hideously far off?

40

Phil from Duluth has another calculating trick. He says that he can calculate $33 \cdot 27$ in his head by taking the square of the sum of the two numbers and subtracting the square of their difference, and then dividing the answer by 4: $\frac{(33+27)^2-(33-27)^2}{4}$. This is quicker, Phil points out, because the numbers are easier—

$$\frac{60^2-6^2}{4} = \frac{60^2}{4} - \frac{6^2}{4} = 900 - 9$$

is a lot easier to calculate in one's head than $33 \cdot 27$.

a. Is Phil correct that this method always gives the right answer? Will this always work with any two numbers? Prove your answer.

b. Is Phil correct that this method is always quicker? Give some examples to back up your position.

41 A terminating decimal is a decimal like $.3764$, which ends. A repeating decimal is one that repeats forever, like $.862862862\ldots$.

a. How can one express any terminating decimal as a fraction? Show that you can do so with $.3674$ and $.1864597$.

b. It isn't obvious whether repeating decimals can be expressed as fractions or not. Still, if $.862862862\ldots$ were to be equal to a fraction, it would have to be close to $\frac{862}{1000}$. See if you can find a fraction which appears to do better.

c. Will the idea of part b work with any repeating decimal? Try it out on $.9166591665\ldots$ and $.333\ldots$.

d. Let $x = .333\ldots$. By multiplying both sides of this equation by 10, one gets a new equation. By combining equations, prove why $.333\ldots = \frac{1}{3}$.

e. Now prove that $.862862862\ldots$ is a fraction as well by a similar method to part d. Will this method let you convert any repeating decimal into a fraction? Does it work on $.9166591665\ldots$, for example?

42 Don't use a calculator for this problem.

a. Factor $x^2 - x + 12$.

b. Solve $x^2 + 4x + 2 = 0$ by completing the square.

c. Divide $5.\overline{3} \div 8$.

d. Simplify $\sqrt{50a^4b^3}$.

e. If $(x-2)(x+2) = 5$, find $x^4 - 4x^2$.

43 Ask a friend to think of a three-digit number in which the digits are all the same (such as 777). Tell them that you don't want to know the number, but that you would like them to add up the digits, multiply by 37 and tell you the answer. Why would they think you are being a wiseacre? Can you prove it always "works"?
(Hint:
$777 = 700 + 70 + 7 = 7\,(100 + 10 + 1)$.)

44 If I have a larger number divisible by 3 and subtract a smaller number divisible by 3, will the answer necessarily be divisible by 3? Try some examples. Explain why what you have found makes sense.

45 If I have a larger number and subtract a smaller number divisible by 3 and get an answer that is divisible by 3 — is the larger number necessarily divisible by 3? Again, try some examples, and explain why your conclusions are justified.

46 After having completed and digested the ideas of Problems 44 and 45, we are now ready to understand why the divisibility test for 3 works. Remember that the divisibility test for 3 states that if the sum of the digits of a number is divisible by 3, then the number itself is as well. (You will prove below that it works for a three digit number, but your proof can be adapted easily to any number of digits.)

a. We can write 741 in the following way: $700 + 40 + 1$, and similarly we can write 6518 as $6000 + 500 + 10 + 8$. How could you write any number "ABC" in the same way, where A is the hundreds digit, B is the tens digit, and C is the units digit?

b. Consider 568, which is not divisible by 3. Notice that when you subtract the sum of its digits ($5 + 6 + 8 = 19$) from itself, the answer you get, 549, is divisible by 3. Does this always work for any 3-digit number?

c. Use algebra to prove the result you found in part b. Your answer to part a. will be helpful here.

d. If $(A + B + C)$ is also divisible by 3, by using what you learned in Problem 45 and parts b and c above, what can you now say about the number "ABC"? Explain!

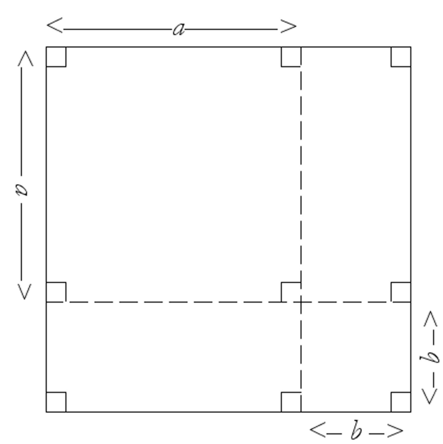
e. If $(A + B + C)$ is not divisible by 3, what, if anything, can we say now about "ABC"? Explain carefully.

47 Pick a 3-digit number. If one subtracts the "reverse" of the number from the number itself, the answer is always a multiple of 9. For example $852 - 258 = 594$, which is a multiple of 9 (and negative numbers "work" as well — e.g. If we had picked 258, $258 - 852 = -594$). Explain why this is so. (Remember that any three digit number "ABC" can be written as $100A + 10B + C$; how do you think you could write its reverse, "CBA"?) Will this work for 4 or 5 digit numbers as well?

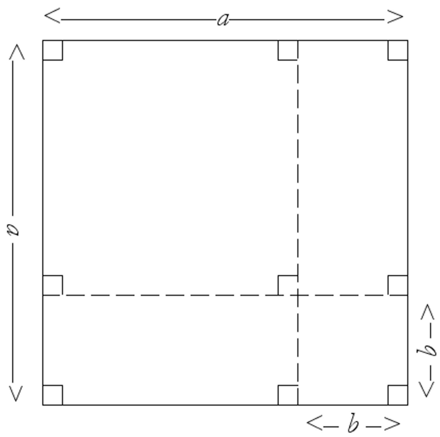
48 Go to the following web page and have fun!: http://digicc.com/fido/ What is going on here? Can you come up with an explanation of how it is done?

49 Look at the following YouTube video on the method of "Vedic Multiplication": www.youtube.com/watch?v=gwaAAEYIW_8 Why does this method work? Will it always work?

50 Explain what the following "Proof without Words" is equivalent to algebraically:



51 Explain what the following (slightly harder) "Proof without Words" is equivalent to algebraically:



Exploring in Depth

52 Is the sum of 3 consecutive integers always divisible by 3? Is the sum of 4 consecutive integers always divisible by 4? Is the sum of 5 consecutive integers always divisible by 5?

53 When is the sum of n consecutive integers divisible by n? Try different values of n to get a feel for the problem, and then see if you can give good reasons for the pattern you see.

54 Prove that the square of an odd number is always one more than a multiple of 8. (Hint: Try factoring part of the expression you came up with to represent the square of an odd number, and also note that with any two successive integers, one of them must be odd and one of them must be even.)

55 Which primes can be expressed as the difference of two squares? Why? Can any be written as the difference of two squares in two different ways? Why or why not?

56 Here's a quick way of multiplying any two 2-digit numbers, like 52 and 58, that have the same tens digit and have units digits that add up to 10.

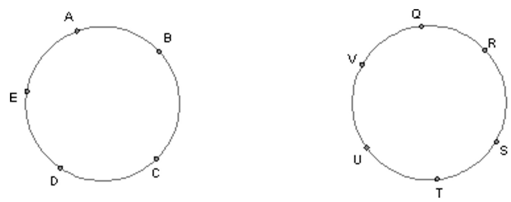
1. Multiply the tens digit by the next largest integer and call it "M" (here, $M = 5 \cdot 6 = 30$).
2. Multiply the units digits of the two numbers together and call it "N" (here, $N = 2 \cdot 8 = 16$).
3. Write the digits of M followed immediately by the digits of N (here, 3016). This gives you the right answer!

Why does this work?

57 Four whole numbers, when added three at a time, give the following sums: 188, 199, 212, and 220. Find the four numbers.

SUMMARY AND REVIEW

1 a. How many colors would it take to color each one of the graphs below? Explain your answer for each graph.



b. What if there were "n" nodes on the circle (instead of the 5 nodes and 6 nodes in the graphs of part a) — then how many colors would it take to color? Answer clearly and completely.

2 Chris draws a graph that has an Euler Circuit. Alonso adds an edge to Chris's graph, and now, although the graph no longer has an Euler Circuit, Alonso says it has an Euler Path. Can Alonso possibly be right? Explain.

3 From Crisler and Froelich: *Discrete Mathematics Through Applications*. The following is a list of chemicals and the chemicals with which each cannot be stored.

Chemicals	Cannot Be Stored With
1	2, 5, 7
2	1, 3, 5
3	2, 4, 5
4	3, 7
5	1, 2, 3, 6, 7
6	5
7	1, 4, 5

a. Draw a graph to represent the situation.

b. How many different storage facilities are necessary to keep all 7 chemicals?

4 Uler says she just drew a graph with 1001 nodes, and exactly 500 of the nodes are of even degree. Do you believe her? Explain.

5 How many colors does it take to color a complete graph with n nodes? Explain.

6 A graph has 6 nodes with degrees 3, 3, 4, 5, 3, and 2. Is it possible for the graph to have an Euler path?

7 For each of the following say whether it's true or false. If false, produce a counterexample or say precisely why.

a. Sum of degrees of any graph is even.

b. Every graph has an even number of nodes of odd degree.

c. If a graph has an Euler path then it must have exactly two vertices/nodes of odd degree.

d. Every graph of 4 or fewer nodes is planar.

e. All the nodes of a graph could be of odd degree.

f. A planar graph could be isomorphic to a non-planar graph.

g. All complete graphs with four nodes are isomorphic to each other.

8 For some graphs that have Euler circuits, it is actually still possible to walk around those graphs in such a way that you get stuck before you've crossed every edge. Draw one such graph, and show (a) how to walk around it so that you cross every edge exactly once, and (b) how to walk around it so that you get stuck before you get to every edge.

9 Simplify $\left(\frac{120x^{-8}y^{7}}{y^{-2}x^{-4}z^{3}\cdot 15}\right)^{-2}$ making sure there are only positive exponents in your simplified answer.

10 Write the fraction in as simplified a form as possible, without using a calculator: $\left(\frac{105/117}{165/363}\right)^{-3}$

11 If x and y are > 0, express $\frac{x^y y^x}{y^y x^x}$, using only a single exponent.

12 Find the prime factorization of 533610. Use a calculator to help! Now, find the prime factorization of 20! (20 factorial), without a calculator.

13 How many of the first 200 positive integers are divisible by all of the numbers 2, 3, 4 and 5?

14 Suppose p and q are prime numbers.

a. What are all the factors of the number pq ?

b. List all the factors of p^{10} .

c. How about p^2q^2 ?

15 Write each of the following numbers in the form $2^a 3^b 5^c 7^d$, where a, b, c, d can be whatever you want.

a. 2100

b. $\frac{1}{21}$

c. 2

d. 1

e. 2.1

16 Write the GCD of $2^a 3^b 5^c 7^d$ and $2^{a+1} 3^{b-2} 5^{c+4} 7^d$, assuming that a, b, c, d have to be integers larger than 1.

17 Is $\frac{50^{24}}{20^{12}}$, simplified, an integer or a fraction? Explain.

18 Simplify the following expressions, and make sure there are no negative exponents:

a. $\frac{36(xy^4)^{-2}}{40x^5y^{10}}$

b. $\left(\frac{(x^2y^{-3})^2}{x^{-2}y^5}\right)^{-1}$

c. . $\left(\frac{16^{-4}x^{-4}y^{-2}}{128^{-2}x^{-3}y^{-3}}\right)^3$

19 If $m \neq 0$, then $\frac{(m^4)^4}{m^4}$ $= \ldots$?

20 If $4^{6x-9} = 64$ then what is the value of x?

21 You ask your friend to do the following calculations:

a. Take a number

b. Add 1

c. Multiply by 3

d. Multiply your answer in (c) by one less than the original number ...

When your friend tells you the answer in d, what's the fastest process you can use to figure out what the original number was?

22 Another number trick:

a. Pick a number.

b. Double your number.

c. Square the result in "b"

d. Subtract 4. Call this answer "Fred."

e. Now take your original number and add one.

f. Multiply this number by one less than the original number. This is "Lilli".

g. What is $\frac{\text{Fred}}{\text{Lilli}}$? Why does this work?

23

You ask your friend to think of a number, then

- Multiply by 6
- Add 9
- Divide by 3
- Subtract 3.

Your friend tells you the answer. What do you need to do to get the original number back so that you can convince your friend you are a mind-reader? Show some algebra to explain why.

24

Which equation represents the following statement?
Twice the difference between a certain number and its square root is 15 more than twice the number.

a. $2N - \sqrt{N} = 15 + N$

b. $2\left(N - \sqrt{N}\right) = 15 + 2N$

c. $2N - \sqrt{N} = 16 + N$)

d. $2N - \sqrt{N} + 15 = N$

e. $2\left(N - \sqrt{N}\right) + 15 = N$

f. $2N - 2\sqrt{N} = 15 + 2N$

25

Simplify the following so that there are only positive exponents: $\left(\frac{168p^{-4}q^5r^{-6}}{42pq^{-2}r^3}\right)^{-2}$

26

Simplify this until it is a single fraction that is reduced as much as is possible:
$\left(\frac{180\cdot 5}{4\cdot 196}\right) \cdot \left(\frac{448\cdot 49}{1800\cdot 15}\right)^2$

27

Simplify by multiplying out and collecting like terms.
(By the way, these specific expressions tend to show up very often in algebra.)

a. $(x+y)^2$

b. $(x-y)^2$

c. $(x+y)(x-y)$

d. $(1+x)(1-x)$

e. $(x+1)(x-1)$

28

Simplify by multiplying out and collecting like terms:

a. $4b(b-9) - b(2-b)$

b. $(z^2y + z^2 - x)(z - x)$

c. $(m+n)(m^2 - n^2)$

d. $-5x(x+3y)^2$

e. $(5t+2)(25t^2 - 10t + 4)$

f. $(z^4 - 3)^2$

g. $(2ab^2 - 5a^2b)(21a^5b^4 - 14a^4b^5 + 7ab^3 - b)$

h. $(2a+3b)^3$

i. $1 - (1 + [1 - (1 + y)])$

29

If $x^2 + 7x + 8 = (x+3)(x+4) + p$ is always true, then $p = ...$

30

Reduce each fraction as much as you can (which may be not at all).

a. $\frac{14a^2-18a}{2a}$

b. $\frac{x^2y-x^7y^2}{x^2y}$

c. $\frac{-12m(m-13)}{3}$

d. $\frac{4x+2y-3}{2}$

e. $\frac{8x^3y^5-6x^4y^3}{x^2y^3}$

f. $\frac{24r^3s^{-2}-36r^{-1}s^5}{r^{-2}s^{-3}}$

g. . $\frac{6x^3-12xy^{\frac{2}{3}}}{x^{\frac{1}{2}}\ y}$

h. $\frac{6f^4g-5f^2g^3}{fg}$

i. $\frac{\left(\frac{2x+2}{x}\right)}{\left(\frac{x}{x+1}\right)}$

j. $\frac{\left(n^2-6n+14\right)^6}{\left(n^2-6n+14\right)^4}$

31

Add the fractions:

a. $\frac{8}{p} + \frac{q}{2}$

b. $\frac{1}{x} + \frac{3}{x^2}$

c. $\frac{2y+3}{6} - \frac{y+3}{4}$

d. $\frac{3}{ab} - \frac{4}{b^2}$

e. $\frac{4}{2x-1} - \frac{3}{2x}$

f. $\frac{2}{x-1} + \frac{2}{x+1}$

g. $\frac{w^2+2w+1}{w} - \frac{(w-1)^2}{w}$

32

What is the result when $3 - 2x$ is subtracted from the sum of $x - 3$ and $5 - x$?
(From Andres *et al*, *Preparing for the SAT Mathematics.* Amsco, 2005.)

33

If $3x - 7 = 5$, then $9x - 21$ =...?

34

If $x^2 + y^2 = 37$ and $xy = 24$, what is the value of $(x-y)^2$?

35

Prove that if one positive integer is 3 more than another, the difference in their squares must be a multiple of 3, but cannot be a multiple of 6.

36 Write each in the form a^b , where a and b are integers not equal to 1:

a. $3^4 9^5 27^6$

b. $4^5 15^{10}$

37 How many 2^3 terms are there on the left side of the equation?

38 Solve for x:

a. $3x^5 = 96$

b. $\frac{2}{3}\ x^3 = 22$

c. $x^{\frac{4}{5}}\ = 37$

d. $6x^{-\frac{1}{6}}\ = 66$

e. $217p^{\frac{2}{9}}\ = 53$

f. $5\sqrt{5x-1} = 7\sqrt{2x+5}$

g. $\sqrt{3x+3} = x + \frac{3}{2}$

39 Rewrite in the form $a^b c^d$, where a, b, c, and d are each integers not equal to 1 and less than 10:
$27^{\frac{4}{3}}\ \cdot 216^{\frac{2}{3}}\ \cdot 32^{\frac{3}{5}}$

40 Simplify the following expressions, making sure there are no negative exponents:

a. $\frac{x^{-\frac{2}{3}}}{x^{\frac{4}{3}}}$

b. $\frac{x^2 y^{\frac{1}{4}}}{y^{-\frac{1}{2}}\ x^{\frac{5}{3}}}$

c. $\left(\frac{16^{\frac{3}{4}}\ x^{\frac{4}{3}}\ y^{-2}}{121^{-\frac{1}{2}}\ x^{-\frac{2}{3}}\ y^{-3}}\right)^3$

41 How many distinguishable 6 letter words can be formed from the letters of “twists”, where each letter can only be used once per word? What if the t’s and s’s were given different colors, so they could be distinguished from each other as well?

42 How many ways can you step right 4 times and left 7 times in a sequence of 11 steps? Put another way, how many ways can you arrange 4 R’s and 7 L’s in a straight line, where the R’s and L’s are indistinguishable from each other?

43 Suppose you’re going to toss a coin 8 times and record the sequence of heads and tails. How many different sequences are possible?

44 Josephine flips a coin 9 times. What is the probability that she’ll get exactly 6 heads? 4 heads? 3 heads?

45 How many ways are there to form a committee of 5 people from a class of 80? (There are at least two different methods to solve this problem)

46 In going packing for a trip, you and your spouse need to pack 4 ties and 6 dresses. You own 7 ties and 10 dresses. How many different ways can you select the ties and dresses for your trip?

47 How many ways can you select 4 letters from the word "fishbowled", if the order of the letters selected is unimportant? What if the order is important?

48 If you start in the lower left corner of an 8x8 chess board, and for each "step" you can move either one square to the right or one square up, how many different ways are there to reach the upper right corner of the board?

49 In how many ways can five girls be chosen from a class of 20 if Sarah Dreadful has to be chosen (so that she won't throw a fit)?

50 Julius goes to a yard sale and sees 8 books on sale. He thinks that he would like to buy either 2, 3, or 4 books. How many different possibilities are there for what books he could buy at the yard sale?

51 Ten people meet at a party, and each pair of people shake hands. How many handshakes are there?

52 Three identical prizes are to be given to three lucky people in a crowd of 100. In how many ways can this be done? What if the prizes were not identical?

53 You plan on forming a Sophomore Prom committee consisting of 8 sophomores. The Sophomore Class has 45 boys and 35 girls in it. How many distinct committees can be formed if

a. there are no restrictions placed on the committee other than there must be 8 sophomores on the committee?

b. there are to be 4 girls and 4 boys on the committee?

c. the committee cannot have Sarah Dreadful on it?

54 A boat has 3 red, 3 blue, and 2 yellow flags with which to signal other boats. All 8 flags are flown in various sequences (1 flag at a time) to denote different messages. How many such sequences are possible?

55 Five boys and five girls stand in a line. How many different arrangements are possible:

a. If boys and girls can be intermixed (or not) in the line?

b. If all the boys stand at the back of the line?

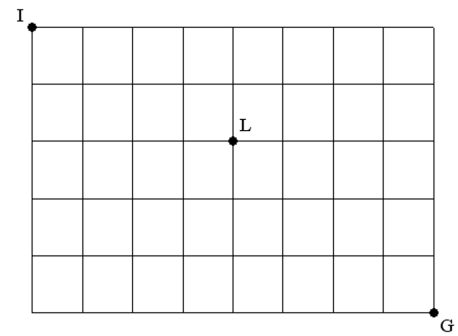
56 Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

57 A "necklace" is a circular string with several beads on it. Two necklaces are the same if they are just rotations, in the plane, of each other. How many different necklaces can be made with 10 different beads?

58 How many 6-digit numbers have at least one even digit?

59 There are five books on a shelf. How many ways are there to arrange some or all of them in a stack? The stack may consist of one book.

60 The map of a town is depicted below. Ian lives at point I; the library is at point L; and the grocery store is at point G. Ian can only walk south or east. The town is many times bigger than the map.



a. How many different ways are there for Ian to walk from his house to the library?

b. How many different ways are there for Ian to walk from the library to the grocery store?

c. If Ian wants to walk from his house to the grocery store by way of the library, how many different ways can he do this?

d. Suppose Ian uses a coin flip to decide whether he's going to go one block south or one block east. After he walks a block, he then tosses the coin again and follows the previous guidelines regarding his movement. If it takes Ian one minute to walk one block, then what is the probability that he will make it from his house to the grocery store in 13 minutes? Assume it takes him no time to flip the coin and interpret the meaning of the outcome.

61 One student has 6 novels and another has 7 novels. How many ways are there for the first student to exchange 3 novels with 3 novels owned by the second student?

62 How many 5 digit numbers have factors of 4 or 5?

63 The equation below shows a series of 1's summing to 14.

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 14$$

a. How many ways can you choose 3 of the plus signs to circle?

b. Clearly explain how your answer to Part a also answers the question below.

c. How many solutions are there of the equation $x + y + z + w = 14$ in positive integers? Note: the solution $(1, 2, 8, 3)$ is different from the solution $(2, 3, 1, 8)$.

64 What is the probability of getting 5 heads out of 10 tosses of a fair coin?

65 Suppose you draw at random 5 cards from a standard 52-card deck.

a. How many possible different 5-card hands are there?

b. How many ways can you pull 5 cards such that you get exactly one ace?

c. How many ways can you pull 5 cards such that you get exactly two aces?

d. How many ways can you pull 5 cards such that you get exactly two aces and three kings?

e. What's the probability of pulling 4-of-a-kind?

66 Suppose you are standing at zero on the real number line. You toss a fair coin and move 1 unit to the left if you get heads. You move one unit to the right if you get tails. You do this 17 more times for a total of 18 coin tosses.

a. After 18 tosses what are the possible numbers you might be at on the real number line?

b. What is the probability that you will be standing at 0 after 18 tosses? At -2? at -3?

67 For fun on a Friday night you and a friend are going to flip a fair coin 10 times. Let H represent the outcome that a flip shows heads and T represent tails. Assume that $\text{Prob}(\text{H}) = \text{Prob}(\text{T}) = 0.5$.

a. You flip the coin 10 times and get the sequence HTHHHTTHHT. Your friend does likewise and gets HHHHHHHTTT. Which of these two sequences was more likely to occur? Justify your response.

b. What is the probability that you will get at least two heads in 10 tosses of the coin?

68 The numbers 1, 2, 3, ..., 25 are placed in random order. What's the probability that the numbers 1, 2, 3 are next to each other?

69 a. Suppose that 14 people are arranged randomly in a line and two of these people are Emma and Eric. What's the probability that there are 3 people standing between Emma and Eric?

a. Suppose that the 14 people in Part a are arranged in a circle. What's the probability that there are 3 people standing between Emma and Eric?

70 Suppose you will roll a standard, fair six-sided die until you get a 2. What's the probability that you will stop rolling after the first roll? What's the probability that you will stop after the second roll? What's the probability that you will stop after n rolls?

71 Eight first graders, 4 girls and 4 boys, arrange themselves at random around a merry-go-round. What is the probability that boys and girls will be seated alternately?

72 Rick writes letters to 8 different people and addresses 8 envelopes with the people's addresses. He randomly puts the letters in the envelopes. What is the probability that he gets exactly 6 letters in the correct envelopes? What about 7 letters?

73 There are 40 people waiting to be selected for a 12-person jury. Of the forty, 20 are African-American, 15 are Hispanic, and 5 are Caucasian.

a. How many different 12-person juries (called "jury panels") can be picked from this group of 40? Show your work.

b. How many jury panels can be picked that have only African-Americans? Show your work.

c. Show that the probability that an all African-American panel would be picked just by chance is approximately .0000225. Show your work.

d. If you were a judge and the two lawyers working a case in your court picked an all African-American jury panel (or a jury panel with no African Americans on it) from the pool of 40 given in the beginning of this problem, would you believe that this panel was created without consideration of race? Explain, making direct reference to the probability in Part c.

74 Suppose Tony's class is trying to form two committees — the Social Committee and the Fund Raising Committee — from the fourteen students in his class. How many possible arrangements are there of the fourteen students into these two committees, if each student is allowed to choose one committee or neither to be on? Show your work.

Park School Mathematics

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