FUNCTIONS AND TRIGONOMETRY

TABLE OF CONTENTS

Mathematical Habits of Mind 2

Habits of Mind: Visualize 4

Habits of Mind: Check For Plausibility 8

Lesson 1: Circular Functions 15

Lesson 2: Getting Comfortable With Trig 25

Lesson 3: Law Of Sines And Law Of Cosines 39

Lesson 4: Modeling With Trigonometry 59

Lesson 5: Making New Functions 75

Lesson 6: Variation And Proportion 97

HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

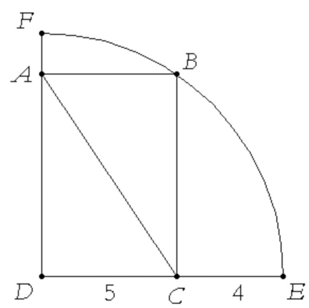
work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

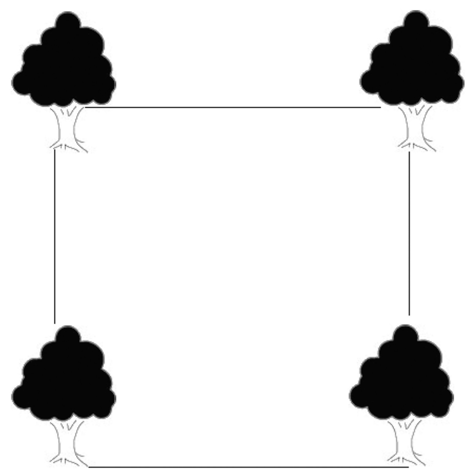
1 In the following picture, rectangle ABCD is inscribed in a quarter-circle. $DC = 5$, and $CE = 4$. Can you figure out the length of diagonal AC?



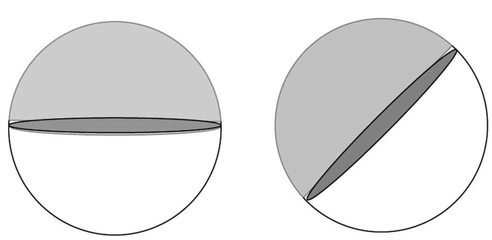
Often a problem that can seem particularly perplexing can be solved by looking at it in a different way. Sometimes the best way to keep track of the different information and variables in a problem is to draw a picture of some sort, to **visualize** the information so that it is in a form that is easier to understand. You saw this last year as you solved a variety of problems, at times by constructing models, and at times by finding clever ways to visually represent things that at first seemed quite nonvisual.

visualize

2 Four trees are planted at each corner of a square park. The city wants to expand the park to twice its current area, but in such a way that the park is still a square, and none of the four trees is in the interior of the park. (The trees cannot be transplanted.) Draw a plan for a new park that meets these criteria.

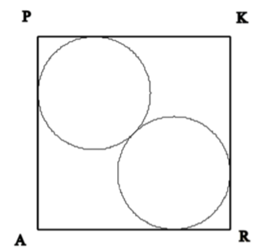


3 The shaded parts of the spheres below are hemispheres.



You throw three darts onto the surface of a globe, each from a randomly chosen direction. What is the probability that all three darts lie in one hemisphere?

4 The circles in the diagram each have radius 1 cm, are tangent to each other and also to the square PARK. Their centers are on the line PR. Find the area of the square PARK.



5 There are many different triangles with side lengths 12 and 13 cm. Which of these triangles has the greatest area?

6 A rectangle and a square are inscribed in congruent circles. The rectangle has a width of 6 and a length of 8. What is the area of the square?

7 Three identical, spherical oranges are placed in a bin as part of a supermarket display. The bin is exactly long and wide enough to have two oranges fit snugly in the bottom, but there's plenty of room to place the third orange on top of these two. If the radius of an orange is 2 inches, find the height of this small stack.

8 A street has parallel curbs 40 ft apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet, and each stripe is 50 feet long. Find the distance between the stripes.

9 Let A and B be any two points in a plane.

a. How many different circles can you draw that go through points A and B? Can you give the radius of the smallest possible circle? Of the biggest?

b. How many different rectangles can you draw with opposite vertices on points A and B?

10 PQ and QR are diagonals of two faces of a cube. Find the measure of $\angle PQR$.

11 A trapezoid is inscribed in a circle of radius 5 cm so that one base is a diameter of the circle, and the other base has length 5 cm. What is the perimeter of the trapezoid?

12 If you start with $\frac{1}{2}$ then add $\frac{1}{4}$, then $\frac{1}{8}$, then $\frac{1}{16}$, and so on, ad infinitum, what do you suppose the answer would be? Draw a diagram that would justify your response.

13 How many sides does a cube have? How about a pyramid?

Can you build a closed 3-dimensional shape out of 4 flat sides? How about out of 3 flat sides? Give examples, or explain why not.

14 Suppose you have a box that has a base of 1 inch by 5 inches and that stands 8 inches tall. How many ½ inch radius spherical balls can you get into this box if you can't let any ball protrude above the top of the box?

15 Craziola, the wacky pizza guy, has decided he wants to cut a pizza into as many pieces as possible, with as few straight cuts as possible. He doesn't care at all if the pieces are of equal size, he just wants to make the most number of distinct pieces. With 1 cut, he produces 2 pieces. With 2 cuts, he creates a maximum of 4 pieces, no matter how crazy the 2 cuts he makes are. How many pieces can he possibly make with 3 cuts? 4 cuts? What about n cuts? Can you find the pattern?

16 Is it possible to arrange six unsharpened pencils so that they all touch each other?

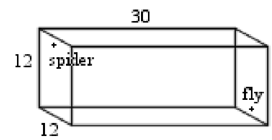
17 Mr. Shimano gives an extremely difficult Japanese test. The highest score was 74% and the lowest 31%. Rather than give a retest, Mr. Shimano decides to raise the 74% to 93% and the 31% to 61%. For a student who scored 57 on the original test, what score do you think Mr. Shimano should give him after the adjustment?

18 A circle is sitting inside an equilateral triangle so that it's tangent to the sides of the triangle in three places. Another equilateral triangle is inside the circle with its three vertices on the circle. If the length of a side of the smaller triangle is 1 cm, find the length of a side of the bigger triangle.

19 Two sides of a triangle measure 6 and 11 cm respectively. If the length of the third side is also an integer, what possible lengths can the third side have?

20 What is the maximum number of acute angles a convex polygon can have?

21 A hungry spider and a fly are in a room 30 feet long, 12 feet wide, and 12 feet high. The spider is on one of the smaller walls, 6 feet from each side and 1 foot from the ceiling. The fly is on the opposite wall, 6 feet from each side and 1 foot from the floor. Assume that the fly does not move (it is paralyzed by fear!) and find the shortest path that the spider can take to eat the fly.



If you were told by a classmate that taking a cup of brackish lukewarm water three times a day for a month would heal your broken leg, you would probably laugh out loud. And even though your laughter might be a bit more subdued were the source of that statement the health segment of the evening news, I bet that you would nonetheless be quite skeptical. On many levels this would seem to you to be highly implausible, and so you might either absolutely dismiss it or yield to that nagging curiosity and check out some other sources.

On the other hand there are a host of other more reasonable sounding claims that you might be prepared to let slide. Examples might include the claim that an increase in oil prices pushes a decline in the Dow Industrial averages, or that there is a strong correlation between wealth and SAT scores. It doesn't seem unreasonable to hold a healthy skepticism for these as well.

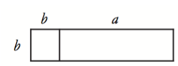
In the context of mathematics, to **check for plausibility** is to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations.

1 Your classmate proposed traveling 240 miles to the beach at an average speed of 80 mph, and the next day traveling home at 40 mph. He pointed out that since you have an average speed of 60 mph, the total 480 miles there and back takes 8 hours. Yet when you went to the beach, it didn't take 8 hours round trip, even though you two drove at precisely the speeds he had proposed. Why?

check for plausibility

It certainly seems at first glance that your classmate is correct, but just because a result seems superficially correct doesn't make it so. When there is an apparent contradiction in a problem, it often pays to make a quick check or two to see if what is being said is plausible. For example, how long does each half of the beach trip take?

2 Martin says, looking at the figure below, that the entire rectangle's area divided by the area of the square inside is equal to $\frac{a}{b} + 1$. Melissa thinks that such a peculiarly simple answer is unlikely to be correct. Martin insists he's right, and that in fact his formula would work for any values of a and b one might choose. How could Melissa check to see if Martin's formula is at least plausibly correct?



One particular way that we check the reasonableness of a solution to a problem is to **examine special and limiting cases**. In the problem above, what would those cases be? Well, one special case would surely be when $a = b$, because you could quickly determine the right answer and check if Martin's formula is correct there. What about if we examined when "a" was small and "b" was large—say $a = 1$ and $b = 100$, or even $a = 1$ and $b = 10000$? Does Martin's formula make sense in those cases as well? Can you come up with another limiting case, and see if his formula is plausible for that case as well?

Once you have checked a solution for all the limiting cases you can think of, if it still seems like a reasonable solution, you might think about how to prove it always works yourself from first principles. Can you come up with Martin's formula?

3 If one looks at $y = x^2$ on the calculator in "ZOOM SQR" mode (so that the scale in the x and y directions are the same and the graph looks most accurate), it appears that the graph is getting so steep, so quickly, that it will eventually become a vertical line. Does it? If so, estimate for what x value it becomes vertical; if not, explain why that can never happen.

4 While teaching in Brazil, Tony was approached by a fellow math teacher who said the following: "Hey Tony, do you know how to prove that all right triangles are similar? I was trying to show my students in class today and I couldn't quite do it." Could you have helped him out?

5 On a 3-D blueprint for an Olympic swimming pool, 1 ft. represents 16 actual feet. In order to determine how much water would be needed to fill the pool, Tim computes the volume from the blueprint, which is 4 ft^3, and then he multiplies by the scale factor of 16 to get 64 ft^3 of water to fill the pool. Tim is a little unsure if that is the correct amount, but it seems right. What do you think?

6 If Jorge told you that 3.16227765 was an exact solution to $x^2 = 10$, how could you determine without a calculator if he is correct, or slightly off?

7 Hero of Alexandria came up with a formula to determine the area of any triangle based solely on the lengths of its 3 sides. Below are 4 formulas, all of which purport to be Hero's formula. In all 4 formulas, a, b, and c are the lengths of the sides, and s is the semi-perimeter, which is equal to half the perimeter.

$$Area = (s-a)\,(s-b)\,(s-c)$$

$$Area = \sqrt{\frac{(s)(a)(b)(c)}{10}}$$

$$Area = \sqrt{(s)(s-a)(s-b)(s-c)}$$

$$Area = \sqrt{3(a+b+c)}$$

a. Which of these formulas do you think is the right one? Why?

b. Try seeing if the formulas give the kind of answers you would expect for various "common" triangles you have experience with.

c. Test to see if "extreme" triangles (ones with very large or small values of some of the sides) also give reasonable results.

d. What units do you typically measure area in? Does that also help you in deciding which formulas are most plausible?

Problem continued on the next page

e. Now pick the formula you are most confident is the right one. Does the fact that it has passed all your “tests” prove that it is correct? If so, explain why. If not, explain how you could become convinced that Hero was in fact correct.

8 Look at these 6 numbers: 1, 3, 6, 9, 11 and 12. Their mean is 7. Subtracting the mean from each of the numbers and adding those together gives us $-6 + -4 + -1 + 2 + 4 + 5$, which equals 0. Juniper is unimpressed, and says that you would always get 0, regardless of the 6 numbers you chose. Sassafras disagrees, and says it is highly dependent on choosing the right 6 numbers; for example, in this case, exactly 3 were above the mean and exactly 3 were below, and also there were no decimals to complicate matters. Who is right?

9 Zargo says that instead of doing lots of intricate calculations, he can find the area of a rhombus by just multiplying the diagonal lengths together. See if you can determine in a minute if his method is plausible.

10 If you draw a line from the vertex of any triangle to the midpoint of the opposite side (i.e. the median), will it be perpendicular to that side, or would it bisect the vertex angle from which it was drawn?

11 A pollster interviewed 100 families, and reported that the mean number of children was 2.037 and the median was 1.8. I do not believe either of these figures. Do you? Why?

12 Brian thinks he remembers that the area of a parallelogram is equal to the product of consecutive sides, but he isn’t quite sure. You can’t remember whether he’s right either, but you know you can check his formula to see if it is plausible. Is it?

13 Show that the formula for the area of a triangle can be viewed as just a special case of the area of a trapezoid.

14 Which is bigger, $4^{\frac{1}{4}}$ or $10^{\frac{1}{10}}$? No calculators allowed! (Hint: try thinking about limiting cases.)

15 Hero’s formula gives us a formula for the area of a triangle based only on the lengths of its 3 sides (see problem 7 in this lesson). No one has yet come up with a formula for the area of a quadrilateral based only on the lengths of its 4 sides. Why do you think that is?

16

a. Veneeta graphs $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 10$ on the same calculator screen using ZOOM STD. What does she notice about how their shapes compare to each other?

b. Grunchik changes the viewing window on his calculator so that it graphs x values from -4 to 4, and y values from -10 to 20. He then graphs the same three equations that Veneeta graphed. What does he notice about how their shapes compare to each other?

c. Veneeta and Grunchik compared their different answers, but aren't sure what to make of them. What do you think?

17

Prashad says that in a quadrilateral ABCD he has examined, AB + BC + CD + DA is equal to 1.8 times the diagonal AC. Evaluate whether what he says is possible or not.

18

a. When an object falls under gravity, its speed increases by a constant amount each second. Two stones are dropped at the same time from a cliff, but one of them is 10 feet higher up than the other at the time of dropping. As they fall, will the distance between them always be the same?

b. Later on, two stones are dropped at the same height from a cliff, but one stone is released one second before the other. As they fall, will the distance between them always be the same?

19

On the same screen as $y = x^2$, graph $y = x$ in ZOOM STANDARD. Then "ZOOM IN" once.

Note that the graph of $y = x$ is "above" the graph of $y = x^2$ for some values of x.

Hermione thinks that those values of x will be the only ones where $y = x$ is above $y = x^2$. What do you think?

check for plausibility

20 Bart is feeling a little sick. Having recently read about simpsonitis, a very rare and debilitating disease, and being somewhat hypochondriacal, he goes to see his physician, Dr. Kalvakian. The doctor checks him out and decides to administer a special blood test for detecting the disease.

This diagnostic test is 98% accurate (returns a positive result) for people who have simpsonitis and 95% accurate (returns a negative result) for people who do not have simpsonitis. Approximately 0.3% of people in the country actually have this disease.

Unfortunately, several days after taking the blood test, Bart receives a phone call from Dr. Kalvakian. The doctor tells him that he tested positive for the disease. Bart asks, "What's the chance that I actually have simpsonitis? I mean, you said that the test was not 100% accurate." Dr. Kalvakian replies, "Well, there's a 98% chance that you have the disease, Bart."

Bart initially has a cow, but then he decides to tell the brainy Lisa what Dr. Kalvakian said. What do you suppose Lisa told Bart in response?

21 Zollywog the crazy Geometry student has come up with a formula for the length of a median in a triangle! He claims that the length of the median m that bisects side a of a triangle (with other sides b and c, of course) is:

$$m = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}.$$

Could this possibly be true?

22 Felipe tells Bradley that he has just come up with a cool fact: a regular polygon of n sides, with distance r from the middle of the polygon to any one of the vertices, will always have an area less than $4r^2$. Bradley is skeptical of Felipe, since as n increases the area keeps getting bigger. Can you resolve their dispute?

23 Jillian says that, for any positive integer x, $\dfrac{420\,(x+1)!}{x}$ will always be an integer. Explain why she is correct.

HABITS

LESSON 1: CIRCULAR FUNCTIONS

Introduction

You've seen sine, cosine, and tangent before. In fact, you've even used them to find missing sides in triangles.

1. How would you define the sine of θ ? Cosine of θ ?

2. Use careful construction of a triangle to find $\sin 37^\circ$. Do not use the sine function on your calculator!

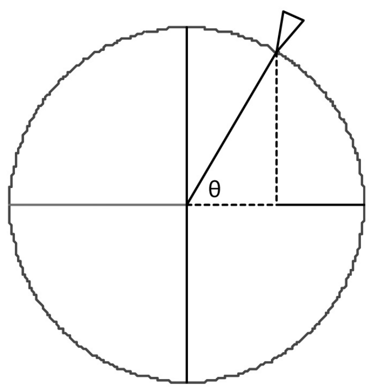
3. Why can't you use a construction to find $\sin 102^\circ$?

Even though we can't construct a triangle to find the sine of 102°, your calculator can tell you an approximate value! In this lesson, we will discover a new way to think about sine and cosine—a way that makes calculating the values of $\sin 102^\circ$, $\cos 1002^\circ$, and $\tan(-10002^\circ)$ possible.

Development

Most of the problems in this section require carefully drawn diagrams.

4 A wheel of radius one foot is placed so that its center is at the origin, and a pin on the rim is at (1, 0). The diagram below shows the wheel after it has been spun an angle θ in a counterclockwise direction.



Now consider the function $P(\theta)$, which outputs the coordinates of the pin after the wheel has been spun an angle θ in a counterclockwise direction. So, for example, $P(0^\circ) = (1, 0)$ and $P(270^\circ) = (0, -1)$. Find $P(\theta)$ when:

a. $\theta = 90^\circ$

b. $\theta = 45^\circ$. Give an exact answer.

c. $\theta = 30^\circ$. Give an exact answer.

d. $\theta = 57^\circ$

e. θ is some measure between 0° and 90°. (Your answer should be expressed in terms of θ.)

f. For values of θ between 0° and 90°, how are $\cos\theta$ and $P(\theta)$ related?

5 Now, let's consider angles greater than 90°.

a. Calculate $P(150^\circ)$, with P being the same position function as in question 4.

b. What is $P(253^\circ)$?

c. Compare these values to the sine and cosine of 150° and 253°.

d. Using the diagram and your calculations, what do you think the tangent of 253° is? The tangent of 150°? Check them on your calculator.

It is the convention for rotations that motion in a counterclockwise direction is considered **positive**, while motion in a clockwise direction is considered **negative**. So if our wheel is spun 57° counterclockwise we would input 57° in our function $P(\theta)$, as we did in problem 1, but if our wheel is spun 57° clockwise we would input -57° in our function $P(\theta)$. So the output for would be the coordinates of the pin after the wheel is spun 57° clockwise.

6 Find:

a. $P(-240^\circ)$. Give an exact answer.

b. $P(-2640^\circ)$. Give an exact answer.

c. $P(-237^\circ)$.

d. $\tan(-237^\circ)$.

e. $P(-\theta)$, where $0^\circ < \theta < 90^\circ$. (Your answer should be expressed in terms of θ.)

7 Now define sine, cosine, and tangent of θ for every value of θ.

8 For angles between 0 and 90 degrees, is using your definition in problem 7 equivalent to using your definition in problem 1? Why or why not?

9 $P(46280^\circ) = (-0.94, -0.34)$. Without using the cosine button on your calculator, find the cosine of 44480°.

10 Let $\cos\theta = -0.4$.

a. For how many angles is that true?

b. How many of these angles are between -180° and 360°?

c. With the help of your calculator find all the angles in part b.

11 Assuming that $\cos 80^\circ = 0.17$, use the symmetry of the circle to find $\cos 100^\circ$, $\cos(-260^\circ)$, $\cos 260^\circ$, $\cos 280^\circ$, $\sin 190^\circ$, and $\sin(-10^\circ)$.

12 Let's revisit question 4, but with a wheel of radius 7 feet instead of 1 foot. The wheel is still centered at $(0,0)$, and still with a pin at $(7,0)$. Let $Q(\theta)$ be the function which outputs the coordinates of the pin after this larger wheel has been spun an angle θ in a counterclockwise direction.

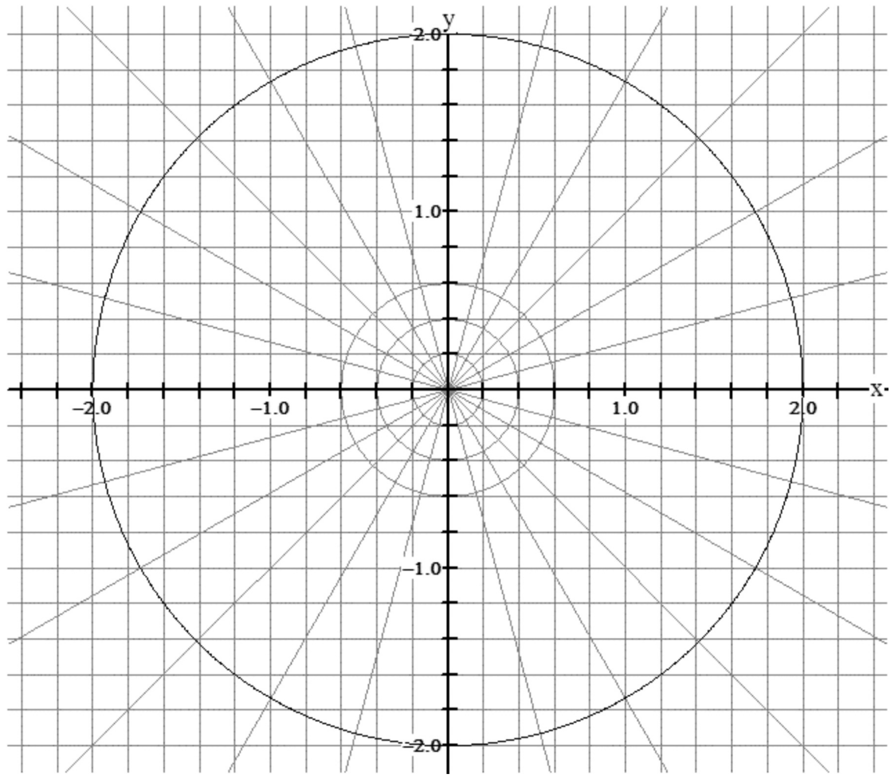
a. What is $Q(48^\circ)$?

b. What is $Q(109^\circ)$?

c. How would you define sine and cosine of θ using this $Q(\theta)$ function? How would you define tangent?

d. How would you define sine, cosine, and tangent using a circle of radius r?

13 In the diagram below, there is a circle of radius 2 inches, with radii drawn at 15° intervals.



a. Use careful estimates and the conclusion of problem 12 to calculate the sines of 30°, 60°, 90°, 120°, ..., 360°.

b. Using your calculations in part a, sketch a graph with θ on the horizontal axis and $\sin\theta$ on the vertical axis. (Use values of θ from $-360°$ to $360°$.)

Surprise! There are actually three more trigonometric ratios (functions) in addition to the three you already know. Here are their names and definitions:

Cosecant of angle θ, written $\csc\theta$, is defined thus: $\csc\theta = \frac{r}{y}$

Secant of angle θ, written $\sec\theta$, is defined thus: $\sec\theta = \frac{r}{x}$

Cotangent of angle θ, written $\cot\theta$, is defined thus: $\cot\theta = \frac{x}{y}$

14 Use the unit circle to find $\csc\theta$, $\sec\theta$ and $\cot\theta$ for $\theta = 240°$.

Practice

15

Use the symmetry of the circle to complete the following chart. Give exact answers. Copy the chart into your notebooks.

θ	30°	45°	60°	315°	–210°	210°	–315°	150°	240°
$\sin\theta$									
$\cos\theta$									
$\tan\theta$									

16

(For this problem use the circle of problem 13.)

a. Use careful measurements and the conclusion of problem 7 to calculate the cosines of 30°, 60°, 90°, etc.

b. Using your calculations in part a, sketch a graph with θ on the horizontal axis and $\cos\theta$ on the vertical axis. (Use values of θ from -360° to 360°.)

17

Determine, without using your calculator, which of the following expressions are the same as $\sin 27^\circ$.

$\sin(180^\circ - 27^\circ)$, $\sin(180^\circ + 27^\circ)$, $\sin(-27^\circ)$, $\sin(360^\circ + 27^\circ)$, $\sin(-207^\circ)$

18

Find, without using your calculator, two of the following expressions which are the same.

$\sin 27^\circ$, $\cos 27^\circ$, $\sin(-153^\circ)$, $\cos(-153^\circ)$, $\cos 63^\circ$

19

Find at least two values for θ that fit the equation $\sin\theta^\circ = \frac{\sqrt{3}}{2}$. How many such values are there?

Problems

20 At constant speed, a wheel rotates once counterclockwise every 8 seconds. The center of the wheel is $(0, 0)$ and its radius is 1 foot. A pin is initially at $(1, 0)$. Where is it 69 seconds later?

21 A wheel whose radius is 1 is placed so that its center is at $(3, 2)$. A pin on the rim is located at $(4, 2)$. The wheel is spun θ degrees in the counterclockwise direction. Now what are the coordinates of that pin? Does your answer work for 90 degrees? 180 degrees?

22 For the following equations use a circle of radius 2 (and your calculator for part b only) to find all solutions θ between 0° and 360°:

a. $\cos\theta = -\dfrac{\sqrt{3}}{2}$

b. $\tan\theta = 6.3138$

c. $\sin\theta = -\dfrac{\sqrt{2}}{2}$

d. $\cos\theta = \cos(251^\circ)$

e. $\sin\theta = \sin 580^\circ$

f. $(\tan\theta)^2 = 3$

23 Find all solutions t between 360° and 720° :

a. $\cos t = \sin t$ (no calculator)

b. $\sin t = -0.9397$

c. $\cos t < \dfrac{\sqrt{3}}{2}$ (no calculator)

In the next few problems you are asked to come up with conjectures or to examine the validity of various statements. You might want to see if the conclusion reached is even plausible, by looking at specific, easy-to-check cases. A simple check will either give the conclusion credence (and thus makes it worth trying to prove) or disprove it instantly.

24 Asked to simplify the expression $\sin(180^\circ - \theta)$, Alex volunteered the following solution:
$\sin(180^\circ - \theta) = \sin 180^\circ - \sin\theta$, and, because $\sin 180^\circ$ is zero, it follows that $\sin(180^\circ - \theta)$ is the same as $-\sin\theta$.

a. Is this conclusion plausible?

b. If it is plausible, try to prove the result. If it isn't, can you come up with a correct way to express $\sin(180^\circ - \theta)$ in simpler form?

c. Answer the same questions for $\cos(180^\circ - \theta)$.

25 Find simpler, equivalent expressions for the following: $\sin(180^\circ + \theta)$, $\cos(180^\circ + \theta)$, $\tan(180^\circ + \theta)$, $\cos(360^\circ - \theta)$, $\sin(360^\circ - \theta)$, $\tan(360^\circ - \theta)$, $\cos(360^\circ + \theta)$, $\sin(360^\circ + \theta)$, $\tan(180^\circ - \theta)$, $\tan(360^\circ + \theta)$

26 Is it the case that $\sin(90^\circ - \theta) = \cos\theta$? Explain. Then write simpler, equivalent expressions for $\cos(90^\circ - \theta),\ \sin(90^\circ + \theta),\ \cos(90^\circ + \theta)$.

27 Is $\sin(-\theta)$ always the same as $-\sin(\theta)$? What can be said about $\cos(-\theta)$?

28 Do the following problems without using a calculator. Explain your reasoning.

a. Which is larger: $\cos 311^\circ$ or $\cos 312^\circ$?

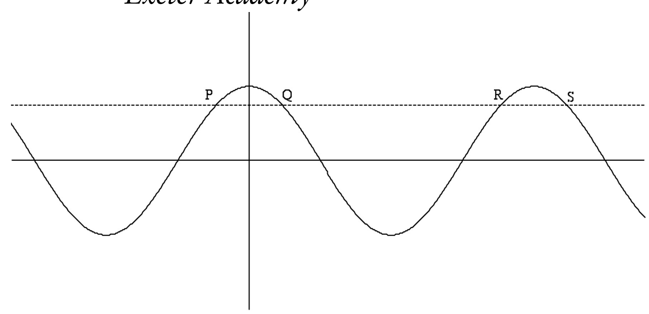
b. Which is larger: $\sin 311^\circ$ or $\sin 312^\circ$?

29 If $\sin A$ is known to be 0.96, then what is $\cos A$? What if it is also known that A is an obtuse angle?

30 Hendrickson is thinking of an angle θ where $\tan\theta > \sin\theta$, and also where $\sin\theta < \cos\theta$. Give two possible values for θ that are in different quadrants.

31 Rodney is running around the circular track $x^2 + y^2 = 10000$, whose radius is 100 meters, at 4 meters per second. Rodney starts at the point $(100, 0)$ and runs in the counterclockwise direction. After 30 minutes of running, what are Rodney's coordinates? *Copyright Phillips Exeter Academy*

32 Below are the graphs of $y = \cos x$ and $y = 0.7431$ (dotted). Given that $Q = (42, 0.7431)$, find coordinates for the intersection points P, R, and S without using a calculator. Use a calculator to check your answers. *Copyright Phillips Exeter Academy*



33 For how many angles in the first quadrant will $\sin\theta$ be rational?

34 One way to find the sine of a 60 degree angle is to draw a circle of radius 2 and draw the related 30°-60°-90° triangle. Find the length of the arc of the circle that is intercepted by this 60 degree angle.

35 Choose an angle θ and calculate $(\cos\theta)^2 + (\sin\theta)^2$. Repeat with several other values of θ. Explain the results. (Note that it is customary to write $\cos^2\theta + \sin^2\theta$ instead of $(\cos\theta)^2 + (\sin\theta)^2$.)

36 For any angle θ, is it true that $1 + \tan^2\theta = \sec^2\theta$? If so, prove it. If not, produce a counterexample.

37 Asked to find an expression that is equivalent to $\cos(\alpha + \beta)$, Laura responded $\cos(\alpha) + \cos(\beta)$. What do you think of Laura's answer, and why?

38 On your calculator, draw a graph of the function $y = \tan x$ with $0^\circ < x < 360^\circ$. Something appears to be wrong here. Say what you think is wrong and explain what's going on.

39 Solve the following without finding the angle θ.

a. Given that $\sin\theta = \frac{12}{13}$, with $0^\circ < \theta < 90^\circ$, find the values of $\cos\theta$ and $\tan\theta$.

b. Given that $\cos\theta = \frac{7}{25}$, with $270^\circ < \theta < 360^\circ$, find $\sin\theta$ and $\tan\theta$.

c. Suppose that $\csc\theta = -\frac{25}{7}$ and $\tan\theta < 0$. Evaluate, in fractional form, the remaining five trigonometric functions of θ.

d. The point $P(-12, 16)$ is on a circle whose center is the origin. Find the cosine of angle θ, the angle between the positive x-axis and the radius to P.

e. The point P is on a circle whose center is the origin, and the x-coordinate of P is five times its y-coordinate. Find the cosine of angle θ, the angle between the positive x-axis and the radius to P.

40 Is it possible for $\sin\theta$ to be exactly twice the size of $\cos\theta$? If so, find such an angle θ. If not, explain why not.

41 Find the three smallest positive solutions to $2\sin\theta = -1.364$.

42 Don't use a calculator for this problem.

a. Reduce: $\dfrac{5ab - 10bc}{ab}$

b. Reduce: $\dfrac{3x^2 - 24}{3x}$

c. Solve for x: $2x^2 - 3x - 2 = 0$

d. Add: $\dfrac{7}{4} + \dfrac{3a}{b}$

e. Subtract: $\dfrac{x}{4} - \dfrac{x-1}{2}$

43 Find all the solutions x, given $0^\circ \leq x < 360^\circ$.

a. $(\cos x - .5)(\sin x + \dfrac{\sqrt{3}}{2}\ ) = 0$

b. $(\cos x)^2 - 5\cos x + 4 = 0$

c. $3(\tan x)^2 + 5\tan x = 0$

d. $2(\sin x)^2 + \sin x = 3$

Exploring in Depth

44 Paul rides a Ferris wheel for five minutes. The diameter of the wheel is 10 meters, and its center is 6 meters above the ground. Each revolution of the wheel takes 30 seconds. Paul's fear of heights kicks in when he is more than 9 meters above the ground. For how many seconds does Paul feel uncomfortable?

45 Jasper's bike has wheels that are 27 inches in diameter. After the front wheel picks up a tack, Jasper rolls another 100 feet and stops. How far above the ground is the tack?

46 You have a circular dartboard. Its target area is defined in an unusual way. Take any point within the dartboard. Draw a straight line that goes through this point and the center of the board. Measure the angle this line makes with the line going from the center to the "easternmost" point on the dart board. If the tangent of this angle is between -0.2 and 0.6, then the point is shaded. Otherwise, it is not shaded. What are your chances of hitting this target area?

47 Given that $\sin\theta = k$, and that $0^\circ < \theta < 90^\circ$, find expressions for $\cos\theta$ and $\tan\theta$.

48 What do the graphs of $y = \sin x$ and $y = \sin 2x$ have in common, and how do they differ? How about the graphs of $y = \cos x$ and $y = \cos mx$, where m is any positive integer?

49 Starting at the same spot on a circular track that is 80 meters in diameter, Andy and Brandon run in opposite directions, at 300 meters per minute and 240 meters per minute, respectively. They run for 50 minutes. What distance separates Andy and Brandon when they finish? Interpret the word *distance* any way you wish.

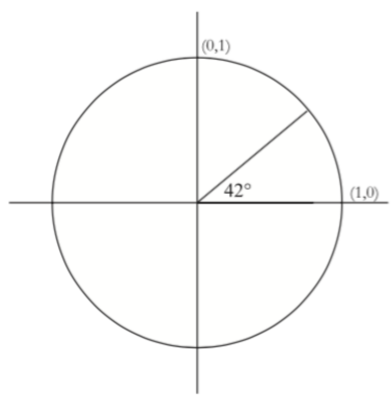
LESSON 2: GETTING COMFORTABLE WITH TRIG

Introduction

In an earlier lesson, you learned how the trigonometric functions sine, cosine, and tangent were defined for angles other than those between 0 and 90 degrees. Here is a quick refresher.

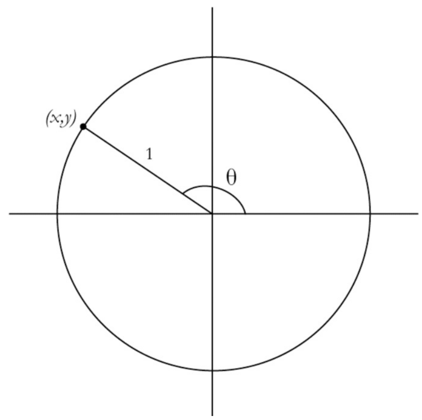
Imagine sitting at the edge of a merry-go-round with radius 1 meter. You've set up a coordinate system with (0,0) at the center of the merry-go-round, and the point (1,0) is due east from (0,0). When the wheel starts to turn counterclockwise, your position is (1,0).

1 If the merry-go-round has rotated 42° from its starting position, find the coordinates of your position.



2 If the merry-go-round has rotated 200° from its starting position, find the coordinates of your position.

This may be enough to remind you of the generalized definitions of sine and cosine. In the figure below, a "spoke" of a circle of radius 1 is drawn so that it makes a central angle of θ with the positive x-axis. The spoke intercepts the circle at the point (x, y). The **sine** of an angle θ is the y-coordinate of this point. The **cosine** of θ is the x-coordinate of this point.



3 Using the figure above, how would you define the tangent of θ?

4 In middle school, you learned that $\sin\theta$ is defined as $\frac{\text{length of the side opposite } \theta \text{ in a right triangle}}{\text{length of the hypotenuse}}$. Where is the right triangle in the figure above, and what are the opposite and hypotenuse?

5 How would you determine the sine, cosine, and tangent of an angle using a circle with a different radius than 1?

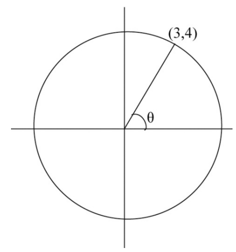
The circle with radius one is better known in mathematical circles as the "unit circle." You'll want to draw it — or a circle with a well-chosen radius — almost every time you solve a trig problem.

Development

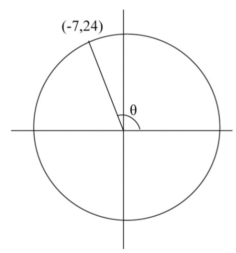
6 Cecilia is on a Ferris wheel and notices that, when she has rotated about 143 degrees from due east, she is 24 meters west and 18 meters above the center of the wheel. Use this information to approximate the sine, cosine, and tangent of 143 degrees. Then see how you did by asking your calculator for these values.

7 For each diagram, find the sine and cosine of the angle θ.

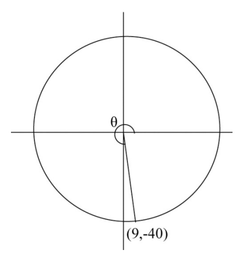
a.



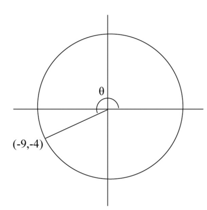
b.



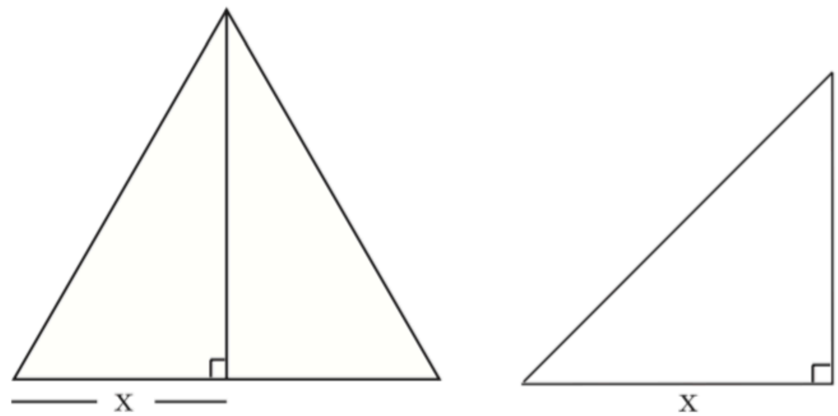
c.



d.



8 Copy the following 30-60-90 triangle and 45-45-90 triangle into your notebook. Then write in the remaining angles and find the remaining sides in terms of x.



Problem 2 may have reminded you that whether the sine or cosine of an angle is positive or negative depends on which **quadrant** the angle is in. This allows you to answer questions like the following.

9 Go back to your 30-60-90 and 45-45-90 triangles and think about positioning them on the unit circle.

a. Name all angles between 0 and 360 degrees that have a sine of $\frac{1}{2}$.

b. Name all angles between 0 and 360 degrees that have a cosine of $\frac{1}{2}$.

c. Name all angles between 0 and 360 degrees that have a tangent of 1.

d. Now explain how to find *all* the angles that have a sine of $\frac{1}{2}$.

10 Solve the equation $\sin\theta = \frac{1}{2}$. How many solutions are there? How many solutions are interestingly different from one another?

Of course, you might also have to solve equations that don't involve sines or cosines of angles in special triangles. For instance, if you had to solve $\sin\theta = .2531$, you would be forced to resort to the inverse sine function on your calculator.

11 Make sure your calculator is in degree mode, then take the inverse sine of .2531 to solve this equation.

12 Draw the "wheel" again, and draw a spoke at the angle that has sine .2531. Remembering that the sine of an angle is the height of a point above the x-axis, indicate the other point on the wheel that would have the same sine.

13 How would you calculate the size of this second angle with sine .2531? Do so.

14 Give *all* the solutions to the equation $\sin\theta = .2531$.

15 Now find *all* the solutions to the equation $\cos\theta = .2531$.

Practice

16 A wheel of radius 2 ft spins around the point $(0, 0)$. A ladybug sticker is initially at the point $(2, 0)$. Find the coordinates of the sticker once the wheel has spun

a. 115°

b. 220°

c. 300°

d. 660°

17 Calculate the exact value of each expression. Do not use a calculator.

a. $\sin 120^\circ$

b. $\cos(-60^\circ)$

c. $\cos 315^\circ$

d. $\tan 150^\circ$

e. $\sin 270^\circ$

f. $\cos 0^\circ$

18 Given that the sine of 138 degrees is about .6691,

a. Find all angles between 0 and 360 degrees that have a sine of .6691.

b. Describe all angles that have a sine of .6691, including those not between 0 and 360 degrees. Can you find a way to write your answer using symbols?

19 Find all solutions to the equation $\sin\theta = .6691$.

20 Find all solutions to the equation $3\sin\theta - 1 = 1.0074$.

21 Find all solutions between 0° and 360° to the equation $\tan\theta = \dfrac{-1}{\sqrt{3}}$.

22 Find all solutions between 0° and 360° to the equation $\cos\theta = -.8834$.

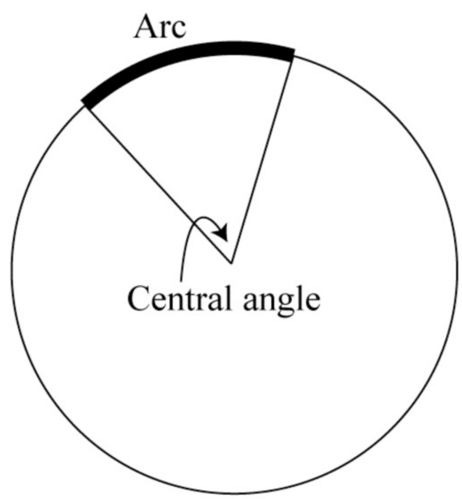
23 In which quadrants is the tangent negative? In which quadrants is the tangent positive?

Further Development

You're used to measuring temperature in degrees Fahrenheit, but you also know that much of the world measures temperature using degrees Celsius. Similarly, the world uses miles and kilometers, ounces and grams, gallons and liters.

When the Babylonians invented degree measure, they chose a 360-degree circle for two reasons. One was that, since there are 365 days in a year, a degree would be very close to one day's progress around the circular path they believed the sun made every year. Another reason is that 360 is divisible by lots of numbers, so we can describe a fourth, a sixth, an eighth, and a tenth of a circle as an integer number of degrees. (Try this with a 100 degree circle and you'll see it doesn't work as well.)

As time went on, though, mathematicians realized that the number 360 didn't really have anything to do with circles, and in fact some of their calculations would be easier if they used a different system. This system was inspired by the close relationship between a central angle of a circle and the arc it intercepts. It was also inspired by the importance of the unit circle.



Specifically, we'd like the measure of a central angle in the unit circle to be literally the same as the length of the arc it intercepts.

24 What is the circumference of the unit circle?

Since a 360 degree "central angle" intercepts the entire circle your answer to Problem 24 will be the equivalent of 360 degrees in our new system.

25 How much of a circle does a 180 degree central angle intercept? So, in this new system, how should we represent a 180 degree angle?

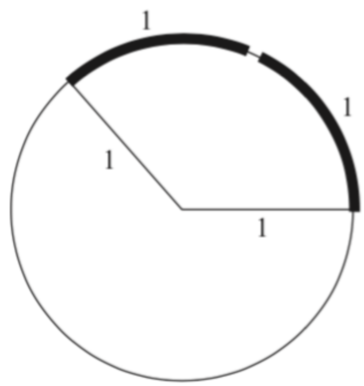
26 Under this new system, what should we call the measure of an angle that intercepts a quarter-circle?

27 What should we call the measure of an angle that intercepts a sixth of a circle?

When you measure angles this way, you are measuring in **radians**. (Try pressing the MODE button on your calculator and notice that there is a radian vs. degree option.) The next problems suggest a reason for the name.

28 How many times does the radius of a circle "fit" around the circumference? (Hint: remember that the circumference of a circle is $C = 2\pi r$.)

29 Say you have an angle that intercepts an arc on the unit circle equal to two of its radii. How many radians is that angle?



If you are about to solve an equation like $\sin\theta = \frac{\sqrt{3}}{2}$, you might not know whether to give your answer in degrees or radians. Your class should agree on which system to use.

Practice

30 If 60 degrees is equivalent to $\frac{\pi}{3}$ radians, how many radians are in

a. 120 degrees?

b. 240 degrees?

c. 30 degrees?

d. 15 degrees?

31 Use your answers to the previous problem to help convert each degree measure to radians.

a. 75 degrees

b. 150 degrees

c. 330 degrees

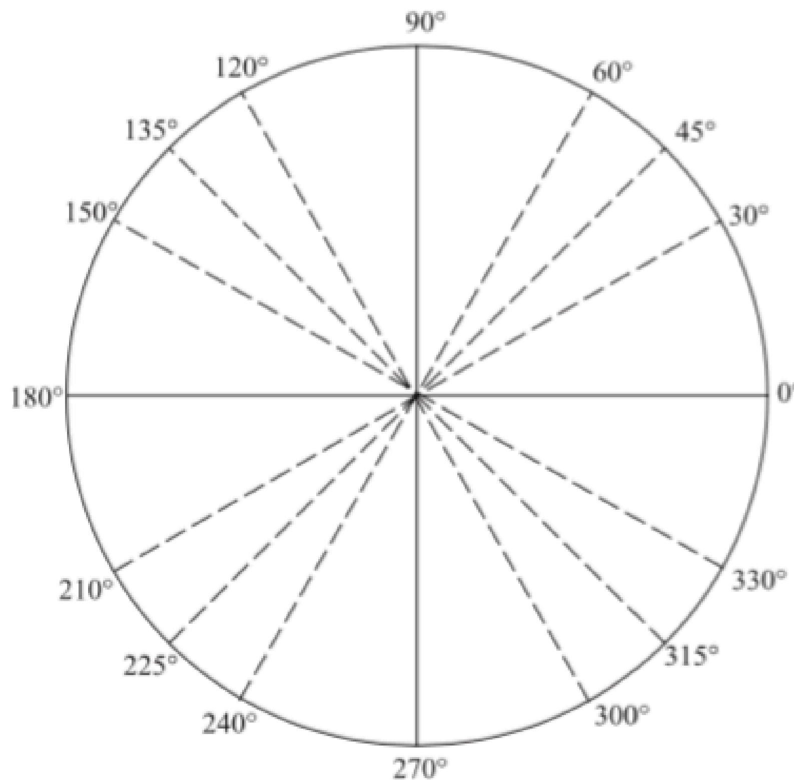
32 If 45 degrees is equivalent to $\frac{\pi}{4}$ radians, how many degrees are in

a. $\frac{3\pi}{4}$ radians?

b. $\frac{7\pi}{4}$ radians?

c. π radians?

33 Copy this unit circle into your notebook and write the radian measure of each angle marked.



34 Give the exact value of each expression.

a. $\sin \frac{\pi}{3}$

b. $\cos \frac{\pi}{3}$

c. $\sin \frac{2\pi}{3}$

d. $\cos \frac{2\pi}{3}$

e. $\tan \frac{7\pi}{4}$

f. $\cos \frac{5\pi}{4}$

Problems

35 Write each set of values in order of smallest to biggest. Do not use a calculator.

a. $\sin 60^\circ, \sin 70^\circ, \sin 80^\circ$

b. $\cos 60^\circ, \cos 70^\circ, \cos 80^\circ$

c. $\tan 60^\circ, \tan 70^\circ, \tan 80^\circ$

36 Write each set of values in order of magnitude (that is, you would write 2 before –10) and then say which are positive and which are negative.

a. $\sin 89^\circ, \cos 89^\circ, \tan 89^\circ$

b. $\sin 269^\circ, \cos 269^\circ, \tan 269^\circ$

c. $\sin 359^\circ, \cos 359^\circ, \tan 359^\circ$

37 Solve each equation for all values of θ. You may use your calculator, but keep in mind that it will not give you all the answers you need.

a. $\cos\theta = .4642$

b. $\dfrac{1}{3}\sin\theta = .2862$

c. $2\tan\theta - 4 = -3.5174$

d. $\sin(\theta - 15^\circ) = .7327$

38 Solve for all values of x. Give answers to Part a using degrees and answers to Part b using radians.

a. $\sqrt{2}\sin(x + 45^\circ) = 1$

b. $\sqrt{3}\tan(x + 4) = 3$

39 Solve for all values of θ.

a. $\tan^2\theta = 3$

b. $\left(\sqrt{3}\sin\theta + 1\right)(2\sin\theta - 1) = 0$

40 Does $\sin^{-1}x$ refer to an angle, or just a number that does not represent an angle? How about $\sin x$?

41 A wheel on a buggy has a radius of one foot. How many degrees has it spun counterclockwise if a chalk mark originally on the wheel's rightmost point is for the first time .8 feet off the ground?

42 Don't use a calculator for this problem.

a. Solve: $x^4 - 9x^2 = 0$

b. Solve: $\frac{3}{27}x^2 = 1$

c. Find the base-ten logs of a million, a billion, and a million times a billion.

d. Solve: $\frac{x+1}{x+3} = 14$

e. Simplify: $\frac{\left(x^2\right)^6}{x^3}$

43 A Ferris wheel has a radius of 132 feet. Its bottom seat is 6 feet off the ground. You are sitting in the seat at the wheel's rightmost point when the wheel begins to spin counterclockwise.

a. What is your height off the ground when the Ferris wheel has rotated $\frac{5\pi}{4}$ radians?

b. What is your height off the ground when the Ferris wheel has rotated $\frac{9\pi}{4}$ radians?

c. What is your height off the ground when the Ferris wheel has rotated $\frac{13\pi}{4}$ radians?

d. You have fallen asleep on the Ferris wheel. When you wake up, you realize that you are level with a tower that you know to be 100 feet off the ground. What are the possibilities for how many radians the wheel has rotated since the beginning of the ride?

44 How many degrees is one radian? How many radians is one degree?

45 Come up with a conversion formula between degrees and radians.

46 Another way of thinking of radian measure is as a ratio. In a circle, it is the ratio of the arc an angle intercepts to the radius of the circle.

a. Draw a circle of radius 5 inches, with an angle inside that intercepts an arc that is $\frac{1}{4}$ the length of the circle. Find the ratio of the arc length to the radius.

b. Repeat part a, but using a circle of radius 7 inches. Your angle will still intercept $\frac{1}{4}$ of the circle.

c. What is the radian measure of an angle that intercepts a quarter-circle?

47 In 10th grade, you found a formula for the length of an arc intercepted by an angle θ: arclength $= 2\pi r \cdot \frac{\theta}{360^\circ}$. Find a similar formula that works when the central angle is measured in radians instead of degrees.

48 In 10th grade, you also found a formula for the area of the sector the angle intercepts: sector area $= \pi r^2 \cdot \frac{\theta}{360^\circ}$. Find a similar formula that works when the central angle is measured in radians instead of degrees.

49 From what we've seen in this lesson, there are often two solutions to the equation $\cos\theta = A$ between 0° and 360°.

a. Are there any values of A for which there would only be one solution? How many values of A?

b. Are there any values of A for which there would be no solutions?

50 Suppose an angle has a sine of .4221.

a. How many angles have this sine?

b. Your calculator tells you that $\sin^{-1}(.4221) \approx 24.97^\circ$. Why do you suppose it only gives you one answer?

c. Try to figure out how the calculator "chooses" the answers it gives for the inverse sine button. What's the biggest answer it will give? What's the smallest?

51 Repeat part c of the previous problem for inverse cosine and inverse tangent. Why do you suppose the range of answers is not the same for the sine and cosine?

52 If $2\theta = 26^\circ + 360^\circ n$, where n stands for any integer,

a. Find all possible values for θ.

b. Did your possible values include $\theta = 553^\circ$? If not, revise your answer to part a.

53 If $3\theta = 150^\circ + 360^\circ n$, where n stands for any integer, find all possible values of θ.

54 Solve each equation.

a. $\sin 2\theta = .1647$

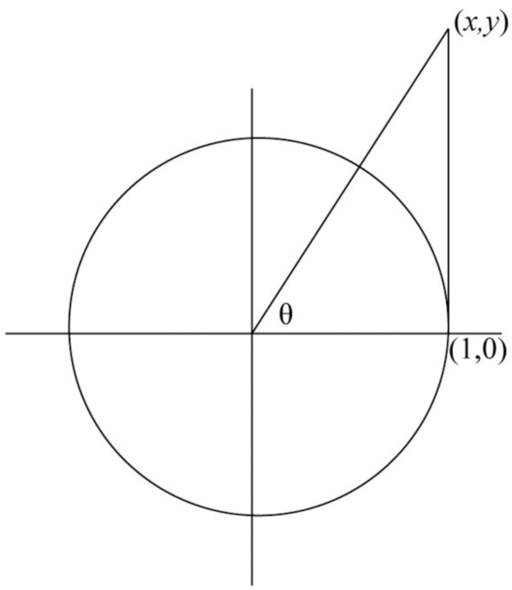
b. $\tan 2\theta + 5 = 42$

c. $2\sin 3\theta = \sqrt{3}$ (no calculator)

d. $\cos 3\theta = \dfrac{1}{\sqrt{2}}$ (no calculator)

Exploring in Depth

55 Find the coordinates of the point labeled (x, y) in terms of θ. Does the diagram suggest a certain word origin?



56 How many solutions does $\sin n\theta = \frac{\sqrt{3}}{2}$ have between 0° and 360°? Answer in terms of n.

57 Refer back to Problem 46, in which you thought of the radian measure of an angle as a ratio. Explain why, unlike degrees, miles, or kilograms, radian measure doesn't require units.

LESSON 3: LAW OF SINES AND LAW OF COSINES

Introduction

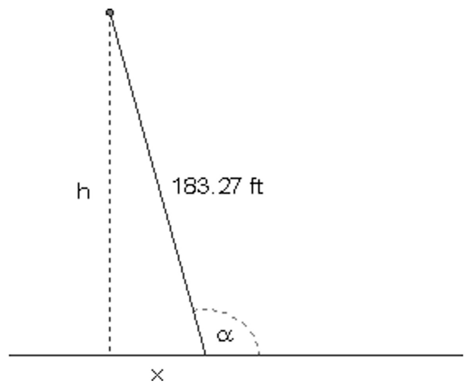
The Leaning Tower of Pisa

The Leaning Tower of Pisa has been leaning almost since the onset of construction in 1173. From ground to top, the tower is 183.27 ft along the lowest side and 186.02 ft along the highest side. The angle of slant of the tower—formed by its shorter slant height and the ground—is 95.5°. An apocryphal tale states that Galileo Galilei (1564-1642), an Italian physicist, mathematician, and philosopher, dropped two cannon balls of different weights from the top of this leaning tower in trying to demonstrate that the descending speed of a falling body is independent of its weight.

1. What is the distance traveled by an object dropped from the top of the Tower of Pisa, on its lowest side, when it hits the ground?

2. What is the horizontal distance from the point where the object hits the ground to the base of the tower?

The Tower of Pisa first acquired a slant after the third floor was built in 1178. More recently, in 1990, it was closed to the public because of safety fears. In fact, the tower was on the verge of collapse, and it was projected that it would have collapsed between 2030 and 2040. However, it has been straightened a bit, but still remains slanted (the tower has been reopened to the public). Thus, the angle of slant of the Tower of Pisa, α, has been changing for centuries. The following is a sketch of the situation.



3 In trying to determine a general expression for the distance, h, traveled by an object dropped from the top of the Tower of Pisa's lowest side when it hits the ground (assuming that the shorter length of the tower has not changed), Gerald found the following expression.

$$h = 183.27 \sin \alpha$$

For the horizontal distance from the point where the object hits the ground to the base of the tower, Gerald found the following expression.

$$x = -183.27 \cos \alpha$$

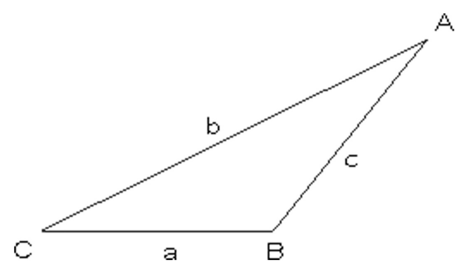
Do you agree with these statements? Explain your answer.

4 You have no more than two seconds after reading the statement of this problem to solve it.

What is the value of $\sin^2\left(\frac{\sqrt[7]{123456}}{\pi + 13}\right) + \cos^2\left(\frac{\sqrt[7]{123456}}{\pi + 13}\right)$?

Development

We follow the convention of labeling the angles of a triangle using capital letters, and the lengths of the corresponding opposite sides with the corresponding lower-case letters. For example, we may label A, B, C the angles of the triangle, and a, b, c the corresponding opposite sides, as in the figure below.

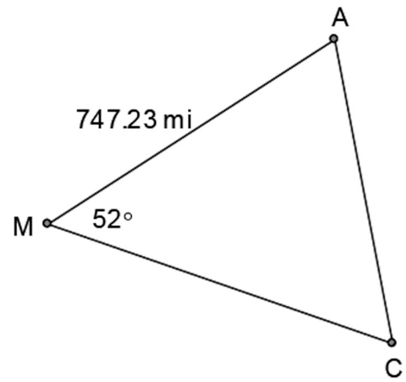


The Bermuda Triangle

The "Bermuda Triangle" or "Devil's Triangle" is an imaginary area located off the southeastern Atlantic coast of the United States of America, which is noted for a supposedly high incidence of unexplained disappearances of ships and aircraft. The vertices of the triangle are generally believed to be Bermuda, Miami (Florida), and San Juan (Puerto Rico).

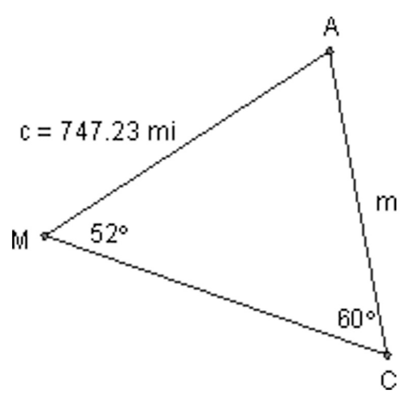
One of the amazing stories from this triangle tells that an aircraft (A in the figure below) was 747.23 miles from the airport of Miami (M in the figure), on the line joining Miami and Bermuda, when its crew received an SOS signal from Cyclops (C in the figure), a ship located at a point on the line joining Miami airport and

San Juan, which was sinking. Flight controllers at the airport were able to estimate the measure of $\angle AMC$ to be $52°$.

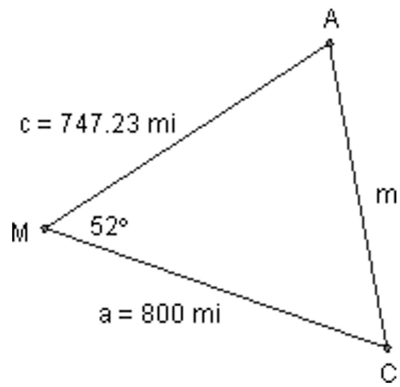


Based on the previous story, Mr. Thomas Howard designed a problem for his mathematics class. The initial situation described in the story, however, contained more than the minimal information required to solve the problem; therefore, he divided the class into two groups, Group I and Group II, and gave each group a problem with different pieces of information, but the same goal to determine how many miles the aircraft had to travel through the purportedly dangerous Bermuda Triangle before reaching Cyclops.

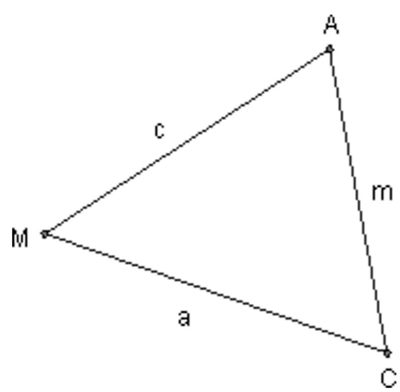
5 Group I: Besides the general information described above, it is known that $m\angle C = 60°$. What is the distance between the aircraft A and the sinking ship Cyclops C? Find a solution to this problem and explain how you arrived at your answer.



6 Group II: Besides the original and general information, it is known that the distance from Cyclops to the airport of Miami is 800 miles, as shown in the figure below. What is the distance between the aircraft A and the sinking ship Cyclops C? Find a solution to this problem and explain how you arrived at your answer.



7 In Problem 5, if the distance given were m and the one to be found were c, the problem could be solved in a similar way. Consider the more general triangle below.



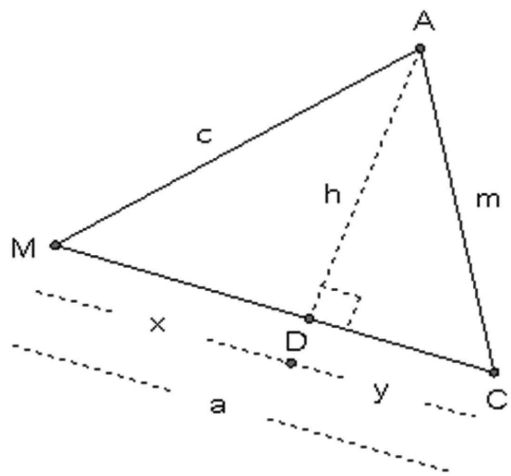
a. Draw the altitude from A to side $\overline{MC}$, and prove that
$$\frac{c}{\sin C} = \frac{m}{\sin M}.$$

Hint: If h is the length of the altitude from A to $\overline{MC}$, find h in ΔADM in terms of c and $\angle M$. Then find $\sin C$ in ΔADC.

b. Now, prove that $\dfrac{m}{\sin M} = \dfrac{a}{\sin A}$

In group II, Rebecca noticed that if the measure of $\angle M$ were 90° rather than 52°, the Pythagorean Theorem would guarantee that $m^2 = a^2 + c^2$. "However," she said, "since $\angle M$ is less than 90°, the Pythagorean relationship among m, a, and c must be adjusted." This adjustment is the result that you are about to find in Part c of the following problem.

8 Consider the triangle in Mr. Howard's problem in a more general form, as shown below, and prove the following.



a. $h = c \sin M$

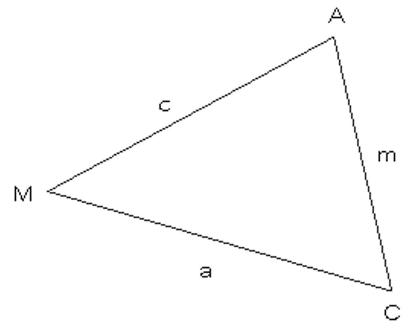
b. $y = a - c \cos M$

c. $m^2 = c^2 \sin^2 M + (a - c \cos M)^2$

9 In part c of the previous problem, expand the square on the right side of the equality. Then use the trigonometric identities that you have learned thus far to find an expression as simple as possible relating m to c, a, and $\angle M$.

As announced above, the expression relating m with c, a, and $\angle M$ that you may have found in the previous problem is the adjustment to the Pythagorean relationship $m^2 = a^2 + c^2$ required when $m\angle M < 90^\circ$. In a similar way, adjustments to the Pythagorean relationship relating a to m and c as well as to that relating c to a and m may be needed when either $m\angle A$ or $m\angle C$ is not 90°, as in the case illustrated above.

10 Consider the general triangle shown below.



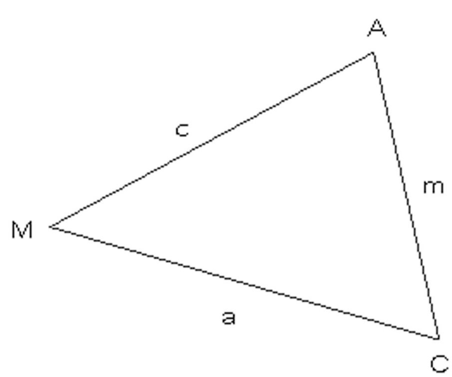
a. If $m\angle A = 90^\circ$, how would a be related to m and c?

b. Conjecture an adjustment to the Pythagorean relationship in part a that may be required when $m\angle A \neq 90^\circ$. Would the proof of your conjecture in this case be quite different from that developed in Problems 8 and 9? Explain.

c. If we had that $m\angle C = 90^\circ$, how would c be related to a and m?

d. Conjecture an adjustment to the Pythagorean relationship in part c that may be required when $m\angle C \neq 90^\circ$. Would the proof of your conjecture in this case be quite different from that required in Part b? Explain.

Summarizing, given a triangle as the one below,



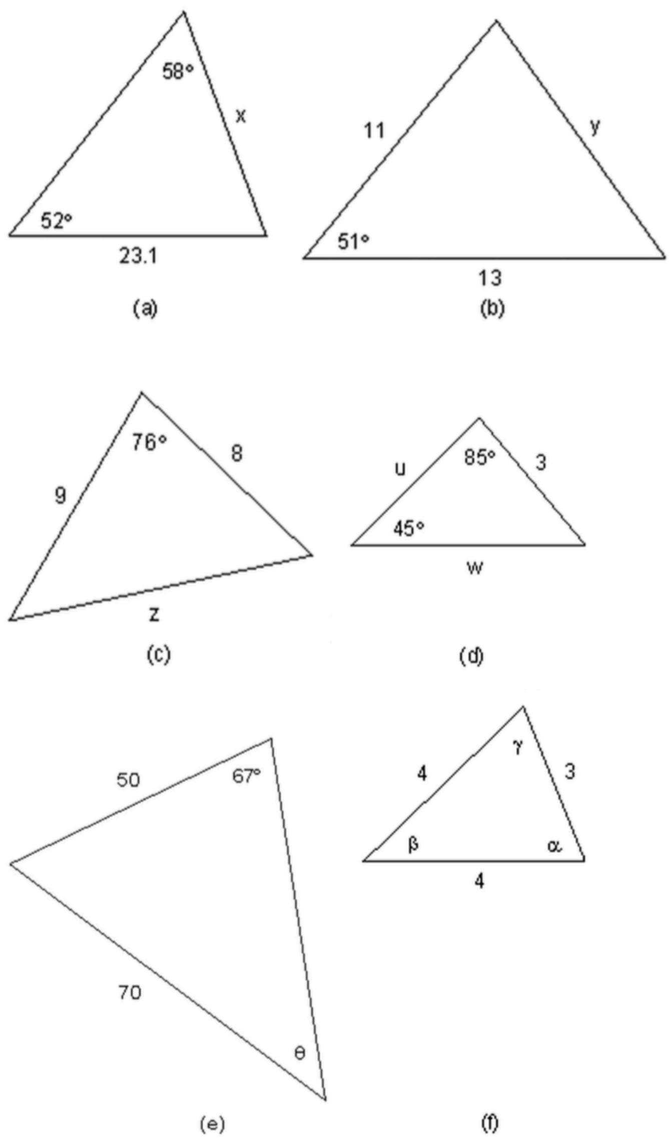
two sets of equalities have been found.

On the one hand, the set of equalities that you may have proven in Problem 7 is known as the **Law of Sines**.

On the other hand, the set of equalities that you may have found in Problems 9, 10b, and 10d is known as the **Law of Cosines**.

Practice

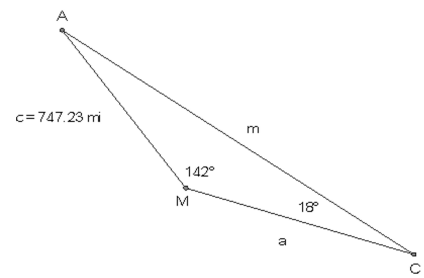
11 Find the indicated side lengths or angle measures in the following figures.



Further Development

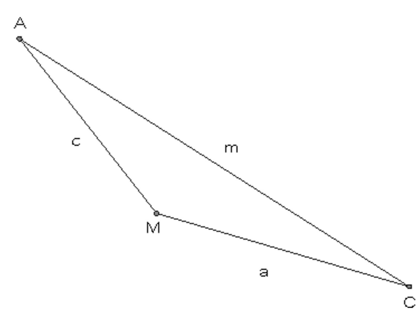
When the students in Mr. Howard's class shared their findings (the Law of Sines and the Law of Cosines) the information contained in these laws was considered "the key" to find missing side lengths or angles in any triangle, when basic information about the triangle is known. However, in reality in all the proofs only acute triangles—triangles with each of their three angles being less than 90°—were used.

12 Regarding Mr. Howard's problem, consider the following situation where $m\angle M = 142^\circ$.



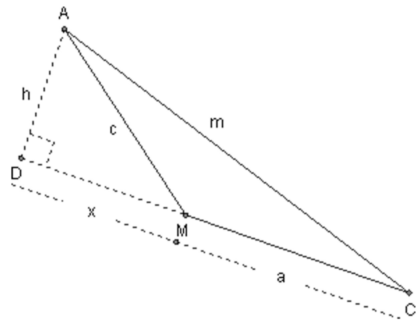
Does the proof of the Law of Sines (in Problem 7), or that of the Law of Cosines (in Problems 8 and 9), support the use of either of these laws to find m in this triangle, which is not an acute triangle? Explain.

13 Consider the following obtuse ΔAMC (that is, a triangle containing an angle greater than 90°).



Prove that $\dfrac{c}{\sin C} = \dfrac{m}{\sin M} = \dfrac{a}{\sin A}$.

14 Consider again the obtuse ΔAMC. Dashed lines have been added to help you prove the equality stated in Part (a).



a. Prove that $m^2 = a^2 + c^2 - 2ac\cos M$.

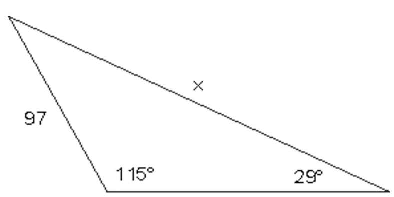
b. Prove that $a^2 = m^2 + c^2 - 2mc\cos A$.

c. Prove that $c^2 = m^2 + a^2 - 2ma\cos C$.

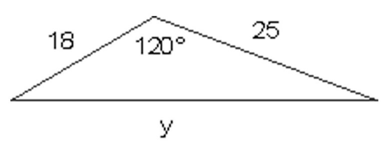
15 Is the Law of Sines or the Law of Cosines worth remembering? Would it be easier always to construct appropriate perpendicular lines and use only trigonometric ratios to solve the problems that may require them?

Practice

16 Find the indicated side lengths or angle measures in the following figures.



(a)



(b)

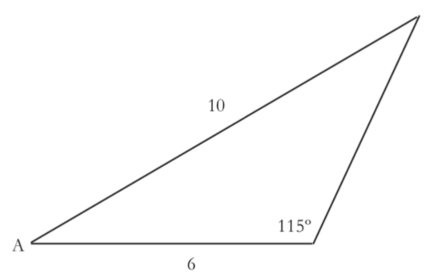
17 Two tracking stations are monitoring the path of a satellite, which has passed to the west of both stations. From station A, the angle of elevation to the satellite is 85.5 degrees. From station B, the angle of elevation to the satellite is 82 degrees. Stations A and B are 65 miles apart.

a. Find the distance from the satellite to tracking station A.

b. Find the height of the satellite above the ground.

18 The distance from Chicago to St. Louis is 440 km, from St. Louis to Atlanta 795 km, and from Atlanta to Chicago 950 km. What are the angles in the triangle with these three cities as vertices?

19 In the figure below, find the measure of angle A.

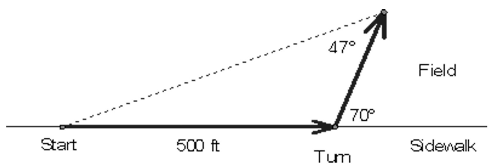


After having done these problems in the Practice section, you may be better prepared to answer question 15, repeated here:

20 Is the Law of Sines or the Law of Cosines worth remembering? Would it be easier always to construct appropriate perpendicular lines and use only trigonometric ratios to solve the problems that may require them?

Problems

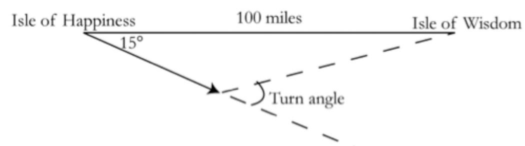
21 Melissa walks along the path shown below: She goes 500 ft along a sidewalk adjacent to a field, then turns 70 degrees, walks a way across the field, and stops. Looking back, she measures a 47 degree angle between her path across the field and her line of sight to her starting point.



a. Find the distance that Melissa walked across the field.

b. How far away is Melissa from her starting point?

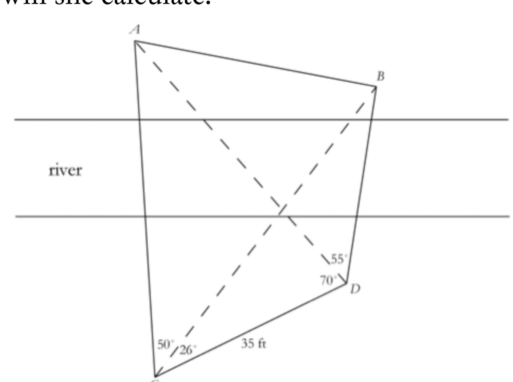
22 A sailboat is attempting to sail between two islands, 100 miles apart. From the very beginning, a wind blows the boat 15 degrees off its course. After the sailboat has been sailing for an hour and a half at 25 mph, it corrects its course so it is sailing straight toward the second island.



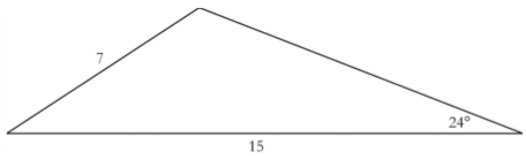
a. By how much does the sailboat need to increase its speed if it wants to arrive at its destination at the same time it would have going 25 mph along the straight path?

b. Find the “turn angle” – the number of degrees the sailboat needed to turn in order to correct its course.

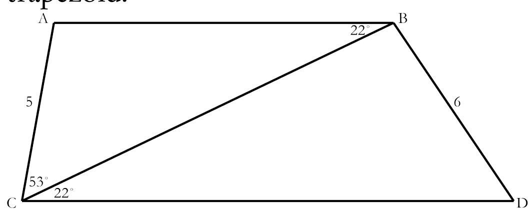
23 Julie needs to find the distance between two trees A and B on the opposite side of a river. On her side of the river, she chooses two points C and D, 35 feet apart. Then she measures the angles shown in the diagram. What distance between the trees will she calculate?



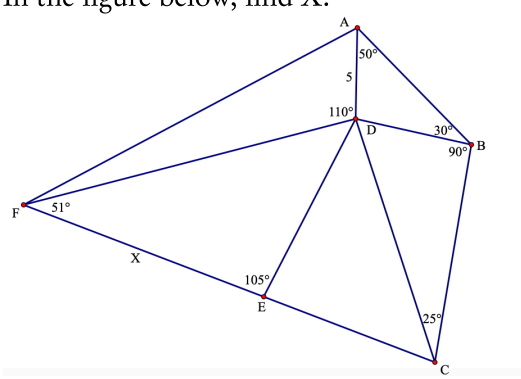
24 Find all three angles of this triangle. Check to be sure that your answer is plausible.



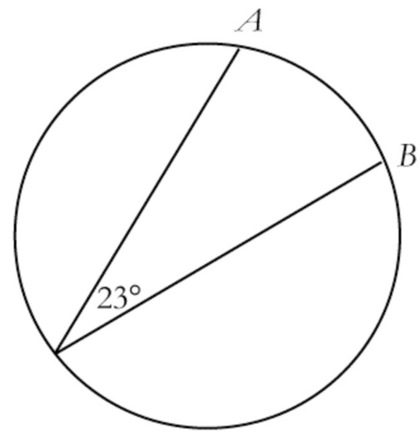
25 Find the measure of all the angles in the trapezoid.



26 In the figure below, find X.



27 In the figure below, the length of the chord drawn from A is 8, and the length of the chord drawn from B is 10. Find the length of arc AB. (Hint: you will need to know the radius of the circle.)



28 The function ThirdSide takes an angle, theta, and outputs the third side of a triangle with sides 3 and 4 and included angle theta.

a. Find $Thirdside(80^\circ)$.

b. What are the minimum and maximum possible values for $Thirdside(\theta)$? Justify your answer in two different ways:

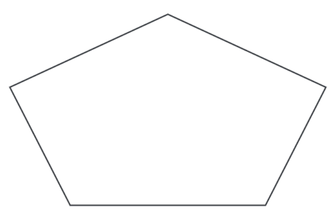
i. by visualizing what different triangles would look like for different values of θ.

ii. by looking at the Law of Cosines formula and seeing how the value of theta affects the value of each term.

Often times, a problem that seems to be hard may be simplified a lot by just drawing a diagram or adding a couple of lines—or even just points—to a diagram already in place. These additions (to the concrete situation given) are implemented to **visualize** and better understand problems which may have been initially confusing. In the following problems, 29 through 33, you will have the chance to use this mathematical habit of mind repeatedly.

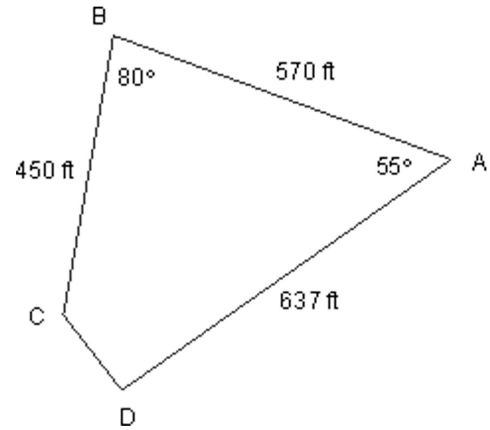
29 A triangle has a 13-inch, a 14-inch, and a 15-inch side. To the nearest tenth of an inch, how long is the median drawn to the 14-inch side? (Recall that a median is a line segment drawn from a vertex of a triangle to the midpoint of the opposite side.)

30 A "half-regular" pentagon isn't perfectly regular, but it does fold perfectly in half (the left half is the same as the right half – it's symmetric across a vertical axis). The half-regular pentagon below has a top angle of 160°, a side length of 8 for the top two sides, a side length of 9 for the base, and a total height of 14 (from the top point down to the base).

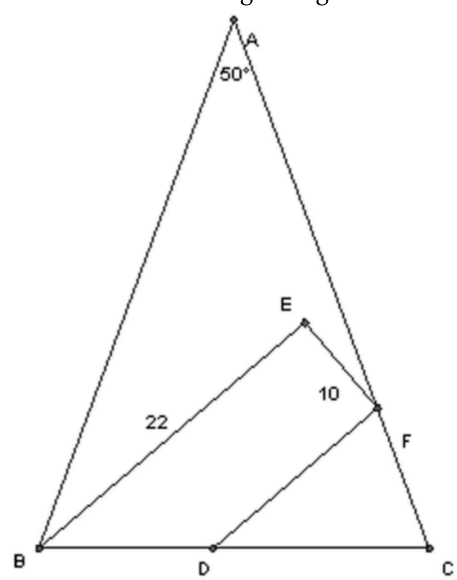


Find the area and perimeter of the pentagon.

31 The diagram below represents a plot of a piece of land. Find the area of the plot.



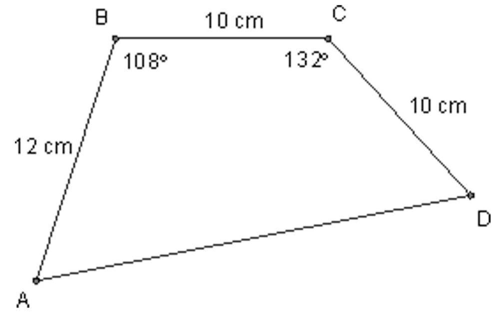
32 In this diagram, ΔABC and ΔCDF are both isosceles. $AB = AC$ and $DF = DC$. $\angle E$ and $\angle EFD$ are right angles.



a. Find all the angles in the diagram.

b. Find AF.

33 The lengths of three sides and the measure of two angles of a quadrilateral are given, as shown in the figure below.

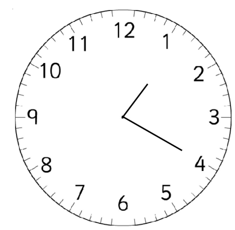


a. Determine the length of the diagonals of this quadrilateral. Round answers to two decimal places.

b. Determine the perimeter of this quadrilateral. Round the answer to two decimal places.

c. Determine the area of this quadrilateral. Round the answer to two decimal places.

34 You're looking at the hour hand and the minute hand on a clock at exactly 1:20. The tips of the hands are 3 inches apart. The hour hand is 2.15 inches long. (Remember – the hour hand is not pointing directly at the 1, since it's after 1:00!)

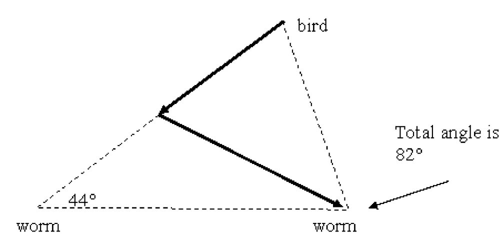


a. What's the angle between the hands?

b. Draw the triangle that this forms, and find the other angles in the triangle.

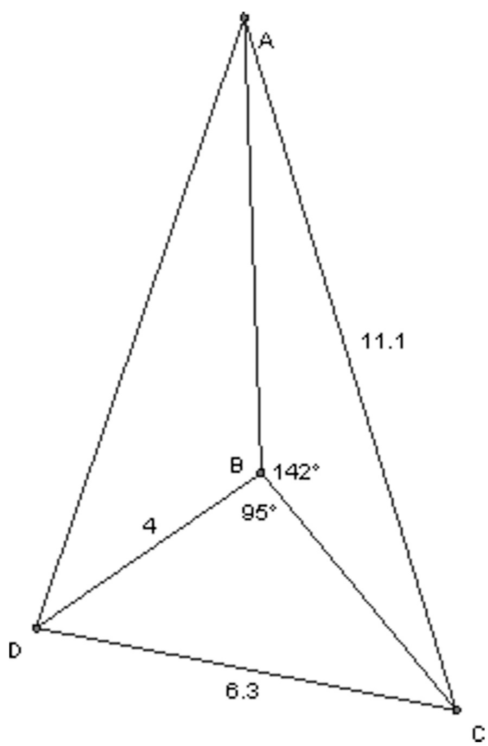
c. Find the length of the minute hand.

35 A bird sees two worms on the ground. The worms are 23 inches apart. The bird flies at the worm on the left, but when it's exactly halfway to the worm, it turns and flies to the worm on the right. The angles are as marked, and the angle marked 82° refers to the entire angle on the right side. Find all the other lengths and angles in the diagram below.

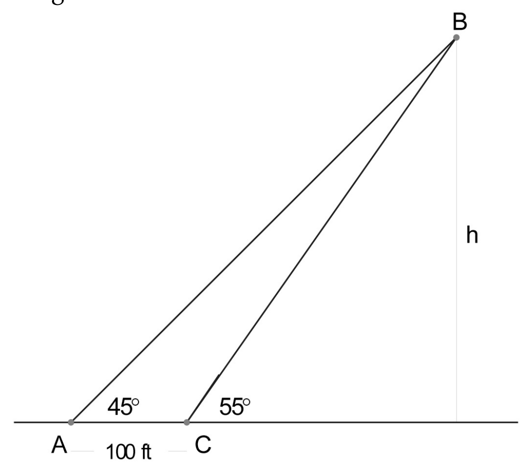


36 Use the law of cosines to determine the angles in a triangle that has sides of lengths 7.3, 23.1, and 15.7. Why would this problem be easier if the sides were 8.1, 23.9, and 16.2?

37 In the figure below, find AB.



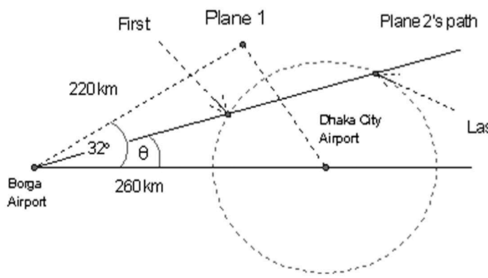
38 A balloon, B, is tethered to the ground by wires $\overline{AB}$ and $\overline{CB}$ as shown in the figure below. How high, h, is the balloon above the ground?



39 In any triangle ABC, prove that $m\angle C = \cos^{-1}\left(\dfrac{a^2+b^2-c^2}{2ab}\right)$.

Exploring in Depth

40 Borga and Dhaka City are two of the main cities in Bangladesh. Borga Airport and Dhaka City Airport are 260 km apart. The ground controllers at Dhaka City monitor planes within a 100-km radius of the airport.



a. Plane 1 is 220 km from Borga Airport at an angle of 32° to the straight line between the airports. Is it within the range of Dhaka City Ground Control?

b. Plane 2 takes off from Borga Airport toward Dhaka City Airport at an angle θ with the line between the airports. If θ is small enough, there is a point when Plane 2 first comes within range of Dhaka City Ground Control, and another point when it is last within range. Is there a value of θ for which Plane 2 is within range of Dhaka City Ground Control at just one point? If so, what is the magnitude of this angle?

c. If $\theta = 15°$, how far will Plane 2 be from Borga Airport when it first comes within range of Dhaka City Ground Control? How far from Borga Airport is it when it is last within range?

41 A triangle has six parts: three sides and three angles.

a. If we know only two out of the six parts of a triangle, is it enough information to precisely describe what triangle it is? Explain.

b. What if we know its three angles? Is it enough information to precisely describe that triangle? Explain.

c. What is the minimal information about the six parts of a triangle needed to precisely describe a triangle?

42 Don't use a calculator for this problem.

a. Reduce: $\dfrac{x^2y + y^2x}{xy}$

b. Reduce: $\dfrac{4x - 20y}{16x + 20y}$

c. Simplify: $\left(\dfrac{84x^5y^2}{14x^{-2}y^4}\right)^{-2}$

d. Rewrite using fractional exponents: $\sqrt{\sqrt[3]{\sqrt{x}}}$

e. If $a = \sqrt{b}$, find a^3 in terms of b.

As you may have explained in Problem 41, in general a triangle is determined by three of its six parts, where at least one of these parts is a side. These are the possibilities. Case SAA: when one side and two angles are known. Case SSA: when two sides and the angle opposite one of those sides are known. Case SAS: when two sides and the included angle are known. Case SSS: when the three sides are known.

43 In the Case SSA described above, what can we say about a triangle for which two sides and the angle opposite one of those sides are known?

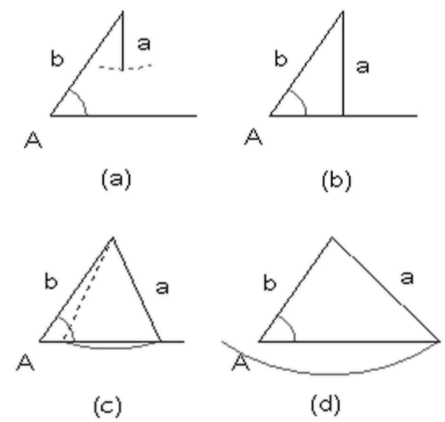
Note: Case SSA is known as the **ambiguous case**. Are there good reasons for this name?

44 Find the side lengths and measures of the angles of $\triangle ABC$ if $m\angle A = 43.1^\circ$, $a = 186.2$, and $b = 248.6$.

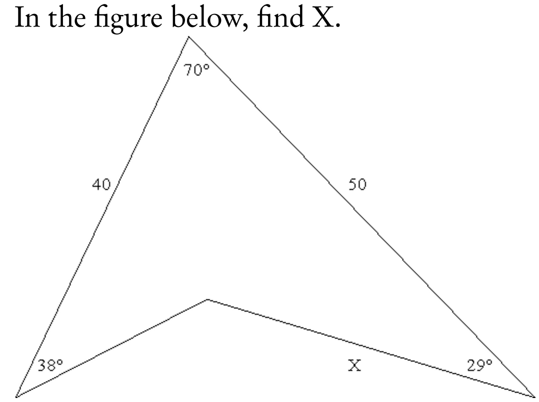
45 Find the side lengths and measures of the angles of the angles of ΔABC if $m\angle A = 42^\circ$, $a = 70$, and $b = 122$.

46 **The Ambiguous Case**

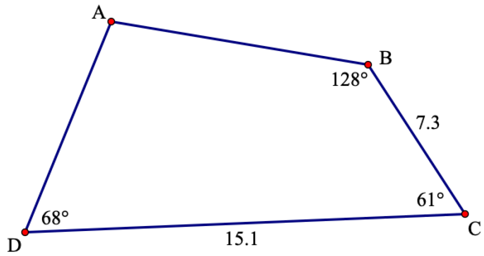
In Case SSA, when two sides and an angle opposite one of those sides are given, it is possible that none, one, or two triangles may exist satisfying the given information. These possibilities are illustrated in the figure below, where $\angle A$, a, and b are the angle and two sides given. For each case, (a) through (d), explain how $\angle A$, a, and b are related.



47 In the figure below, find X.



48 In the figure below, find the lengths of the other two sides of this quadrilateral.



49 **Heron's Formula**

Suppose you have ΔABC.

a. Prove that its area, A, is given by
$A = \frac{1}{2}\, ab \sin C$

b. Prove that
$A^2 = \frac{1}{4}\, a^2 b^2 \,(1 - \cos C)\,(1 + \cos C)$

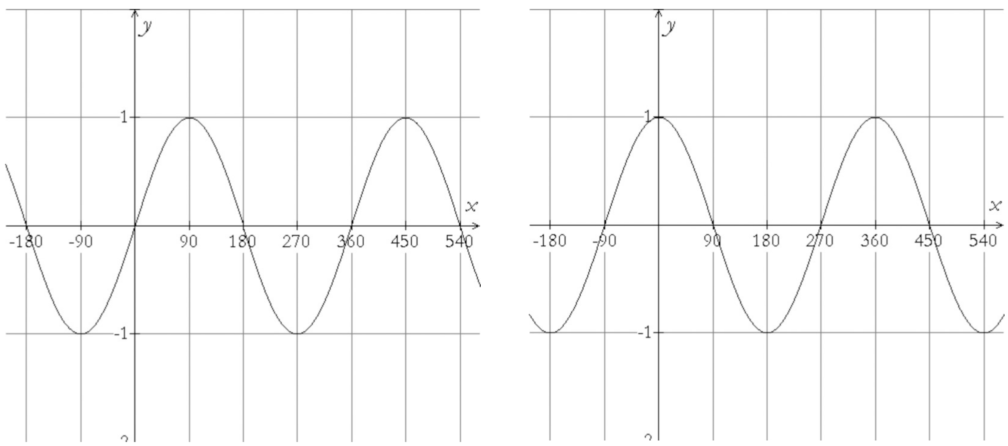
c. Use the Law of Cosines to express $\cos C$ in terms of a, b, and c, and from Part b prove Heron's Formula:
$A = \sqrt{s(s-a)(s-b)(s-c)}$,

where $s = \frac{1}{2}\,(a + b + c)$ is the **semiperimeter** of the triangle.

LESSON 4: MODELING WITH TRIGONOMETRY

Introduction

The graph of the sine and cosine functions are very distinctive looking.



1 Which of the figures above is $y = \sin x$? Which is $y = \cos x$?

2 Using your knowledge of trigonometry, find coordinates for a few of the high and low points for each graph, and of the places they cross the x-axis.

Though you're used to using trigonometry to find missing angles and lengths in diagrams, the special shape of the graph of the sine and cosine functions make them very useful in modeling as well. Graphs that have a shape similar to that of a sine curve are called **sinusoidal** (even if they involve the cosine instead of the sine).

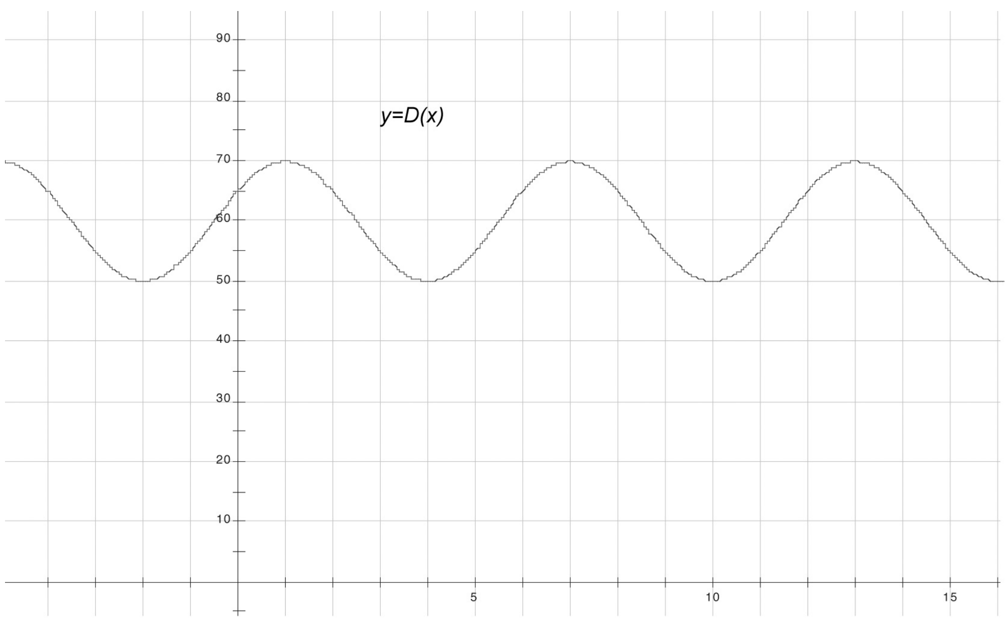
3 Come up with some situations in which a graph with the characteristics of a sine or cosine curve would provide a good model.

In general, sine and cosine curves are used to model situations that are **periodic** – that have some repeating pattern. (You might be familiar with this word from the periodic table, which is a chart of elements arranged so that, every time you begin a new row in the chart, the elements begin to repeat the same pattern of characteristics that was established in the previous row.) Sine and cosine functions are examples of periodic functions, and the distance you need to go on the x-axis before the values of the functions begin to repeat themselves is called the **period**.

4 What is the period of the sine function? How about the period of the cosine?

5 If you are using radians, what is the period of each function?

Not all situations that are periodic will have the same period. Nor will all be the usual sine or cosine graphs. For example, one situation that lends itself well to trigonometric modeling is the depth of a lake at a fixed location, measured over time as the tide comes in and out. Here is a graph of some data for a particular lake. $D\left(x\right)$ measures the depth of the water in inches at time x hours after midnight.



Though the shape of $D(x)$ is similar to that of a sine or cosine curve, clearly its equation is not $y = \sin x$ or $y = \cos x$. However, we can get the equation for this graph by doing various transformations of sine and cosine curves, much like we can take the graph of $y = x^2$ and use transformations to turn it into any parabola we choose. In order to figure out how to take a curve like $y = \sin x$ or $y = \cos x$ and turn it into the curve above, we'll **take things apart** by looking at each of the transformations separately.

Development

Let's first think about how we could change the period of a sine or cosine curve.

6 What type of transformation would changing the period be — a vertical stretch, vertical shift, horizontal stretch, or horizontal shift?

Suppose we wanted to cut the period of the sine function in half — make it 180° instead of 360°. In other words, the sine function starts to repeat itself after 360°, but we want to find a different function that repeats itself after 180°. And put a third way (!) a partial table of values for the sine function looks like this ...

x	30°	60°	90°	120°	150°	180°	210°
$\sin x$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

...but for our new function the table of values should look like this, taking on all the same values but twice as quickly.

x	15°	30°	45°	60°	75°	90°	105°
The new function's values	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

7 Come up with a function that looks like $y = \sin x$ but has half the period. If necessary, try a few formulas and test your conjectures.

8 Now try to find a formula for a function like the one above, except it has a period of 90°. Now find one with a period of 720°.

9 In terms of n, what is the period of the function $y = \sin nx$?

10 a. Write an equation for a function related to the sine function that has period P.

b. Does your idea work for cosine as well? Why or why not?

11 See if you can write an equation for a function that has the same period as the function in the tide example. You could pick either sine or cosine... for consistency, let's use cosine.

In addition to the change in period, the tide function differs from the sine and cosine functions in other ways. For one thing, it is vertically stretched as compared to those functions.

12 What is the height of the usual cosine function, $y = \cos x$, measured from the x-axis to its highest point?

13 How many times its normal height do you need to stretch a cosine curve in order to get the same height as the curve for the depth of water function, $D(x)$?

14 Conjecture an equation for a cosine-based function that would be vertically stretched by the right amount. If you've done it right and also changed the period, the graph of your function should be exactly the same shape as $D(x)$?.

The **amplitude** of a sine or cosine curve is defined as *half* the vertical distance from the top to the bottom of the curve. This was the quantity you found in Problem 12.

15 Why do you suppose the amplitude is defined as *half* the distance, rather than the entire distance from top to bottom?

16 What's the amplitude of $f(x) = \sin x$? What's the amplitude of $D(x)$?

17 Write an equation for a sinusoid curve that has amplitude A.

At this point, our graph has the right shape; we just need to translate it over to the right location. This is something you learned how to do when studying quadratics, though if you've forgotten, try experimenting with your calculator when answering the next few questions.

18 By how much do we need to vertically shift the graph?

19 Write an equation for a graph with the same period, amplitude, and vertical shift as $D(x)$ (again using cosine).

We could make this graph coincide perfectly with $D(x)$ by shifting it horizontally.

20 To remind yourself how to do a horizontal shift, first try writing equations that shift the graph of $y = x^2$ three units to the right, then three units to the left.

21 Now use the same technique to shift the graph of $y = \cos x$ by 90 degrees to the right, then 90 degrees to the left.

22 It's not obvious how to do a horizontal shift at the same time that you change the period of a graph. First make a sketch of $y = \cos 90x$. Then write an equation that you could use to shift this graph two units to the right. Check by using your calculator to graph your equation, and refine your equation if necessary.

23 Determine the required horizontal shift needed to turn your graph in Problem 19 into the graph of $D(x)$.

24 Now that you have all the tools, write an equation for $D(x)$.

Having this model allows us to make predictions. For example ...

25 Use your equation for $D(x)$ to predict the depth of the water at 10:15 pm.

26 Find all of the times today that the water will have a depth of 65 inches.

a. Algebraically — by solving an equation.

b. Graphically — by using your calculator.

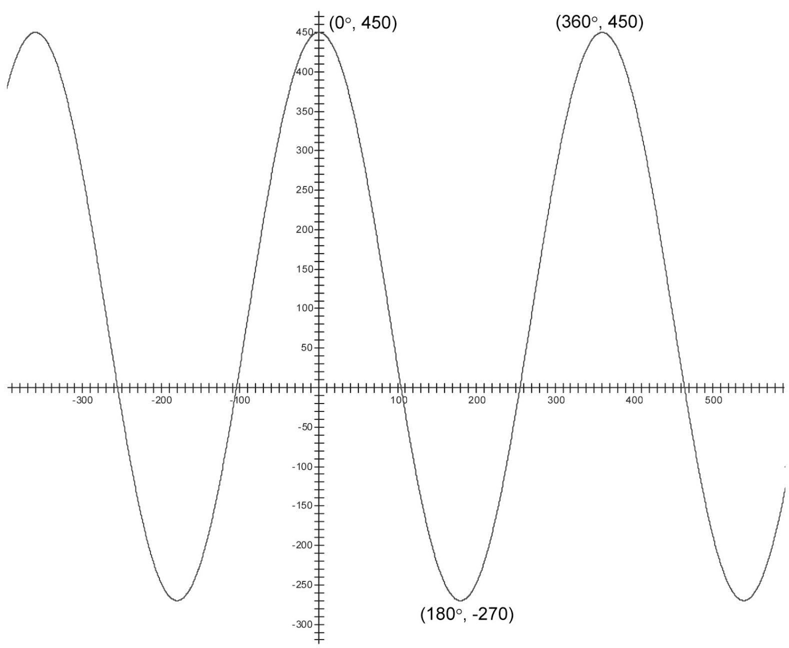
Practice

27 For each graph below,

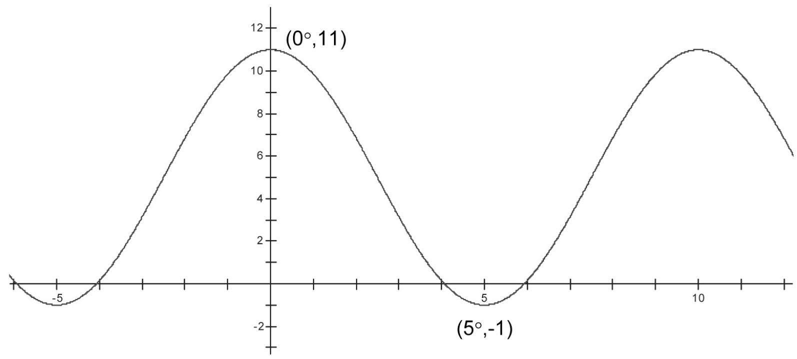
a. say what the period, amplitude, vertical shift, and horizontal shift is, assuming that the graph is a transformation of $y = \cos x$.

b. write an equation for the graph. Check it by graphing the equation on your calculator.

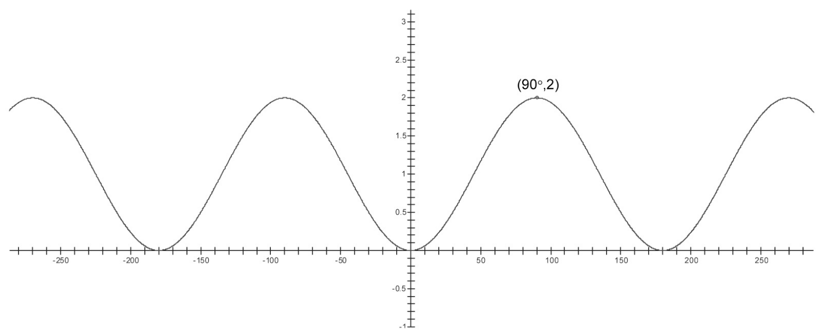
i.



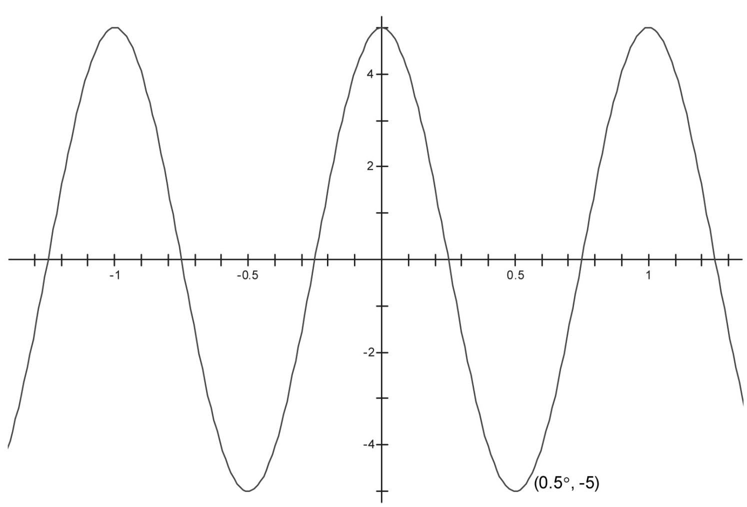
ii.



iii.



iv.



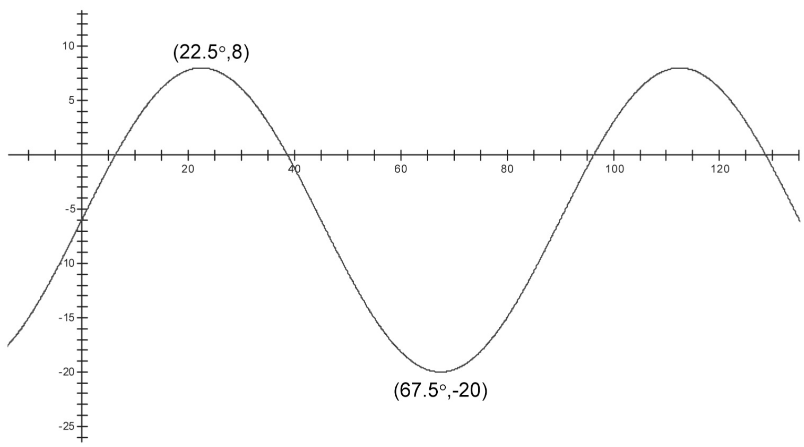
28

For each graph below,

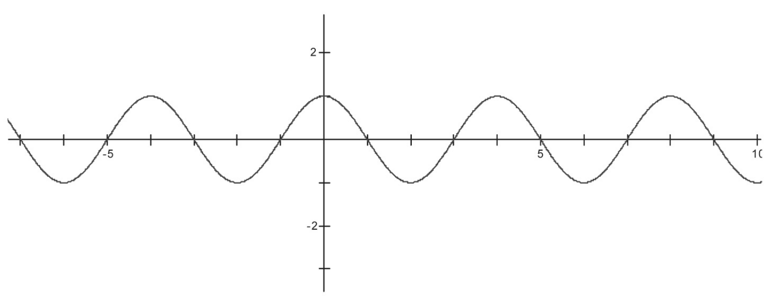
a. say what the period, amplitude, vertical shift, and horizontal shift is, assuming that the graph is a transformation of $y = \sin x$.

b. write an equation for the graph. Check it by graphing the equation on your calculator.

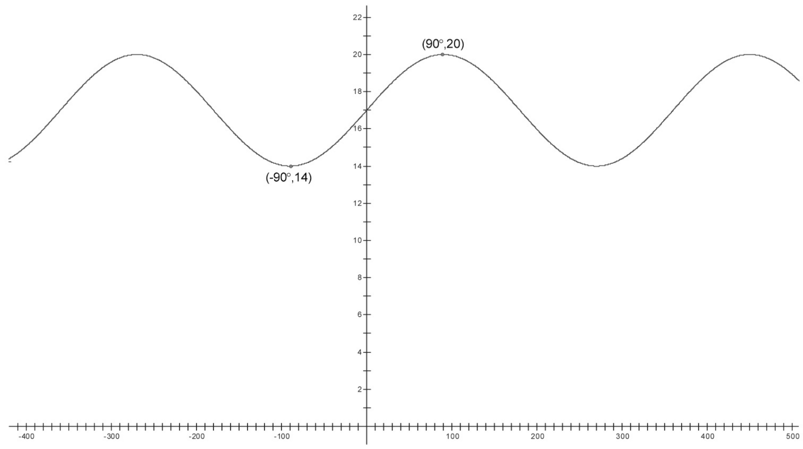
i.



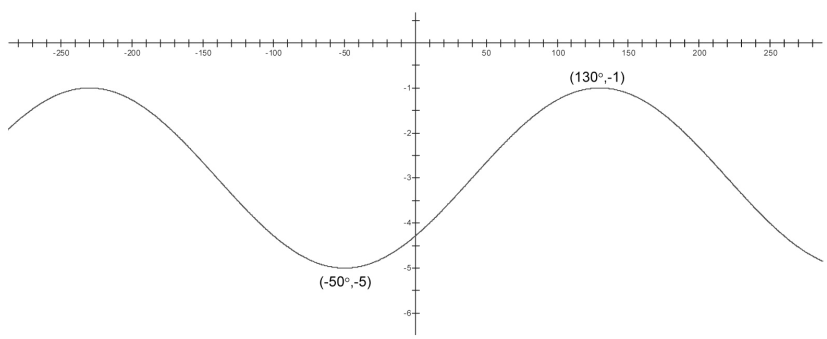
ii.



iii.


(90°,20)
(-90°,14)
22
20
18
16
14
12
10
8
6
4
2
-400
-300
-200
-100
100
200
300
400
500

iv.


(130°,-1)
(-50°,-5)
-250
-200
-150
-100
-50
50
100
150
200
250
-1
-2
-3
-4
-5
-6

29 Find another equation for the function graphed in part i of the previous problem, this time using cosine instead of sine.

30 By identifying the period, amplitude, and shifts, draw a graph of each equation without the aid of your calculator (except to check).

a. $y = 2\sin x + 4$

b. $y = \frac{1}{2}\cos(2(x - 180))$

c. $y = \cos(3\,(x + 60))$

d. $y = \sin(3x + 60)$

31 How would you describe to someone the difference between what you do to change the period and the amplitude of a trigonometric graph?

32 Find an equation for a sinusoidal function that has a peak at the point $(2, 9)$, and has its first trough following that peak at $(7, -3)$.

33 Write a sinusoidal equation that has each of the following characteristics:

a. Period 36, amplitude .5, horizontal shift 5, vertical shift -7.

b. Period 6, amplitude 3, has a high point at $(7, 10)$.

34 Zinadine argues that in the function $D\,(x)$ the vertical shift should really be 69, because the high point of an untransformed cosine graph is 1 unit above the x-axis, but the high point of $D\,(x)$ is 70 units above the x-axis. Set him straight.

Problems

Note to the eager student: though most references to angles in this section use degree measure, feel free to solve the modeling problems using radians. In fact, your teacher may insist!

35 Write an equation for a function involving sine or cosine that has a period of 40 degrees.

36 Write an equation for a function involving sine or cosine that has a period of 91 degrees.

37 Write an equation for a function involving sine or cosine that has a period of 4π radians.

38 What would be the period — measured in radians — of the function $y = \sin \frac{2\pi}{3} x$?

39 Which of these equations shifts the graph of $y = \sin x$ or $y = \cos x$ six units to the right? How do you know?

a. $y = 4\cos(24(x-6)) - 7$

b. $y = 2\sin(3x-6) + 4$

40 Sketch the following transformations: first shift the graph of $y = \cos x$ 90 degrees to the right. Then horizontally compress the resulting graph towards the y-axis by a factor of two.

a. Write an equation for the resulting graph.

b. Can you "see" from the equation that there was a shift of 90 degrees?

c. Describe the transformations that must be done on $y = \sin x$ in order to get the graphs of $y = \sin(Ax - B)$ and $y = \sin(A(x-B))$. To what extent does order matter?

41 Graph the following equations:

a. $y = \sin(10x - 60)$

b. $y = \cos(4x + 128)$

42

Don't use a calculator for this problem.

a. Solve: $\dfrac{9}{x+5} = x - 5$

b. Evaluate: $\log 50 + \log 4 - \log 2$

c. Simplify $\sqrt{\dfrac{y^3}{x^2}}$

d. Solve the equation $x^3 (x-1)(x-2)^2 = 0$. What is the degree of this equation?

e. Find k if $2^{2015} - 2^{2014} + 2^{2013} - 2^{2012} = k \cdot 2^{2012}$

43

You are at the Inner Harbor on June 22. At 11:30 am, low tide, you find that the depth of the water at the end of a pier is 0.5 ft. At 5 pm, high tide, the water is 1.1 ft deep.

a. Find an equation for depth as a function of time that has elapsed since 12 midnight at the beginning of June 22.

b. Use your equation to predict the depth of the water at 8 pm on June 23.

c. At what time does the first low tide occur on June 23?

d. What is the earliest time on June 24 that the water will be 0.85 ft deep?

44

In August, in Baltimore, the temperature can be modeled by the equation $y = -11\cos\left(\frac{\pi}{12}(x-2)\right) + 74$, where x stands for the number of hours past midnight on Aug. 1, and y stands for the temperature (in Fahrenheit) at time x.

a. Use algebra to find all times at which the equation predicts that the temperature will be *more than* 80°.

b. What does this equation imply that the maximum temperature will be? Explain your reasoning carefully. (And don't appeal to your graphing calculator)

45

A spring is attached to the ceiling of a room. You attach a weight to the bottom of the spring, then let go. Some time later, you start timing the oscillations of the spring. 0.4 seconds after you start the time, the spring stretches as far as it will go, to a distance of 24 inches above the floor. It then bounces back up, to a highest point of 50 inches above the floor, at 2 seconds, and continues to oscillate sinusoidally.

a. Sketch the graph of this function.

b. Find an equation for distance from the floor as a function of time.

c. What was the distance of the weight from the floor when you started the clock?

d. What is the distance of the weight from the floor after 4 seconds?

e. When does the weight first reach a height of 40 inches?

46 At time zero, someone puts a chalk mark at the top of your unicycle tire, and you immediately begin pedaling in such a way that the wheel makes two complete revolutions per second. The radius of your wheel is 1 foot.

a. Write an equation giving the height (measured from the ground) of the chalk mark as a function of time since you began pedaling. Be sure to show your work in finding the equation.

b. Use the equation to find the height of the chalk mark 2.3 seconds after you began pedaling.

c. Use the equation to find two times at which the chalk mark will be .8 feet above the ground. Then describe *all* the times at which the chalk mark will be .8 feet above the ground.

47 What might the graph of $y = -\sin x$ plausibly look like? Without using your calculator, check a few values in your head.

48 Try graphing $y = -\sin x$ on your calculator. Then try graphing $y = \sin(-x)$. Describe how these graphs differ from the graph of $y = \sin x$ using the language of transformations.

49 Use the results of the previous problem to predict what the graphs of $y = -\cos x$ and $y = \cos(-x)$ will look like. Then test your prediction by graphing them on your calculator. Describe how these graphs differ from the graph of $y = \cos x$ using the language of transformations.

50 Make a table of values for the function $y = \sin^{-1} x$. Then graph it, taking care to choose a scale appropriate for your values for x and y in the table. Does it appear to be visually related to the graph of $y = \sin x$ in any way?

51 (Hint: visualize!) Make up a sinusoidal function $f(x)$ that has exactly four solutions to the equation $f(x) = \frac{1}{2}$ between 0° and 360°. Then make one up that also has four solutions to $f(x) = \frac{1}{2}$ but also two solutions to $f(x) = 5$ between 0° and 360°.

52 Go back to the function $D(x)$, for which you should have found the equation $D(x) = 10\cos(60(x-1)) + 60$. Recall that x, the time, is measured in hours and $D(x)$, the depth of the water, is measured in inches.

a. Approximately how many feet per hour was the water level changing at 6 am?

b. How about at 3 am? Explain the sign of your answer.

c. At what times during the day would you expect the rate of change to be greatest? How about the smallest?

53

Let $f(x) = 2\sin x$ and $g(x) = 8\sin x$.

a. Is there an x-value at which the slope of g will be four times greater than the slope of f? Test your conjecture.

b. Why is it plausible that the slope of g would always be four times greater than the slope of f?

54

The website http://ptaff.ca/soleil/?lang=en_CA creates a "daylight hour graph" for major world cities.

a. Make one for Baltimore on today's date. Then, using x to represent days of the year and y to represent the number of hours of daylight on that day, write an equation to match the graph you see.

b. Imagine what the daylight hour graph for Vancouver would look like. Would you expect it to be different from the Baltimore graph in any of the following respects: period, amplitude, horizontal shift, vertical shift? Then create a graph for Vancouver and see if you are right.

55

Here is a table showing the number of daylight hours on the first of each month in the cities of Juneau and Fairbanks, Alaska. (Data from http://www.absak.com/library/average-annual-insolation-alaska)

City	Jan	Feb	Mar	Apr	May	June
Juneau	6:30	8:15	10:34	13:13	15:43	17:48
Fairbanks	4:00	6:55	10:07	13:35	17:01	20:33

	July	Aug	Sept	Oct	Nov	Dec
Juneau	18:10	16:30	14:01	11:30	8:55	6:53
Fairbanks	21:25	18:11	14:39	11:19	7:51	4:43

a. Write equations for the hours of daylight in both Juneau and Fairbanks. Use month of the year for x. Note that the numbers in the table are given in "hours:minutes" form, so you'll have to decide how to convert them to decimal notation.

b. According to the equation/graph you've made, what is the day of the year that receives the least daylight? Does this seem right? What data would you need in order to make a more accurate graph?

c. Graph the two equations on your calculator in order to estimate when Juneau and Fairbanks have the same number of hours of daylight. What time of year is this?

Exploring in Depth

56 Critics of the theory of global warming say that changes in the Earth's temperature are cyclic. The graph below shows the level of CO_2 in the atmosphere (which directly affects temperature).

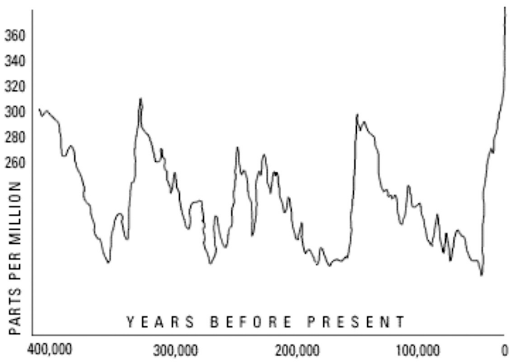


image from http://www.pbs.org/wgbh/warming/etc/graphs.html

a. Approximate some data points, and use these to come up with an equation giving the CO_2 level vs. year.

b. Use this equation to predict what the CO_2 levels should be in the early 21st century, and compare your answer with the graph.

57 (from Foerster's *Precaclulus*) The hum you hear on some radios when they are not tuned to a station is a sound wave of 60 cycles per second.

a. Is the 60 cycles per second the period, or is it the frequency? If it is the period, find the frequency. If it is the frequency, find the period.

b. The **wavelength** of a sound wave is defined as the distance the wave travels in a time equal to one period. If sound travels at 1100 ft/sec, find the wavelength of the 60-cycle-per-second hum.

c. The lowest musical note the human ear can hear is about 16 cycles per second. In order to play such a note, the pipe on an organ must be exactly half as long as the wavelength. What length of organ pipe would be needed to generate a 16-cycle-per-second note?

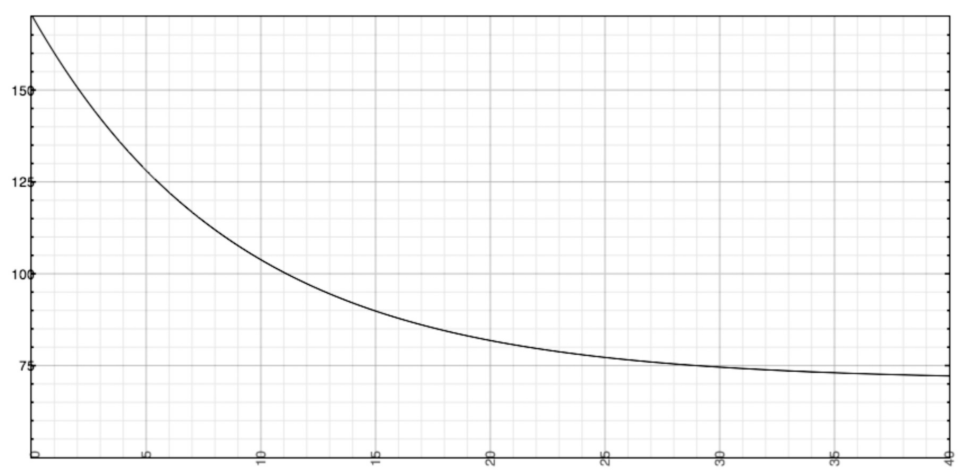
58 Make the following graph: put radian measure on the x-axis. On the y-axis, plot the value for the *slope* of $y = \sin x$. Plot enough points to get a general sense of the curve, then find an equation for it.

59 Repeat the previous question, this time plotting the slope of $y = \cos x$. Find an equation for this graph as well.

LESSON 5: MAKING NEW FUNCTIONS

Introduction

In previous lessons, you may have used exponential, logarithmic, and power functions to model data. For instance, if you had evidence that the data could be modeled by an exponential function, then your task was to find values for a and b in the function type $f(x) = ab^x$. However, one problem in the Exponential Functions lesson involved a kind of exponential function for which it wouldn't work simply to find a and b. A 180° cup of coffee is left out to cool in a 70° room. Below is a graph showing the temperature of the coffee over time.

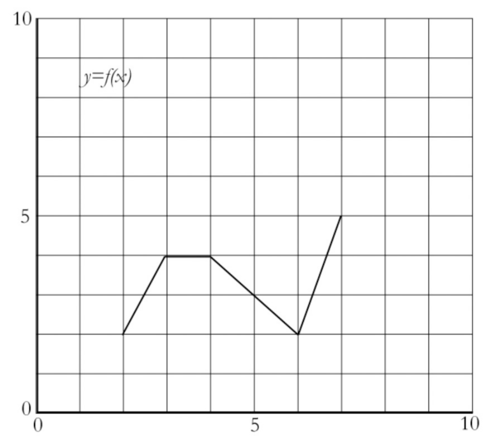


A function that does a decent job of modeling this data is $f(x) = 99.5(.895)^x + 71$. You can think of the function as a translation of 71 units up of the more familiar function $g(x) = 99.5(.895)^x$.

It's easy to imagine changing the vertical shift of g in order to produce lots of different functions, perhaps corresponding to cooling curves in rooms of different temperatures. More generally, in modeling data it's useful to be able to start with basic function types and use transformations in order to control the shape and position of those graphs. We will begin our study of this technique by examining more abstract types of functions.

Development

Below is a graph of a function f. Note that f is not one of the function types you have studied before. It does not have a nice formula. But its graph is still enough for you to answer questions about input and output.



A table of values for the function might look like this:

x	1	2	3	4	5	6	7	8
$f(x)$	---	2					5	---

1 Fill in the four missing values.

Note that we say "a table of values" "might" look like this because there are far more points on the graph of this function. If you wanted to, you could use $x = 3.5$ in the table, or even $x = \pi$. The dashes in the table are there because the function is not defined for certain inputs: specifically, for $x < 2$ and $x > 7$. Another way of saying this is that the **domain** of the function is $2 \leq x \leq 7$.

2 Using the same function f as in problem 1, draw the following graphs. If you need help getting started, remember that an expression like " $f(4) + 2$ " means "two more than the value of $f(4)$."

a. $y = f(x) + 2$

b. $y = \frac{1}{2}\, f(x)$

c. $y = -f(x)$

In Problem 2, you saw examples of a vertical translation, a vertical contraction, and a reflection over the x-axis. Horizontal transformations merit a little more attention.

3 Much as we can define the function $y = f(x) + 2$, we can define the function $y = f(x - 2)$.

a. Find the output of the function $y = f(x - 2)$ for the input $x = 5$.

b. Make a table of values with the headings x and $y = f(x - 2)$, and fill it in. Take care to think about which values of x it makes sense to use.

4 Using the values from your table in Problem 3, graph the function $y = f(x - 2)$. How was the graph of $y = f(x)$ transformed to make this new graph?

5 What is the domain of the function you graphed in Problem 4?

6 Conjecture a formula (like $y = f(x - 2)$) that translates the graph of $y = f(x)$ three units to the left. Then make a table and/or graph to see if you are right.

7 Now consider the function $y = f(2x)$. First find the output when $x = 2$. Then make a table of values with the headings x and $f(2x)$, and fill it in. Take care to think about which values of x it makes sense to use.

8 Using the values in your table from Problem 7, graph the function $y = f(2x)$. Describe the transformation of the graph of $y = f(x)$ this represents.

9 What is the domain of the function you graphed in Problem 8?

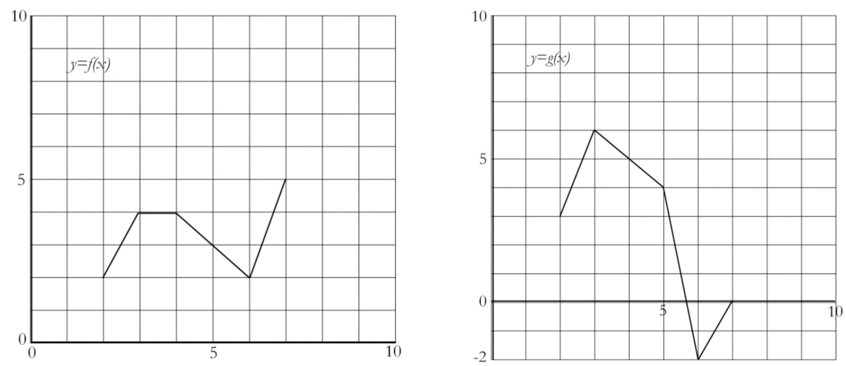
10 Conjecture a formula that horizontally stretches the graph of $y = f(x)$ so that each point is three times farther away from the y-axis.

11 Use a similar process to graph $y = f(-x)$. What transformation of the graph of $y = f(x)$ does this represent?

12 Find a formula for $f(-x)$ if $f(x) = x^2 + 2x - 8$. Then sketch the graphs of $y = f(x)$ and $y = f(-x)$ on the same axes.

Using transformations is one way to build a new function from a known function. You can also combine two functions to get a third.

13 Let f be the function from the introduction (shown below, on the left), and g be the function defined by the graph below, on the right.



a. The function h is defined by the equation $h(x) = f(x) + g(x)$. For example, to find the value of $h(3)$, you would add together the values of $f(3)$ and $g(3)$. Find $h(3)$, then sketch a graph of h.

b. Now sketch a graph of $j(x) = f(x) - g(x)$.

The following exercise illustrates another way to combine functions, known as **composition** of functions. Composition is when you "chain" two functions together, by taking the output of one function and using it as the input to a second function. To see why this might be useful, consider the following situation.

14 Howie is chewing Super-Bubble gum and blows a super-bubble. The radius (in inches) of his bubble increases with time (in milliseconds) in a way given by the equation $r(t) = 2t^{0.25}$. What is the volume of Howie's bubble 5 milliseconds after he blows it?

In order to solve the previous problem, you had to do two things. First, you had to find the radius of the bubble at 5 milliseconds. Then you had to use the radius you found in order to find the volume of the bubble. In function language, if V is the familiar function $V(r) = \frac{4}{3}\pi r^3$ giving the volume of the bubble in terms of the radius, then you first found $r(5)$, and then found $V(r(5))$.

15

Using the functions r and V as in the previous problem,

a. find $V(r(3))$.

b. find a formula, in terms of t, for $V(r(t))$.

16

Let functions f and g be defined as in problem 13.

a. Find $f(g(3))$.

b. Find $g(f(3))$.

c. Find $f(g(5))$.

d. Sketch a graph of $g(f(x))$.

17

The tribble population grows exponentially, and the level of the squeaky noise they collectively make is directly proportional to the size of their population. Let the tribble functions be defined as follows:

P outputs the tribble population at time t.

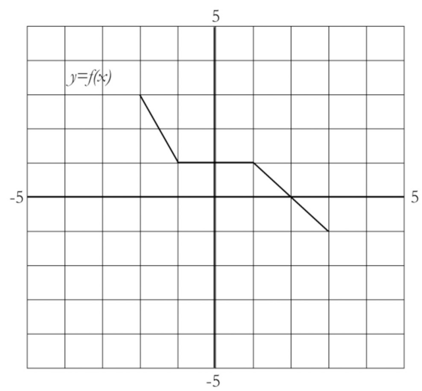
N outputs the noise level produced by a given number of tribbles.

D outputs the noise level given a time t.

Express one of these functions as a composition of the two others.

Practice

18 Let a function f be defined by the graph below. In your notebook, sketch a graph of each of the following equations.



a. $y = f(x - 4)$

b. $y = f(x) - 2$

c. $y = f(-x)$

d. $y = -f(x)$

e. $y = f(\frac{1}{2}x)$

19 Without using your calculator, sketch a reasonable graph of $j(x) = \frac{1}{x}$. Identify the coordinates of three points on the graph. Then, without using your calculator, graph the following equations, including the images of the three points you identified. If the asymptotes move away from the axes, indicate them with dashed lines.

a. $y = j(x) + 1$

b. $y = j(x) - 0.5$

c. $y = 0.4j(x)$

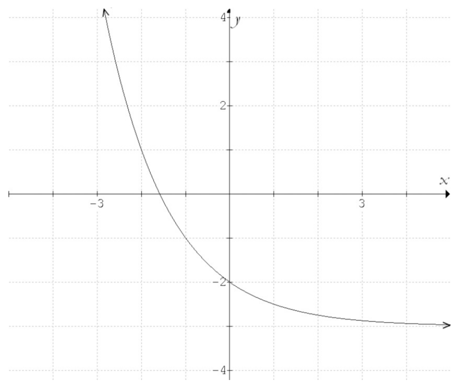
d. $y = -1.5j(x)$

e. $y = 2j(x - 2)$

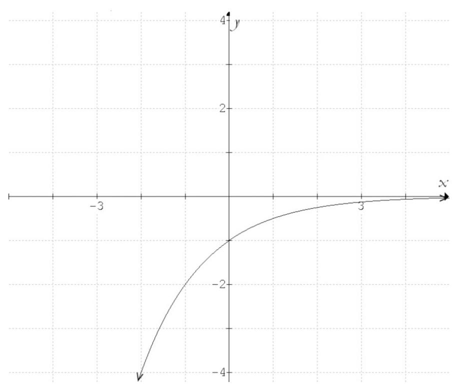
20 Write equations for each of the graphs you sketched in the previous problem.

21 Each graph below is a transformation of the graph of $y = 0.5^x$. For each graph, write an equation that models it.

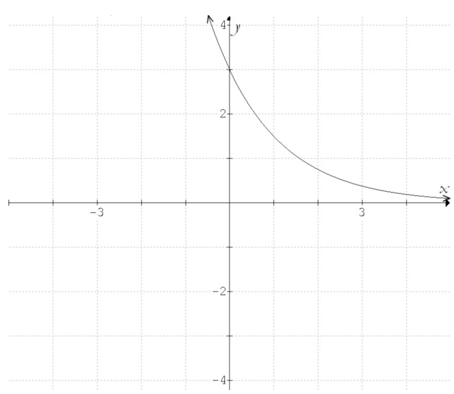
a.



b.

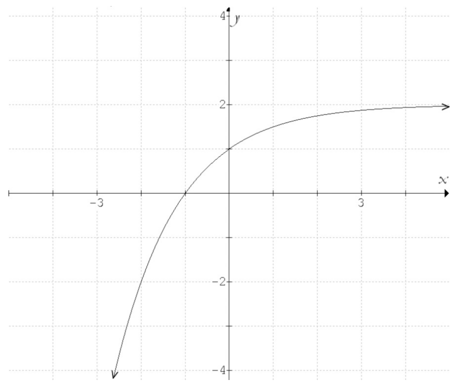


c.

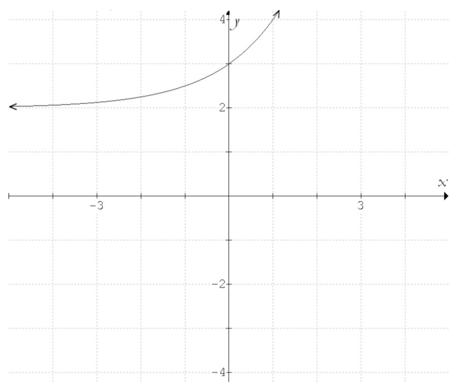


Problem continued on the next page.

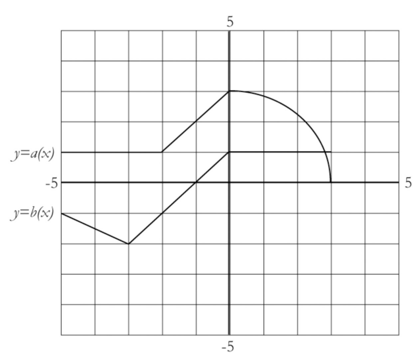
d.



e.



22 Let a and b be functions defined by their graphs below.



Sketch a graph of...

a. $y = a(x) + b(x)$

b. $y = a(x) - b(x)$

23 Let $f(x) = 3x$ and $g(x) = 2x + 1$. Sketch the graph of $h(x) = f(x) + g(x)$. What is its formula?

24 Let c and d be functions defined by their tables below.

x	0	1	2	3	4	5
c	5	-1	3	1	0	3
d	4	2	7	4	7	1

a. Find $c(d(3))$

b. Find $c(d(5))$

c. Find $d(c(5))$

d. If possible, find $c(d(2))$ and $d(c(2))$.

25 Let $f(x) = x^2$ and $g(x) = x^3$.

a. Write a formula for $f(g(x))$.

b. Compare your answer in part a to the formula for $f(x)g(x)$.

26 Give the domain...

a. of the square root function.

b. of the reciprocal function ($f(x) = \frac{1}{x}$)

Problems

27 Let $f(x) = x^2 + 2x - 8$.

a. Without using your calculator, graph $f(x)$ and $f(x-2)$ on the same set of axes.

b. By looking at your graph, find a formula for the function $g(x) = f(x-2)$.

c. Try the following alternative method for finding a formula for $f(x-2)$: Start with the fact that $f(x-2) = (x-2)^2 + 2(x-2) - 8$, then simplify. Why should this method work?

28 Find a formula for $f(2x)$ if $f(x) = x^2 + 2x - 8$. Then sketch the graph of $y = f(2x)$.

29 $y = f(x)$ is an exponential function that has a y-intercept of 2 and is asymptotic to $y = 0$. Find the y-intercept and asymptote of $y = -2f(x)$.

30 $(2, 5)$ is a point on the graph of the function $y = g(x)$. Identify the coordinates of the corresponding point on the graph of...

a. $h(x) = \frac{1}{2}\, g(-x)$.

b. $h(x) = -3g(x)$.

c. $h(x) = 6 - g(-x)$.

d. $h(x) = g(x-5) + 1$.

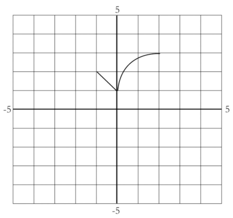
e. $h(x) = g(-2x)$.

f. $h(x) = c \cdot g(x) + b$.

g. $h(x) = -g(cx)$.

31

Let g be the function defined by the graph below.

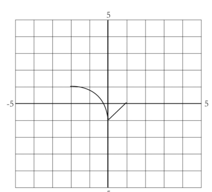


For the graphs in parts i through vi,

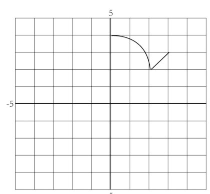
a. come up with a series of transformations that will turn the graph of g into the graph you see.

b. write an equation for the transformed function (for example, the first function shown below is $y = g(-x) - 2$).

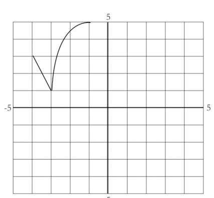
i.



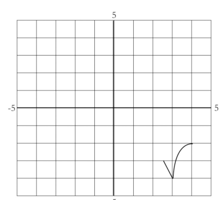
ii.



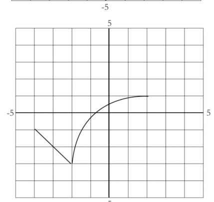
iii.



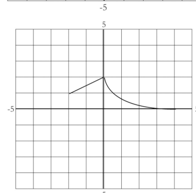
iv.



v.

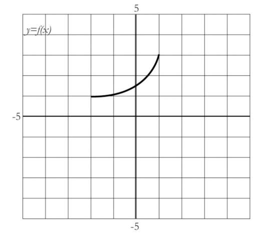


vi.



32

Let f be the function defined by the graph below.

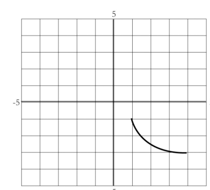


For each graph in parts i through vi,

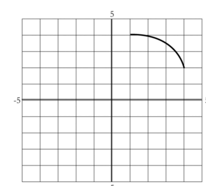
a. come up with a series of transformations that will turn the graph of f into the graph you see.

b. write an equation for the transformed function.

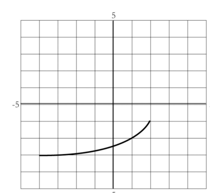
i.



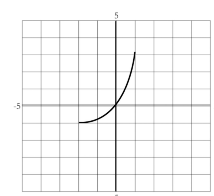
ii.



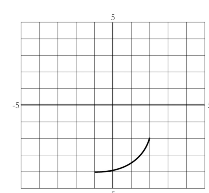
iii.



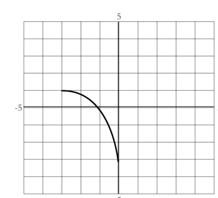
iv.



v.



vi.



33 (continuation of problem 17) When tribbles first appear on the Enterprise, there are four tribbles, collectively squeaking at a noise level of 6 decibels. Five hours after they appear on the Enterprise, there are 30 tribbles, squeaking at a noise level of 45 decibels.

a. Find a formula for the function giving population in terms of time, $P(t) = \ldots$. Assume that it is an untranslated exponential and therefore has the form $P(t) = ab^t$.

b. Find a formula for the function giving noise level in terms of population, $N(P) = \ldots$. (Recall that the noise level is directly proportional to the size of the tribble population.)

c. Now find a formula for $N(P(t)) = \ldots$. What does this formula tell you?

d. How loud will it be 24 hours after the arrival of the tribbles?

e. How long did it take until the noise level aboard the Enterprise reached an unbearable 150 decibels?

34 (Courtesy of *Functions Modeling Change*, Connally, et al) The number of pounds of fertilizer, $n = f(A)$, needed to fertilize a lawn is a function of the surface area A of the lawn, in m^2. Match each story (a-c) to one expression (i-iii).

a. I figured out how many pounds I needed and then bought 2 extra pounds just in case.

b. I bought enough fertilizer to fertilize my lawn and my neighbor's lawn, which just happens to be the size of mine.

c. I bought enough fertilizer to cover my lawn and my flower bed; the flower bed measures 2 square meters.

i. $2f(A)$
ii. $f(A+2)$
iii. $f(A)+2$

35 a. Sketch the graph of $y = |x|$. The point $(0, 0)$ is called the **vertex** of this graph.

b. Use transformation reasoning to predict what the graph of $y = |x - h| + k$ would look like, including the location of the vertex.

c. Investigate the effect of the constant a on the graph of $y = a\,|x - h| + k$. Does it affect the vertex?

36 Without using your calculator, sketch an accurate graph of $y = \sin(\frac{\pi}{2}\,x)$. Include at least two periods. On the same axes, sketch the graph of $y = x$. Now, by using what you know about function addition, sketch a graph of $y = x + \sin(\frac{\pi}{2}\,x)$.

37

Sally was asked to find $a(\frac{\pi}{4})$ in the function $a(x) = \sin(x-3)+2$, and found the answer 0.75. Without using your calculator in any way, decide if Sally's answer is plausible.

Earlier in this lesson, you learned that the **domain** of a function is the set of allowable inputs. Similarly, the **range** of a function is the set of outputs that a function might produce. For example, the range of the function $y = x^2$ is $y \geq 0$, since you can get any non-negative number you want by squaring some number, but you will never square a number and get an answer that is less than zero.

38

Give the range of each function.

a. Sally's function a from the previous problem

b. f(x) = $\frac{1}{x^2}$

c. $g(x) = e^x$

d. $h(x) = \log x$

e. $i(x) = 2^x + 3$

f. $j(x) = \frac{1}{x}$

39

Give the domain and range of each function. Visual thinking is encouraged. Thinking about transformations should obviate the need for your graphing calculator.

a. $f(x) = \frac{1}{x+2}$

b. $g(x) = \frac{1}{(x+2)^2}$

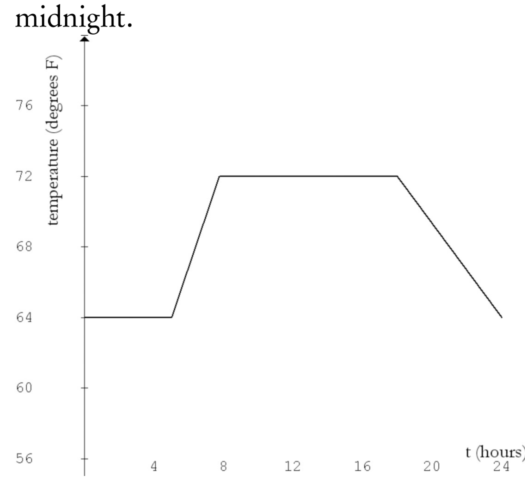
c. $h(x) = \log(x+1)$

d. $i(x) = 2(x-3)^2 - 4$

e. $j(x) = -2\cos x - 4$

40

The function T graphed in the figure below gives the winter temperature in °F at a high school at time t hours after midnight.



a. Graph $c(t) = 142 - T(t)$.

b. Give the range of T and of c.

c. Explain why c might describe the cooling schedule of the school during the summer months.

41 An investigation into slope...

a. Find the slope of the line that heads in the same direction as the curve defined by $y = \log x$ at the point (10,1).

b. Make a prediction about the slope of the line heading in the same direction as the curve defined by $y = 3\log x$ at the point (10,3), and see if you are right.

42 Don't use a calculator for this problem.

a. Solve: $\frac{2}{x-1} = x + 3$

b. Evaluate: $\log_2 4^7$

c. By what factor does $\frac{w^5 x^2}{y^3 z^7}$ increase if w doubles and y is halved?

d. Solve for x: $\sqrt{4x} + x = 3$

e. Find all values of x so that

$$25^{-2} = \frac{5^{\frac{48}{x}}}{\left(5^{\frac{26}{x}}\right)\cdot\left(25^{\frac{17}{x}}\right)}$$

43 The greatest integer function takes an input and outputs the greatest integer that is less than or equal to that input. The greatest integer of x is written $\lfloor x \rfloor$. Here are some examples: $\lfloor 2.7 \rfloor = 2$, $\lfloor -4.5 \rfloor = -5$, and $\lfloor 5 \rfloor = 5$.

a. Sketch a graph of the greatest integer function. When you're done, ask your teacher about "open circle" notation.

b. Find the domain and range of the greatest integer function.

44 The greatest integer function is nicknamed the "floor function," which suggests the existence of a "ceiling function." The ceiling function takes an input and outputs the least integer greater than x. "Ceiling of x" is written $\lceil x \rceil$.

a. Sketch a graph of $y = \lceil x \rceil$.

b. Write a formula for $\lceil x \rceil$ in terms of $\lfloor x \rfloor$. Is one graph a translation of the other?

45 Sketch a graph of $y = \lfloor x \rfloor + x^2$ for $-2 \le x \le 2$.

46 Define the **signum function** as follows: $\text{sgn}(x) = 1$ if x is positive, -1 if x is negative, and 0 if $x = 0$. Sketch the graph of...

a. $y = \text{sgn}(x)$

b. $y = \text{sgn}(x) - 2$

c. $y = \text{sgn}(x) + 1$

d. $y = x^2 + \text{sgn}(x)$

47 The function © tells you how many siblings someone has, so ©(Malia Obama) = 1 and ©(Chelsea Clinton) = 0.

a. Find ©(you).

b. Give the domain and range of the function ©.

c. The function $\uparrow$ has the formula $\uparrow(x) = x^2 + 1$. Which makes sense: ©($\uparrow(x)$) or $\uparrow$(©(x))?

48

Below are tables for three functions, f, g, and h. These tables are complete tables — that is, the functions are not defined for any inputs other than the inputs given here.

x	-5	-2	1	2	3	7
$f(x)$	2	1	2	3	1	7

x	0	1	2	3	4	5
$g(x)$	-5	1	3	7	2	-2

x	-3	0	1	2	3	7
$h(x)$	0	1	2	3	4	5

a. Which compositions of functions are possible? That is, can you compute $f(g(x))$ for all values of x? How about $g(f(x))$? Find all pairs that work, paying attention to order.

b. Choose one of the compositions from part a and make a table of values for it.

c. Does $f(f(x))$ exist? How about $g(g(x))$ or $h(h(x))$?

49

An investigation into composing functions...

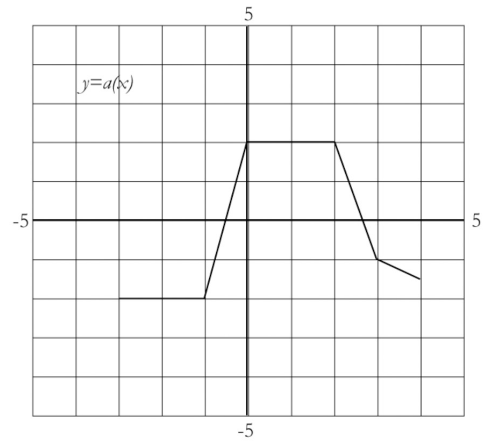
a. Make up tables for two functions, f and g, such that $h(x) = f(g(x))$ exists. Then make a table for the function h.

b. How do the domain and range of the function h relate to the domain and range of the functions f and g?

c. What (helpful) advice would you give to someone else in the class if they were trying to come up with two functions that could be composed?

50

Let the function a be defined by the graph below.

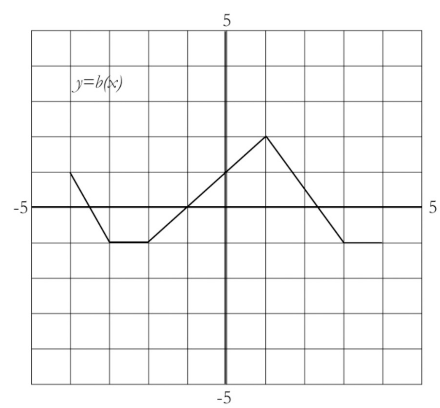


a. Sketch the graph of $y = |a(x)|$.

b. Sketch the graph of $y = a\left(|x|\right)$.

51

Let the function b be defined by the graph below.



a. Sketch the graph of $y = -\,|b(x)|$.

b. Sketch the graph of $y = b(|-x|)$.

52 a. Plot the graph of a function where, for all x, $f(-x) = f(x)$. Use the domain $-5 \le x \le 5$.

b. Plot the graph of another function that meets the criteria given in part a.

c. Make a conjecture about the graph of any function for which $f(-x) = f(x)$

53 a. Plot the graph of a function where, for all x, $f(-x) = -f(x)$. Use the domain $-5 \le x \le 5$.

b. Plot the graph of another function that meets the criteria given in part a.

c. Make a conjecture about the graph of any function for which $f(-x) = -f(x)$.

54 Are there any standard functions you know of that have the properties you conjectured in problems 52 and 53? What are they?

55 a. Plot the graph of function where, for all a and b, if $f(a) = b$, then $f(b) = a$. For example, if $f(2) = 4$, it must also be true that $f(4) = 2$. Use a domain of $-5 \le x \le 5$.

b. Plot the graph of another function that meets the criteria given in part a.

c. Make a conjecture about the graph of any function for which $f(a) = b$ implies $f(b) = a$. If necessary, make up more examples to test or refine your conjecture.

56 Let the function g be defined by the following chart.

x	-1	0	0.5	1	2	3	4
$g(x)$	-4	-3	-1	1	1.5	2	4

a. Plot the points on a graph.

b. From the information you have, make a table and graph of the inverse function for g. The standard notation for this function is "g^{-1}," read "g inverse."

c. Describe any patterns you notice.

d. The graph of g^{-1} is a transformation of the graph of g. Describe this transformation in precise language.

57 Find a formula for the inverse function of $f(x) = x^3$, and graph both f and f^{-1} on your calculator. Is the transformation you see the same one as in the previous problem? What evidence do you have for this?

58 The function f from problem 1 does not have an inverse. Why not? Does the function g from problem 13 have an inverse?

59 Which functions below DO have inverses?

a. $f(x) = x^2$

b. $g(x) = \sqrt{x}$

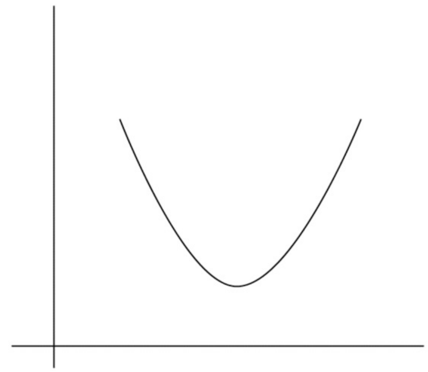
c. $h(x) = x^3$

d. $i(x) = 2^x$

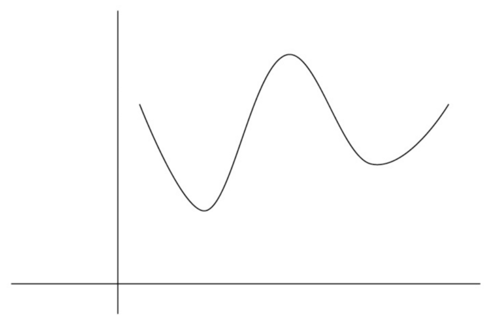
e. $j(x) = x^6$

60 Which functions graphed below have inverses?

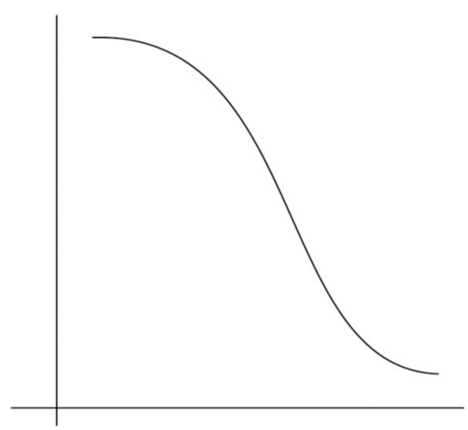
a.



b.

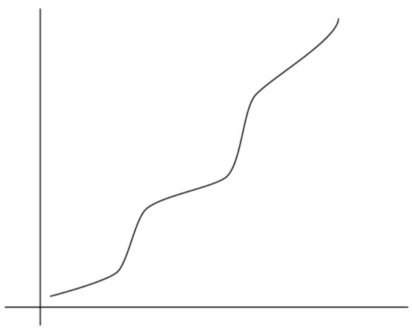


c.

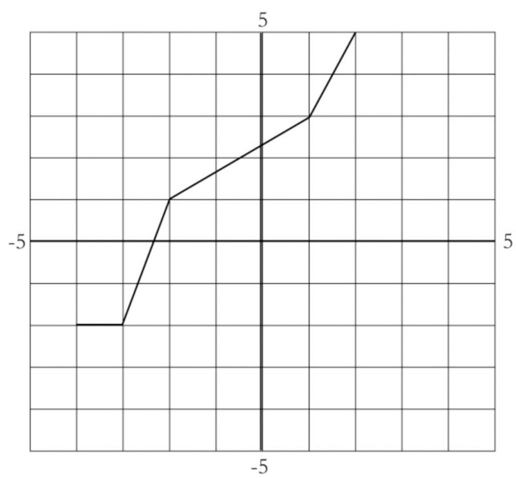


Continued on the next page

d.



61 The following questions refer to the function sketched below.



a. Find the domain and range of the function

b. Sketch the inverse of the function on the same axes

c. Find the domain and range of the inverse.

62 Find the domain and range of

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

63 The radius of a circle is initially 3 cm. It increases by two centimeters each hour.

a. Find a formula for the area of the circle in terms of time.

b. Sketch three graphs, labeling the axes appropriately:

i. $y = r(t)$ (radius of the circle vs. time)
ii. $y = A(r)$ (area of the circle vs. radius), and
iii. $y = A(t)$ (area of the circle vs. time).

c. When the circle has a radius of 7 cm, what is the rate of change of the area of the circle with respect to the radius? Make sure that your answer includes units.

d. At time t = 2 hours, what is the rate of change of the area of the circle with respect to time? Make sure that your answer includes units.

Exploring in Depth

64 Graph $f(x) = x^2$ and $g(x) = 2x$ on the same axes.

a. Make a rough sketch of what you think a graph of $y = f(x) + g(x)$ would look like. Then see if you are right by graphing the function $y = x^2 + 2x$.

b. Using the techniques of this lesson, write an equation for the graph you drew in part a. Is the equation equivalent to $y = x^2 + 2x$?

65 Graph the linear function $f(x) = 2x - 6$.

a. Reflect the line you've drawn over the line $y = x$.

b. Find an equation for the reflected line. Verify that it is the graph of f^{-1} .

66 If f and g are inverse functions, find $f(g(x))$.

67 Let $f(x) = x^2$, $g(x) = x - 2$, and $h(x) = 5x$. Write each of the following functions as a composition of f, g, and/or h.

a. $a(x) = 5(x - 2)$

b. $b(x) = 5x^2$

c. $c(x) = x^2 - 2$

d. $d(x) = (5x)^2 - 2$

e. $e(x) = 5(x - 2)^2$

f. $z(x) = (5x - 2)^2$

68 Find the domain and range of each function.

a. $a(x) = \sqrt{x - 5}$

b. $b(x) = \sqrt{x^2 + x - 6}$

c. $c(x) = \frac{x+5}{x-3}$

d. $d(x) = \frac{x^2}{(x+1)(x-2)}$

e. $e(x) = -x^2 - 7$

f. $f(x) = \frac{3x}{x}$

You have been using the "inverse sine" function for years, but thinking about it now, you can see that the sine function does not have an inverse! There are many ways to get an output of 1: when x is 90°, 270°, 450°, etc if you are working in degrees, or when x is $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, etc if you are working in radians. Thus, if someone says, "find $\sin^{-1} 1$," you wouldn't know which of the values listed above to give in response.

The reason we can still talk about the inverse sine function is that we are thinking about only a limited set of inputs (angles) that you can plug in to the function $f(x) = \sin x$. Limiting the allowable inputs in this way is called a **domain restriction.**

69 Use your calculator to take the inverse sine of some angles. Notice the kinds of outputs it gives you. Can you figure out exactly which angles belong to the restricted domain of the sine function?

70 In your notebook, sketch a graph of the sine function, showing a few periods, and show in bold the part of the graph that is produced by using inputs from the restricted domain. Does the restriction seem like a logical choice? Would there be other ways to do it?

71 Do the $\cos^{-1}$ and $\tan^{-1}$ functions use the same domain restriction as $\sin^{-1}$? Play around with your calculator to see. If not, what are the other domain restrictions?

72 Let $f(x) = \sqrt{x}$ and $g(x) = x^2$. Do $f(g(x))$ and $g(f(x))$ have the same domain? First answer the question by looking at the algebra, then check your answer by having your calculator graph the unsimplified formula for each composition.

73 Let $f(x) = 2x$ and $g(x) = \sqrt{16 - x^2}$.

a. Find $f(g(x))$ and $g(f(x))$.

b. Give the domain and range of each of the functions in part a.

74 Let $f(x) = x^2$ and $g(x) = 4 - x$.

a. Find $f(g(x))$ and $g(f(x))$.

b. Give the domain and range of each of the functions in part a.

75 Trigonometric functions are not the only common functions that require a domain restriction in order to take their inverse. $f(x) = x^2$ does not have an inverse because for a given output, like 4, there are multiple inputs, -2 and 2, that produce that output. However, if you restrict the domain of f to allow only values of x greater than or equal to zero, then the function has an inverse. You can think of the square root function as the inverse of $f(x) = x^2$ on this restricted domain.

a. For each function below, suggest a domain restriction that would allow the function to have an inverse.

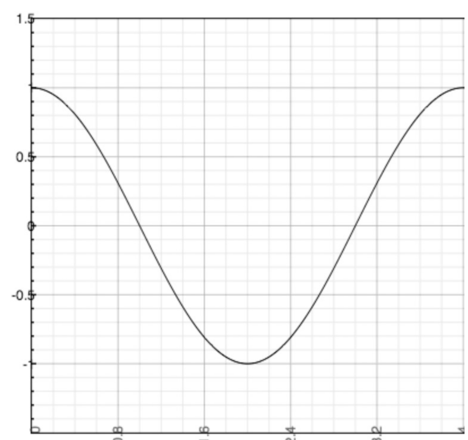
b. For functions i, ii, and iii, find an equation for the inverse of the function (with its domain suitably restricted).

i. $a(x) = (x-4)^2 + 3$

ii. $b(x) = \frac{1}{x^2}$

iii. $c(x) = \frac{1}{(x+3)^2}$

iv.



76 If $g(x) = \cos x$, find (using no calculator)...

a. $g(g^{-1}(\frac{1}{2}))$

b. $g^{-1}(g(45^\circ))$

c. $g^{-1}(g(37^\circ))$

77 Will it always be true that...

a. $\cos(\cos^{-1} a) = a$?

b. $\cos^{-1}(\cos a) = a$?

LESSON 6: VARIATION AND PROPORTION

1 You might recall from Physics class that Newton's 2nd Law of Motion tells us that the net force on an object divided by its mass equals its acceleration, i.e., $a = \frac{F_{net}}{m}$. Let's look at the case of a person pushing a bobsled and its passengers on the ice.

a. If the mass of the bobsled and passengers is 300 kg and the person pushes so that there is a net force of 900 Newtons on the bobsled, what is the acceleration of the bobsled (the units are in meters/sec^2)?

b. If the net force on the bobsled is tripled, does the acceleration triple?

c. If, instead, the net force remained at 900 N but the mass of the bobsled and passengers was tripled from part a, what would happen to the acceleration?

d. What would be the effect on the acceleration of tripling the net force and the mass from part a at the same time?

e. What would happen to the acceleration if the net force were the same as part a, but the mass of the bobsled and passengers was cut in half?

f. Finally, what would be the effect on the acceleration if one both doubled the net force from part a and halved the mass at the same time?

2 As you know, the circumference of a circle and the area of a circle of radius r can be found through the formulas $C = 2\pi r$ and $A = \pi r^2$.

a. If one circle has a radius of 1 meter and another has a radius of 3 meters, how many times bigger is the circumference of the second circle? How many times bigger is its area?

b. If one circle has a radius of 2.687 meters and another has a radius 3 times as big, how many times bigger is the circumference of the second circle? How many times bigger is its area?

c. Based on parts a and b, what do you think you can conclude in general? Prove your answer by considering two circles, one with radius X and the other with radius $3X$, and by doing some minor algebra.

d. The volume of a sphere of radius r can be found by using the formula $V = \frac{4}{3}\pi r^3$. If you have two spheres, where the larger sphere has a radius 5 times that of the smaller sphere, how many times bigger must the larger sphere's volume be? Prove that your answer is correct no matter what the radius of the smaller sphere is.

e. If one sphere is $\frac{1}{4}$ the radius of another, what fraction of the larger sphere is the volume of the smaller sphere? Again, prove your answer works for any two spheres that have this relationship.

3 The braking distance of a car is the minimum distance a car going at speed v can stop in (by using the brakes, of course). By using some fundamental physics, one can calculate the braking distance, assuming a car of average weight and a dry road, by the equation $d = .06v^2$, where v is in miles per hour and d is in feet.

a. If Jessup hits the brakes while driving in his jalopy at 20 mph, what is his braking distance? Alternatively, if Tess takes 384 ft. to stop, how fast was she going in her Miata?

b. Julie says that if she goes 40 mph she will stop in twice the distance than if she was going 20 mph. Is Julie correct? If so, show why. If not, how many times bigger is the 40 mph braking distance than the 20 mph braking distance?

c. Determine how fast Julie would *actually* have to go to stop in twice the distance.

d. How many times faster than 20 mph would she have to go to stop in 16 times as much distance as she would have stopped in at 20 mph?

e. How many times faster than 20 mph would she have to go to stop in M times as much distance as she would have stopped in at 20 mph?

4 Given a cube of side length x,

a. What is its volume?

b. What is its surface area?

c. If the length of the cube were to change to $5x$, then by what multiple would its volume increase?

d. If the length of the cube were to change to $5x$, by what multiple would its surface area increase? Why doesn't the "6" in the surface area formula affect your answer?

5 In the previous problem you examined the volume and surface area of a cube; you found that they were related to the side length x by a cubic (x^3) and a quadratic ($6x^2$). Is this true of more complicated shapes? Let's see.

a. Say that there is an irregularly shaped drawing that looks like, say, Big Bird.

Obtain a copy of this drawing from your teacher. How could you approximate its area in square centimeters to a reasonable degree of accuracy with a ruler?

b. Now if you doubled all the dimensions of this drawing (i.e. its height, its width, the distance between Big Bird's eyes — everything), how would its area change? Why? What if you scaled it up by a factor of 5 instead?

6 At the Macy's Day parade last year, there was an enormous balloon of Spongebob Squarepants that was 40 ft high. Toy copies of this balloon were being sold on the parade route and were 3 inches high.

a. Assuming the other dimensions were proportionally reduced, what is the scale factor of the copy in comparison to the original balloon?

b. Approximately how many of the toy copies would it take to fill the 40 ft. balloon?

The types of relationships between variables that you have been exploring in the previous problems are special cases of two broad categories: direct proportionality and inverse proportionality.

y is **directly proportional** to x when $y = kx$, where x and y are variables and k is a constant not equal to 0. Another way of saying this is that y **varies directly** with x. For example, in problem 2, where $C = 2\pi r$, we say that C is directly proportional to r, or that C varies directly with r.

y is **inversely proportional** to x when $y = \frac{k}{x}$, where x and y are variables and k is a constant not equal to 0. Another way of saying this is that y **varies inversely** (or **indirectly**) with x. For example, in problem 1c, where $a = \frac{900}{m}$, we say that a is inversely proportional to m, or that a varies inversely with m.

7 Answer the following questions by using the definitions of direct and inverse proportionality given above.

a. If $WZ = 100$, are W and Z directly or inversely proportional, or neither?

b. If $\frac{C}{T} = 64$, do C and T vary directly or indirectly, or neither?

c. If $P = Q + 20$, are P and Q directly or inversely proportional, or neither?

d. Suppose Z is directly proportional to S. As S increases, must Z also increase?

e. Suppose M varies inversely with R. As R increases, must M decrease?

Just as it is useful to say that "A varies directly with B", we can apply this language to a wider range of expressions as well. For example, in problem 4, where $A = 6x^2$, we can say that A varies directly with x^2, as A is equal to a constant times x^2. Alternatively, we could also have said in words that surface area is directly proportional to the square of the side length of the cube.

8 In the following questions, either write an equation (which may have an unknown constant k in it) based on the sentence given, or write a sentence based on the equation given.

a. The weight of a human is directly proportional to its volume.

b. The surface area of a sphere varies directly with the square of its radius.

c. From problem 3, $d = .06v^2$.

d. The intensity of light varies inversely with the square of the distance from the light source.

e. $G = \frac{5}{7w^3}$, where G are "galumphs" and w are "wagdoodles".

f. The height of an extra-terrestrial varies directly with the square root of the length of its yellow antennae.

Practice

9 If a large pizza is 1.5 times the diameter of a small pizza, how much more should it cost, assuming that price is based on area?

10 The width of a rectangle is 12 ft and the length is 24 ft. By what *multiple* does the area change if:

a. The width is doubled and the length is tripled?

b. The width is halved and the length is quadrupled?

c. The width is multiplied by some number A and the length is multiplied by some number B?

d. How would your answer to part a be affected if the original width and length of the rectangle were 100 ft and 500 ft?

11 You have a rectangular fish tank with dimensions 12 ft x 10 ft x 8 ft Which side, if you double it, will give you the biggest volume?

12 In the study of waves one discovers the relationship between the velocity, wavelength, and frequency of any wave: $v = \lambda f$. For example, if the wavelength is 4 meters and the frequency is 6 cycles/second, then the velocity would be 24 meters/second.

a. Say that you were looking at a different wave that had twice the wavelength and half the frequency. How many times bigger or smaller would the velocity be than in the example?

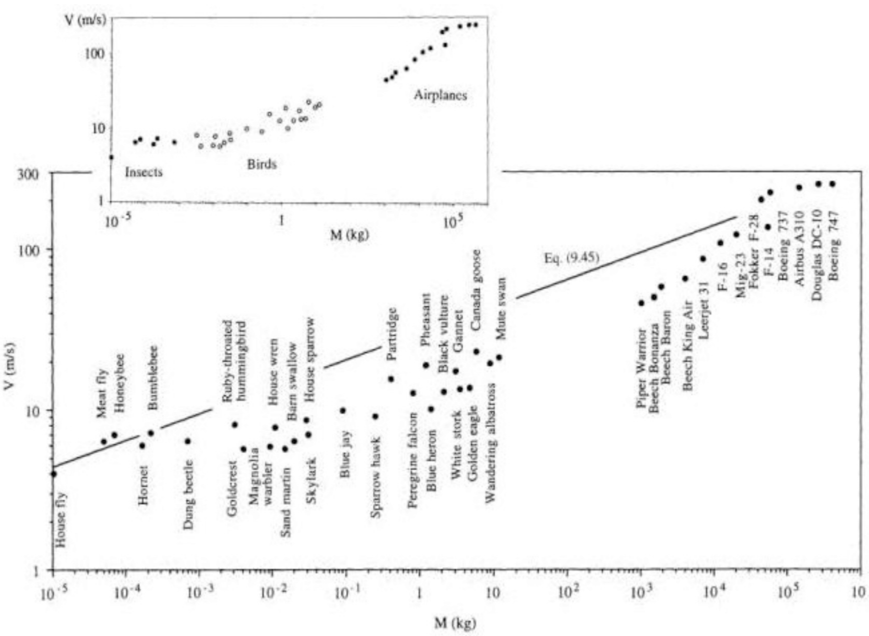
b. Say instead that you were looking at a different wave that had 2.5 times the wavelength and 4 times the frequency. How many times bigger or smaller would the velocity be than in the example?

c. Finally, say that you were looking at a different wave that had 6 times the velocity and half the wavelength of the initial example. How many times bigger or smaller would the frequency be?

13 The heat emitted per second (Q) due to radiation from an object is directly proportional to the fourth power of temperature T (in $^\circ K$) of the object. If the temperature of a quantity of metal triples, how much greater will its heat output be per second as a consequence?

14 Biologists have determined a good approximation of the relationship between the mass of a flying object and its optimum cruising velocity. "Optimum cruising velocity" is defined as the velocity at which an object travels the most distance for a given amount of energy it expends; the easiest analogy is with cars, where the most fuel efficient speed to drive turns out to be 55 mph.

The relationship is $M = \frac{V^6}{729000000}$, where V is the optimum cruising speed measured in meters/second and M is the mass of the object measured in kilograms. This relationship holds for birds, insects, and even planes!



Source: http://en.wikipedia.org/wiki/
File:Allometric_Law_of_Body_Mass_vs_Cruising_Speed_in_Constructal_Theory.JPEG

a. If one bird's V is double another's, how many times more massive is it? What if its V were triple another's?

b. If one bird's M is double another's, how many times faster is its optimum velocity?

15 a. If $R = PS$, are R and S in direct or inverse variation, or neither?

b. If $R = P^3S$, are P^3 and S in direct or inverse variation, or neither?

c. If $R^4 = PS$, are R^4 and P in direct or inverse variation, or neither?

16 In each question below, first find the constant of proportionality so that you can then answer the question being asked.

a. If y is directly proportional to x, and $y = 24$ when $x = 6$, what is y when $x = 13$?

b. If y varies inversely with x, and $y = 8$ when $x = 5$, what is y when $x = 80$?

c. If y varies directly with x^2, and $y = 63$ when $x = 3$, what is y when $x = 10$?

d. If y is inversely proportional to x^3, and $y = 12$ when $x = 2$, what is y when $x = 6$?

17 a. A cylinder has a base radius of 4 cm and a height of 10 cm. How will the volume of the cylinder be affected if you scale up the cylinder in all dimensions by a factor of 10? How will the surface area be affected?

b. A cylinder has a base radius of R cm and a height of H cm. How will the volume of the cylinder be affected if you scale up the cylinder in all dimensions by a factor of 10? How will the surface area be affected?

18 The area of a standard ghost in Mac-Pan is about 6.5 square units. What is the area of a ghost that has been scaled up by a factor of 2?

19 a. A beach ball has 8 times the volume of another beachball. How many times bigger is its circumference?

b. A beach ball has K times the volume of another beach ball. How many times bigger is its circumference?

20 Daddy bear, Mommy bear, and Preteen bear are all perfect copies of each other, except that each is .9 times the scale of the previous bear. How many more times than Preteen bear does Daddy bear weigh?

21 In the ideal gas equation from Chemistry, $PV = nRT$. R is an unchanging constant, but the other 4 quantities can change. Which of these pairs of quantities are in direct variation? Which pairs are in inverse variation?

22 Write an equation that predicts the number of rotations a wheel makes in a mile, given that you know the diameter of that wheel in inches (and note that 5280 feet = 1 mile).

23 You have an enormous vat of punch, 10 gallons (160 cups), which you have made for the Kiwanis-Elks Club-Rotary-March of Dimes annual dinner. You're expecting a big turnout. Write an equation relating the number of people who show up and the number of cups of punch each person can have.

Problems

24 You have two cans of soup. The large can is 3 times the scale of the small can. For each feature of the cans listed below, write how many times bigger it would get for the large can.

Feature	How many times bigger?
Example: Height of can	3 times
Amount of soup in can	
Amount of metal used to make the can	
How many people you can feed with the soup	
Number of calories if you eat all the soup	
Amount of ink needed to print label	
How many cans it takes to stack up to the ceiling	
Time it takes to open it with a can opener	
Number of peas in the soup	
How much heat rises off the can if it's hot	

25 For the following, make an educated guess as to whether the relationship between the given variables is approximately direct, inverse, or neither.

a. Average income in a town vs. average education level

b. Time it takes to read a book vs. how interesting the book is to you

c. Monthly sales of a video game vs. time the video game has been on the shelves

d. Time it takes to read a book vs. number of pages in the book

e. The number of hours of daylight vs. day of the year

f. Birth rate in a country vs. average income per family in that country

g. Pressure on a tire vs. the amount of air in a tire

h. Price of a new electronic device vs. number of people who want to buy one but are unable to due to limited availability

i. Number of diagonals in a polygon vs. number of vertices of the polygon.

26 It turns out that, to a very good degree of accuracy, the mass of an animal is inversely proportional to the fourth power of its resting heart rate. Given that a typical adult human has a resting heart rate of about 72 beats/minute and a mass of 160 pounds, what would you estimate the heart rate of a mouse (.055 pounds) and dinosaur (70000 pounds) to be?

27 You have a set of six Russian dolls. Each is a perfect copy of the others, except that each is only .8 times the scale of the previous one. If it took 3 oz of paint to paint the largest doll, how much paint was needed to paint the second-largest doll? How much paint was needed to paint the smallest doll?

28 The force on a car that is moving in a circle is described by the centripetal force equation: $F = \frac{mv^2}{r}$, where m is the mass of the car in kilograms, v is the velocity of the car in meters/second, r is the radius of the circle the car is moving in, and F is the force in Newtons. Suppose the force on a car is 12000 Newtons as it travels in a circle.

a. If the radius of the circle the car is traveling in doubles, what will the new force on the car be?

b. If the speed of the car doubles, what will the new force on the car be?

c. If the speed of the car doubles at the same time that the radius of the circle it is moving in doubles, what will the new force on the car be?

d. If the force on another car is 27000 Newtons as it travels in a circle, and the radius of the circle is suddenly cut to a ninth of the original radius, how much slower would the car have to travel for the force to stay unchanged?

29 The gravitational attraction of two objects is described by the equation $F = \frac{k}{d^2}$ (where d is the distance in meters between the centers of the objects, k is a constant based on the masses of the objects, and F is the force of attraction between them, measured in Newtons). For brevity, one often says that Gravity is an "inverse square" force, which just means that the force is inversely proportional to the distance squared.

a. If initially the force between a person and a planet is Q_0 and the distance between their centers is d_0, and then the person moves so that the distance between their centers is doubled, what is the new Force Q_1 between the objects, in terms of Q_0?

b. What if the distance were tripled instead of doubled? Now what would the force be in terms of Q_0?

c. Lastly, what if the distance were halved? What would the force be in terms of Q_0 then?

d. The force between a typical person and the Earth is about 980 Newtons. Given that the radius of the Earth is 6370 km and that the space shuttle orbits 390 km above the Earth's surface, determine what the force would be on the same typical person, if they were at the height above the Earth that the space shuttle is when it is in orbit.

30 According to Dr. Killjoy's research, how long a couple stays married can be predicted with absolute certainty: the length of the marriage is directly proportional to the number of times per day the couple holds hands and is inversely proportional to the square of the number of nasty looks per day they exchange. As an example, he says that a couple that holds each other's hands 6 times per day and exchanges 3 nasty looks will be married for 16 years.

If Dr. Killjoy is correct, how long will a couple remain married that holds hands twice a day but exchanges 4 nasty looks a day as well?

31

The so-called kinetic energy of a bullet is equal to $\frac{1}{2}mv^2$, where m is its mass in kg and v is its speed in meters/sec. The higher the kinetic energy of the bullet, the more its ability to cause damage. A typical rifle can shoot a .0042 kg bullet at 965 meters/sec.

a. If a second rifle can fire bullets that are twice as heavy but only at half the speed, how would the kinetic energy of the bullets of this rifle compare to those of the original rifle?

b. If a third rifle fires bullets that are half as heavy as the original rifle, but twice the speed, how does the kinetic energy of the bullets of the third rifle compare to those of the original rifle? To the second rifle?

c. If two rifles use bullets of the same mass, how many times faster must the muzzle velocity (i.e. speed at which the bullet leaves the gun) of one be than the other for it to be able to cause 20 times as much damage?

d. Two rifles are compared to see how far they can penetrate in to a target (this turns out to be directly proportional to the damage it can inflict). One has bullets with one-third the mass of the other, but its bullets still penetrate 4 times as far in to the target as the other's. How many times faster is the muzzle velocity of the rifle with the lighter bullets?

32

A beam that is supported horizontally at one end and free at the other can be deflected an amount D at the free end according to the equation $D = \frac{FL^3}{3K}$, where F is the force applied, L is the length of the beam, and K is a constant based on the shape and stiffness of the beam. Assume K does not change in all parts of this problem.

a. If the beam is currently being deflected an amount D_1, how much more force would you have to apply to triple the deflection?

b. If the beam was half the length it currently is, how much harder would one have to push to yield the same deflection D_1?

33

The number of alligators observed in a Martian swamp increases according to the equation $A = k \cdot 2^t$, where t is the number of days after the first alligators were observed, and k is a constant.

a. If there are 400 alligators in the swamp in the initial observation, how many alligators are there after 1 day? 2 days? 3 days? 6 days?

b. There are A_1 alligators at time $t = 13$. At what time will there be double this number of alligators?

c. If there are A_1 alligators after time t_1, how many alligators will there be after time $2t_1$? After time $3t_1$? After time nt_1?

34 The Richter scale indicates the intensity (I) of an earthquake by relating it to the amplitude (A) of waves at its epicenter in the following way: $I = \log_{10} A$. (Units are ignored in this problem for simplicity.)

a. What amplitude of waves would give an intensity of 1?

b. How many times bigger than in part a would the amplitude of the waves have to get for the intensity to be 2?

c. How many times bigger than in part a would the amplitude of the waves have to get for the intensity to be 6?

d. If the amplitude of the waves were double that of the waves in part a, what would the intensity be?

e. If the amplitude of waves at some time is A_1, and at some later time is $2A_1$, how did the intensity change between the two times?

35 The total cost C of producing N cars from scratch, where one has to first build the factory and the assembly line before producing a single car, follows the equation $C = 6000N + 12,000,000$. The factory can produce tens of thousands of cars in a single year. What affect does doubling the number of cars produced, from N_1 to $2N_1$, have on the total cost C? Justify your answer with a specific example or two.

36 An ideal gas obeys the equation $PV = nRT$. If the pressure is P_1 at some point in time, what will the new pressure be if, simultaneously, V is halved, T is tripled, R remains the same, and n is doubled?

37 The pull of a magnet turns out to be inversely proportional to the cube of an object's distance from the magnet; that is, $Pull = \frac{k}{d^3}$, where k is some constant.

Sarah, who is 24 feet from a magnet and holding a piece of iron, feels 512 times less Pull from the magnet than Isaac does, even though he is holding an identical piece of iron. If the magnet, Isaac and Sarah are all in a straight line in that order, how far apart are Sarah and Isaac?

38 "The Attack of the 50 Foot Woman!" was a campy "sci-fi" thriller from the 1950's, about a woman who, due to an alien encounter, grows to be a 50 ft giant, wreaking havoc on the metropolis she lives in. Could humans prosper at such heights if, say, our DNA gave our body permission to "keep growing"? Let's check it out.

a. Here are two important facts: 1) The weight of any animal is directly proportional to its volume, and 2) The strength of a bone is directly proportional to the cross-sectional area of the bone. Write equations that represent these relationships.

b. A 50 ft woman is 10 times taller than a 5 ft woman, indicating that one could roughly think of her as a 5 ft woman scaled up by a factor of 10 in all dimensions. How many times more would the 50 ft woman weigh than the 5 ft woman?

c. Since all dimensions are scaled by a factor of 10, how many times stronger would the 50ft. woman's bones be than the 5 ft woman's?

d. Bones in the leg break if they are under too much strain/pressure, that is, if there is too much weight above pressing down on the bones. Using your answers to parts b and c, determine how the pressure (i.e. $\frac{\text{Weight}}{\text{Cross}-\text{sectional area}}$ of the leg bones) on the 50ft woman's leg bones compares to the 5 ft woman's.

e. Thus explain why animals that are small would have difficulty if they grew too large, even given that their DNA permitted it. That is, why aren't there 50 ft women?

39 A spherical balloon becomes bigger and bigger as it is filled with more and more air, although it always retains its spherical shape. It starts out at $t = 0$ as a balloon with a volume of 10 cubic centimeters. With every second, 30 cubic centimeters more of air is pumped into the balloon.

a. Write an equation for the volume of the balloon as a function of time.

b. Write an equation for the volume of the balloon as a function of its radius.

c. Using parts a and b, write an equation relating time and radius of the balloon.

d. Now, using algebra, find radius as a function of time.

40 Examine the following crazy formula: $y = \frac{ab^2c^3\sqrt{d}}{ef^2g^3\sqrt{h}}$. Initially, y equals some unknown number based on current values of a, b, c, d, e, f, g, and h.

What happens to y if...

a. b is tripled?

b. g is halved?

c. d is quadrupled?

d. h is multiplied by 16?

e. c is doubled AND e is doubled?

f. d is multiplied by 9 AND g is tripled?

Exploring in Depth

41 The speed on a given planet with which an object must be propelled from its surface straight up so that it would never come down is called its escape velocity, determined by the equation $v_{esc} = \sqrt{\frac{2GM}{R}}$, where v_{esc} is the escape velocity in meters/second, G is a fixed constant, and M and R are the mass and radius of the planet in kilograms and meters.

a. The **escape velocity** on the surface of the Earth is 11,200 meters/second (you can see why we need rockets to achieve such velocities). If the Earth were twice as massive but the same size, what would the escape velocity then be? How many times more massive would the earth have to be for the escape velocity to be twice what it is now?

b. The moon has a radius that is .273 times the Earth's. It has a mass that is .0123 times the Earth's. What is the escape velocity on the surface of the moon?

42 Don't use a calculator for this problem.

a. Evaluate: $\log_3 45 + \log_3 2 - \log_3 10$

b. Solve for x: $x^2 = 10 - 3x$

c. Evaluate: $\frac{(2^3 3^2)^5}{4^7 3^8}$

d. Expand: $(2x - 3)^3$

e. Simplify: $\sqrt{6} \cdot \sqrt{35} \cdot \sqrt{21}$

43 Kleiber's Law for Animal Metabolic Rates is as follows: $q^4 = km^3$, where m is the mass of the animal in kilograms, k is a constant, and q is the metabolic rate.

a. If one animal is twice the mass of another, how many times bigger is its metabolic rate?

b. If one animal has 3 times the metabolic rate of another, how many times bigger is its mass?

c. If one animal has half the metabolic rate of another, how many times less massive is it?

44 The pressure exerted by an object A on another object B is directly proportional to A's mass and inversely proportional to the area of that part of A that rests upon B. (Remember, too, that the mass of an object is directly proportional to the cube of its height, as long as the other two dimensions change proportionally with height.)

Suppose A is a brick and C is another brick that is similar (in the mathematical sense) to A and made of the same material. Both bricks are resting on a table.

a. If C is two times as big (in each direction) as A, then how much pressure does C exert on the table in comparison to A?

b. If C's surface area is twice as big as that of A, then how much pressure does C exert on the table in comparison to A?

45 Physics tells us that an object falling in a vacuum will, because of gravity, get faster and faster — there's no limit on its top speed as it falls. In practice, though, the Earth's atmosphere causes **drag**. Because of this, a falling object (like, say a skydiver) will initially speed up, but eventually will reach its **terminal velocity** — its constant top speed, where the drag force perfectly cancels out gravity, so that there's no more acceleration.

According to Wikipedia, the square of the terminal velocity of an object is directly proportional to its mass and inversely proportional to its cross-sectional area.

Suppose you dropped two spherical balls, both made of identical material, from an airplane.

a. If the radius of the larger of the two balls is sixteen times the radius of the smaller, then which ball has a higher terminal velocity? How many times faster is it than the other one?

b. If, instead, the larger of the two balls weighs twice as much as the smaller, then which ball has the higher terminal velocity? How many times faster is it than the other one?

c. Now suppose that two balls, this time made of *different* materials, turn out to have the same terminal velocity. If the radius of the larger ball is twice the radius of the smaller, then which ball is made of a denser material? How do their densities compare? (Note: density = mass/volume.)

Park School Mathematics

This curriculum was written by members (past and present) of the upper school math department at the Park School of Baltimore. This was possible thanks to Park School's F. Parvin Sharpless Faculty and Curricular Advancement (FACA) program, which supports faculty every summer in major curricular projects. In addition to the support of FACA by the Nathan L. Cohen Faculty Enhancement fund and the Joseph Meyerhoff FACA fund, this project was also funded by grants from the E. E. Ford Foundation, the Benedict Foundation, Josh and Genine Fidler, and an anonymous donor.