BOOK 6: GEOMETRY AND PROOF

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

HABITS

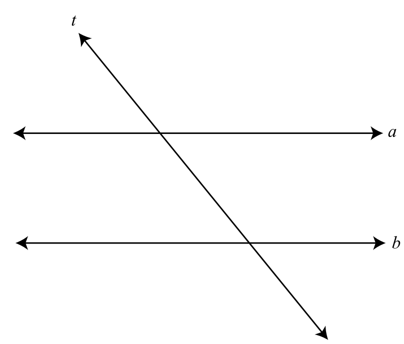
LESSON 1: PARALLEL LINES AND FIRST PRINCIPLES

Introduction

You have already seen how coordinate geometry works, with the focus on slope and distance. Now you will be considering an earlier geometry, where there are no coordinates, so lines don't have particular slopes, nor do we determine numerical distances between points. It's just our reasoning and imagination that enable us to discover valuable relationships among the figures we draw.

In this lesson, you will start to use this reasoning and imagination to help uncover the structure of geometry. Some things you know about geometry are simple and you may be happy to take them on faith. Others are less obvious, but can be explained by a careful argument. Still others may remain mysterious for now.

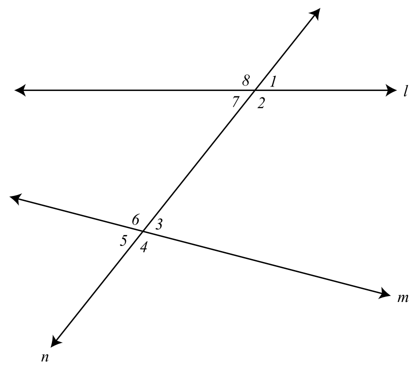
To give you some tools, let's start with the figure below, which you have probably seen in various contexts. The figure shows parallel lines a and b and line t crossing them. The line t is called a transversal, as it "moves" across the other pair of lines.



1 In the figure above, label all the acute angles you believe to be equal in measure with the letter x. Then label all the obtuse angles you believe to be equal in measure with the letter y. Are there any angles that you haven't labeled?

2 What is the relationship between the value of x and the value of y? Make an argument that doesn't depend on knowing the specific value of x or of y.

Some of these pairs of angles come up often enough so as to deserve special mention. In the diagram below angles 1 and 3 are **corresponding angles**. Angles 3 and 7 are **alternate interior angles**. Angles 2 and 3 are **same-side interior angles**. Notice that lines l and m need not be parallel, yet the names still apply.



3 Name another pair of angles in the above figure that are

a. Corresponding.

b. Alternate interior.

c. Same-side interior.

4 In the figure showing lines l and m not parallel, you'll notice that there are no longer so many pairs of equal angles. But there is a fact that carries over from the previous figure. What is it, and how do you know for sure?

5 If you have two parallel lines crossed by a transversal, do you think that the pairs of corresponding angles will always be equal? How about the pairs of alternate interior angles? Try to find an exception.

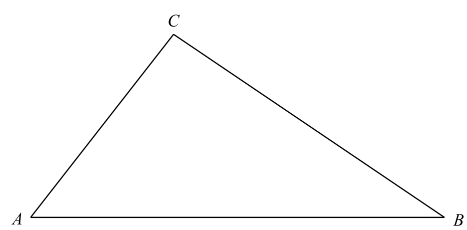
6 Draw another two parallel lines crossed by a transversal. The same-side interior angles probably don't look equal. Is there anything else you can say about them? If you're not sure, take some measurements.

Development

The results you found/reviewed in the introduction are basic results in geometry. You may have learned them before, and you may consider them so obvious that you wonder why anyone would point them out. The reason is that we can use those simple facts to convince ourselves of some results that are not so simple!

For instance, you have probably been raised to believe that there are 180 degrees in a triangle, even though that fact is not visually obvious. But the few results we've stated so far in this chapter can serve to convince us once and for all.

Below is a triangle, meant to be general enough to stand for any triangle. It would be nice to use some of your findings so far to prove that the sum of its angles is 180 degrees. However, there are no parallel lines in this picture, so there doesn't appear to be much we can do.



7 Add something to the diagram so that you do have some parallel lines.

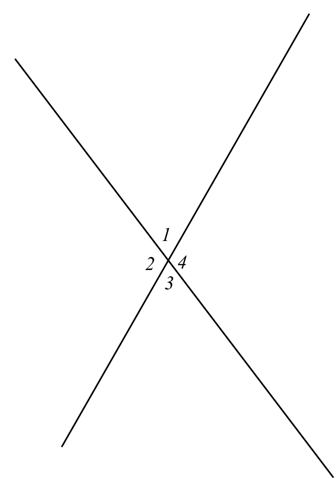
8 Now that you have some parallel lines, label as many angles as you can that you know to be equal.

9 Now finish the proof that the angles of a triangle add up to 180 degrees.

You have probably known for a long time that the angles of a triangle add up to 180 degrees — in that sense, you did not learn any new facts by doing this proof. However, you may not have realized how strongly the result depends on alternate interior angles (or whichever angle pair you decided to use) being equal when lines are parallel. In fact, there are non-Euclidean geometries in which the alternate interior result is not true, where the alternate interior angles result is not true — and, sure enough, in those geometries, the angles of a triangle can actually add up to more or less than 180 degrees!

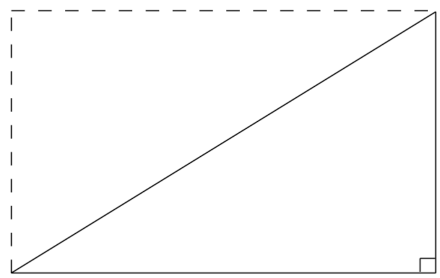
The ancient Greeks appreciated how important it was to uncover the structure of geometry, and very often looked for proofs of statements they were pretty sure were true using statements they already knew to be true. They tried to reduce the number of statements they accepted without proof to the smallest number possible. The most famous Greek geometer, Euclid, was able to reduce his number of assumptions to five mathematical statements and five "common notions" — like "two things equal to the same thing are equal to each other." He was able to prove a huge amount of geometry from these simple assumptions.

10 In the spirit of Euclid, try to prove that angles 1 and 3 below are equal. What facts are you relying on?

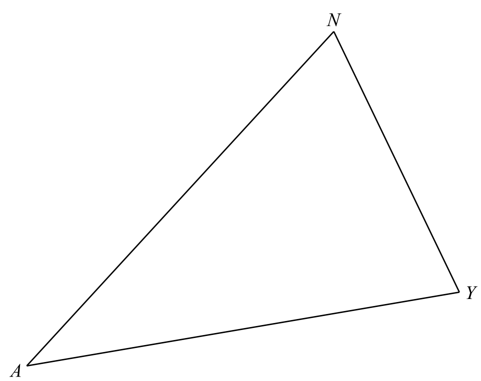


The rest of this Development consists of examples of some other argument styles that do not involve coordinates.

Another fact you have known for a long time is that the area of a triangle can be found by the formula Area $= \frac{1}{2} \cdot$ base $\cdot$ height. You might also have seen the following picture, designed to show why the formula is true for right triangles.



But alas, not all triangles are right, and we'd still like to be able to use this familiar formula with any triangle. So below is triangle ANY.



11 Add something to the diagram so that you can use what you know about the area of rectangles and right triangles to find its area. Is your formula equivalent to Area $= \frac{1}{2} \cdot$ base $\cdot$ height?

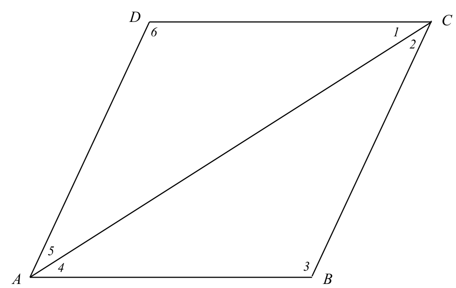
12 And last, but not least, a Physical Challenge:

a. Using only a compass and a tool for drawing straight lines, construct an equilateral triangle. Your tool for drawing straight lines may not be a ruler, or anything else that can take measurements.

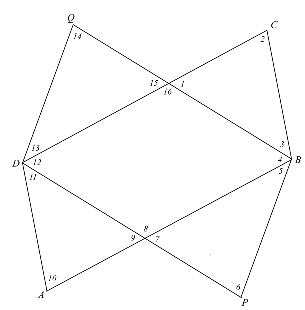
b. Come up with an argument that would convince a friend that the triangle you constructed really has to be equilateral and, if they did the same thing, their triangle would be equilateral, too.

Practice

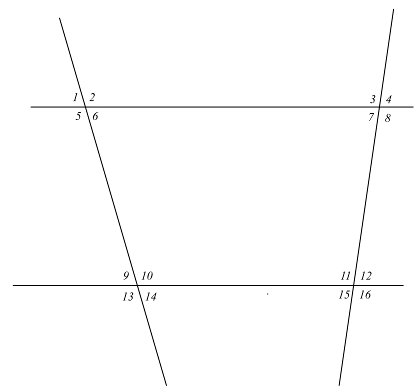
13 Given that $\overline{AB}$ is parallel to $\overline{CD}$, which pairs of angles must be equal?



14 In the figure below, ABCD and PBQD are parallelograms. Which of the numbered angles must be the same size as the angle numbered 1?
(*Copyright Phillips Exeter Academy*)



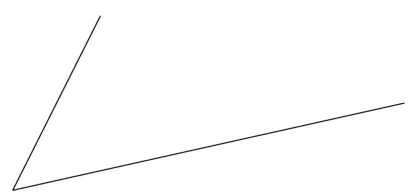
15 Given the following figure, where only the horizontal segments are parallel, name three angle pairs that are equal and three angle pairs you can conclude are **supplementary: that is, sum to** 180°.



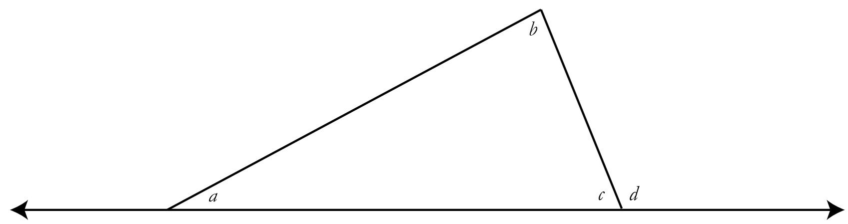
16 Assuming that alternate interior angles are equal when lines are parallel, prove that corresponding angles must be as well.

17 In problem 11, triangle ANY was meant to stand for any triangle, but notice that triangle ANY was drawn to be acute. Try the argument again with an obtuse triangle, and see if you can still obtain the formula $\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$.

18 The two line segments below are "half" of a parallelogram. Trace it into your notebook. Then use a compass and straightedge to draw in the remaining two line segments. How do you know for sure that what you've got is a parallelogram?



19 In the figure below, the angle *d* is an **exterior angle** of this triangle. Determine experimentally if there is any relationship between d and the measure of any angles of the triangle. Does your observation apply even for an obtuse triangle (a triangle where one angle is greater than 90 degrees)?



20 Prove the result you found in the previous problem.

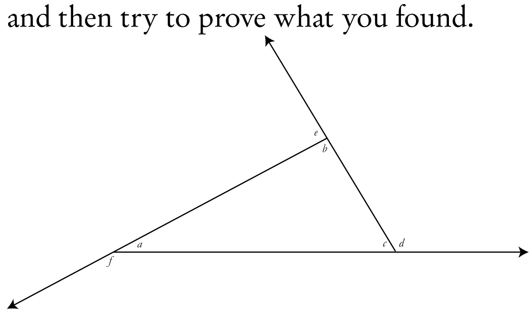
Problems

21 Now that you've proved that the sum of the measures of the interior angles of any triangle is 180°, experiment with the sum of the measures of the interior angles of any quadrilateral. How about other polygons? Prove what you find.

22 How many regular polygons have interior angles that are integers? How can you be sure your answer is correct?

23 On the bus, Rafael asked his three friends what they had figured out for the sum of the interior angles of a 500-sided polygon. Lemuel said he got 8640. Julia got 89640, and Marla got 899640. Rafael didn't have any of his books, pencil, or paper, but he was still able to determine who was likely to be right. Can you?

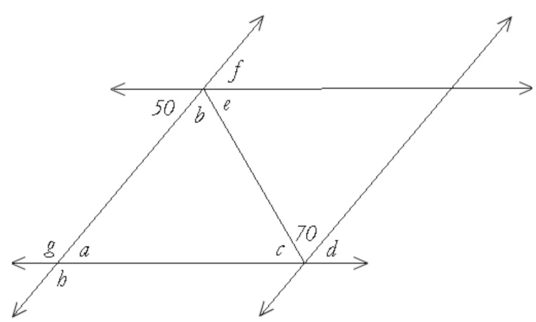
24 You know the sum of the interior angles of a triangle, but how about the sum of the exterior angles of any triangle? (see the figure below — the exterior angles are d, e, and f.) Take measurements, if necessary, and then try to prove what you found.



25 What do you suppose would be the sum of the exterior angles of a quadrilateral? Try it out on some different quadrilaterals. See if you can prove it.

26 Draw any triangle ANY. Recall that a **median** is a line segment from a vertex of a triangle to the midpoint of the opposite side. A median will split a triangle into two smaller triangles. Try this with a few different triangles. Try to draw a triangle so that the left triangle created by the median visibly has a larger area than the right.

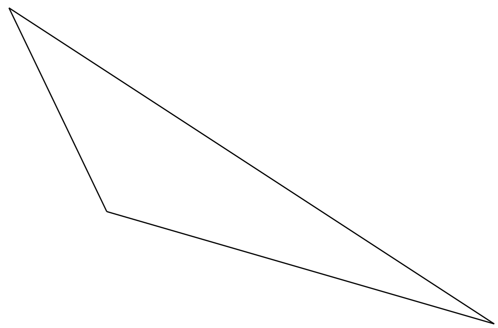
27 Given the following figure, where both pairs of lines that appear parallel are assumed parallel, find all the angles represented by letters.



28 Try to prove or disprove the statement "If two lines intersect the same line, they must intersect each other."

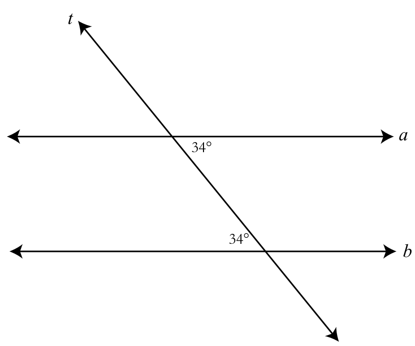
29

Here is a triangle. Using only a compass and straightedge, make an exact copy of the triangle in your notebook. No tracing allowed.



30

You know from the beginning of the chapter that, if lines are parallel, then alternate interior angles created by a transversal are equal. So, does this imply that in the figure below, must lines a and b be parallel?

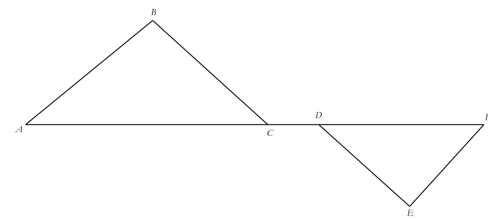


31

Though it seems obvious, one of Euclid's theorems was "If two lines are parallel to a third line, they are parallel to each other." Prove this statement. Remember the tools you have to prove that lines are parallel.

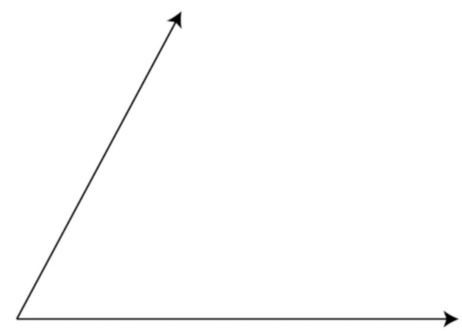
32

In the figure below, (which means the measure of angle BCA equals the measure of angle FDE). Also, the points A, C, D, and F all lie on a straight line. Is this enough evidence to claim that $\overline{BC}$ must be parallel to$\overline{DE}$? Explain.



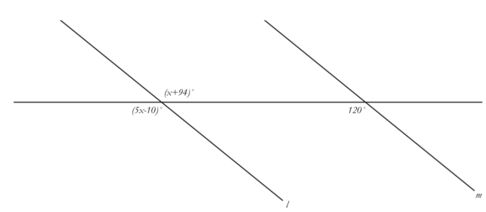
33

Here is an angle. Using only a compass and straightedge, find a way to copy the angle into your notebook. (Again, no tracing allowed.)

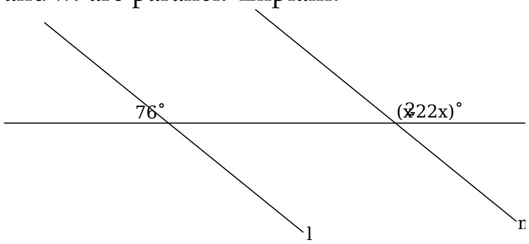


34

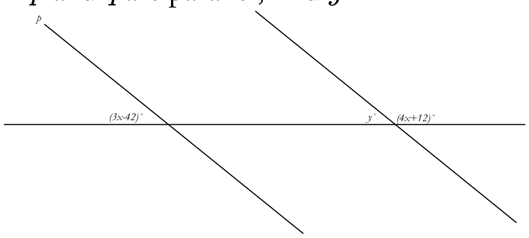
Would you claim that l and m are parallel in the following figure? Explain.



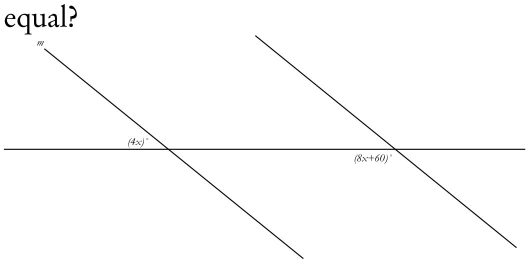
35 In the figure below, what would x need to equal if you could claim correctly that l and m are parallel? Explain.



36 If p and q are parallel, find y.

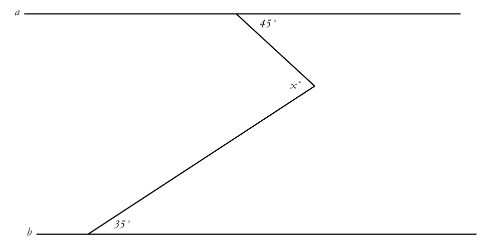


37 If m is not parallel to n, what can't x equal?

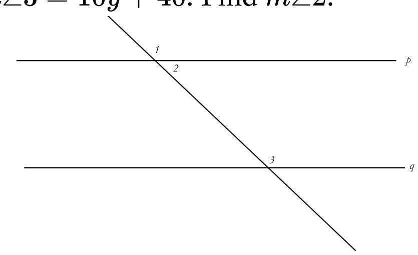


38 The shorter side of a rectangle is 10 units. If the diagonal is two units more than the longer side of the rectangle, what is the area of the triangles created by the diagonal?

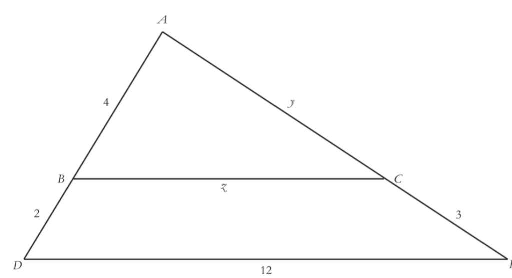
39 If a is parallel to b, find x. (Hint: this is a problem for which geometric tinkering will prove useful.)



40 Assume line p is parallel to line q. You are given that $m\angle 2 = x + 15$, and $m\angle 3 = 10y + 40$. Find $m\angle 2$.



41 Given that $\overline{BC}$ is parallel to $\overline{DE}$, what can you conclude about triangles ABC and ADE? What would be the length of $\overline{AC}$? $\overline{BC}$?



42

Don't use a calculator for this problem.

a. Simplify $\sqrt{\frac{3}{4}}$

b. Simplify $y^2 - \left(\frac{1}{2}\, y\right)^2$

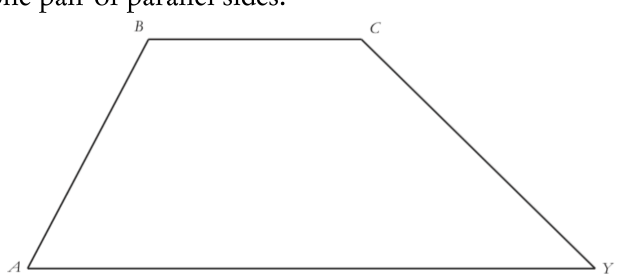
c. Factor $x^2 + 3x + 2$

d. Reduce the fraction $\frac{x^2-8x+16}{x^2-11x+28}$

e. What value of x satisfies $2^5 + 2^5 + 2^5 + 2^5 = 2^x$?

There are a number of other two-dimensional forms that we can find the areas of now that we know the area formulas for rectangles and triangles. Here are some.

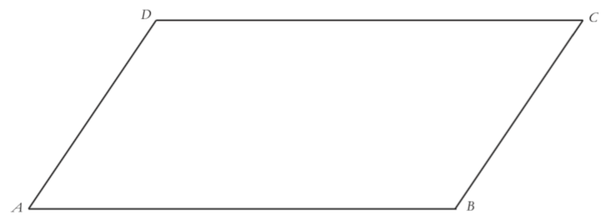
Draw a triangle ANY and a line parallel to $\overline{AY}$ so that it cuts through the sides $\overline{AN}$ and $\overline{NY}$. Let $\overline{BC}$ represent the parallel line intersecting the triangle. Now erase what's above $\overline{BC}$. You're looking at what is called a trapezoid — a four-sided figure with one pair of parallel sides.



43

Knowing what you do about the areas of triangles and rectangles, come up with a formula for the area of a trapezoid. You may want to label BC, the top base, as b_1 and AY, the bottom base, as b_2. What other dimensions of the trapezoid might be useful?

If we now consider any parallelogram $ABCD$, not just those with right angles, we can consider whether there is a unique formula for the area of any and all parallelograms, including rectangles and squares.

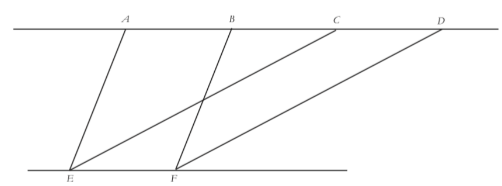


44

Partition the parallelogram (divide it up) into shapes that you already know the area of. By summing the areas of these shapes, find an area formula that should work for any parallelogram.

45

Given the image below where it is assumed the horizontal segments are parallel, what can you conclude, if anything, regarding the areas of parallelograms $ABFE$ and $CDFE$?



46

If the lengths of the sides of a parallelogram are doubled, what seems plausible regarding the area of the new parallelogram? Can you make an argument that your conjecture is the case?

Exploring in Depth

47 A certain trapezoid has bases of length 10 and 24, sides of length 13 and 15, and a height of 12. If you were to make an enlarged, scale copy of the trapezoid whose area was 4 times that of the original, then what would the measurements of the larger trapezoid be?

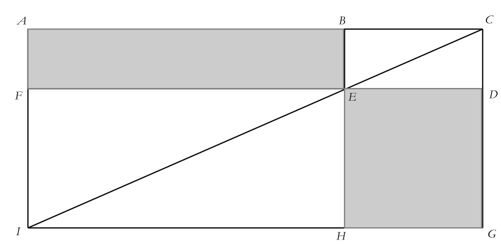
48 Let $ABCD$ be a square, and $EFGH$ another square whose area is twice that of square $ABCD$. Prove that the ratio of AB to AC is equal to the ratio of AB to EF.

The ancients who studied form naturally included others besides the Greeks. There were the Egyptians who designed and built pyramids and came up with a very interesting formula for the area of a circle (see if you can find it on Google and see what an impressive fit it provides). The Babylonians also studied form, and they came up with a formula for the area of a quadrilateral.

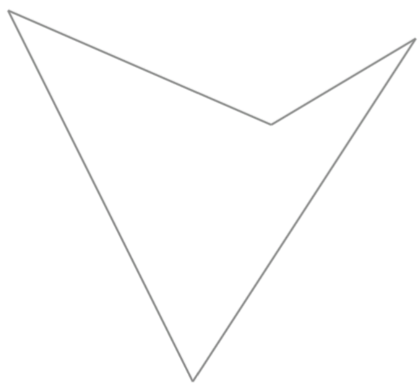
49 The Babylonians' formula for the area of a quadrilateral is Area $=\frac{(a+b)(c+d)}{4}$, where a and c are the lengths of one pair of opposite sides of a quadrilateral, and b and d are the other. Does their formula always, sometimes, or never work? (Try this with quadrilaterals you actually know the area of.)

50 Earlier in this lesson, you probably decided that, if you have lines crossed by a transversal in such a way that the alternate interior angles created are equal, then those lines must be parallel. Here's a suggestion for arguing that your conjecture must be the case. Suppose that you had alternate interior angles that were equal, but the lines were NOT parallel. Do some geometric tinkering to derive some interesting things.

51 If $BCDE$, $FEHI$, and ACGI are rectangles, can you logically conclude that quadrilaterals $ABEF$ and $EDGH$ have the same area?



52 The polygons you have studied in this unit have mostly been **convex**. When you pick any two points inside a convex polygon and connect them with a straight line segment, the line segment always stays within the polygon. If there are two points where the line connecting them must go outside the polygon, then the polygon is called **nonconvex**.

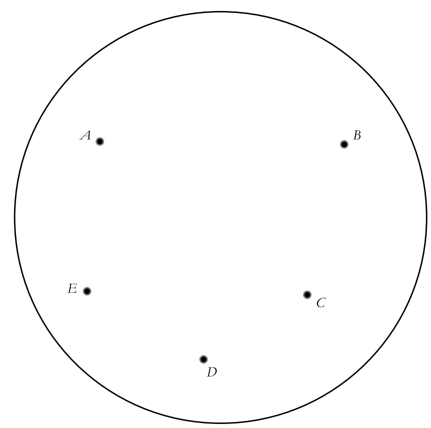


a. Show that the polygon above is nonconvex.

b. Draw in the exterior angles of this polygon. See Problem 24 if you need a reminder.

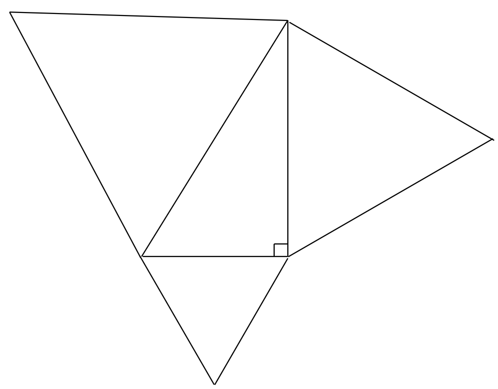
c. Do the exterior angles of a nonconvex polygon still add up to 360 degrees? Experiment.

53 First remind yourself of the result in problem #31. Then consider the following image. As you have seen in Euclidean geometry parallel lines are infinite lines in the same plane that don't intersect. Given the circle as the entire universe, lines AB and DE are said to be parallel, as are lines AC and DE, as neither pair has any intersection points within the circle. However, lines AB and AC are not parallel. How does this finding fit with your finding in problem 31?



54 Draw a good-sized equilateral triangle and pick any point inside the triangle. From that point draw the three perpendiculars to the three sides of the triangle. Let the lengths of the three perpendiculars be a, b, and c. If $d =$ height or altitude of the triangle, show that $a + b + c = d$. (Quite a surprise?)

55 Draw a right triangle. On each of the three sides of the triangle draw equilateral triangles where the side of the original triangle is also a side of the new triangle. Each of those equilateral triangles has an area. Does it seem that the sum of the areas of the equilateral triangles on the legs of the right triangle is equal to the area of the equilateral triangle on the hypotenuse? Can you prove this? (Hint: what do you know about 30°-60°-90° triangles?)



56 Draw any line segment on a piece of paper. Now come up with a compass-and-straightedge construction to *bisect* the segment — cut it exactly in half.

LESSON 3: GEOMETRIC PROOF

Introduction

C'est Mathematique! is a French expression meaning that something is so certain, it is definite. Indeed, the 19th-Century French mathematician Pierre-Simon Laplace imagined a demon that could predict every future event because it knew the exact position of every particle in the universe, and the mathematical laws that governed the motion of those particles.

Though no such demon exists, we can see why Laplace might have imagined one. Our mathematical knowledge does seem more certain than some other facts we believe to be true about politics or human nature. For instance, maybe you think that passing laws to ban handguns will reduce homicides. Then you read a study that says that actually these bans don't have that effect. You might change your mind about whether banning handguns is a good idea... until you read another study that says that the bans *do* reduce accidental death among children. You'll always have to remain open to changing your opinion based on new information.

Likewise, in the history of science, people once thought that the atom was the smallest particle that existed (the word "atom" comes from a Greek word meaning "indivisible"). Now we know that atoms are made up of protons and neutrons, and in recent decades scientists have discovered quarks. In the scientific community, people are constantly revising and improving accepted theories.

In mathematics, on the other hand, we still accept the results of ancient texts. Archimedes, Pythagoras, and Euclid are credited with many of the results that we still use today. You have probably already **proven** things mathematically in such a way that you're sure they must be true. In this lesson we'll look at what it means for a proof to be absolutely airtight — with no possibility you'll want to revise your opinion later.

Development

Let's think about the following two statements:

The diagonals of a parallelogram bisect each other.

The diagonals of a parallelogram bisect the angles from which they're drawn.

1 Make some sketches to determine whether you think each of these statements is true or false. Recall that "bisect" means "cut exactly in half."

When we do proofs in this section, we're going to focus on making them airtight. We'll use the following standard. Any time we make a statement as part of a proof, we need to be able to give a reason for that statement. That reason, in turn, needs to be a statement about geometry that we *already* know to be true.

Here are some statements about geometry you already know to be true:

In an isosceles triangle, the angles opposite the congruent sides are congruent.

When parallel lines are cut by a transversal, same-side interior angles add up to 180 degrees.

Vertical angles are equal.

If two sides and the angle between them in one triangle are congruent to two sides and the angle between them in another triangle, then the triangles are congruent. (SAS congruence theorem)

If the three sides of one triangle are all related to the sides of another triangle by the same proportion, then the triangles are similar. (SSS similarity theorem)

In a parallelogram, opposite sides are congruent.

2 Make a list of statements about geometry that you know to be true, including the six statements above. To get ideas, try paging through your notes from the past few weeks, or through the lessons earlier in this textbook. Which results seem worth remembering? Also think about some basic shapes you know, like quadrilaterals, and write down some facts you know to be true about each one.

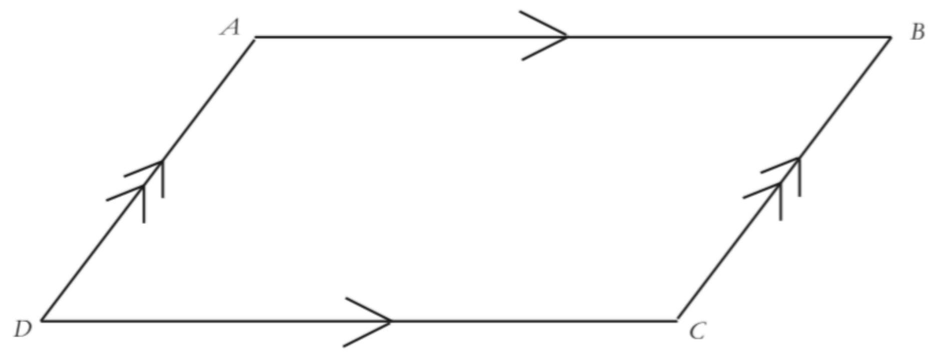
As you may recall from the first lesson of this chapter, the Greek mathematician Euclid did something very similar to what you are about to do. He boiled down his geometric knowledge to a few assumptions and then proved the rest of what he knew about geometry using those few things. He called the facts he assumed without proof "postulates." Any fact that he had proved was a "theorem."

3 Consolidate the list you made in the previous problem with the rest of the class. Discuss which items on it are postulates and which are theorems. Now separate a blank page of your notebook into two columns. Label one column "postulates" and the other "theorems." Write the items in the class list in the appropriate column. Leave yourself lots of room in both columns to add to the list, probably over multiple pages.

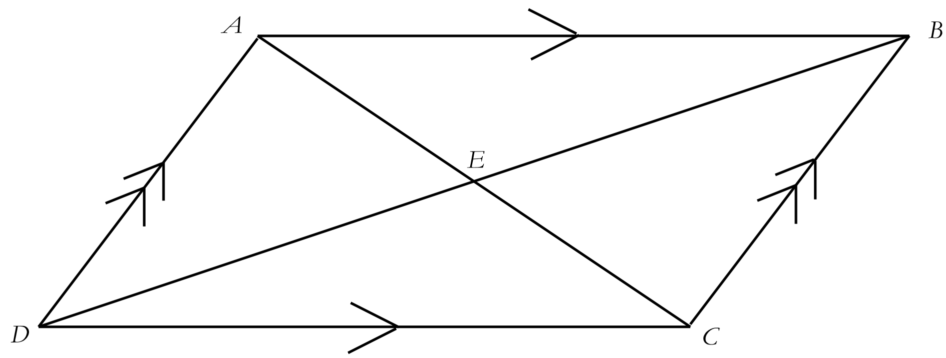
Hang on to this list. You'll be referring to it for the remainder of this chapter.

Now let's return to the problem of proving the statements from the beginning of this lesson. If you did your job carefully, you'll have found that the first statement was true — the diagonals of a parallelogram bisect each other.

Before writing our airtight proof of this fact, it's important to know where to start. Since we're proving something about a parallelogram, it's all right to assume that we *have* a parallelogram. So let's draw one...



... and label the vertices so that we can talk about them later. So, specifically, $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$. Those are two things we know for sure. It's also important to be sure of what we want to show about this diagram. Let's draw in some diagonals and call their intersection point E.



Then what we want to show is that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$.

The comic book hero G.I. Joe used to say "knowing is half the battle." In the case of proof, knowing what information you have to start with and where you want to end up with is half the battle. Really!

What we need now is a chain of reasoning, beginning with the facts that $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$ and ending with the statement that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$. In this chain of reasoning, every step needs to be justified with some reason we *know for sure*.

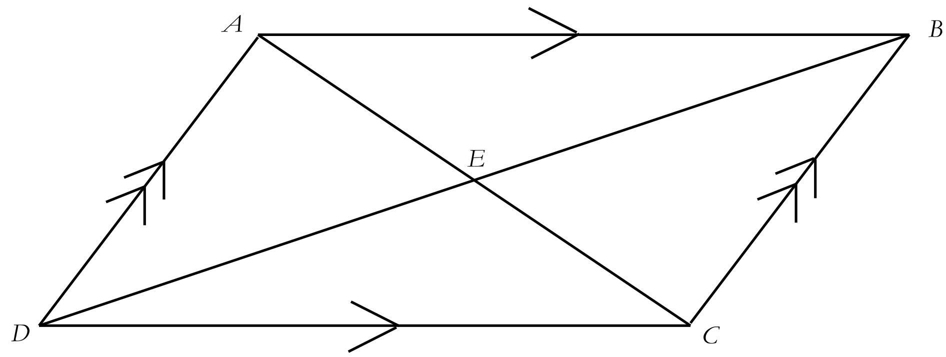
4 Try this yourself before reading the proof on the next page.

When thinking about what will help us to write this proof, let's think backwards a little bit. We want to show that certain lengths are the same. Look back at your list of postulates. What tools do you have for showing that lengths are the same? Not the results about parallel lines. Not vertical angles or linear pairs. Possibly some statements you have about parallelograms or rhombuses, but in this case the lengths we're interested in are diagonals rather than sides of those shapes. That really only leaves one thing that could help — congruent triangles.

If we can show that certain triangles are congruent, then we can conclude that the lengths making them up are also congruent. That should be our strategy in writing this proof.

5 Find some congruent triangles in this diagram that you think would help. Do you know for sure they're congruent?

Depending on the way you answered question #5, you may have an airtight proof already. Let's write one out as an example of a chain of reasoning in which all the steps are justified.



Given: $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$.

Goal: Prove that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$.

$\angle DAE \cong \angle BCE$ because they are alternate interior angles of the parallel lines $\overline{AD}$ and $\overline{BC}$.

$\angle ADE \cong \angle CBE$ for the same reason.

$\overline{AD} \cong \overline{BC}$ because opposite sides of a parallelogram are congruent.

Therefore, $\triangle AED \cong \triangle CEB$ by the ASA congruence theorem.

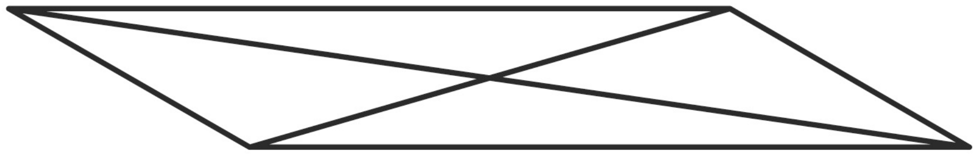
$\overline{AE} \cong \overline{EC}$ because they are corresponding parts of congruent triangles.

$\overline{BE} \cong \overline{ED}$ for the same reason.

Are we done? YES! We've accomplished our goal, showing that the diagonals cut each other in half.

That proof might have been written with a little more care than you are used to. Your class will have to decide exactly how much writing is expected in a proof. But notice that every step in the argument comes from your list of postulates, so that anyone who accepts the postulates can't disagree with your conclusion — there is no "wiggle room" in this proof.

Meanwhile, you should have found that it is *not* true that the diagonals of a parallelogram bisect the angles. You may remember from last year that an example designed to show that something isn't true is called a **counterexample**. Here's what a counterexample to this statement might look like:



You can see from this picture that it doesn't look true that the angles are bisected, and so it isn't worth your time to look for a proof.

In order to start the proof that the diagonals of a parallelogram bisect each other, we had to know exactly what the word "parallelogram" meant. That's how we knew to start with the statements "$\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$". Whenever we write a proof, we'll have to be very precise about what we mean by the objects we're talking about. For instance, last year you worked with the definition of a square: a quadrilateral with four congruent sides and four right angles.

6 Come up with a definition for each shape. Then compare notes with your class if you haven't already done so.

a. A rectangle

b. A parallelogram

c. A rhombus

d. A trapezoid

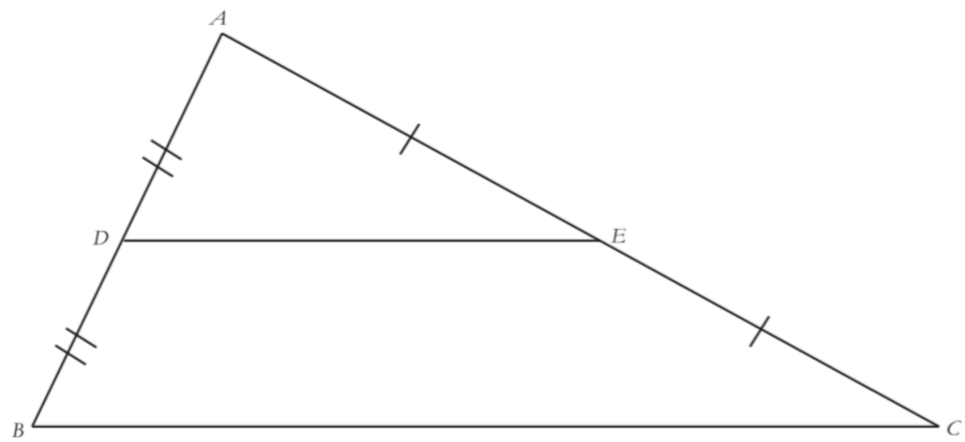
e. A kite

f. A circle

7 Now begin a section in your notebook labeled "definitions," again leaving space for you to add definitions as you consider new shapes. Start with the definitions you came up with in Problem 6.

8 Considering the shapes in the previous question might remind you of some *more* things you know to be true about geometry. Add them to your list of postulates and theorems (which should they be?), and confer again with your classmates.

Much as you can use congruent triangles to prove things about diagrams, you can also use similar triangles.



9 The diagram has been marked and labeled for you.

a. Based on the diagram, write what is given.

b. Can you remember the two things you showed about $\overline{DE}$ and $\overline{BC}$ last year? If not, try drawing this diagram with triangles of a few different shapes, and take some measurements in order to make a conjecture.

c. With an eye towards proving these results, prove that there are similar triangles in this picture (The three similarity theorems are AA, SSS, and SAS). What is the scale factor?

d. Finish proving what you wrote in part b about $\overline{BC}$ and $\overline{DE}$.

You've proven a couple of new things so far in this lesson. The interesting thing is, not only do you know those things, but you have proofs of them. Write the new things you know under the column labeled "theorems" on your list. Whenever you prove something new that you think may come in handy later, or that you just think is interesting or important, add it to this theorem list.

By the way, is there anything under "postulates" that you could prove and make a theorem?

Practice

10 In this problem, you will prove that the diagonals of a rectangle are congruent.

a. Draw a rectangle that can stand for any rectangle and label the vertices.

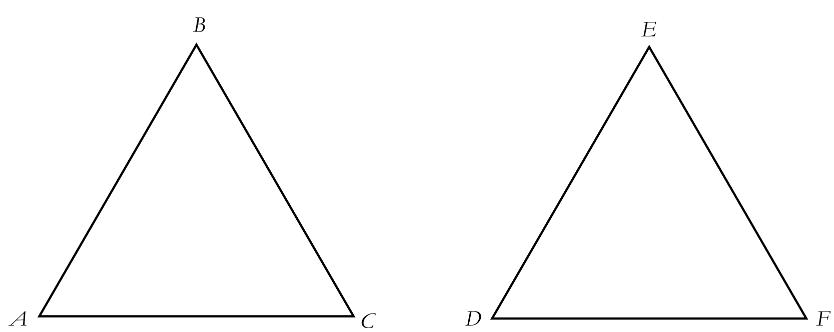
b. In terms of your diagram, write what is given and what is to be proved.

c. Finish the proof.

In the following five problems, you should mark up the diagram with the starting information before you attempt a proof.

11 Given: $\overline{AB} \cong \overline{EF}$, $\angle A \cong \angle E$, $\angle C \cong \angle D$

Prove: $\overline{BC} \cong \overline{DF}$

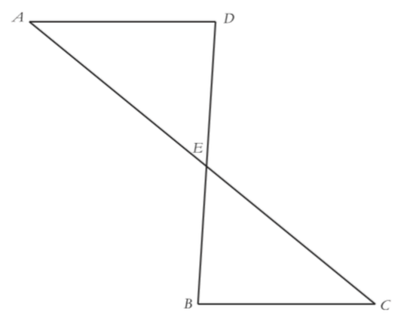


12

Given: $\overline{AE} \cong \overline{CE}$, $\overline{BE} \cong \overline{ED}$

Prove: $\overline{AD} \parallel \overline{BC}$

(In this problem and in the remaining problems in this lesson, assume that lines that appear to be straight in a figure really are straight.)

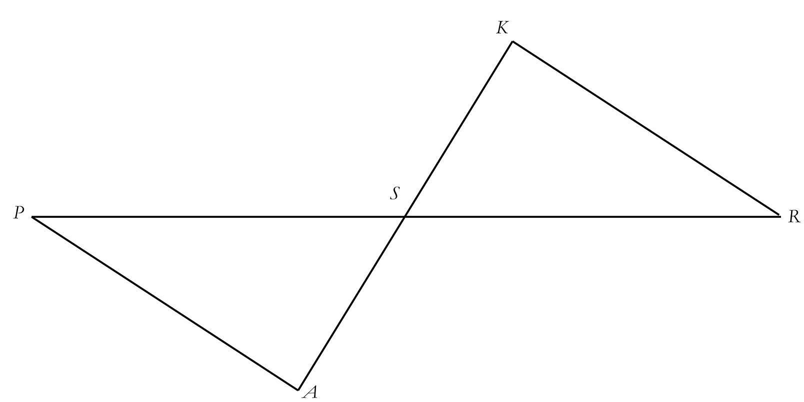


13

There is a joke about a mathematician asked to describe an algorithm for boiling an egg. She responds, "you take a pot, fill it with water, put in an egg, put the pot on the burner, turn the burner on, and wait 20 minutes." "All right," you say, but what if the pot is already sitting on the burner, filled with water?" "Then," she says, "you take the pot off the burner and empty the water, and then you've reduced the problem to the previous case."

a. Suppose, in the diagram below, that $\overline{PA} \cong \overline{RK}$, $\angle A \cong \angle K$
Prove: The length of $\overline{KA}$ is twice the length of $\overline{KS}$.

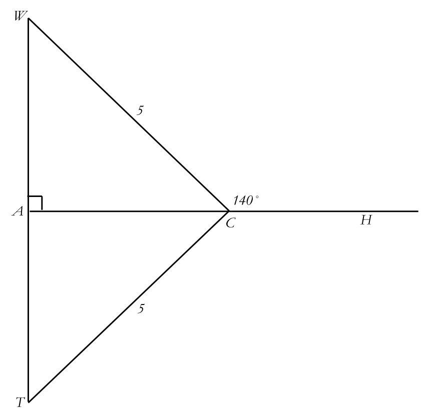
b. Now suppose instead that $\overline{PA} \cong \overline{RK}$, $\overline{PA} \parallel \overline{KR}$ Prove: The length of $\overline{KA}$ is twice the length of $\overline{KS}$.



14

Given: the angles and sides as marked; W, A, and T are points on a line segment.

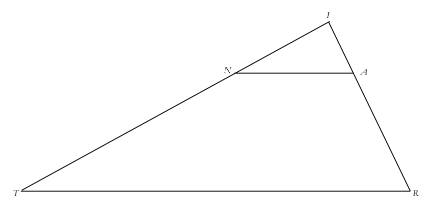
Prove: $m\angle T = 50^{\circ}$



15

Given: $TI = 3NI$, $RI = 3AI$

Prove: $\overline{TR} \parallel \overline{NA}$ and $TR = 3NA$



16

Which of these statements appear true, and which are false? If they are true, you don't need to prove them, but if you think they are false you should provide a counterexample.

a. If one isosceles triangle has side lengths of 5 cm and 5 cm, and so does another, then those triangles are congruent.

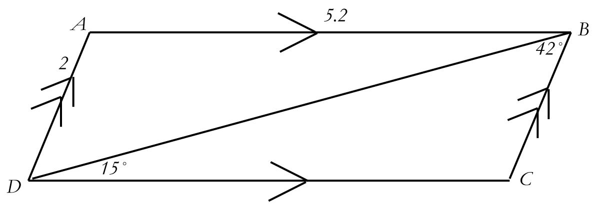
b. In an obtuse triangle, label the longest side c and the other two sides a and b. Then $a^2 + b^2 < c^2$.

c. The three medians of a triangle always cross in a single point.

d. If someone tells you three positive integer lengths, you will always be able to draw a triangle that has those lengths.

For each figure, determine as many missing sides and angles as you can. (You don't need to write a formal proof of your answers.) You will not be able to find *all* of the sides and angles. Avoid using trigonometry.

17



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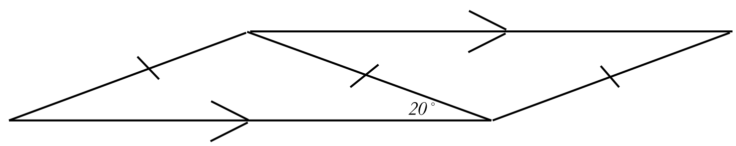
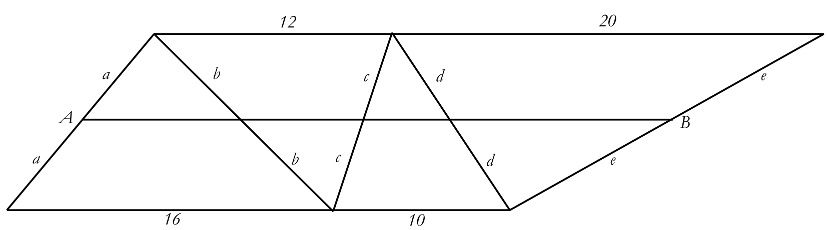
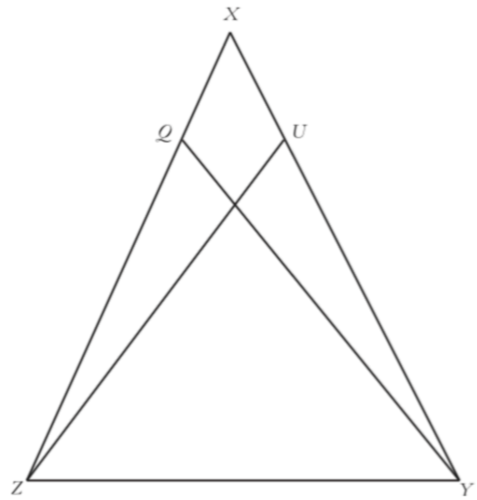


19 In the diagram below, lines that appear parallel are parallel. Find the length of the line segment .



Going Further: Longer Proofs; Pitfalls When Writing Proofs

Sometimes the only way you can see that a statement must be true is to talk yourself through a series of steps in proof-like form. It would be hard to come to know the statement to be proven below, in any other way. In addition, the statement's proof deals with some complications that you will sometimes come across.



Given: $\overline{XQ} \cong \overline{XU}$ and$\angle XQY \cong \angle XUZ$.
Prove:$\angle UZY \cong \angle QYZ$.

"Working backwards and forwards" will be very important in the planning stages of this proof. At first glance, it seems that we should try to prove triangles QYZ and UZY congruent, since those are the triangles that contain the two angles we want to prove congruent.

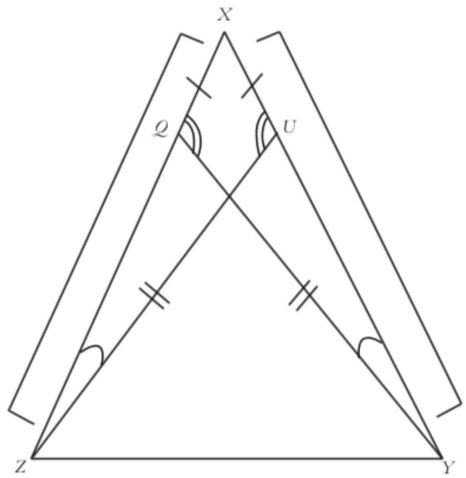
20 Find as much evidence as you can that $\triangle QYZ \cong \triangle UZY$.

In the previous problem you should have found a (shared) side and an angle, which by themselves are not enough to prove the triangles congruent. So we need to find another strategy. It may be that, using some other pair of congruent triangles in the figure, we can find that third piece of information needed to prove $\triangle QYZ \cong \triangle UZY$. Or, perhaps the solution is something completely different. Instead of doing all of our planning backwards, let's return to thinking forwards and just look for any pair of triangles that might be congruent.

21 Name some other pairs of triangles that might be congruent in the figure. Try to find enough evidence for their congruence.

The triangles that we can actually prove congruent may have been hard to spot. As a hint, they share angle X. If you haven't already, prove that these two triangles are congruent.

Now that we have congruent triangles, we can mark all of their sides and angles congruent, as follows:



22 Now we can return to the question of how to prove $\angle UZY \cong \angle QYZ$. Using the information we've derived, argue that these angles are congruent.

One way to complete the proof is to notice that, since we now know $\triangle XYZ$ is isosceles, $\angle XZY \cong \angle XYZ$. Since these angles are both composed of two parts, and the parts $\angle XZU$ and $\angle XYQ$ are congruent to each other, that means the remaining parts must be equal as well. In other words, $\angle UZY \cong \angle QYZ$! Though this kind of part/whole reasoning doesn't appeal to theorems or postulates that are most likely on your list, it is a common type of geometrical reasoning. Your class can decide what kind of language it wants to use for these types of arguments.

After all that, we have used congruent triangles to figure out why $\angle UZY \cong \angle QYZ$. However, much of the argument was probably done orally, in combination with a marked-up diagram. It is now time to

23 Write a formal proof of the fact that $\angle UZY \cong \angle QYZ$.

In the proof you just wrote, you took pains to make sure that each statement was correctly justified. However, it is easy to be sloppy when writing proofs. Two examples of proofs follow that illustrate some mistakes that are easy to make.

Consider the proof below showing that, if a quadrilateral has two pairs of congruent, opposite angles, then the quadrilateral is a parallelogram.

> First, draw a quadrilateral that can stand for any quadrilateral with congruent, opposite angles. Label the vertices $ABCD$, where $\angle A \cong \angle C$ and $\angle B \cong \angle D$.
>
> Given: Quadrilateral ABCD, $\angle A \cong \angle C$, $\angle B \cong \angle D$.

Prove: $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{AD}$.

Draw the diagonal $\overline{AC}$.

$\angle B \cong \angle D$, as given.

$\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$, because opposite sides of a parallelogram are congruent.

Therefore, $\triangle ABC \cong \triangle CDA$ by the SAS congruence theorem.

Then $\angle DAC \cong \angle BCA$ since they are corresponding parts of congruent triangles.

That makes $\overline{BC} \parallel \overline{AD}$ since alternate interior angles are congruent.

Similarly, $\angle BAC \cong \angle DCA$, again since they are corresponding parts of congruent triangles.

So $\overline{AB} \parallel \overline{CD}$.

Therefore, $ABCD$ is a parallelogram.

24 This proof may seem to work quite well. However, it contains a major error that invalidates the proof entirely. What is this error?

25 This type of error is sometimes called **circular reasoning**. Why do you suppose this phrase is used?

26 Do you still think it's true that, if a quadrilateral has two pairs of congruent, opposite angles, it must be a parallelogram?

Finally, here is an example of a proof that just can't be right: if a triangle is isosceles, then it must be equilateral. You have seen lots of isosceles triangles that are not equilateral, so you know that this proof must contain a mistake! The challenge is to find out what that mistake is. Remember that, if you agree with every step of a proof, you have no choice but to accept its conclusion.

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$.

Prove: $\overline{AC} \cong \overline{BC}$ (which would also make $\overline{AC}$ congruent to $\overline{AB}$).

Draw the median from C and label the point where it hits $\overline{AB}$ as D.

$\overline{AD} \cong \overline{DB}$ because $\overline{CD}$ is a median.

Also, $\angle CDA \cong \angle CDB$ because, as a median, $\overline{CD}$ splits $\overline{AB}$ into two right angles.

$\overline{CD} \cong \overline{CD}$ because any line segment is congruent to itself.

Therefore, by SAS congruence, $\triangle CAD \cong \triangle CBD$.

Since they are corresponding parts of congruent triangles, $\overline{AC} \cong \overline{BC}$.

Therefore, all three sides of $\triangle ABC$ are congruent, making it equilateral.

27 Find the flaw in the proof above.

Practice

28

Which of the following proofs are "airtight"?

a. Prove that, in a quadrilateral, if a pair of opposite sides is both congruent and parallel, then the quadrilateral is a parallelogram.

Proof: Draw segments $\overline{AB}$ and $\overline{CD}$ so as to be congruent and parallel. Connect points A and C. Then connect points B and D. Because $\overline{AB}$ and $\overline{CD}$ are congruent and parallel, the line segments $\overline{AC}$ and $\overline{BD}$ have to slope in the same direction. Therefore, $\overline{AC} \parallel \overline{BD}$, and so ABCD is a parallelogram.

b. Prove that the diagonals of a rhombus bisect each other.

Proof: We've already proven, and put on our theorem list, that the diagonals of a parallelogram bisect each other. A rhombus is a type of parallelogram. Therefore, any statement that applies to a parallelogram also applies to a rhombus, and so the diagonals of a rhombus bisect each other.

c. Prove that the diagonals of a parallelogram bisect each other. (We've already proved this, but here is an alternate version of a proof.)

Proof: Draw parallelogram $ABCD$, and add diagonals that cross at E. We're going to show that triangles AED and CEB are congruent. The two pairs of angles needed for congruence are $\angle DAE \cong \angle ACB$ and $\angle ADB \cong \angle DBC$. Those pairs really are congruent because they are both pairs of alternate interior angles created by parallel lines and a transversal. Also, sides $\overline{AD}$ and $\overline{BC}$ are congruent because they are opposite sides of a parallelogram. So now we have enough to show triangles AED and CEB are congruent, by ASA. Therefore, the diagonals do bisect each other: $\overline{DE} \cong \overline{EB}$ and $\overline{AE} \cong \overline{EC}$ as they are corresponding parts of congruent triangles.

29 The following are situations in which you might want to add an extra line to a diagram to help you with a proof. For each situation, say whether there could be multiple lines that fit the description, exactly one line that would fit the description, or whether a line that fits the description might not exist at all.

a. Given line line AB and point C not on it, draw a line segment from C to AB.

b. Given line AB and point C not on it, draw a perpendicular line segment from C to line AB.

c. Given line AB and point C not on it, draw a line through C parallel to line AB.

d. Given parallelogram $ABCD$, draw diagonal AC so that it bisects angles A and C.

e. Given triangle ABC, draw a line segment from points D and E, the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively, so that $\overline{DE}$ is parallel to and half the length of $\overline{AC}$.

f. Given triangle ABC, draw the perpendicular bisector of side $\overline{AB}$ from C.

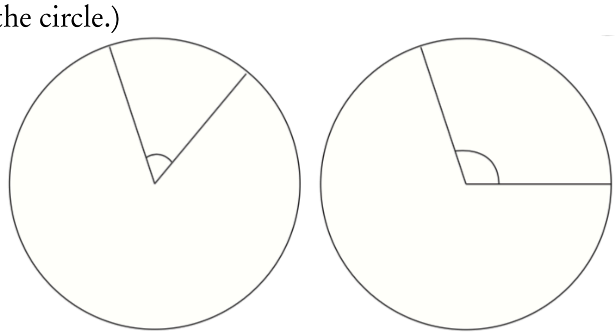
Problems

30 At the beginning of a lesson, you found that the diagonals of a parallelogram do not always bisect the angles from which they are drawn. How about for a rhombus? Prove it or find a counterexample.

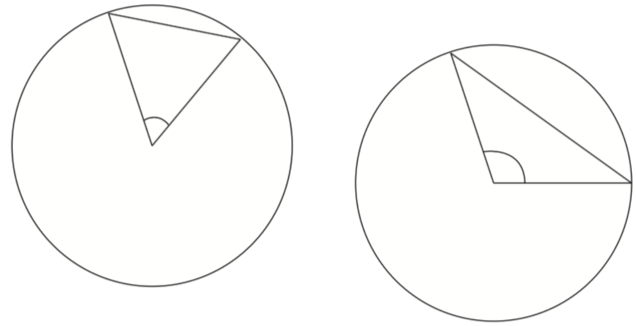
31 For which of the following shapes — square, rectangle, parallelogram, trapezoid, kite, rhombus — are the diagonals always perpendicular to each other? Investigate with sketches.

32 For the shapes in the previous problem where the answer is "yes," prove your response. Before beginning each proof, carefully state the given and what is to be proved. Hint: how can you possibly show that an angle is 90 degrees if you don't know the measures of any other angles in the figure?

The angles below are called **central angles** of each circle. (The vertex of each angle is on the center of the circle.)

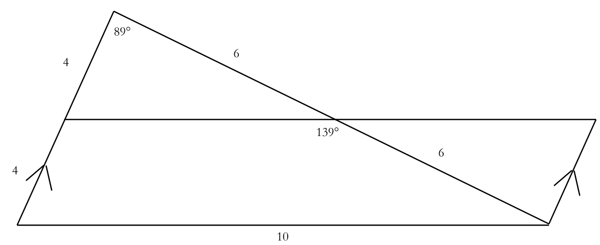


If you use a straight line segment to connect the points where the angle meets the circle, you get a **chord** of the circle.

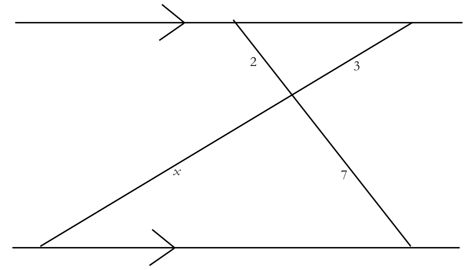


33 Prove that if a circle contains two central angles of the same size, then their corresponding chords are congruent.

34 Determine as many missing lengths and angles as you can. For each angle or length you find, give the reason you used to figure it out.

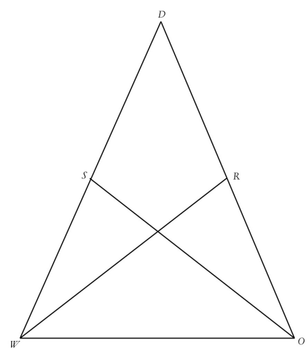


35 Find x. Write out the reasoning you used.



36 If you make an "X" shape in between two parallel lines, as in the figure for the previous problem, will the resulting triangles always be similar? Either prove it, or draw an "X" shape that creates non-similar triangles.

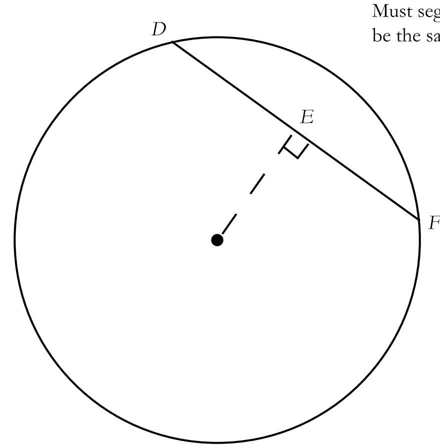
37 Given: S is the midpoint of $\overline{WD}$, R is the midpoint of $\overline{DO}$. Also, $\overline{WD} \cong \overline{DO}$.
Prove: $\overline{WR} \cong \overline{SO}$



38 Emmy and Sophie are standing in the center of the free throw circle on a basketball court. They want to measure their distance to the special free-throw line they put in for Emmy's little brother — it's a chord of the circle positioned close to the hoop.

Emmy says, "It's easy — we just measure along a perpendicular path from where we are standing to the special free-throw line. After all, that's how you find the distance from a point to a line.

Sophie says, "You're right. And I think the line we measure along will not only be perpendicular to the special free-throw line, but it will *bisect* the line as well."



Must segments DE and EF be the same length?

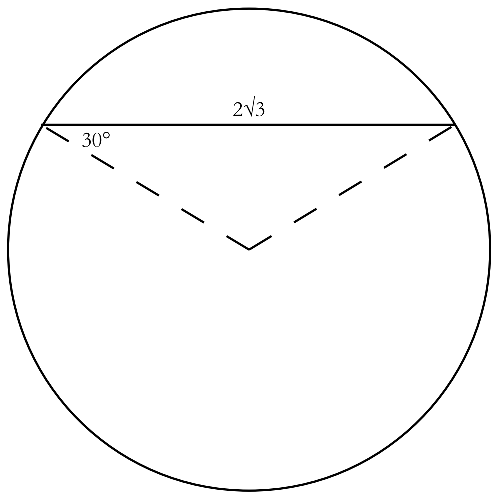
Is Sophie right? Prove it or find a counterexample.

Suppose that Emmy and Sophie realize that they don't have a ruler, but they remember that the radius of the circle they're standing in is 10 feet, and the length of the special free-throw line is 14 feet. Help them calculate their distance from the line.

39 What's the distance from the center of a 41-foot radius circle to an 80-foot chord?

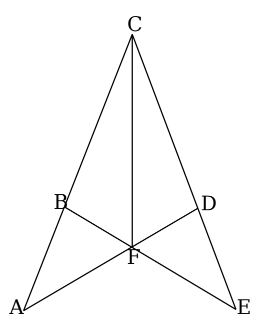
40

Someone saw this circle with chord of length $2\sqrt{3}$ drawn in. They connected the endpoints of the chord to the center of the circle. Find as many lengths and angles in this picture as you can. Also find the measure and lengths of each of the 2 arcs that the chord separates the circle into.



41

Given: $\overline{CB} \cong \overline{CD}$ and $\overline{BA} \cong \overline{DE}$
Prove: $\angle BCF \cong \angle DCF$



42

Don't use a calculator for this problem.

a. Simplify $\frac{x^{-3}y^{-4}}{x^2y^{-6}}$

b. Simplify $x^2 + \left(\frac{3}{4}\,x\right)^2$

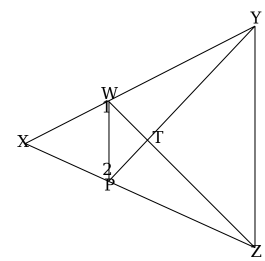
c. Reduce $\frac{3x^2}{6xy-9x^3}$

d. Reduce $\frac{x^2-1}{x^2-2x+1}$

e. Solve for x if $x^2 + 1728x = 1729$

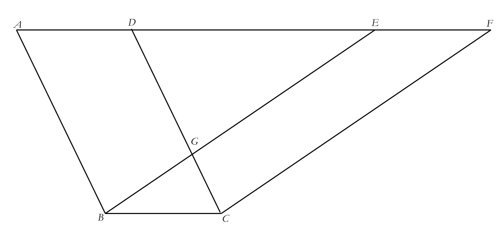
43

Given: $\overline{PT} \cong \overline{WT}$ and $\angle 1 \cong \angle 2$
Prove that $\triangle ZYT$ is isosceles



44

Given: $ADCB$ and $EFCB$ are parallelograms.
Prove: $\triangle AEB \cong \triangle DFC$

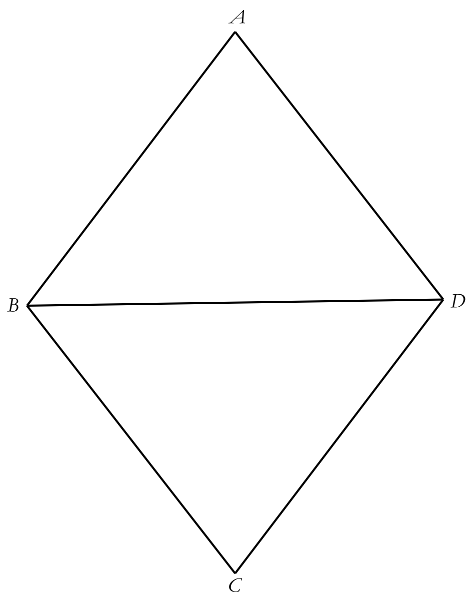


45

Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{DC}$

a. Prove that the two triangles are congruent.

b. Can you then show that $\overline{AD} \cong \overline{BC}$?



46

Let ABC be any triangle. Choose point P somewhere on side AB of the triangle, and draw a line through P parallel to $\overline{CB}$. Where this line intersects $\overline{AC}$, label the point Q.

a. Prove that $\frac{AP}{AB} = \frac{AQ}{AC}$.

b. Prove that $\frac{AP}{PB} = \frac{AQ}{QC}$.

47

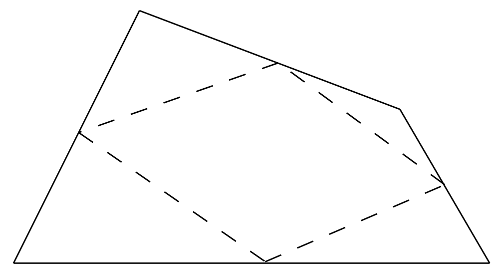
Gina did the previous problem and wonders if it's possible that $AP = x - 2$, $AB = x - 1$, $AQ = x - 3$, and $AC = x$. What should you tell her? Provide two explanations:

a. The first explanation should involve solving an algebraic equation.

b. The second explanation should involve considering a proportion.

48

The diagram below was formed by starting with any old quadrilateral, then connecting the midpoints of each side.



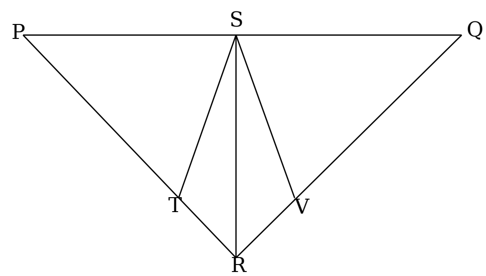
a. Try this several times on your own, with many different quadrilaterals. Form a conjecture about what always happens.

b. Prove your conjecture. (Hint: this is a problem where adding extra lines is helpful.)

Exploring in Depth

49 Given: $\overline{RS}$ is the perpendicular bisector of $\overline{PQ}$, and $\angle PST \cong \angle QSV$

Prove: $\overline{ST} \cong \overline{SV}$



50 Draw triangle ABC, and choose X to be a point on $\overline{AB}$. Draw $\overline{CX}$. Now add two lines to the picture that are parallel to $\overline{CX}$: one through A and the other through B. Y is the point where the first added line intersects line CB. (You may need to extend $\overline{CB}$ to find this point.) Z is the point where the second added line intersects line AC.

Now you have the line segments $\overline{AY}$, $\overline{CX}$, and $\overline{BZ}$. This problem will lead you towards proving a relationship between the lengths of those three segments.

a. Find two pairs of similar triangles in this figure, each pair involving the segment $\overline{CX}$.

b. Your two pairs of similar triangles allow you to write two proportions. Using these proportions, and facts such as $AB = AX + XB$, find an equation that relates $\overline{AY}$, $\overline{CX}$, and $\overline{BZ}$.

c. Prove that your answer is equivalent to $\frac{1}{AY} + \frac{1}{BZ} = \frac{1}{CX}$.

LESSON 4: LOGIC AND GEOMETRY

Introduction

Lewis Carroll, of *Alice in Wonderland* fame, made the following set of statements in his less famous book, *Symbolic Logic*."

No potatoes of mine, that are new, have been boiled. All my potatoes in this dish are fit to eat. No unboiled potatoes of mine are fit to eat.

1 What, if anything, can you conclude from these three statements?

Development

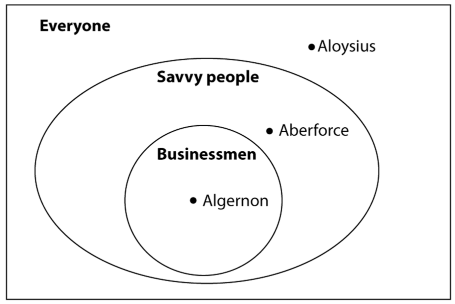
It's a well-known fact that all squares are rectangles, but not all rectangles are squares. We can express these statements in the form below.

If a figure is a square, then it is a rectangle. (true)
If a figure is a rectangle, then it is a square. (false)

You may recall that these statements are called **converses** of each other, as is true of any pair of statements of the form

If A then B.
If B then A.

You may have seen Venn diagrams at some point in your life. They are handy when thinking about logic. For instance, the statement "All business people are savvy" (or "If a person is a businessman, then he is savvy") can be represented by the following diagram:



In this example, Algernon is a savvy businessman, Aberforce is savvy but not a businessman, and Aloysius is neither savvy nor a businessman.

2 Draw a similar diagram illustrating the statement "All savvy people are businessmen."

a. How would you write the statement in "if/then" form?

b. Do the diagrams make it clear that the statements "All businessmen are savvy" and "All savvy people are businessmen" mean different things?

You've seen that if/then statements have converses, and that just because the if/then statement is true doesn't necessarily mean the converse is. Here are two other types of statements you can get from an if/then statement.

Take the statement "if A then B".
"If not-A then not-B" is called the **inverse**.
"If not-B then not-A" is called the **contrapositive**.

3 Pick three "test" statements of the form "if A then B." Then use them to decide if the inverse and/or the contrapositive must be true when the original statement is. If you're not sure, try drawing Venn diagrams.

Some statements on your list of theorems and postulates can be written in if/then form as well, and it is worth investigating if their converses are true. For example, you probably have the following statement:

Opposite sides of a parallelogram are congruent.

4 Write a pair of if/then statements expressing this statement and its converse.

5 If you were going to prove the converse, what would be the given? What would be the result that you are trying to prove?

6 Go ahead and prove the statement “if a quadrilateral has both pairs of opposite sides congruent, then it is a parallelogram.” (Hint: this is a problem where adding something extra to the diagram is useful.)

Another way of saying what you just proved is that having both pairs of opposite sides congruent is **sufficient** for a quadrilateral to be a parallelogram. In other words, if you have a quadrilateral and you know its opposite sides are congruent, that’s enough — you don’t need to know anything else about it to know that it’s a parallelogram.

On the other hand, the original statement on your list, “Opposite sides of a parallelogram are congruent,” provides a statement that is true about all parallelograms. This kind of statement is called a **necessary** condition. For if you’ve got a quadrilateral without congruent opposite sides, you know you definitely don’t have a parallelogram. You’re lacking one of the necessary criteria.

Conditions aren’t always both necessary and sufficient. For example, having four right angles is necessary for a quadrilateral to be a square, but having four right angles is not sufficient for a quadrilateral to be a square. The quadrilateral might be only a rectangle and still have four right angles.

Also, conditions can be sufficient without being necessary. For example, a closed shape’s having three 60-degree angles is sufficient for the shape to be a triangle, but it is certainly not necessary that a triangle have three 60-degree angles.

What’s more, necessary and sufficient conditions don’t have to be about shapes. For example, at Park, playing soccer for all four years is sufficient to fill your athletic requirement. Also, being a senior is necessary for taking Senior Studio — you can’t take Senior Studio without being a senior.

7 Is having four congruent sides necessary, sufficient, or both for being a square?

8 Is having four congruent sides necessary, sufficient, or both for being a rhombus?

9 Is playing soccer all four years a necessary condition for Park students to satisfy their athletic requirement? Is being a senior at Park sufficient for taking Senior Studio?

Practice

10 Poet Li-Young Lee says "every wise child is a sad child." According to him, is being a wise child sufficient for being a sad child?

11 Is being seventeen sufficient for having a provisional driver's license in Maryland?

12 Is being an even number necessary for being divisible by eight?

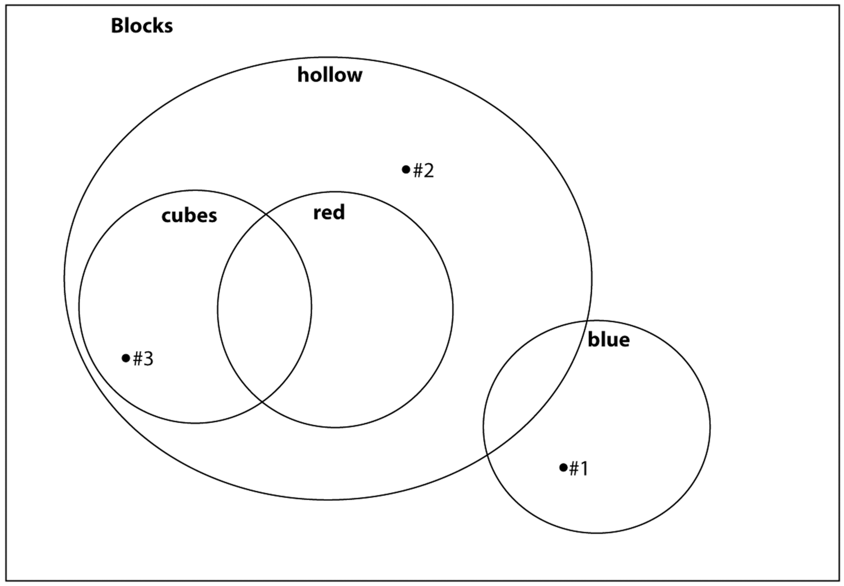
13 Is being a pair of numbers' LCM...

a. ...sufficient for being divisible by each of those numbers?

b. ...necessary for being divisible by each of those numbers?

14 Is being divisible by 108 a sufficient condition for a number to be divisible by 8? By 6? by 54?

15 You're organizing your little sister's blocks. While doing so, you make the following Venn diagram to classify them. Say whether each statement is true.



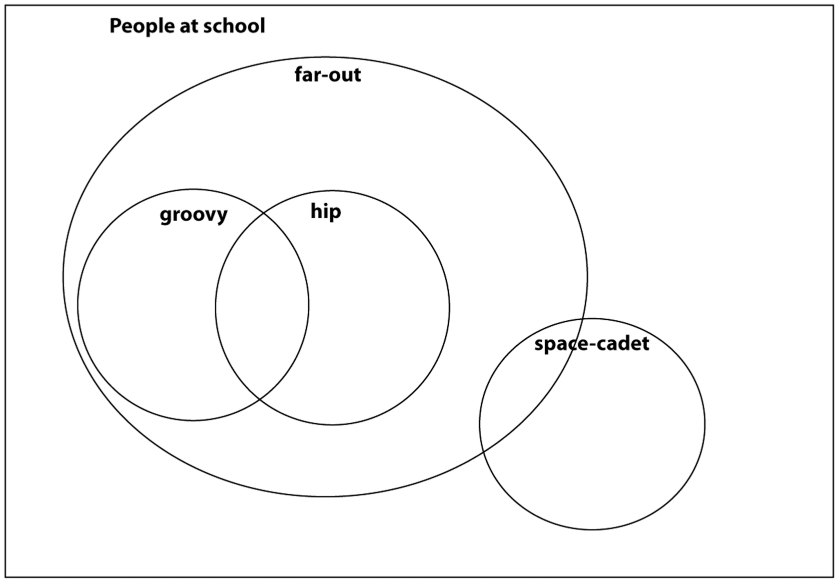
a. If a block is blue, then it is hollow.

b. If a block is red, then it is hollow.

c. If a block is hollow, then it is red.

d. Three of your sister's blocks are shown in the Venn diagram. Describe each block according to the characteristics described in the picture.

16 Bill and Ted come up with the following classification system of the students attending their high school.



According to their scheme,

a. Is being hip necessary for being far-out?

b. Is being groovy sufficient for being far-out?

c. Is being a space-cadet necessary for being far-out?

d. Is being far-out necessary for being hip?

17 Draw a diagram (perhaps a Venn diagram or a "tree" diagram) that clearly indicates which quadrilaterals are types of other quadrilaterals (for instance, a square is a type of rectangle).

Problems

18 Draw a Venn Diagram and use it to see what you can conclude from the following Lewis Carroll puzzle.

All babies are illogical.
Nobody is despised who can manage a crocodile
Illogical persons are despised.

19 The statement “no circle is a parallelogram” can’t be represented in a Venn diagram as a circle within a circle.

a. Write “no circle is a parallelogram” as an if/then statement.

b. Draw a Venn diagram that you think captures the meaning of the statement.

c. Write the converse, inverse, and contrapositive of that statement. Does the same diagram work for the contrapositive?

20 In the last lesson you proved that if you have two central angles of the same size in a circle, then their corresponding chords are congruent.

a. Is the converse of this statement true?

b. Either prove the converse or provide a counterexample.

21 In the last lesson you proved that if you draw a line from the center of a circle perpendicular to a chord, the line would bisect the chord.

a. Write the converse of this statement in a form that you might be able to prove.

b. Is the converse true?

c. Either prove it or provide a counterexample.

22 Euclid’s fifth postulate is famous for being more complicated than his others. The postulate is similar to the statement

If, when two lines are cut by a transversal, the alternate interior angles are not equal, then the lines will meet.

Euclid also proved a theorem which states

If, when two lines are cut byatransversal, the alternate interior angles are equal, then the two lines are parallel.$

a. How are these two statements related to one another? (Converse? Inverse? Contrapositive?)

b. Did Euclid need to prove the second statement separately, or would it have been enough to appeal to the first statement?

23 Dr. Gordon says, "Say you have a statement like 'if p then q'. If this statement is true, then p is sufficient for q and q is necessary for p."

a. Test this condition out by trying some examples of statements (like, "If you live in Baltimore, then you live in Maryland.")

b. Explain, in language that an eighth-grader could understand, why Dr. Gordon's claim is true.

24 In the 2008 presidential campaign, some people were worried about whether Obama could still be a good president even though he lacked military experience. At the same time, General Wesley Clark pointed out that, though McCain had impressive military experience, that experience didn't necessarily prepare him well for the presidency.

a. Are Obama's critics arguing that military service is necessary to become a good president, or sufficient for being a good president?

b. Is Wesley Clark arguing that military service is not necessary to become a good president, or that it is not sufficient?

25 Decide if each of the following is a sufficient condition for a quadrilateral to be a parallelogram. You can think of these as "tests" to decide if the shape is really a parallelogram.

a. Both pairs of opposite angles are congruent.

b. When you draw a diagonal, a pair of alternate interior angles is congruent.

c. One pair of opposite sides and one pair of opposite angles are congruent.

d. Two pairs of adjacent (next to each other) angles are supplementary.

e. One pair of opposite sides is both parallel and congruent.

26 Two of the statements in the previous problem are true. Write a proof of each of them. (If you're stuck on a proof, **work backwards**; what do you need to show in order to show that something is a parallelogram? What tools do you have to show those things?)

27 Draw a Venn diagram to illustrate the following statements:

a. If you can make it through *War and Peace*, you're a superstar.

b. If you're not a superstar, you can't make it through *War and Peace*.

c. In either diagram, can you have not read *War and Peace* and yet still be a superstar?

28 Being an ack is a necessary condition for being a kook. Being an ook is a sufficient condition for being an ack. There are 22 acks, 7 ooks, and 10 kooks. Of the acks, 8 are neither ooks nor kooks. Now, supposing you pick a random ack, what is the probability that it will be an ook, but not a kook?

29 Investigate the following:

a. Draw a bunch of quadrilaterals where the diagonals are perpendicular and where one diagonal bisects the other (but not necessarily the other way around). It may help to start with the diagonals, then draw in the rest of the shape around them. What shape do these always turn out to be?

b. Prove that all quadrilaterals with perpendicular diagonals, one bisecting the other, are this shape.

c. Did you just find a necessary condition for a quadrilateral to be this shape, a sufficient condition, or both?

30 Name some necessary conditions for a shape to be a rhombus.

31 Are the conditions you named in the previous problem on your theorem list already? If the class agrees, add them to your theorem list.

32 Suppose that someone asked you to prove that, if a quadrilateral is not a parallelogram, at least one pair of opposite sides must not be congruent. Explain why you wouldn't have to do a lot of work, even though this statement is not on your theorem list.

33 In 9th grade, you learned about a made-up rule, $a \triangle b$, which takes the first number, adds the second number to it, adds the first number to the sum, then takes that whole answer and multiplies it by the second number. For example, $7 \triangle 2$ is 32.

a. Also in 9th grade, a mysterious fellow named John claimed: "To get an odd number for your answer from $a \triangle b$, you need to input odd numbers for a and b. Did John get it right?

b. Did he identify a necessary or a sufficient condition for getting an odd number, or was his condition both necessary and sufficient?

34 The symbol "&" from the 9th grade means that you take the first number and then add the product of the numbers. About this symbol, John claimed: "To get an odd number for your answer from $a\&b$, you need to input an odd number for a and an even number for b."

a. Did John get this one right?

b. Did he identify a necessary or a sufficient condition for getting an odd number, or was his condition both necessary and sufficient?

35 **Rational numbers** are those numbers that can be expressed as a fraction (where both numerator and denominator are integers and the denominator doesn't equal 0, or a repeating decimal (like 4/3, .256256256..., and 8.000000...). Many **roots** (such as square roots, cube roots, and higher roots) are not rational. Some roots of integers work out to be integers themselves, like $\sqrt[4]{16} = 2$. However, roots of integers that don't themselves work out to be integers are never rational; instead, they can be expressed as non-repeating decimals. Examples of these are $\sqrt{2}$ and $\sqrt[3]{7}$.

Since all numbers can be expressed as repeating or non-repeating decimals, can we conclude that all numbers are either rational or the root of an integer?

36 Recall from your study of quadratic equations that, so long as $a \neq 0$, the equation $ax^2 + bx + c = 0$ has as its solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

a. What is the converse of this statement? Is it true?

b. If the converse is true, then prove it. If it is not true, find a counterexample.

37 Update your theorem list with some of the sufficient conditions you have proven in this lesson.

Exploring in Depth

Many arguments have this type of structure:

- If a Park Student is in fall semester of 10th grade, then they are taking a "Writing about..." class.
- Gwendolyn is in fall semester of 10th grade.
- Therefore, Gwendolyn is taking a "Writing about" class.

Aristotle, a Greek philosopher who lived close to the time of Euclid, called this a **syllogism**. A syllogism has the form

- If A then B
- A
- Therefore, B.

In this example, the statements "if A then B" and "A" are called **premises** and "therefore, B" is called the **conclusion**. Note that if an argument really works, if you agree with the premises you should also agree with the conclusion. (And if you don't, that's a sign that something is wrong with the argument!)

Another form of argument can lead you to form an "if/then" statement as the conclusion. For example,

- If a Park Student is in fall semester of 10th grade, then they are taking a "Writing about..." class.
- If a student is taking a "Writing about" class, then they have to write a self-reflection.
- Therefore, if a Park student is in fall semester of 10th grade, then they have to write a self-reflection.

This type of argument has the form

- If A then B
- If B then C
- Therefore, if A then C.

Note the important thing that allows you to conclude the final step is that "B" occurs as the "then" clause of the first statement and the "if" clause of the second statement.

Boiling arguments down to their logical structure can be handy, especially when you start using the equivalence of the contrapositive. For example, recall the Lewis Carroll puzzle of problem #18:

a) All babies are illogical.
b) Nobody is despised who can manage a crocodile.
c) Illogical persons are despised.

Suppose you have managed to rewrite the puzzle this way:

(from a and c) If you are a baby, then you are despised

(from b) If you can manage a crocodile, then you are not despised.

Right now the premises are written as

If A then B.

If C then not-B.

It's difficult to know what we can conclude just now, but if we replace the second premise with its contrapositive, "if B then not-C," we can make things match up in the right way:

If you are a baby, then you are despised.
If you are despised, then you cannot manage a crocodile.

This allows us to easily see that we can conclude

If you are a baby, then you cannot manage a crocodile.

...which is the solution to the puzzle.

For exercises 38-41 and 43-46, say what, if anything, you can conclude from the premises given.

38 If you are on Park's basketball team, then you are over six feet tall. If you are over six feet tall, you can't go on the Pirate ride at Six Flags.

39 If you live in Baltimore, you should go see an Orioles game. If you don't like baseball, then you shouldn't go see an Orioles game.

40 If it snows a lot, they'll cancel school. If they don't cancel school, I'll scream.

41 If Tom committed the murder, he was in the library at 10 last night. If Tom wasn't in the library at 10 last night, then he was at home.

42 Don't use a calculator for this problem.

a. Simplify $\sqrt{4a^2}$

b. Factor $x^2 + 3x - 28$

c. Find x if $\frac{17x}{3x+2} = 5$

d. Expand $(x + 2)^3$

e. Simplify $\left(\frac{w^5x^{-2}y}{(wx^2)^2y^{-1}}\right)^2$

43 If Jean committed the murder, she was in the library at 10 last night.
If Jean bought groceries on her way home, then she wasn't in the library at 10 last night.

44 All brilliant mathematicians have a heart of gold.
If you study for hours, then you don't have a heart of gold

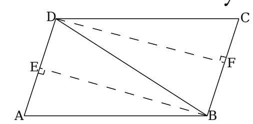
45 If you don't file a form, you can't get good service.
All people who get good service are content.

46 This puzzle comes from Lewis Carroll, author of *Alice in Wonderland*.
(note: "affected" means "written in a snobby, insincere way")
(other note: assume that all poems are either ancient or modern)

No affected poetry is popular among people of real taste.
No modern poetry is free from affectation.
No ancient poem is on the subject of soap-bubbles.
All your poems are on the subject of soap-bubbles.

47 Part c of problem #25 might have seemed true. ("One pair of opposite sides and one pair of opposite angles is sufficient for a quadrilateral to be a parallelogram.") What goes wrong when you try to attempt a proof? Does this suggest a method for finding a counterexample?

48 In fact, P. Halsey published the following "proof" that one pair of opposite sides and angles in a quadrilateral was sufficient for it to be a parallelogram, challenging his readers to find the flaw. Can you?



(source: Jim Loy's puzzle pages: http://www.jimloy.com/geometry/quad.htm)

Given: $\overline{CD} \cong \overline{AB}$, $\angle A \cong \angle C$.
Prove: $ABCD$ is a parallelogram.

Draw perpendiculars from points B and D that intersect the opposite sides of the quadrilateral at E and F, respectively. Then $\Delta DCF \cong \Delta BAE$ by AAS. Since the triangles are congruent, $\overline{AE} \cong \overline{CF}$.

Also, from the triangle congruency we know $\overline{BE} \cong \overline{FD}$ as well. This means that we can now show $\Delta DEB \cong \Delta BFD$ by Hypotenuse-Leg. From this, we can conclude $\overline{DE} \cong \overline{FB}$.

Now, the lengths of $\overline{DE}$ and $\overline{EA}$ add up to the length of $\overline{DA}$, and likewise the lengths of $\overline{BF}$ and $\overline{FC}$ add up to the length of $\overline{BC}$. Since $\overline{DA}$ and $\overline{BC}$ are made up of the same-sized "pieces", they must themselves be congruent. Therefore, $ABCD$ is a parallelogram.

LESSON 5: BUILDING UP NEW THEOREMS

Introduction

In this chapter, you have proven many properties of triangles and quadrilaterals. You've known about some of these properties for a long time; others may have been new to you. You have also set about creating a structure for the geometry you know, choosing certain facts to consider as basic and showing how other facts depend on those. At this point, you are well beyond the basics, and you have probably already begun to make conjectures about ideas you haven't thought about before.

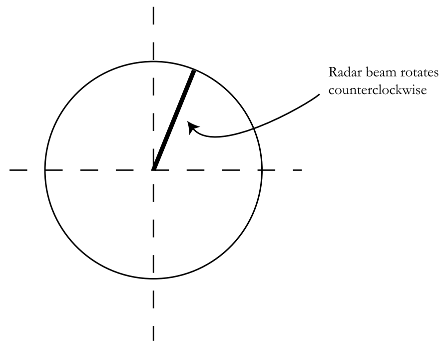
1 In your notebook, use a compass to draw a circle. Then draw two chords of two different lengths that cross somewhere inside the circle. Label the endpoints of one chord A and B, and the endpoints of another chord C and D. The chords cross at point E. Now measure $\overline{AE}$, $\overline{EB}$, $\overline{CE}$, and $\overline{ED}$. Try calculating $AE \cdot EB$ and $CE \cdot ED$. What do you find?

In this section, most ideas that come up will probably be new, and many, like the result above, will be unexpected. You will be in a good position to make some conjectures and to prove many of the conjectures you make.

After looking at parallel lines, triangles, and quadrilaterals, we'll now turn our attention to circles. Many of the theorems you've already established will prove useful in investigating circles. Circles themselves often prove useful in analyzing other shapes, as you've probably realized when doing compass-and-straightedge constructions. In the development, most of your time will be spent learning the substantial vocabulary that has to do with circles. Then you'll learn and prove a few basic properties of circles. In the problems section, you will have freer rein to conjecture and prove more things about circles, and about geometry in general.

Development

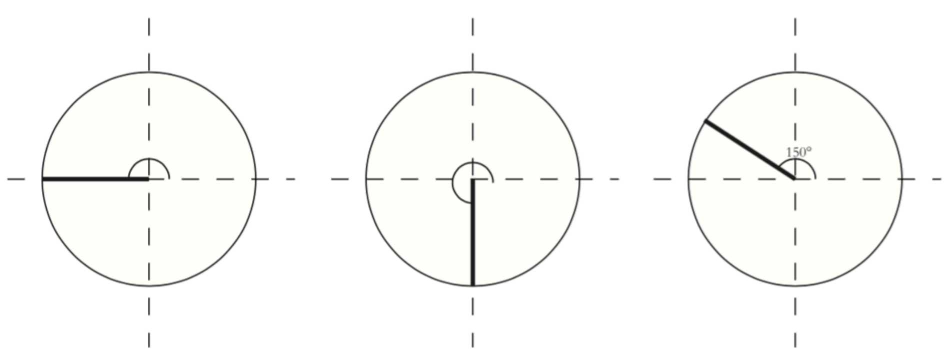
Your submarine has a rotating radar beam that is used to get a picture of the surrounding area. Your radar screen looks like this,



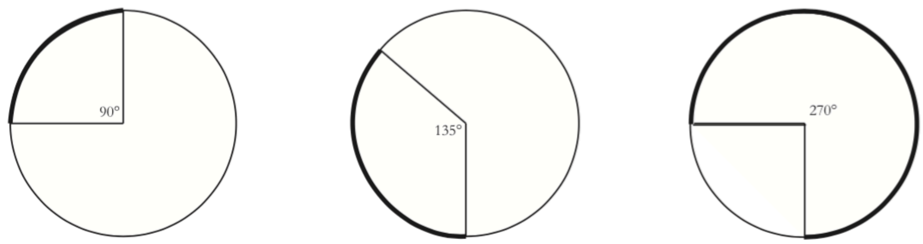
with the bold line segment showing the motion of the radar beam. As the beam rotates, the screen updates based on what the beam finds.

2 Why do you think the radar screen is in the shape of a circle, rather than, say, a rectangle?

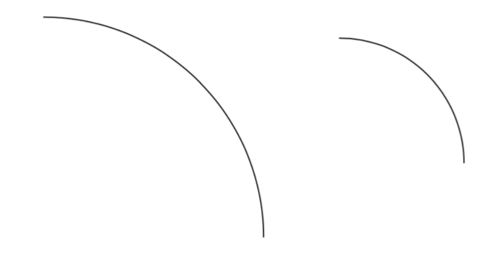
3 Say that you know that the circle the screen makes has a circumference of 360 millimeters. What's the length of the edge of the screen that has been updated since the beam was pointing due east? Answer this question for each picture below.



Recall that the 150° angle in the rightmost picture on the previous page is called a **central angle**. As you can see, there's a direct relationship between a central angle and the arc it **intercepts**, even if the circumference of the circle is not 360 millimeters.



Because of this relationship, people also talk about the angular **measure** of an arc. The measures of the arcs above are 90°, 135°, and 270° . So both of the arcs below have measure 90°, even though their lengths are very different.

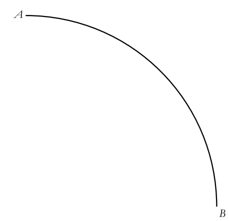


4 What's the measure of the arc that borders the portion of the radar screen between east and northwest?

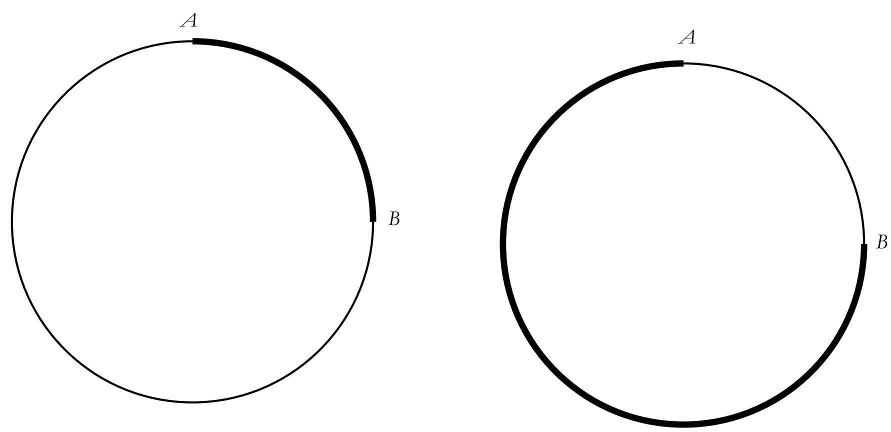
5 What's the measure of the arc that borders the portion of the radar screen between north and south-southeast?

6 If an arc of a circle has measure 47°, what's the measure of the arc you'd need to form the rest of the circle?

To refer to an arc, you use the two points where the arc begins and ends. Here's arc AB, or, better yet, $\overset{\frown}{AB}$.



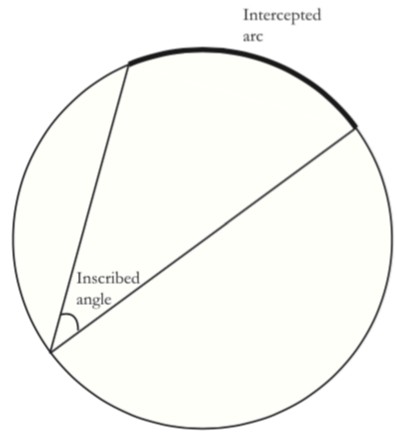
Of course, when we're talking about arcs that are parts of circles, naming arcs this way has a little problem...



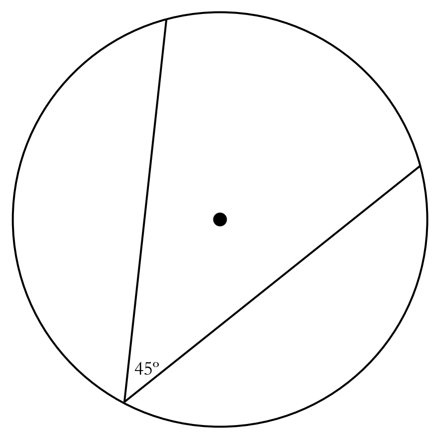
...because how would we know whether we meant the arc that looks like it's about 90°, or the one that looks like it's about 270°? When you don't specify, it's assumed that you mean the smaller, or **minor** arc.

7 Draw a circle in your notebook. Pick points Q and Z anywhere on the circle. Use measuring tools (and perhaps a bit of ingenuity) to find the measure of arc QZ. Then describe a general strategy for finding arc measure.

Another type of angle in a circle is called an **inscribed** angle. An inscribed angle has its vertex *on* the circle, not at the center. Inscribed angles can intercept arcs, too.



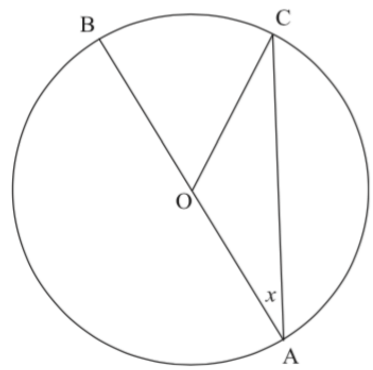
8 Here's a 45° inscribed angle. Does it look like it also intercepts a 45° arc?



9 There is a conjecture to be found here. Draw some more circles with inscribed angles, and find it.

Now that you have a conjecture, it's time to look for a proof. The following problems will help you.

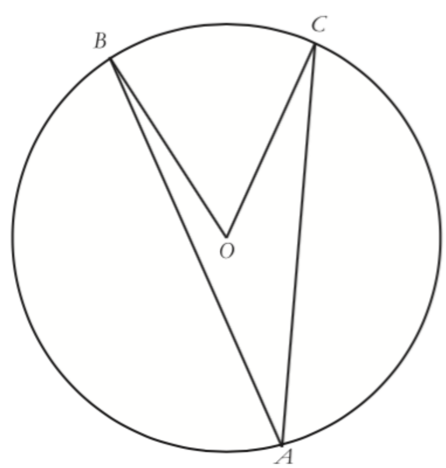
10 Below is a picture of an inscribed angle BAC, which intercepts the same arc as central angle BOC.



a. Using geometric tinkering, find the measure of $\angle BOC$ in terms of x.

b. The proof that you've done does not work for the diagram in problem 8. What goes wrong?

Here is another diagram, one that looks more like the situation in problem 8.



Since we already know something about when one of the sides of an angle is a diameter, let's draw in the diameter and label the angles this way:

11 Using this diagram, prove that the central angle is still twice the inscribed angle.

12 Between problems 10 and 11, will your proofs work for all inscribed angles? If so, great. If not, draw a picture of an inscribed angle, along with its associated central angle, for which the proofs will not work.

Practice

13 For this problem, think of an analog clock (a clock with hands).

a. Find the angle determined by the hands of a clock when it is 4 o'clock. What's the measure of the arc formed by the edge of the clock between the 12 and the 4?

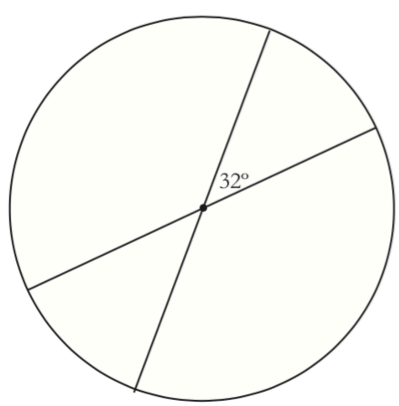
b. Now imagine that you draw a line from the 12 to the 8 and then from the 8 to the 4. What's the angle formed at the 8?

c. What's the measure of the arc formed by the edge of a clock between the 7 and the 11?

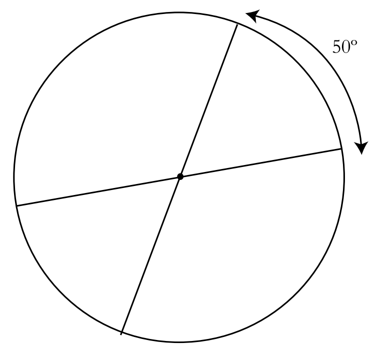
d. Create an inscribed angle by starting at the 7, going to any time on the clock not between 7 and 11 o'clock, and then going to the 11. How will this compare with your angle in part c?

14 For each figure, find as many angles and arc measures as you can. A dot indicates the center of the circle.

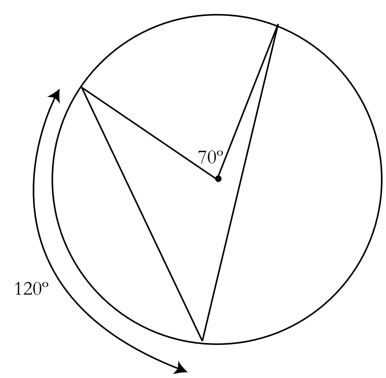
a.



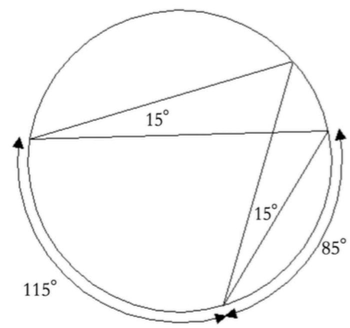
b.



c.

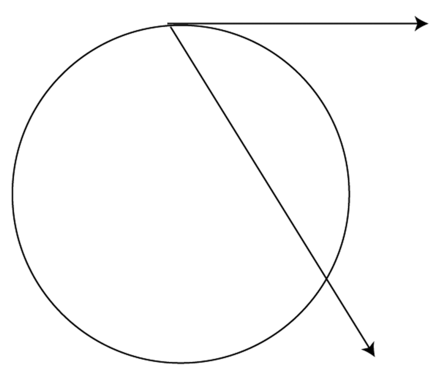


d.



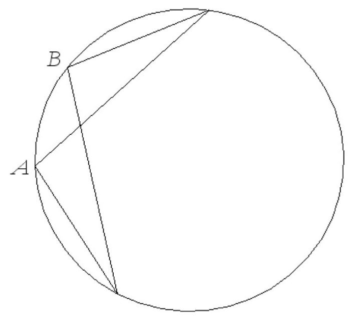
15 What can you say about two inscribed angles that intercept the same arc? Why?

16 Below is an angle formed not by two chords, but by a chord and a tangent segment. Note that this angle still can be said to intercept an arc. Make a conjecture about the measure of a chord-tangent angle and the measure of its intercepted arc.

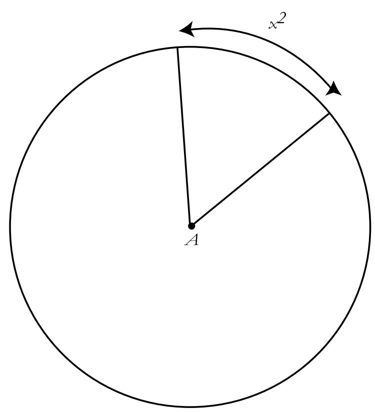


Problem continued on next page

17 In the diagram below, $m\angle A = \left(x^2\right)^\circ$ and $m\angle B = (2x + 120)^\circ$. Find x.



18 Let $m\angle A = (2x + 24)^\circ$. Find x.



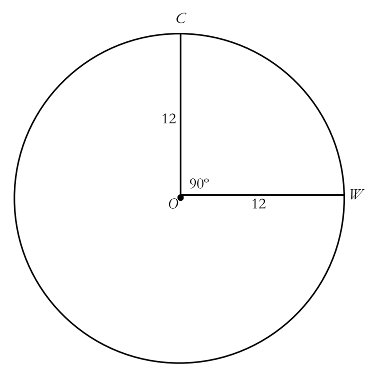
19 Q is the center of the circle, and $m\angle A = 25^\circ$. Find the measure of $\overset{\frown}{AB}$.

20 Circle O has radius 12. Find

a. The measure of $\overparen{AB}$.

b. The actual length of $\overparen{AB}$.

c. The *area* of the **sector** of the circle COW. (What do you suppose a **sector** is?)

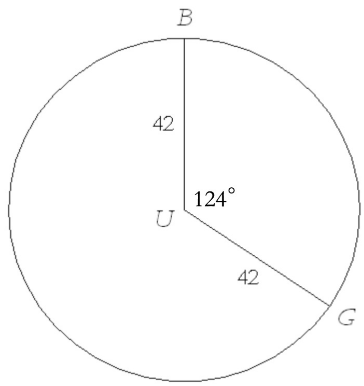


Problems

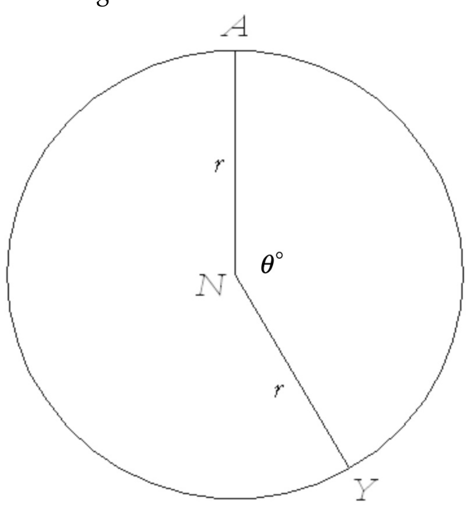
21 The circle below has radius 42. Find

a. The length of $\overset{\frown}{BG}$

b. The area of sector BUG



22 The circle below has radius r, and the central angle is θ. Find

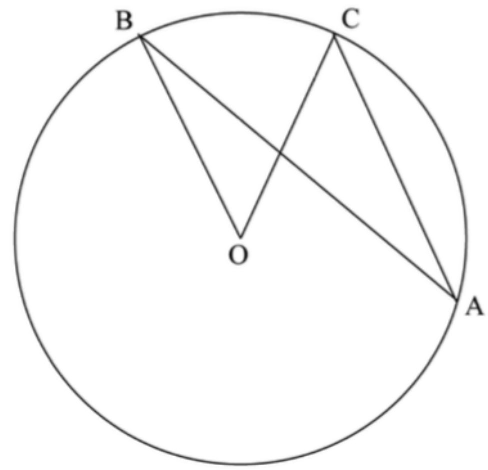


.

a. The length of $\overset{\frown}{AY}$.

b. The area of sector ANY.

23 Here's another diagram showing a central angle and an inscribed angle that intercept the same arc.



a. Why don't the methods of proof you used in problems 10 and 11 work for this inscribed angle/central angle pair?

b. Prove that $m\angle BOC$ is twice $m\angle BAC$. As is often the case, you'll need to act on the diagram first.

c. Now have you proved this result for all inscribed angles, or are there other positions for inscribed angles for which none of your proofs would work?

d. When you need to do separate proofs depending on the situation, that is called "proof by cases." For example, if you wanted to prove a result in number theory, you might do different proofs depending on whether your original number was even or odd, but you would have to do both proofs in order to prove that the result always worked. Describe in words the three cases you considered when you did the inscribed angle proof.

24

Draw line segment AB, then accurately draw its perpendicular bisector.

a. Pick a few points on the perpendicular bisector and label them C, D, E, etc. Compare CA vs. CB. Does this result seem to hold up consistently?

b. Investigate whether the converse of your conjecture in part a is true. If you are having trouble figuring out what the converse should be, try rewriting your original conjecture in "if/then" form.

25

The radius of a circle is 16.5 inches.

a. What's the area of a 90-degree sector of the circle?

b. What's the area of a 1-degree sector of the circle?

c. What's the area of a 53.5-degree sector of the circle?

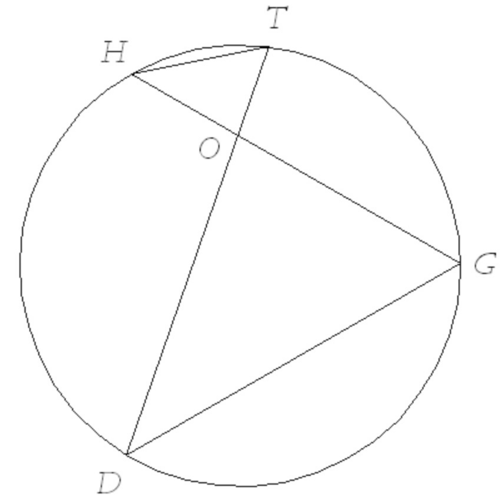
d. What's the area of an n-degree sector of the circle?

26

In your notebook, draw $\angle XYZ$. Suppose you begin to walk a path "inside the angle," starting at Y. You walk in such a way that you are always the same distance away from $\overrightarrow{YX}$ and $\overrightarrow{YZ}$. Using precise mathematical language, make a conjecture about your path.

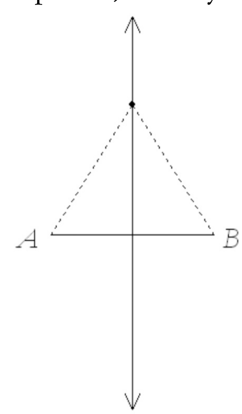
27

Prove that $\triangle HOT \sim \triangle DOG$.



28

Here is a figure designed to help you with a proof. Two proofs, actually.



a. Use the figure to prove that points on the perpendicular bisector of a segment are equidistant from the endpoints of the segment. Be careful in selecting the given.

b. Now use the figure to prove the converse. Again, be careful in selecting the given and what is to be proved.

29 If you had only a compass and straightedge, but no ruler, suggest a method for drawing in the perpendicular bisector of any line segment AB. No fair folding the paper.

30 You cut a pizza into 10 congruent slices.

a. If the area of one slice is 6.5 square inches, what's the radius of the pizza?

b. If instead the perimeter of one slice is 17 inches, what's the radius of the pizza? (If you're stuck, you could write an equation first: $17 = \ldots$)

31 Can the area of a circle ever be an integer? Why or why not?

32 Here is Problem 1, reprinted:

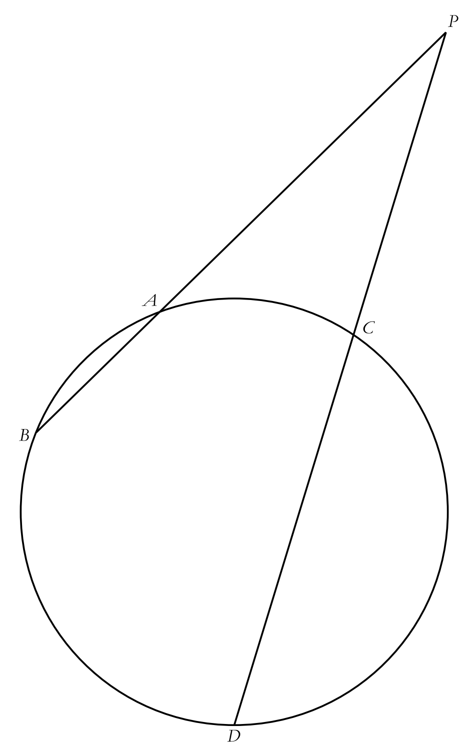
In your notebook, use a compass to draw a circle. Then draw two chords that cross somewhere inside the circle; they needn't cross at the center of the circle. Label the endpoints of one chord A and B, and the endpoints of another chord C and D. The chords cross at point E. Now measure segment $\overline{AE}$, $\overline{EB}$, $\overline{CE}$, and $\overline{ED}$. Try calculating $AE \cdot EB$ and $CE \cdot ED$. What do you find?

Prove the conjecture you made in Problem 1. Planning this proof is a great opportunity to **work backward**. What geometry tools do you have to show that two products equal one another?

33 Is it possible for the lengths of the four segments formed by two intersecting chords in a circle to be 4 consecutive integers?

34 Suppose that you have a triangle inscribed in a circle in such a way that one of the triangle's sides is the circle's diameter. What kind of triangle must you have, and why?

35 There is a theorem very similar to the one dealt with in Problem 32 about line segments drawn from a point outside a circle to a circle. (You could also think of this as a case where lines cross outside of a circle, instead of inside it.) As in the case of crossed chords, there is a product of lengths that remains constant. Which lengths are they?



36 State and prove a theorem based on your findings in Problem 26.

37 In the previous problem, did you prove a result corresponding to Problem 26, or did you prove the converse? Either way, state and prove the converse of the theorem you proved in the previous problem.

38 Suggest a way to construct an angle bisector using only a compass and straightedge.

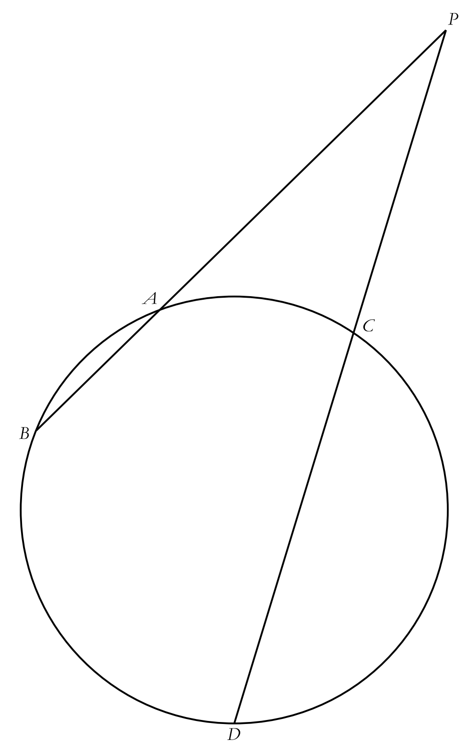
39 Given triangle ABC, let F be the point where segment BC meets the bisector of angle BAC. Draw the line through B that is parallel to segment AF, and let E be the point where this parallel meets the extension of segment CA.

a. Find the four congruent angles in your diagram.

b. How are the lengths EA, AC, BF, and FC related?

c. How are the lengths AB, AC, BF, and FC related?

40 In the figure below, prove that $PB \cdot PA = PD \cdot PC$. Here are a few hints.

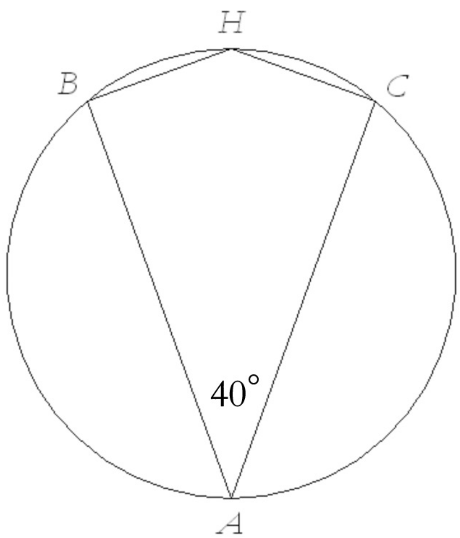


a. Show that the result you want is equivalent to $\frac{PB}{PC} = \frac{PD}{PA}$.

b. Part a is a sign that you should be looking for similar triangles where $\overline{PB}$ and $\overline{PD}$ are corresponding sides, and so are $\overline{PC}$ and $\overline{PA}$. Add some lines to the diagram so that triangles exist with these sides.

c. Now prove the result.

41 BACH is a kite inscribed in a circle. Find all the angles of the kite. (Hint: find some arc measures along the way.)



42 Don't use a calculator for this problem.

a. Solve for x if $\frac{2x-4}{x^2+3x+1} = 0$

b. Simplify $\sqrt{72}$

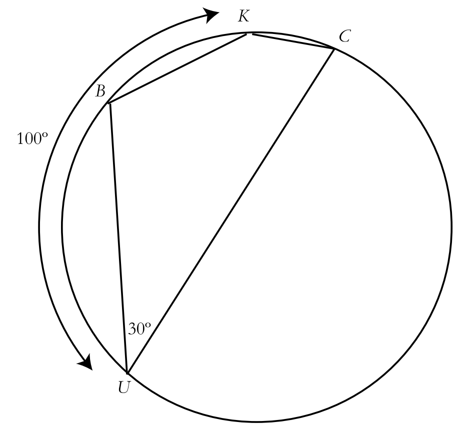
c. At what point do the lines $y = 2x - 4$ and $y = -\frac{4}{7}x + 5$ cross?

d. Solve for x if $x^2 + 6x - 2 = 0$

e. If $x + y = 1729$ and $\frac{1}{x} + \frac{1}{y} = 1729$, find xy.

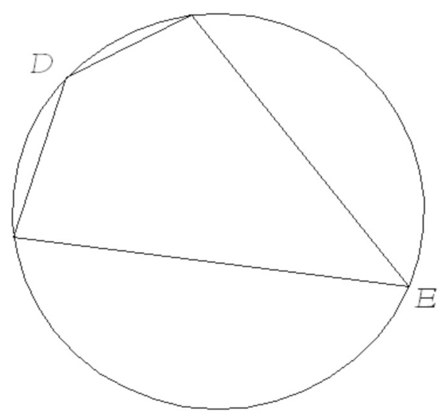
43 Draw several triangles of varying shapes and sizes, then construct the perpendicular bisectors of each side of each triangle. Coincidence? You be the judge.

44 BUCK is not a kite. Find all of its angles.



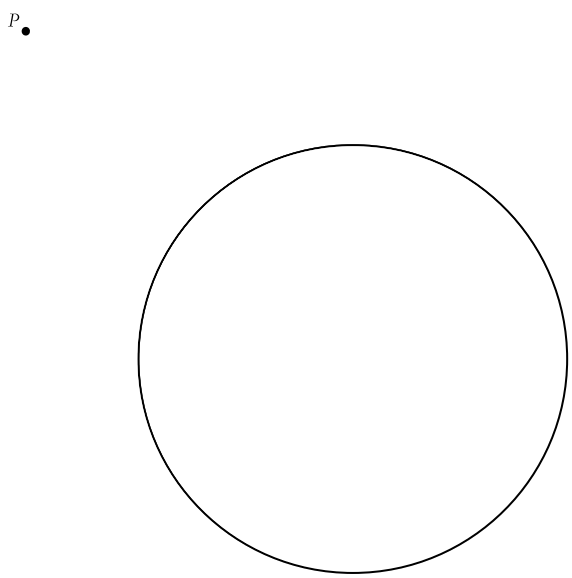
45 Make a generalization about the opposite angles of a quadrilateral inscribed in a circle.

46 Let $m\angle D = \left(x^2 - 10x + 60\right)^\circ$ and $m\angle E = (8x + 85)^\circ$. Find all possible values of x.



47 You know results about central angles and their intercepted arcs, as well as inscribed angles and their intercepted arcs. How about angles with their vertex not on the center or on the edge of a circle, but inside the circle? Make a conjecture. Hint: One way to get angles with their vertex inside a circle is to draw a pair of crossed chords.

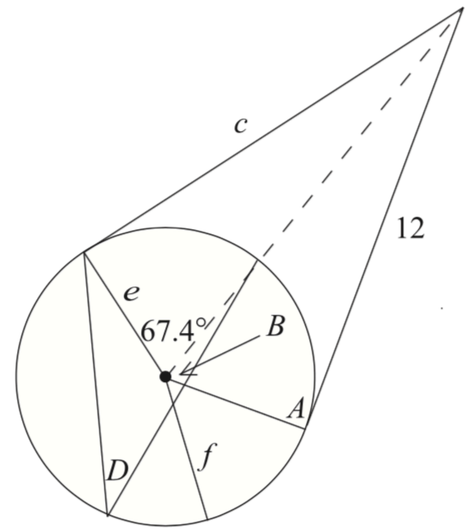
48 How many lines can you draw through point P that are tangent to the circle? Try drawing lines from a point other than P. What about a point inside or on the circle?



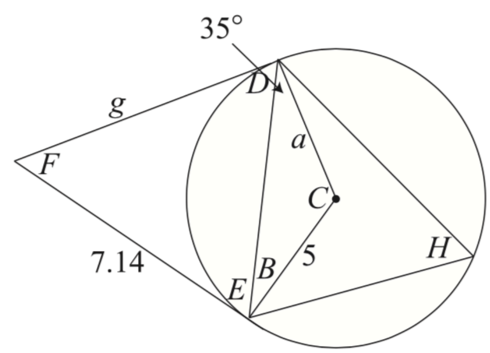
49 Rotate the diagram above so that P is on the bottom and the whole figure looks like an ice-cream cone. Make a conjecture about the lengths of the tangents from P.

50 Prove the conjecture you made in the previous problem. Do you need to have additional information? How can you create some?

51 The dotted line segment has a length of 13. Find the lengths and angles marked with letters. Give a reason each time you deduce something. The center is marked with a dot; lines that appear to be tangents are tangents.

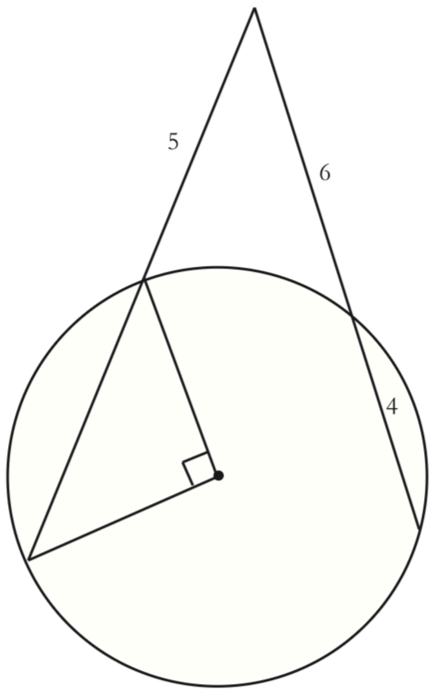


52 Find the lengths and angles marked with letters. (Uppercase for angles; lowercase for lengths) Give a reason each time you deduce something. The center is marked with a dot; lines that appear to be tangents are tangents.



Exploring in Depth

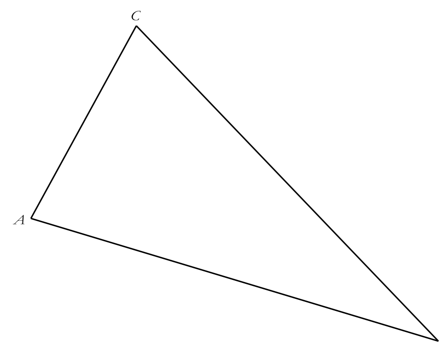
53 Find the exact value of the radius of the circle below.



54 Prove that the measure of an angle formed by crossed chords in a circle is the average of the measures of the two arcs intercepted by the two angles of that size.

55 Find the probability that 3 points chosen at random from n points evenly spaced on a circle will form a right triangle. (Hint: simplify the problem.)

56 Draw in the perpendicular bisectors for sides AB and BC of the triangle below. Then label the point where they cross D.

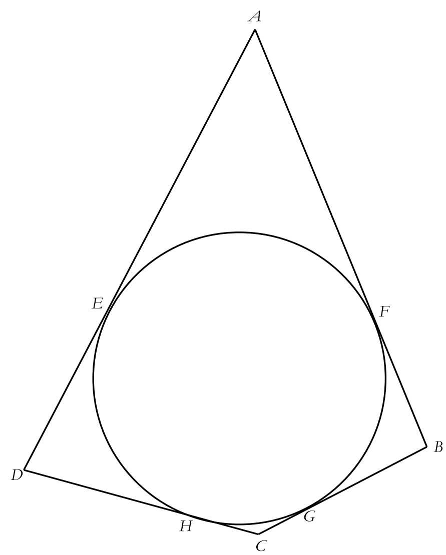


a. Is the point D closer to A or B?

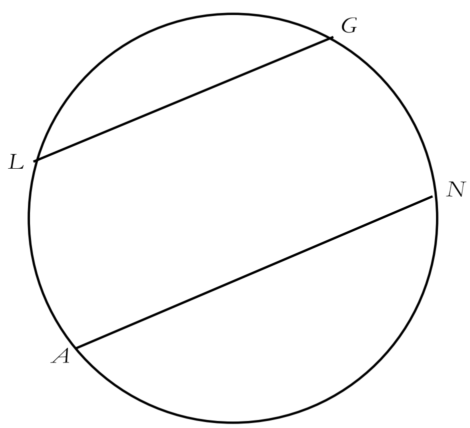
b. Is the point D closer to B or C?

c. Suppose, now, that you were going to draw the perpendicular bisector of $\overline{AC}$ as well. Can you make an argument that it should pass through the point D, even before you draw it?

57 In this figure, $AB = 20, BC = 11, \text{and} DC = 14.$ Find AD. (Hint: pick a side length to call x, then try to write the other lengths in terms of x.)



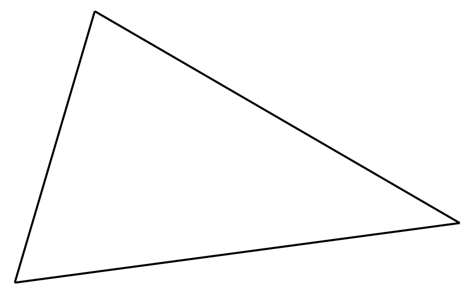
58 Given: $\overset{\frown}{NG} \cong \overset{\frown}{LA}$ Prove: $\overline{AN} \parallel \overline{GL}$



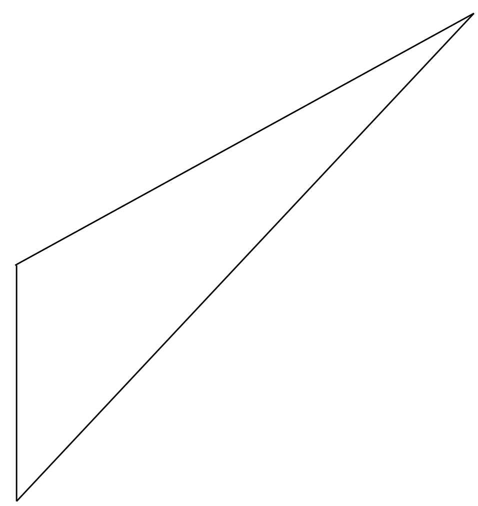
You know that every triangle has three different heights (or "altitudes").

59 Use a ruler and protractor to draw in the three different altitudes for each triangle:

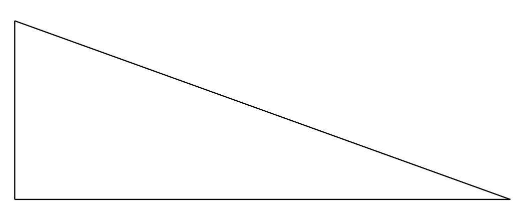
a.



b.



c.



60 Come up with an argument for why, in a right triangle, the three altitudes must all meet in one point.

61 In each of the triangles in problem 59, the three altitudes should have met in one point. Do you think this is true for all triangles? See if you can find a counterexample.

62 Draw a large triangle, then draw its midpoint triangle. Draw the perpendicular bisectors of the sides of the large triangle. These lines have a significance for the midpoint triangle as well. What's the significance, and why is this always the case?

63 Can you now prove that the three altitudes of any triangle meet at one point? Do so or explain why there still might be counterexamples.

64 Remind yourself of the result about angle bisectors you proved earlier in this lesson. Use it to prove that the three angle bisectors in a triangle meet at one point.

65 Prove that an angle formed by a chord and a tangent to a circle is half the measure of its intercepted arc.

66 Draw a point P outside of a circle. Then draw two line segments from P: a tangent segment that hits the circle at point A, and a segment that intersects the circle twice, at point B (farther from P) and C (closer to P). Prove that $PA^2 = PB \cdot PC$.

Park School Mathematics

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