BOOK 3: INVESTIGATING SHAPE AND SIZE

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

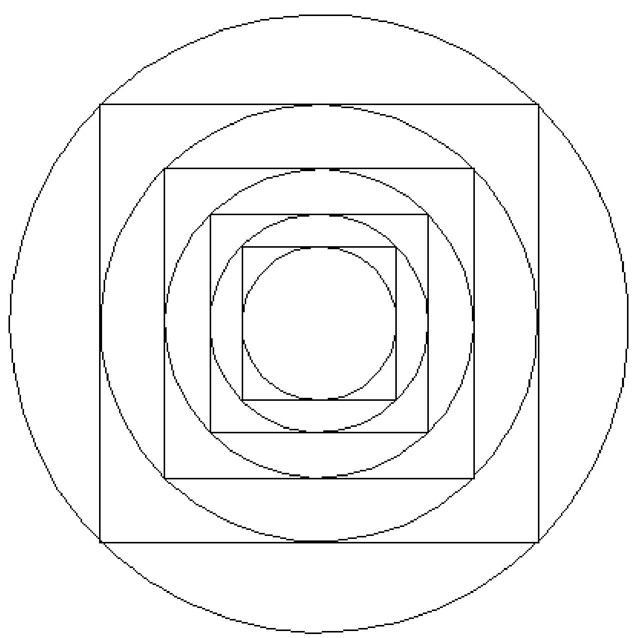
re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

When you have a diagram, it is sometimes helpful to **Fill in all the Information you Can**, even if you think it has nothing to do with the problem you have to solve. This is another example of **tinkering**. Doing this will help you to see patterns and to notice things about the diagram that you might not have noticed otherwise. It is a wonderful thing to do when you are stuck on a problem.

1 The radius of the smallest circle is one unit. What is the ratio of the area of the largest circle to the area of the smallest circle?

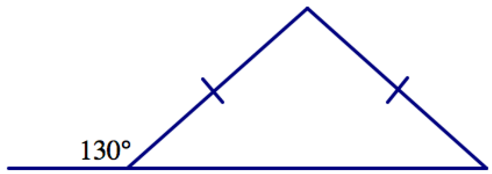


geometric tinkering

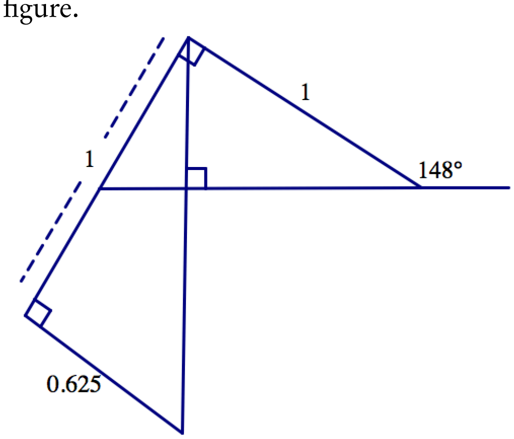
In the problem on the previous page, rather than trying to figure out the ratio of the two areas immediately, it makes more sense to figure out some smaller, manageable thing about the figure, and see where that leads us. A clear place to start is with the smallest circle. Once we find the area of that circle, we can then find the square that surrounds it. Can we then find the area of the second largest circle? By looking at the smallest parts, we can see the pattern which allows us to solve the original problem.

In your previous experience with geometry, you probably learned some basic facts about angles, lines, and triangles. For instance, two angles that form a line add up to 180 degrees, and so do the three angles of a triangle. You may also know that, in an isosceles triangle, the angles opposite the sides that are the same length are also the same. These facts will help you with many of the problems that follow.

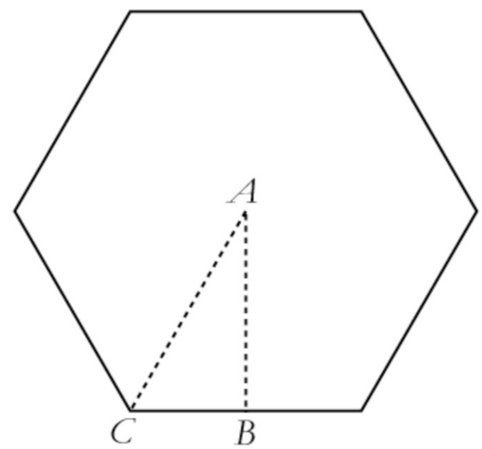
2 The two marked lengths are congruent. Find the measures of all the angles in the figure.



3 Find the measures of all the angles in the figure.

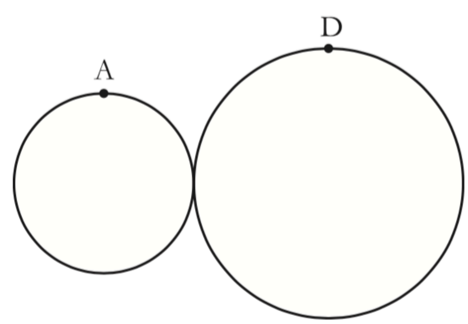


4 The area of this hexagon is 60. Find the product of the lengths of the segments AB and BC.

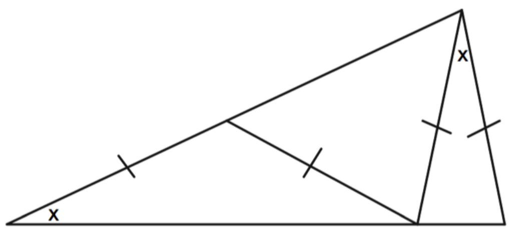


5 A smaller circle has radius 2 and a larger circle has radius 3. *A* and *D* are points on the tops of these circles. The circles are touching in the middle, as shown.

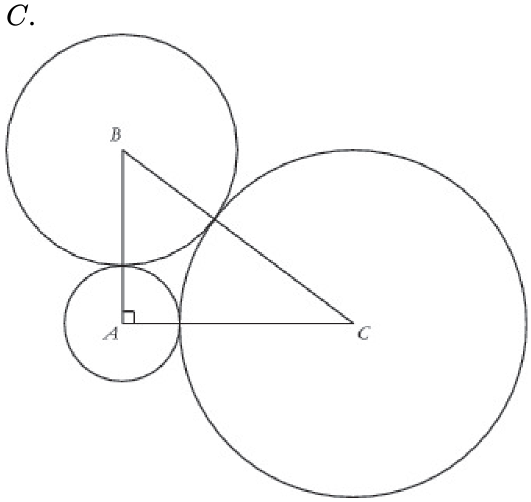
Find the distance from *A* to *D*.



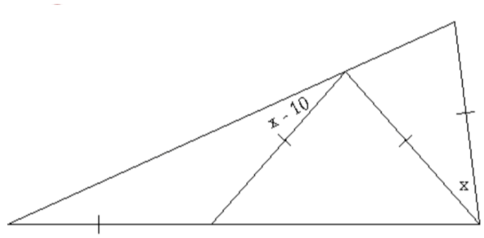
6 Given that the four marked lengths are congruent, find x.



7 In the following diagram, $BC = 5$, $AC = 4$, and the circle B has four times the area of circle A. Find the area of circle C.



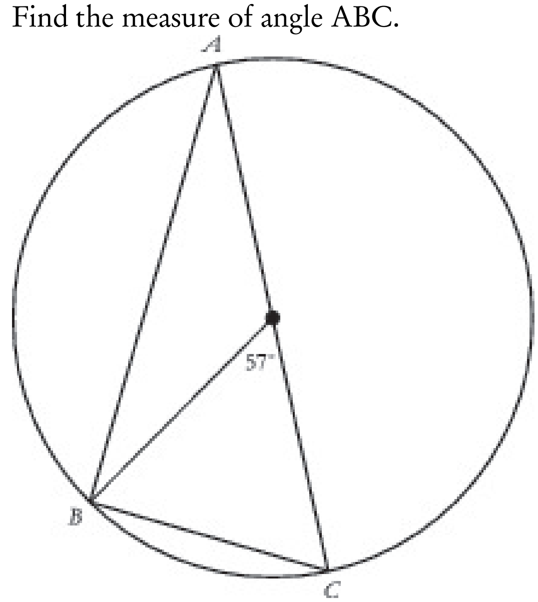
8 Find x.



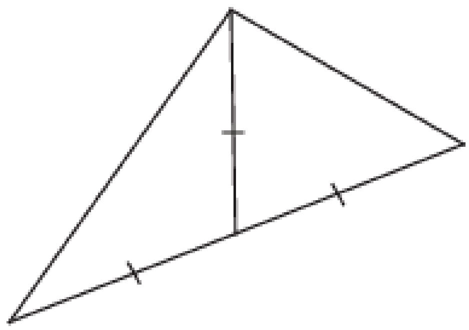
9 Draw a square with vertices A, B, C, and D, in that order. Then draw an equilateral triangle with vertices A, B, and E, where E is a point outside the square. Connect points E and C with a line segment. Find the measure of angle ECB.

10 Repeat the previous problem, but this time place point E inside the square. Again, find the measure of angle ECB.

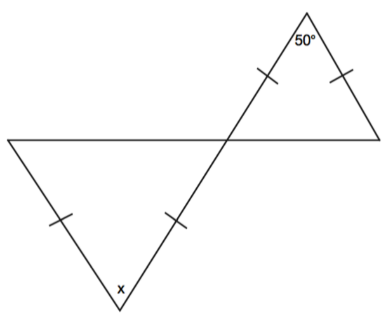
11 Find the measure of angle ABC.



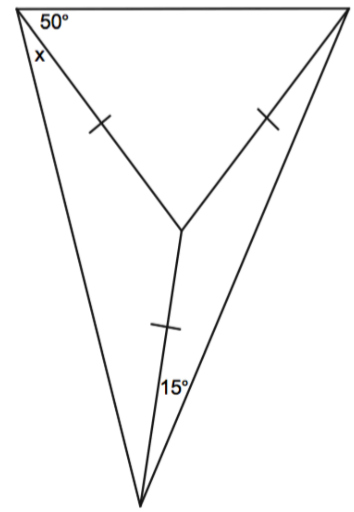
12 Prove that the large triangle below is a right triangle. It may help to label some angles x and other angles y (give angles the same letter if you know they are the same!)



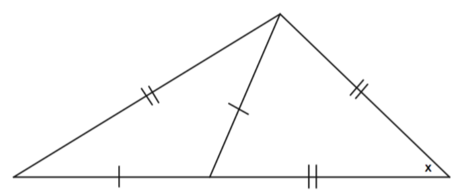
13 Find x.



14 Find x.



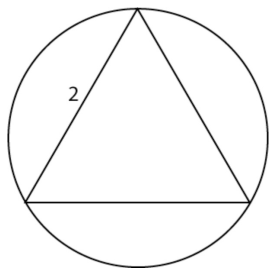
15 Find x.



16 A rectangular solid always has 12 edges (count the edges of a tissue box), and 6 faces (count the 2D rectangles that form the tissue box.) Each face of a rectangular solid has two diagonals, formed by connecting opposite corners.

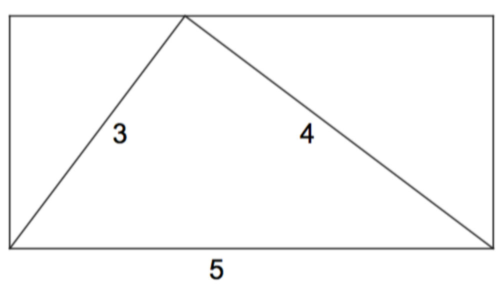
Find the dimensions of a rectangular solid where the 12 edges and the diagonals on all 6 faces are integers.

17 The triangle below is equilateral. Find the area of the circle.

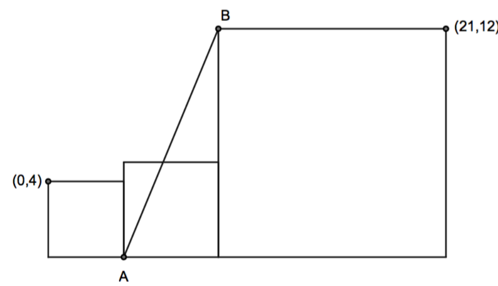


18 The triangle below is the famous "3-4-5" right triangle; famous because it is one of the simplest triangles that satisfies the Pythagorean theorem: $3^2 + 4^2 = 5^2$.

As you can see, the 3-4-5 right triangle has been placed in a rectangle. What is the height of the rectangle?



19 Three squares are lined up along the x-axis as shown, and the points with coordinates (0,4) and (21, 12) are labeled accordingly. Find AB. *Copyright mathleague.com*.

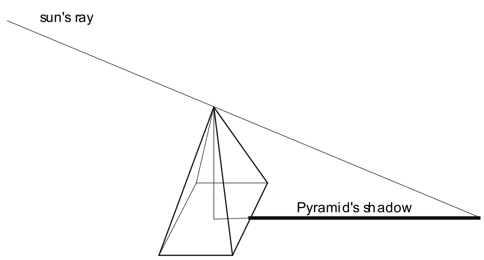


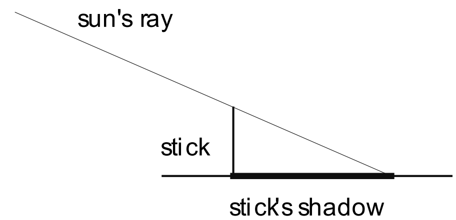
HABITS

LESSON 1: SIMILAR TRIANGLES

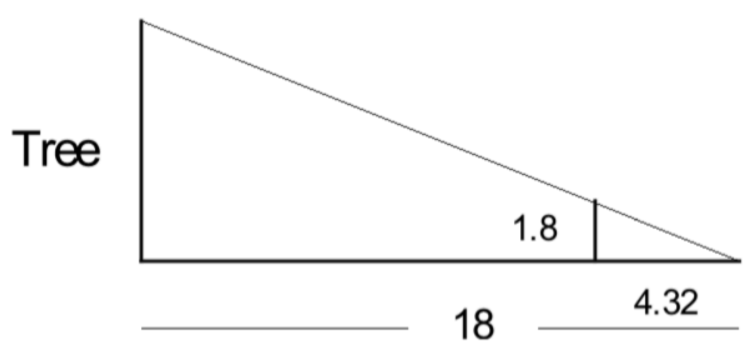
Introduction

Thales of Miletus (625-547 B.C.) is generally regarded as the first of the Seven Wise Men of antiquity. Among other things, he is known for having calculated the height of the Great Pyramid in Egypt using the length of its shadow when compared to a stick in the ground and its shadow at the same time of the day.

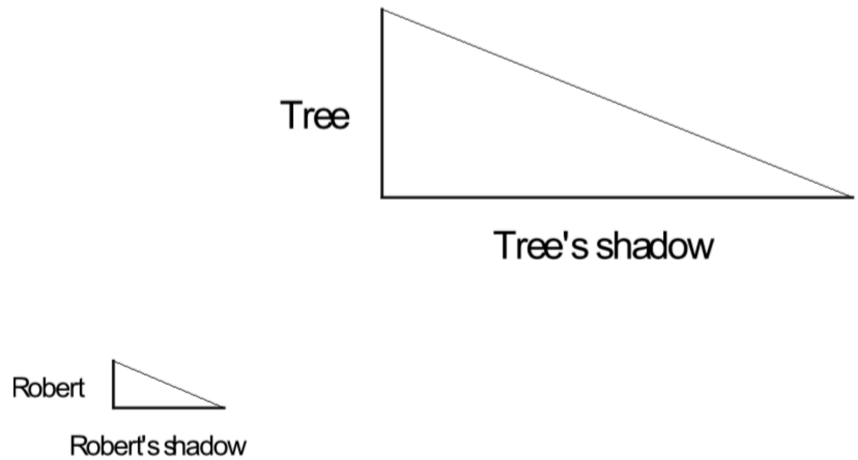




1 Inspired by Thales' method, Robert and Mary calculated the height of a tree. At a certain time of day, Robert stood at a point such that the tip of his shadow coincided with the tip of the tree's shadow. Then they measured both the shadow of Robert, who is 1.8 m tall, and the shadow of the tree. They found that Robert's shadow was 4.32 m and the tree's shadow was 18 m long. With this information they were able to calculate the height of the tree to be 7.5 m. Could you explain how they might have found the height of the tree?



2 **Taking things apart** in the figure in Problem 1, we have two triangles. One right triangle whose legs are the tree and its shadow, and another right triangle whose legs are Robert and his shadow (see figure below). What can you conjecture about the angles in these two triangles? Explain your answer.



Development

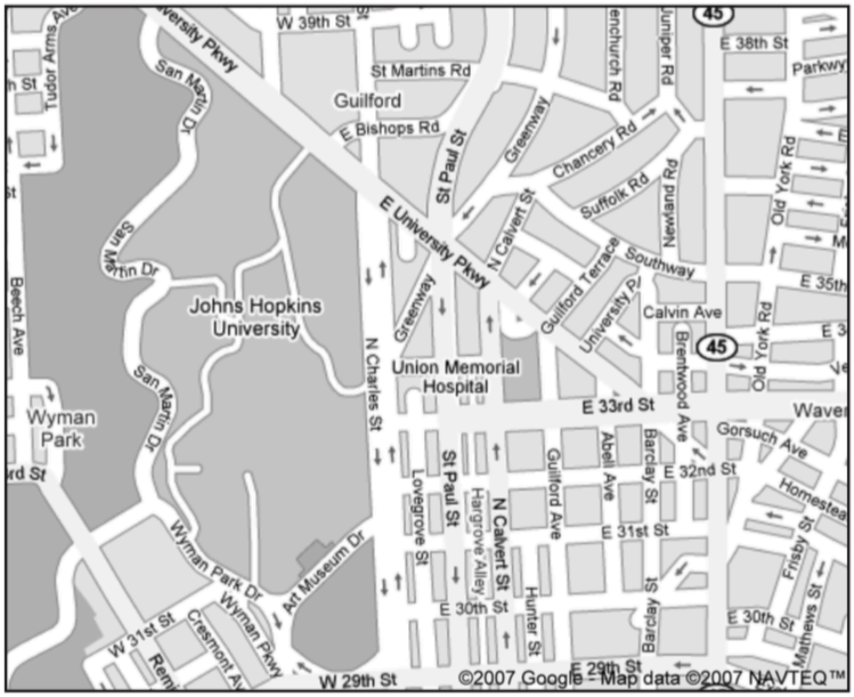
3 Kristi is sitting in the back seat of her parents' car, driving from Omaha, NE to visit her grandparents in Hastings. The scale on her map is 1 inch = 20 miles. She's using a Bazooka gum wrapper, which she knows is 1.5 inches long, to estimate distances.

a. When leaving her parents' garage, Kristi first opens the map, she notices that there are about five gum wrapper lengths from Omaha to Hastings. How many miles will her trip be?

b. How many "gum wrapper lengths" will Kristi measure between her position on the map and Hastings if she is 100 miles away from Hastings?

c. Much later in the trip, Kristi notices that the distance remaining on the map is about a third of a gum wrapper. How many miles are remaining?

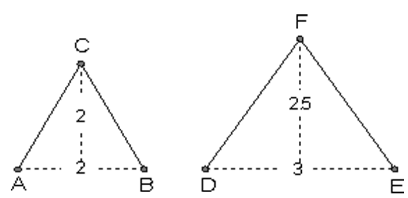
4 In Baltimore, Maryland, Union Memorial Hospital is the largest building in the triangular region bounded by E University Pkwy, N Calvert St, and E 33rd St (the higher on the map below). If the Union Memorial Hospital's left side, the side on N Calvert St, is 250 meters long, use the map below to estimate the other two side lengths of this triangular region. You can either measure with a ruler or measure with some other object, like Kristi.



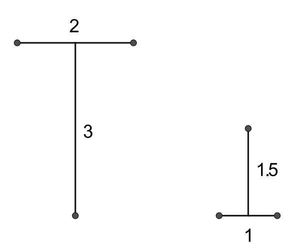
In Middle School, you may have learned how to scale down (or up) a figure. In fact, you may now realize that in doing dilations or contractions, you are changing the scale of a figure. When you scale a figure by a factor of r, your new figure will have lengths r times the corresponding lengths of the original figure. This can also be expressed by saying that the sides of the second figure are **proportional** to the corresponding sides of the first figure by a factor of r. In this case, we say as well that the second figure is a **scaled copy** of the first figure by a factor of r.

5 Below there are some pairs of figures. In Part b, the segments in each figure are perpendicular. Determine whether or not one is a scaled copy of the other. Explain.

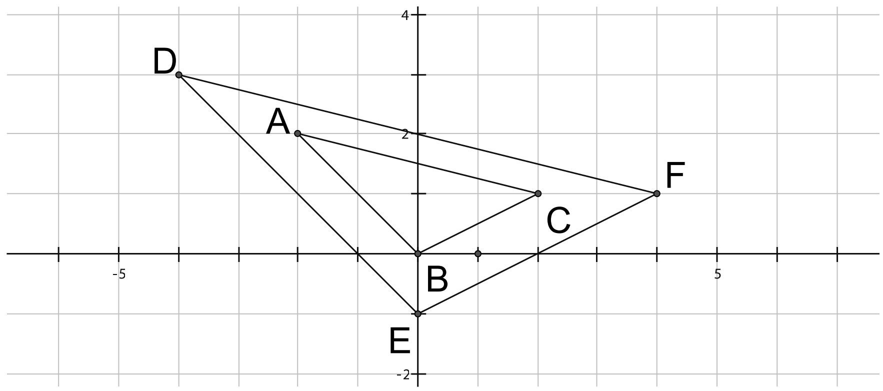
a.



b.



6 Two triangles are given in the picture below.



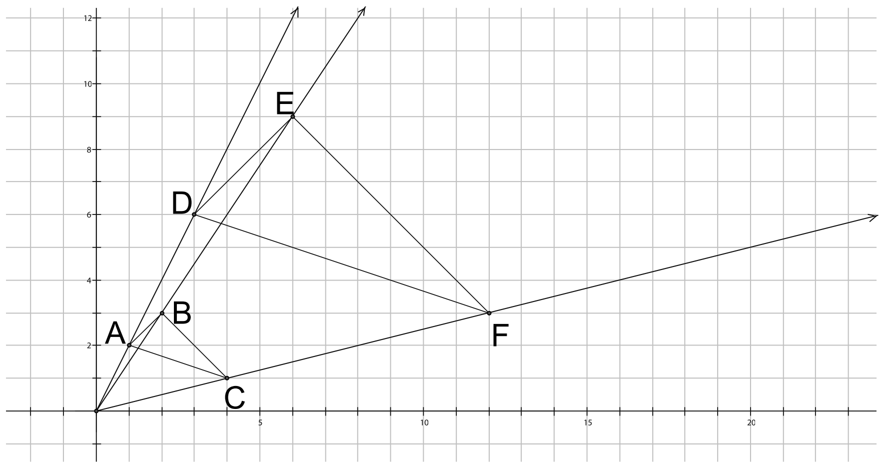
a. Take whatever measurements and do whatever calculations are necessary to check whether or not the two triangles are scaled copies of each other. If they are, determine the scale factor.

b. Determine the coordinates of the triangle $\triangle A'B'C'$ obtained when you translate triangle $\triangle ABC$ 1 unit down.

c. Based on your answer to Question (b), what can you conclude about $\angle B$ and $\angle E$?

d. What can you say about the two other pairs of angles $\angle A$ and $\angle D$, and $\angle C$ and $\angle F$? Explain.

7 Triangle $\triangle DEF$ is the image of under a dilation centered at the origin $O\,(0, 0)$.



a. Are triangles $\triangle ABC$ and $\triangle DEF$ scaled copies of each other? If so, what is the scale factor? Explain.

b. How are the angles of triangle $\triangle ABC$ related to the angles of $\triangle DEF$? Explain.

In our study of dilations, you may have noticed that the angles of the image of a figure under a dilation have the same measurements as the corresponding angles of the original figure. What's more, you can see in Problem 7 that the following relationship holds:

side length of image = (scale factor) · (side length of corresponding side in original figure)

The next two problems use the important vocabulary word **congruent**. Two angles are congruent if they have the same measure, just as two line segments are congruent if they have the same length.

8 Discuss the following statement with the members of your group.

"If the three angles of a triangle are congruent to the three angles of a second triangle, then the triangles are scaled copies of each other."

If you disagree, give a counterexample showing that this statement is false.

9 Now examine the following statement.

"If two angles of a triangle are congruent to two angles of another triangle, then the triangles are scaled copies of each other."

If you disagree, give a counterexample showing that this statement is false.

The word **similar** is used in mathematics to describe two triangles that are scaled copies of each other. Similarity plays an important role in the design of large or small objects such as automobiles, airplanes, or integrated circuits. As a matter of fact, before building an automobile, engineers sketch out scale drawings and use them to build scale models which they then use to run tests. Also, as we saw in the introduction of this section, Thales of Miletus may have used similar triangles to calculate the height of the Great Pyramid in Egypt, more than 2500 years ago. Similarity is thus a mathematical instrument that allows us to zoom in or zoom out when examining the physical world.

10 Thus far, you have discovered several features of similar triangles.

a. Why does it make sense to use the word "similar" to describe two triangles that are scaled copies of each other?

b. If two triangles are similar, how are their angles related? How are their sides related?

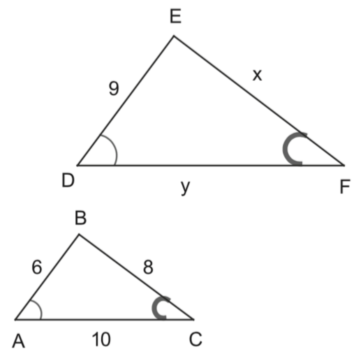
c. If you are given a $\triangle ABC$, describe a procedure for drawing a $\triangle A'B'C'$ similar to it.

Practice

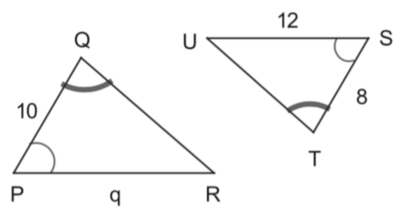
In many of the problems that follow, you will see a convention that will be used throughout your high school career. If two angles are marked with arcs that are the same style (both bold, or double instead of single), that means those angles are congruent.

11 Calculate the lengths of the sides marked by small letters. Angles marked equally have the same angle measurement.

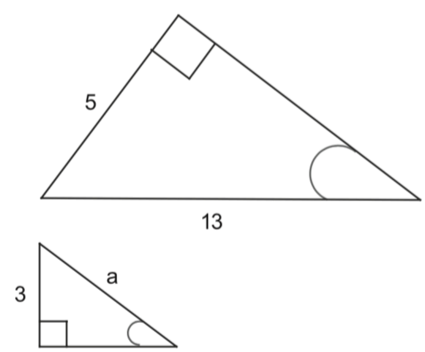
a. Calculate x and y.



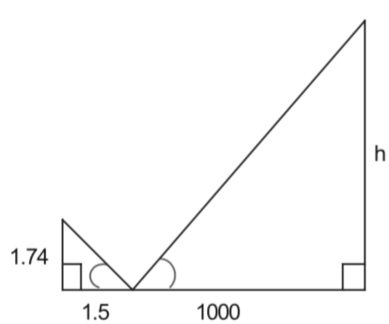
b. Calculate q.



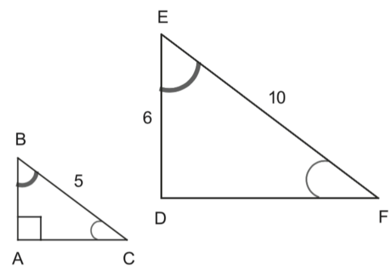
c. Calculate a.



d. Calculate h.



12 In the figure below, angles marked equally have the same angle measurement.



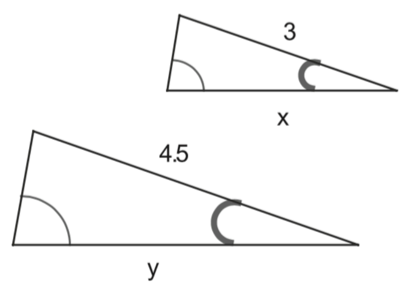
a. Find the perimeter of triangle $\triangle ABC$.

b. Triangle $\triangle DEF$ is a scaled copy of $\triangle ABC$. What is the scale factor?

c. What is the ratio of the perimeter of $\triangle DEF$ to the perimeter of $\triangle ABC$?

d. What is the ratio of the area of $\triangle DEF$ to the area of $\triangle ABC$?

13 In the figure below, what is the ratio of x to y?



14 Claudia and Rodolfo were walking together on a sunny day. Claudia is 1.5 meters tall and casts a 2.1-meter shadow. How tall is her taller brother Rodolfo if he casts a shadow at the same time which is 0.42 meters longer?

15 Examine the following statement.

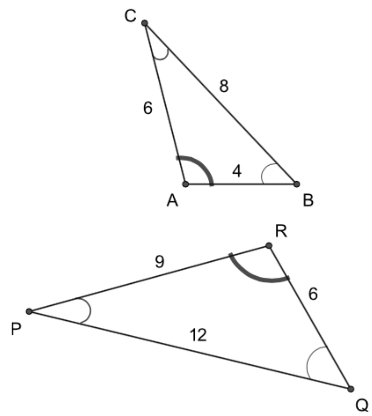
"If two triangles have one pair of angles congruent, the triangles are similar."

If you disagree, give a counterexample showing that this statement is false.

Further Development

16

Jasmine has two triangles with equal angle measurements.



The sides of one triangle are 4, 6, and 8.
The sides of the other are 9, 6, 12.
She says that the triangles are not scaled copies because:

$$\frac{4}{9} = 0.444..., \ \frac{6}{6} = 1, \text{ and } \frac{8}{12} = 0.666...$$

Do you agree with Jasmine? Thoroughly explain your answer.

In two similar triangles, angles with the same measure are called **corresponding angles**, and sides opposite corresponding angles are called **corresponding sides**. As a matter of fact, in the previous problem, $\angle A$ and $\angle R$ are corresponding angles, and $\overline{BC}$ and $\overline{PQ}$ are corresponding sides. Given misunderstandings such as that of Jasmine's in the previous problem, it is important to state explicitly what pair of angles have the same measurement when describing similar triangles. A natural way of doing this is by listing their corresponding angles in the same order. Thus, regarding the triangles in the previous exercise, we say that $\triangle ABC$ and $\triangle RQP$ are similar, and write $\triangle ABC \sim \triangle RQP$. The symbol $\sim$ is read "is similar to."

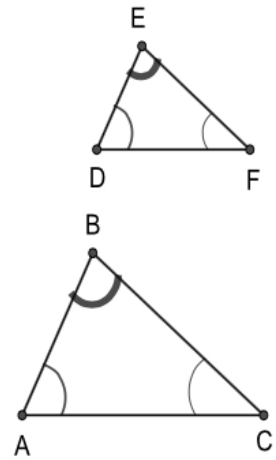
17

$\triangle BUD \sim \triangle PAT$

$BU = 7, UD = 8, BD = 10$, and $PT = 12$.

Draw the figure and find PA.

18 Let $\triangle DEF \sim \triangle ABC$.



a. List all pairs of corresponding sides.

b. Let k be the scale factor, that is, the lengths of the sides of $\triangle DEF$ are k times the lengths of the corresponding sides of $\triangle ABC$. So, for example, $AB = k \cdot ED$.

Write two more equations involving k, each one using a different pair of sides.

c. Solve each of the three equations for k.

d. In part c, you should have found three quantities that all equal k. Translate this into a statement about side lengths in similar triangles.

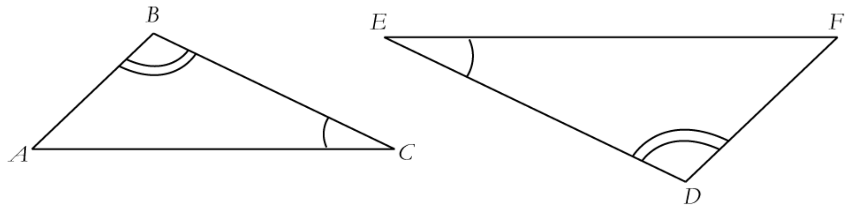
In the previous problem you may have concluded that if $\triangle DEF \sim \triangle ABC$, then

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$$

These equalities are also used to express that in two similar triangles corresponding sides are proportional. These equalities allow us to relate corresponding sides of similar triangles without explicitly mentioning the scale factor.

19

In the following figure, angles marked equally have the same angle measurements.



a. Take into account the convention stated in the paragraph above problem 17 to determine whether or not $\triangle ABC \sim \triangle DEF$. Explain your answer.

b. Describe the relationship between these two triangles using the $\sim$ notation.

c. Below, you have a list of equalities between two ratios relating to the triangles above. Determine which ones are true and which ones are false, if any. Explain.

i) $\dfrac{BC}{DE} = \dfrac{CA}{EF}$

ii) $\dfrac{BC}{EF} = \dfrac{AC}{DF}$

iii) $\dfrac{BC}{DE} = \dfrac{BA}{DF}$

iv) $\dfrac{AB}{AC} = \dfrac{DF}{EF}$

v) $\dfrac{BC}{AB} = \dfrac{DE}{EF}$

vi) $\dfrac{BC}{AC} = \dfrac{DE}{FE}$

vii) $\dfrac{AC}{BC} = \dfrac{FE}{DE}$

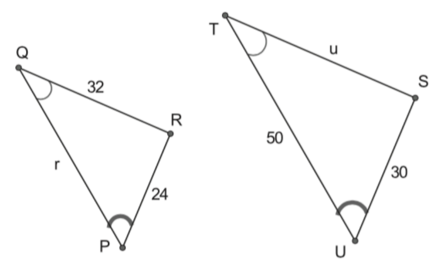
viii) $\dfrac{FD}{DE} = \dfrac{AB}{BC}$

ix) $\dfrac{AB}{FD} = \dfrac{AC}{DE}$

x) $\dfrac{FE}{AC} = \dfrac{FD}{BC}$

Practice

20 In the following figure, angles marked equally have the same angle measurements.

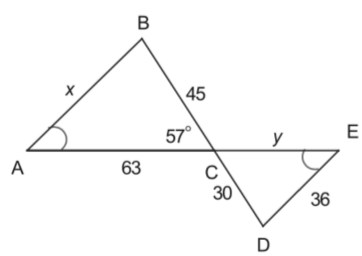


a. Is $\triangle PRQ \sim \triangle STU$? Explain your answer.

b. Describe the relationship between these two triangles using the $\sim$ notation.

c. Find r and u.

21 In the following figure, angles marked equally have the same angle measurements, and $\overline{BD}$ and $\overline{AE}$ intersect at C.



a. Describe the relationship between these two triangles using the $\sim$ notation.

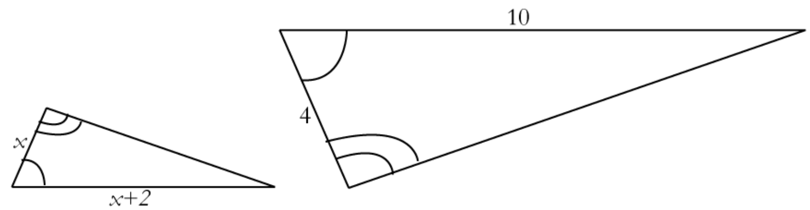
b. Find x and y.

c. In this problem, you needed to use the fact that $\angle BCA$ was congruent to $\angle DCE$. Is this assumption justified?

22

In each of the following exercises, pieces of information about a pair of triangles are given. A figure is provided for exercise in Part a. In each case, find x.

a. Angles marked equally have the same angle measurements.



b. $\triangle PQR \sim \triangle STU$, $PQ = 3$, $QR = x$, $ST = x$, and $TU = 7$.

c. $\triangle ABC \sim \triangle DEF$, $AB = x + 1$, $BC = 50$, $DE = 2$, and $EF = x + 1$.

23

Felipe looks at a dot on a perpendicular wall 20 ft. away. If he looks .1 degrees to the right, he will be looking at a point on the wall about 3/8 of an inch to the right of the dot. Later Felipe (carefully) looks at the center of the Sun, which is 93 million miles away. If he looks at a sunspot that is .1 degrees to the right, how far away from the center of the Sun will he be looking? Would you estimate it was about:

a. 16 miles to the right

b. 1,600 miles to the right

c. 16,000 miles to the right

d. 160,000 miles to the right

e. 16,000,000 miles to the right?

Now check your estimate by actually figuring out the answer.

Problems

24 Let $\triangle ABC \sim \triangle DEF$ and $k = \frac{DE}{AB}$ be the scale factor.

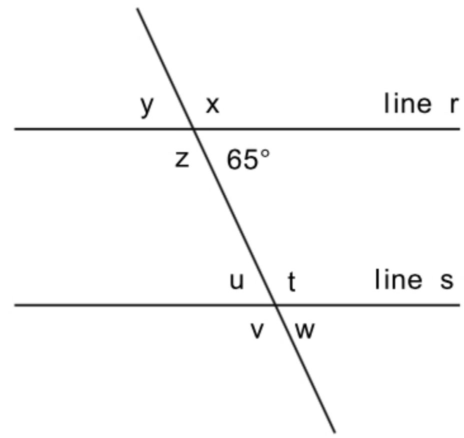
a. What is the ratio of the perimeter of $\triangle DEF$ to the perimeter of $\triangle ABC$? Justify your answer.

b. What is the ratio of the area of $\triangle DEF$ to the area of $\triangle ABC$? Justify your answer.

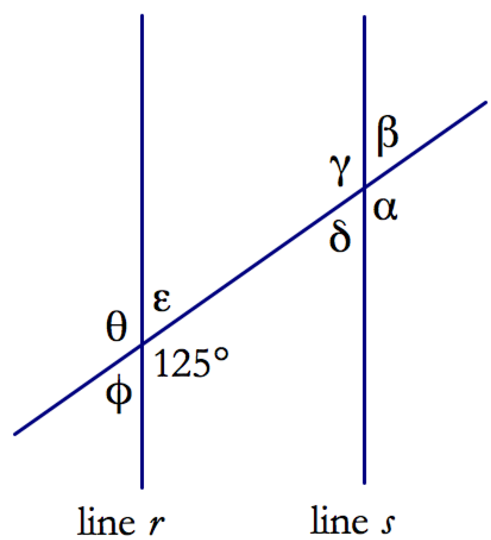
When studying similarity, it is important to determine when two angles in a figure have the same measurements. The following exercises will refresh or make clear some ideas regarding the angles formed when a line, called a **transversal**, intersects two parallel lines.

25 If line r is parallel to line s, written $r \parallel s$, determine the angle measurements of the remaining angles shown in each of these figures.

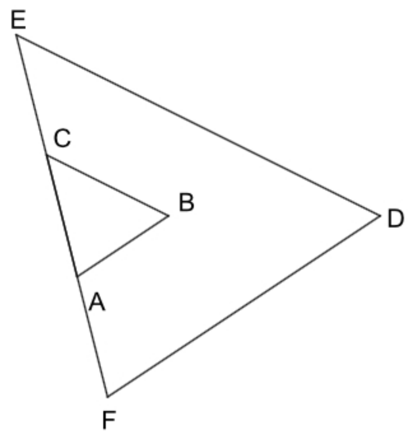
a.



b.



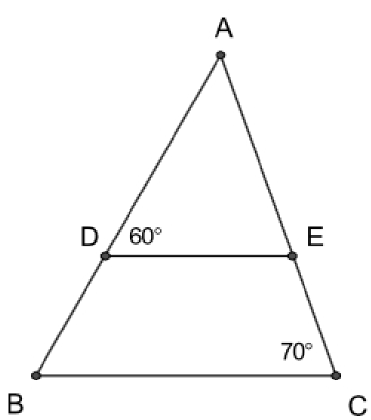
26 In the following figure, $\overline{AB} \parallel \overline{FD}$, $\overline{BC} \parallel \overline{DE}$, and points A and C lie on $\overline{EF}$.



Are these two triangles similar? If so, say why and express their similarity using the $\sim$ notation.

27 In the following figure $\overline{BC} \parallel \overline{DE}$.

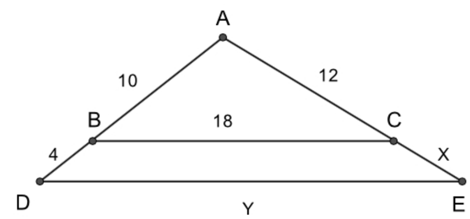
a. Determine all the angles in the triangles below.



b. Identify a pair of similar triangles in this figure. Using the $\sim$ notation, describe their relationship.

28 Given a triangle PQR, let point S be on $\overline{PQ}$ and point T on $\overline{PR}$ such that $\overline{ST} \parallel \overline{QR}$. Determine whether there is any relationship between the triangles PST and PQR. Explain your answer.

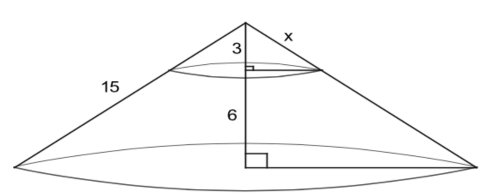
29 In the figure below $\overline{BC} \parallel \overline{DE}$. Find x and y.



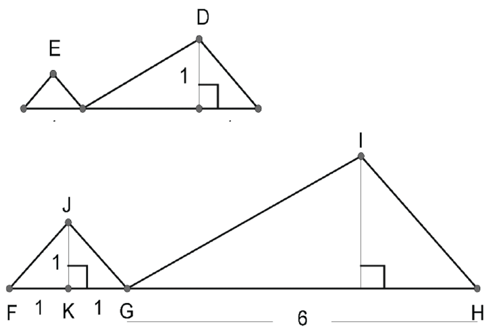
30 In a $\triangle ABC$, $BC = 9$ and point D lies on $\overline{AB}$ 4 units from A and 2 units from B. Furthermore, E lies on $\overline{AC}$ and $\overline{DE} \parallel \overline{BC}$. Find the length of $\overline{DE}$.

31 In a trapezoid, the lower base is 15, the upper base is 5, and each of the two legs is 6. How far must each of the legs be extended to form a triangle?

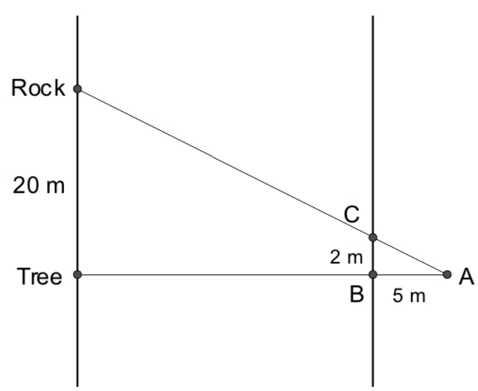
32 The height of the larger cone below is 9, and its slant height is 15. If the height of the smaller upper cone is 3, find its slant height x.



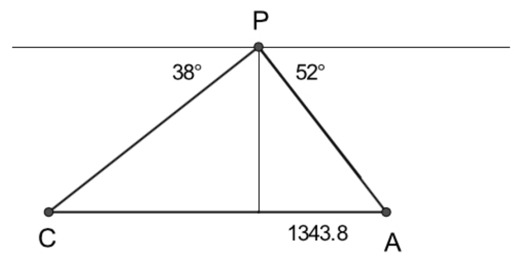
33 The shapes below are known to be scaled copies of each other. Calculate the missing side lengths for each of the two shapes.



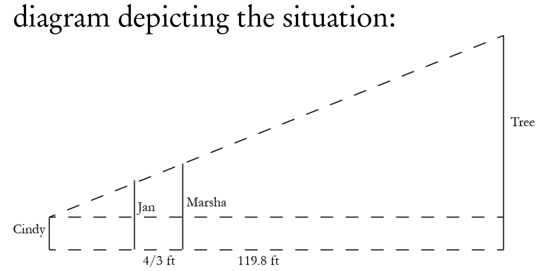
34 In order to estimate its width, Amy, Bruce, and Carmen stood at the vertices of a right triangle ABC on one side of a river, with Bruce standing at the right angle. Their distances from one another are indicated in the figure below. If it is known that the distance between the tree and the rock shown on the other side is 20 meters, what is the width of the river?



35 An airplane P is flying at an elevation of 1720 m, directly above a straight highway. Two cars, C and A, are moving on the highway on opposite sides of the plane, and the **angle of depression** from airplane to car C is 38° and to car A is 52°, as shown in the figure below (the dashed line is the imaginary horizontal line through P). If the second car is at a horizontal distance of 1343.8m to the right of the airplane, how far apart are the cars at this time?



36 Cindy, Jan, and Marsha have come up with an ingenious method for measuring the height of a tree. Marsha stands at a distance she measures to be 119.8 feet away from the tree. Jan walks 4/3 more feet away from the tree than Marsha. Cindy then finds a spot in which she can crouch down and see Jan's head, Marsha's head, and the top of the tree all in one continuous sightline. Cindy (when crouching) is 2.5 feet tall, Jan is 5 feet tall, and Marsha is 5.5 feet tall. Here is a diagram depicting the situation:

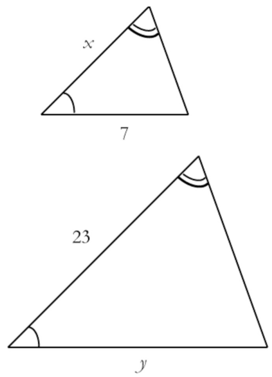


a. How far away from Jan did Cindy go in order to make this work?

b. How tall is the tree?

37

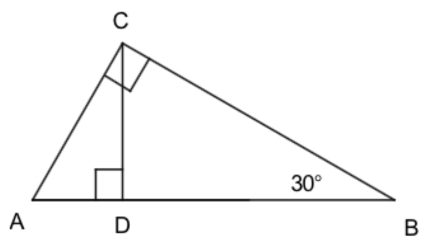
The two triangles below are similar but not congruent:



Can x and y both be integers, neither of them equal to 1? Why or why not?

38

In the figure below, right has been split into two smaller right triangles.

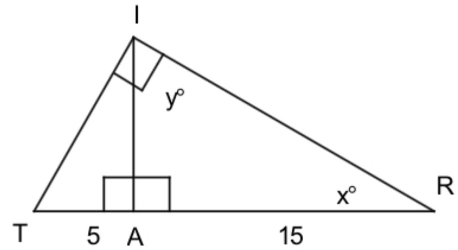


a. Find all the angles in the figure.

b. Name all pairs of similar triangles.

39

Regarding $\triangle IRT$ shown below, do the following.



a. Mark in another angle whose measure is x° and another whose measure is y°.

b. Name all pairs of similar triangles. Hint: Take things apart.

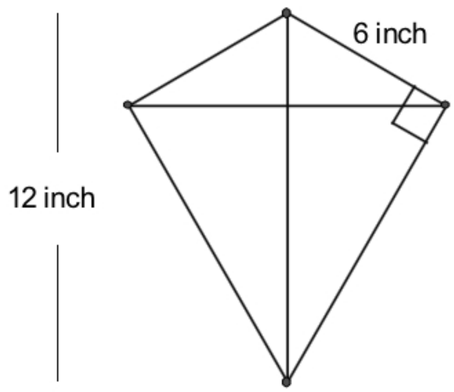
c. Calculate the length of $\overline{IA}$. Hint: Use similarity.

d. Calculate the length of $\overline{IR}$.

e. Calculate the length of $\overline{IT}$.

f. Check your answers by using Pythagoras' theorem on $\triangle TRI$.

40 Andrew wanted to cover the upper part of his kite with yellow paper and the bottom part with blue paper (see figure). The leftmost and rightmost angles of the kite are right angles. Taking into account that the diagonals of a kite are perpendicular, how much paper of each color does he need?



41 $\triangle ABC$ is an equilateral triangle of side length equal to 1, that is, each side of this triangle is one unit long. D, E, and F are the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. Determine the side lengths of $\triangle DEF$. Thoroughly explain your work.

42 Don't use a calculator for this problem.

a. Add: $3\frac{5}{8} + 7\frac{7}{8}$

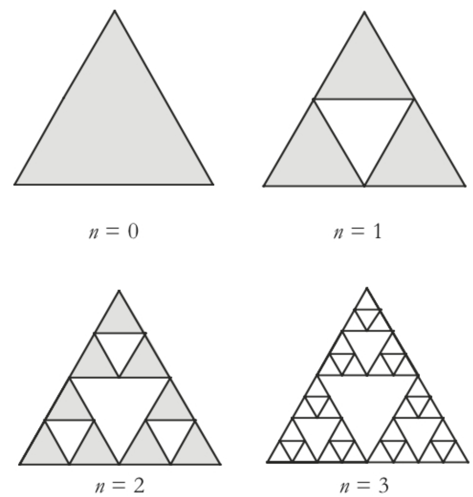
b. Factor: $4x^2 - 16$

c. Factor: $14x^2y^3 - 28x^4y^2 + 7x^2y$

d. Reduce: $\frac{2x+12}{2x+6}$

e. Are lines with slopes $\frac{42a}{-14b}$ and $\frac{7b}{21a}$ parallel, perpendicular, or neither?

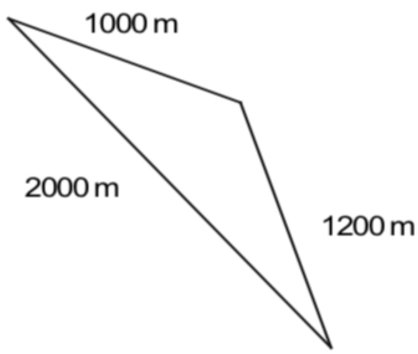
43 Consider an equilateral triangle with sides of length 1. This triangle is considered to be stage number 0 of the Sierpinski triangle. Then the central triangle obtained by joining the midpoints of each side is removed. This is considered to be stage number 1 of the Sierpinski triangle. Stage number 2 is obtained when from each of the three remaining triangles the central triangle is removed, as was done on the initial triangle when going from stage number 0 to stage number 1 (see figure below).



Appropriately shade the last triangle above to examine stage number 3. Then complete the table.

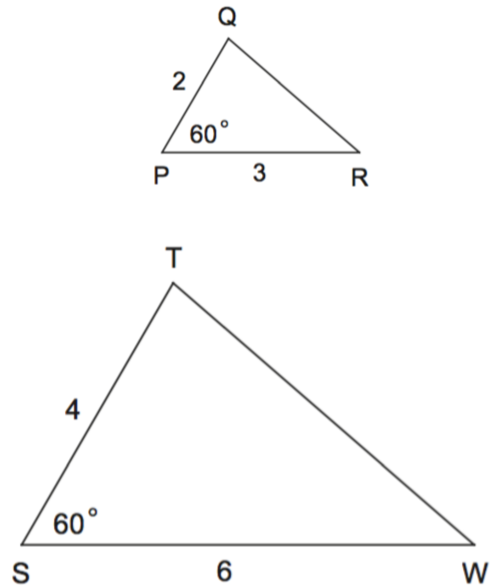
Stage number	0	1	2	3	n
Number of shaded triangles					
Total perimeter of shaded triangles					
Total area of shaded triangles					

44 Thinking of his will, Mr. Maury wants to leave a portion of his farm to each of his four children. This farm is a triangular piece of land whose dimensions are 1000, 1200, and 2000 meters. However, Mr. Maury wants to be as fair as possible and leave pieces of equal size and shape to his children. Do you think that it is actually possible to split Mr. Maury's farm into four pieces of equal size and shape? Thoroughly explain your answer.



45 Is the result you found in Problem 44 true for any triangle? That is, is it always possible to divide a triangle into four triangles of equal size and shape? Explain.

46 **Case SAS**



Regarding triangles PQR and STW above, we have the following:

$m\angle P = 60^\circ$, $PQ = 2$, $PR = 3$,
$m\angle S = 60^\circ$, $ST = 4$, and $SW = 6$.

Are these two triangles similar? How could you justify your answer?

47 For each statement below, draw a picture illustrating it. Then decide whether or not this statement is true. For each "Case" with which you disagree, give a counterexample showing that it is false.

AA Case
If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

SSS Case
If the three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are similar.

SAS Case
If the following conditions are met:

i) Two sides of one triangle are proportional to two sides of another triangle. That is, the lengths of two sides of one triangle are a constant times the lengths of two sides of the other triangle.

ii) The included angles are congruent. That is, the angle formed in one of the triangles by the two sides that are proportional to two sides of the other triangle is congruent to the corresponding angle in the other triangle.

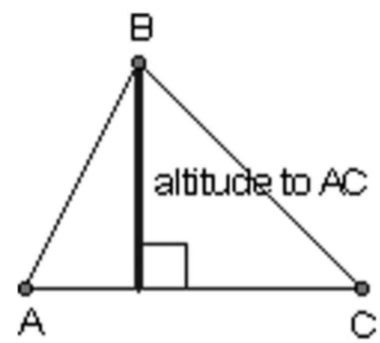
Then the two triangles are similar.

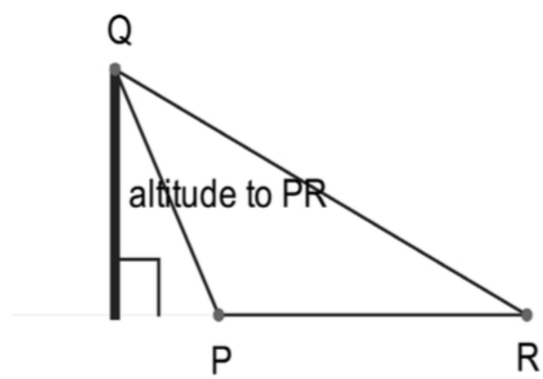
SSA Case
If two sides of one triangle are proportional to two sides of another triangle and the angle opposite one of those sides of the first triangle is congruent to the corresponding angle opposite one of those sides of the second triangle, then the two triangles are similar.

48 A certain crop of black-eyed peas has been genetically altered such that each pea is flat and perfectly circular, and its "eye" runs in a straight line through the center of the pea. The $PeaEye$ function takes any pea and divides its circumference by the length of its eye. If pea #1 is twice the size (total area) of pea #2, then which is bigger: $PeaEye\,(pea\#1)$ or $PeaEye\,(pea\#2)$?

Exploring in Depth

In a triangle, a segment drawn from a vertex of a triangle perpendicular to its opposite side, or to the extension of its opposite side, is called an **altitude** to this side of the triangle.



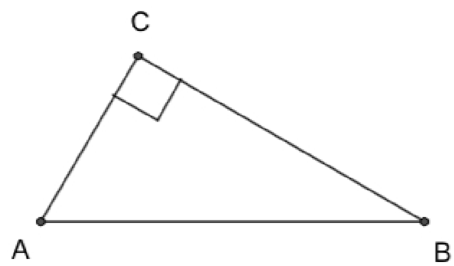


If in a right triangle the altitude to the hypotenuse is drawn, the hypotenuse is divided into two segments. The lengths of these segments and the length of the altitude to the hypotenuse are related. This relationship has already been revealed in Problem 39. In the following problems, we will try to make this relationship more explicit.

49 In the right $\triangle ABC$ below, draw the altitude $\overline{CD}$ to the hypotenuse. Then:

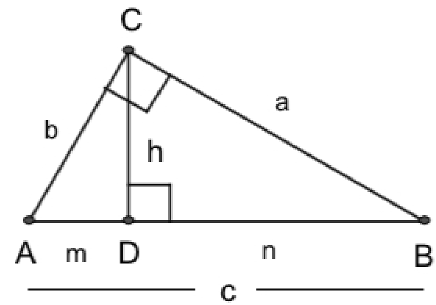
a. Mark in all the angles that turn out to be congruent.

b. Name all pairs of similar triangles. Justify your answer.



Very likely, you have been using the Pythagorean Theorem for quite some time now. However, have you ever created a formal proof of this theorem? You might not have known enough until now.

50 Regarding a right triangle as the one below, answer the following questions.



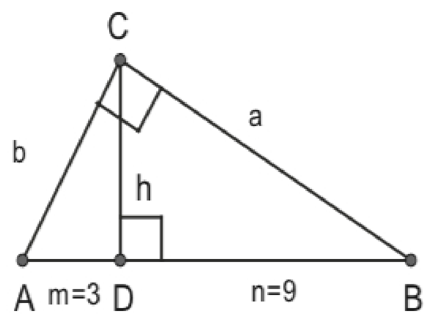
a. Write an equation relating, h, the length of the altitude to the hypotenuse, with m and n, the two lengths into which this altitude divides the hypontenuse. Hint: Consider triangles $\triangle ADC$ and $\triangle CBD$.

b. Write an equation relating the legs, the hypotenuse, and the length of the altitude to the hypotenuse. In other words, write an equation involving a, b, c, and h in the above triangle. Hint: Consider triangles $\triangle ABC$ and $\triangle ACD$

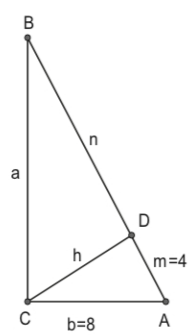
c. $\overline{AD}$ is called the **projection** of leg $\overline{AC}$ onto the hypotenuse $\overline{AB}$. Also, $\overline{DB}$ is the projection of leg $\overline{CB}$ onto the hypotenuse $\overline{AB}$. How is each leg related to the hypotenuse and its projection onto the hypotenuse? In other words, write an equation relating a, c, and n in the triangle shown above. Then write an equation, relating b, c, and m. Hint: Consider $\triangle ABC$ and $\triangle CBD$, and $\triangle ABC$ and $\triangle ACD$.

d. Use Part c above to prove the Pythagorean Theorem. That is, prove that $a^2 + b^2 = c^2$.

51 In the figure below, $\triangle ABC$ is a right triangle and $\overline{CD}$ is the altitude to the hypotenuse. As shown in the figure, the lengths of the projections of the legs b and a over the hypotenuse are, respectively, $m = 3$ and $n = 9$. Find h.



52 In the figure below, $\triangle ABC$ is a right triangle and $\overline{CD}$ is the altitude to the hypotenuse. As shown in the figure, the lengths of the projections of the legs b and a onto the hypotenuse are, respectively, $m = 4$ and n. As usual, c is the length of the hypotenuse $\overline{AB}$.

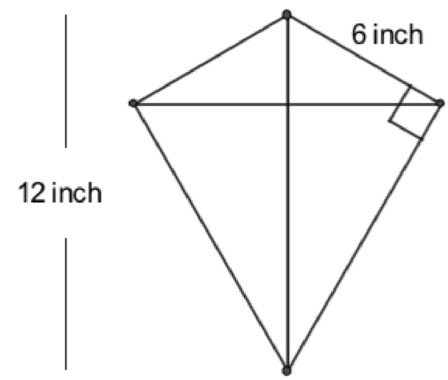


a. Find AB.

b. Find h.

53 Let us recall Andrew's kite problem (#40).

Can you use the ideas developed in the problems of this section to find out how much paper of each color Andrew needs to cover his kite?



LESSON 2: INTRO TO TRIGONOMETRY

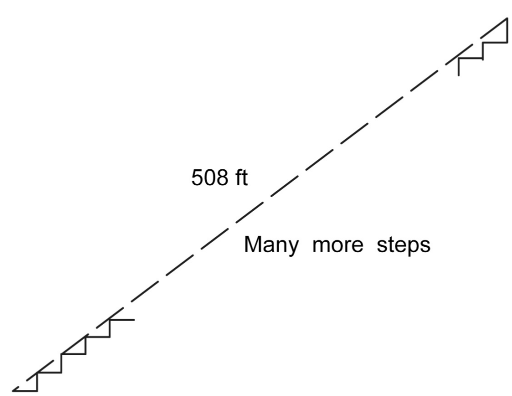
Introduction

The escalator at the Wheaton metro station in Silver Spring, Maryland, is the longest escalator in the Western Hemisphere. If you ride it you will travel a diagonal distance of 508 feet, which will take two minutes and forty-five seconds. You're about to build a bigger one – your goal is to make an escalator with a diagonal distance of 520 feet. Before you can start, though, you need to do some planning. You need to figure out how much space along the ground your escalator will take so that you can prepare the foundation. Standard escalator construction rules require that:

- the steps each be 1.2 feet long

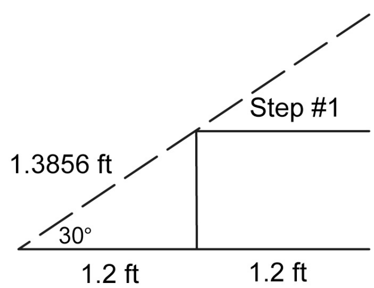
.

- The measure of the angle between the ground and the diagonal path of the escalator must be no more than 30°



The Wheaton Escalator

1 You start to build your escalator and measure a diagonal distance of 1.3856 feet per step. What is the horizontal distance, along the ground, you'll need to clear in order to build your escalator with a diagonal distance of 520 feet?



Your Escalator

Development

Your solution to the escalator problem probably involved ratios or scale factors in some way. Here's one way of thinking about the problem:

2 If you haven't already, explain how you know that $\frac{1.2}{1.3856} = \frac{\text{the required horizontal distance}}{520}$.

3 Use a similar method to find the height of the Wheaton escalator.

4 Imagine a large right triangle ABC with the right angle at A and angle $C = 30°$. Based on your work in problems 1-3:

a. What would you expect the ratio of AC to BC to be? Why?

b. How about the ratio of AB to BC?

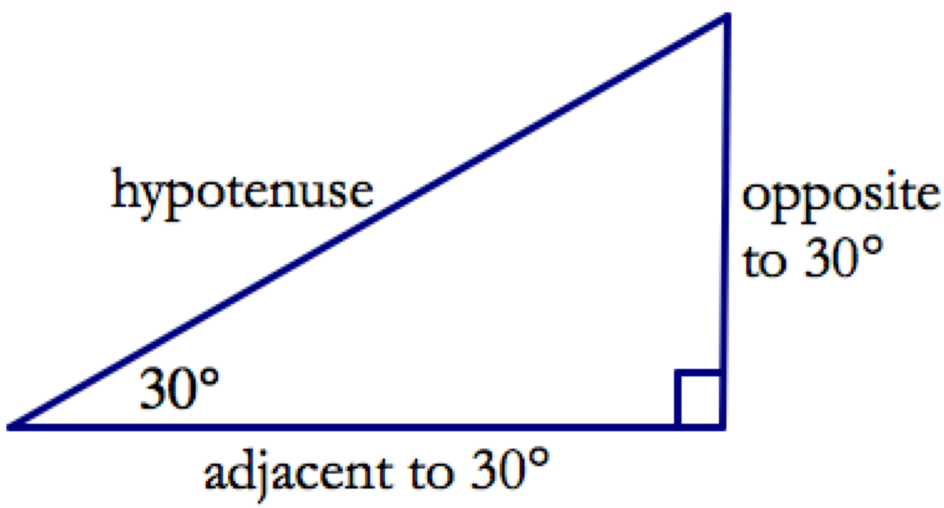
c. Go ahead and check your thinking by using a protractor and ruler to accurately draw such a triangle, and measure the sides. Convince yourself that these ratios are the same no matter how large or small the right triangle is, provided the measure of angle C remains 30°.

As problem 4 emphasizes, certain ratios will always stay constant in a right triangle with an angle measuring 30°. In the escalator example, the ratio of the vertical distance to the diagonal distance is called the **sine** of 30°. The ratio of the horizontal distance to the diagonal distance is called the **cosine** of 30°.

A third ratio is called the **tangent** ratio. In the escalator example, this is the ratio of the vertical distance to the horizontal distance.

5 If you asked your calculator to tell you the sine of 30° what do you suppose it would give you? How about the cosine of 30°? Tangent of 30°?

If you think of the escalator as a triangle with a 30° angle and a 90° angle, you can label the sides as follows:



There are two sides that form the 30° angle. One is of course the **hypotenuse**; the other is called the side **adjacent** to the 30° angle. The vertical side of the triangle is called the side **opposite** to the 30° angle, since it is all the way across the triangle from the angle.

6 For each of the following, use a protractor to draw a RIGHT triangle with the angle given. Then use a ruler to measure the lengths and calculate the sine, cosine, and tangent of each angle.

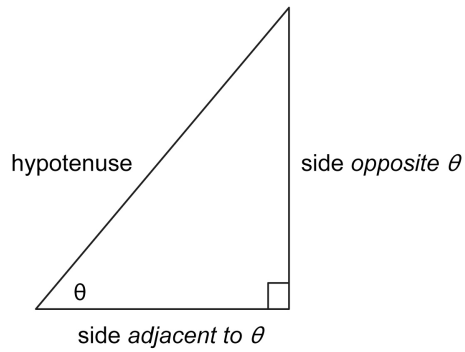
a. 25°

b. 63°

c. 42°

7 For each part in the previous problem, use the SIN, COS, and TAN buttons on your calculator to see how close your estimates were.

Here's the picture in general. We're using θ ("theta") to stand for any angle, just like we can use x to stand for any number.



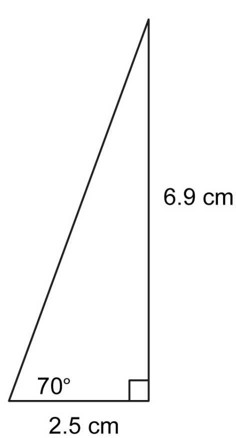
If θ is an angle in a right triangle, but not the right angle, then here's how we define the three trigonometric ratios:

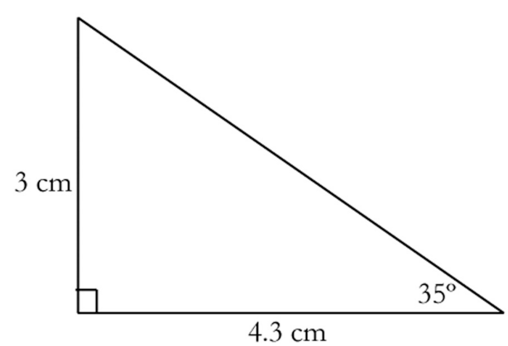
$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

8 Using the two triangles below, find the sine, cosine, and tangent of a 70° angle, a 20° angle, a 35° angle, and a 55° angle.





9 Did you notice any repeated answers in your calculations above?

a. Make a conjecture based on your findings.

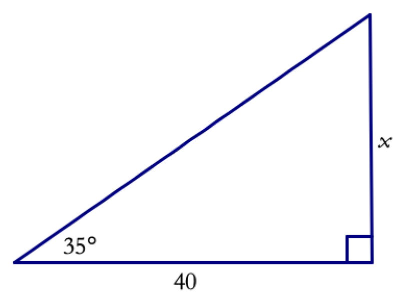
b. Try out your conjecture on a triangle that you create.

c. Can you explain why this is going on?

In the similar triangles lesson, and also in the escalator problem, you were able to find missing lengths in certain triangles because you had a second, similar triangle that allowed you to find the scale factor. One reason that sine, cosine, and tangent ratios are so important is that they help you find lengths in triangles even when you don't have a second triangle to help you get the scale factor.

10 Here is a problem that allows you to "solve a triangle" using only trigonometry.

a. Use this triangle to write an expression for the tangent of 35 degrees. Your expression will, of course, contain the letter x.



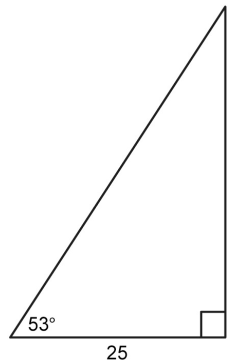
b. Using the equation "$\tan 35^\circ = \ldots$", find x. (You should know the value of $\tan 35^\circ$ from a previous problem)

c. Find the length of the hypotenuse of the triangle.

In the previous problem, you were lucky that you already knew the tangent of 35°. You are not always so lucky. However, your calculator can help you out.

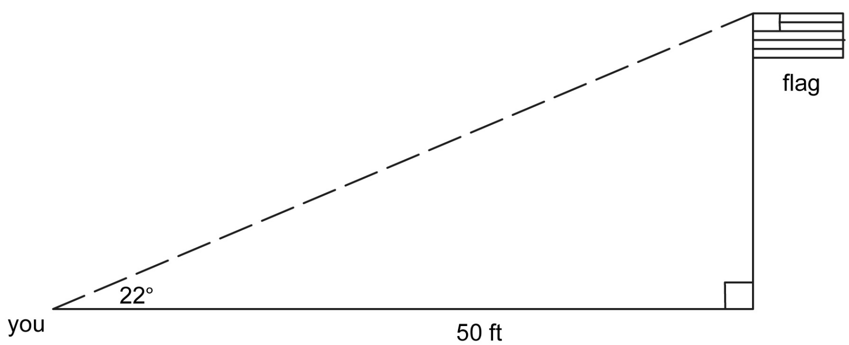
11 Press the MODE button on your calculator and make sure "degree" is selected. Then type "tan(35)". The number on your screen should look familiar.

12 Find both of the missing sides in the following triangle:



Often, trigonometry is used to find lengths of triangles in the "real world" that are too difficult to measure directly.

13 You'd like to know how tall a flagpole is, but you can't reach the top to measure it. So instead you use your tape measure to measure 50 ft along the ground. Standing 50 ft away from the flagpole, you can see the top of the flagpole by looking up at a 22° angle. How tall is the flagpole?

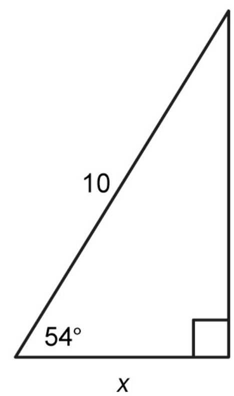


14 In the previous problem, you might have ignored your height. Say your eyes are at a level of 5 feet 6 inches above the ground. Taking this into account, figure out how tall the flagpole *really* is.

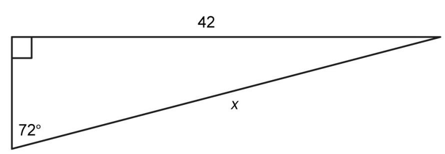
Practice

15 Find the length x in each triangle.

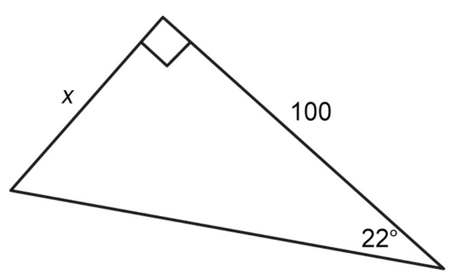
a.



b.



c.



16 Stephen can see the head of the Loch Ness Monster if he looks up at an angle of 35°. He's standing on the level floor of a shallow lake, and he's six feet tall. If it is 20 yards along the ground from where he's standing to where the Loch Ness Monster is standing, how tall is "Nessie"?

17 Think about the location of Stephen's eyes. Is your answer to the previous question actually a little bit too big, or a little bit too small?

18 A 20-ft ladder is placed against a wall at an angle of 72 degrees with the ground.

a. How far from the base of the wall is the ladder?

b. How high does the ladder reach on the wall?

Problems

19 Your fifth-story apartment window looks out over a courtyard 50 feet long. To see your friend in the eighth-story window across the courtyard, you need to tilt your head up at a 31° angle. How far apart are the floors positioned in your apartment building? (Assume they're spaced evenly).

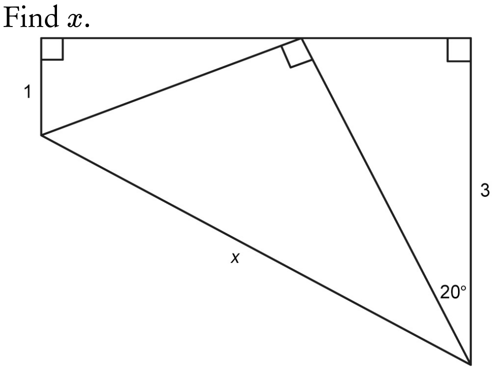
20 Say that, for some angle θ, $\sin\theta = \frac{5}{8}$. Mark's triangle has a 90° angle and an angle of θ. The side opposite θ is 45 cm long. How long is the hypotenuse of Mark's triangle?

21 The sine of 25 degrees is about .4226. What angle has a cosine of about .4226?

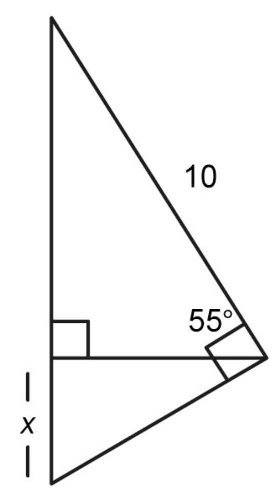
22 John looks at a building whose base is 100 ft. away from where he is standing. He sees a friend leaning out of a window at an angle of elevation of 30 degrees. His friend yells to him that the people in the building are part of a "flash mob" (see http://en.wikipedia.org/wiki/Flash_mob), and that a mutual friend will be appearing at a window that is twice as high up. So John looks at the window at an angle of elevation of 60 degrees, but their mutual friend never appears. Why?

In problems 23-27, **geometric tinkering** will prove very helpful.

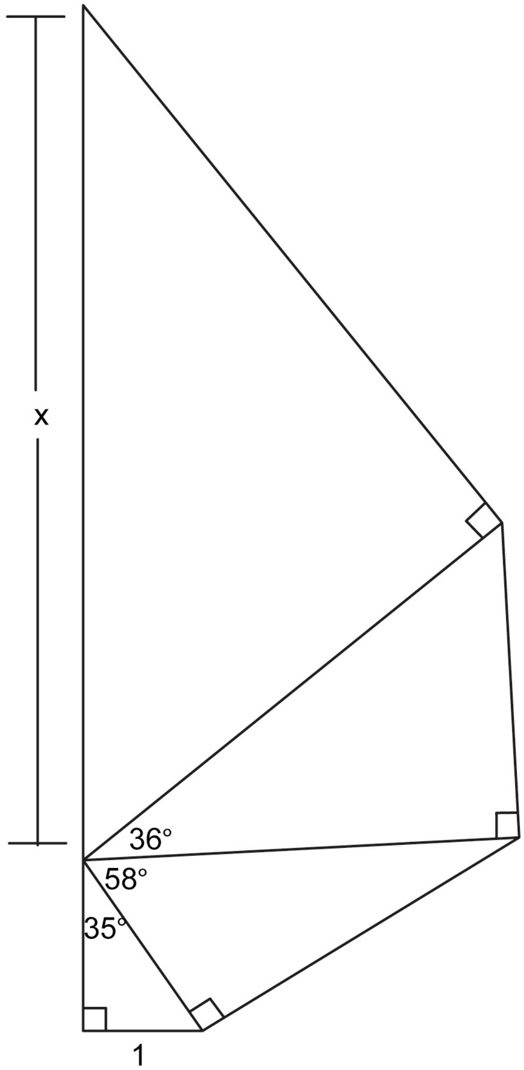
23 Find x.



24 Find x.

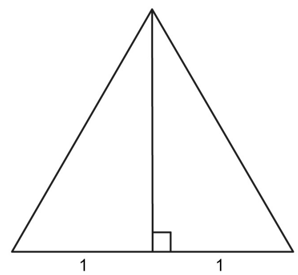


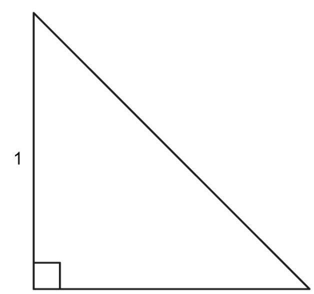
25 Find x.



Your calculator gives you approximate answers when you ask it to tell you the sine, cosine, or tangent of an angle. (See http://www.mathsisfun.com/irrational-numbers.html for more information and a juicy, though probably apocryphal, tidbit.) Sometimes you would rather know the answer exactly. Some geometry can help you do this.

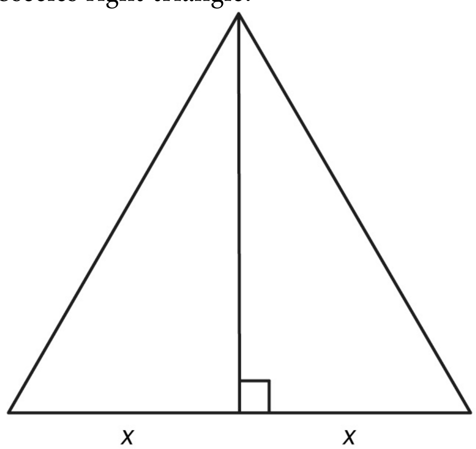
Below are two triangles. The first one is an equilateral triangle. The second is an isosceles right triangle.

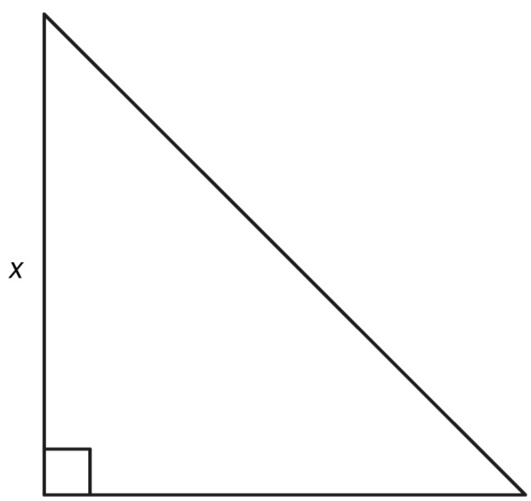




26 Find all the sides and angles in the figures on the previous page, including the height of the equilateral triangle. Do not use a calculator, and do not convert your answers to decimals.

Here are two more triangles. Again, the first is equilateral and the second is an isosceles right triangle.





27 Now find all the sides and angles. You'll have to give some of your answers in terms of x.

Your answer to the previous exercise turns out to be tremendously useful. For instance, let's say that you have a triangle that you know to have angles of 30, 60, and 90 degrees. We'll call this a **30-60-90 triangle**. Normally, you'd need to know two sides of a right triangle in order to figure out the third side of a triangle. But the result in the previous exercise can help you find *both* missing sides.

28 For example, say you have a 30-60-90 triangle with its shortest side of length 3. Find the lengths of the other two sides.

29 Suppose you have a 30-60-90 triangle and the hypotenuse has length 5. What are the lengths of the other two sides?

30 Find the lengths of the legs in a 45-45-90 triangle if the hypotenuse has length $7\sqrt{2}$.

31 If you haven't already, make a good sketch in your notebook of a 30-60-90 triangle with shortest side x, and label its other sides in terms of x. Then do the same thing for a 45-45-90 triangle.

32 Copy the following chart into your notebook and use the diagrams you drew in problem 31 to help find exact values for the sine, cosine, and tangent of the following angles.

θ	Sine of θ	Cosine of θ	Tangent of θ
30°			
45°			
60°			

33 Sally spies Bigfoot, who is a bit too close for comfort. With her trusty protractor in hand, she notices that, if she looks up at a 60-degree angle, she can just see the bald spot on the top of his head. When Bigfoot sees her protractor, he shies and runs away. Sally then measures along the ground to the spot where he was standing, and finds that he was 10 meters away. Five-foot-tall Sally can calculate *exactly* how tall Bigfoot is. Can you?

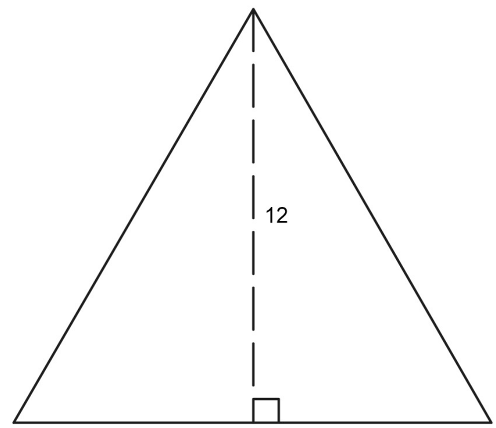
34 Now that Sally knows how tall Bigfoot is, she decides to spy on him from a distance. If she can now see the top of his head by looking up at a 30° angle, exactly how far away is she from Bigfoot?

35 Kenya computes the sine of 45 degrees from her isosceles right triangle and gets $\frac{1}{\sqrt{2}}$, while Victoria computes the sine of 45 degrees from her isosceles right triangle and gets instead $\frac{\sqrt{2}}{2}$. Are these answers actually the same answer, or not? Explain.

36 Are there other right triangles besides 30°-60°-90° triangles for which one of the sides is half the length of the hypotenuse?

37 Recall that an **altitude** of a triangle is a line segment drawn from the vertex of a triangle at a right angle to the line containing the opposite *side*.

The three altitudes of an equilateral triangle all have length 12 cm. How long are its sides?



38 An isosceles triangle has two 65° angles. The altitudes drawn from those angles are 12 cm. What are the lengths of the three sides of the triangle? Answer to the nearest tenth of a millimeter.

39 Suppose you have a rectangle with dimensions a and b. The Oblongness function is defined by:
$\text{Oblongness}(a, b) = \frac{\max(a,b)}{\min(a,b)}$.

a. Draw rectangles with the following oblongnesses: 3, 10, 1.618.

b. For which pairs of values of a and b is $\text{Oblongness}(a, b)$ as small as possible?

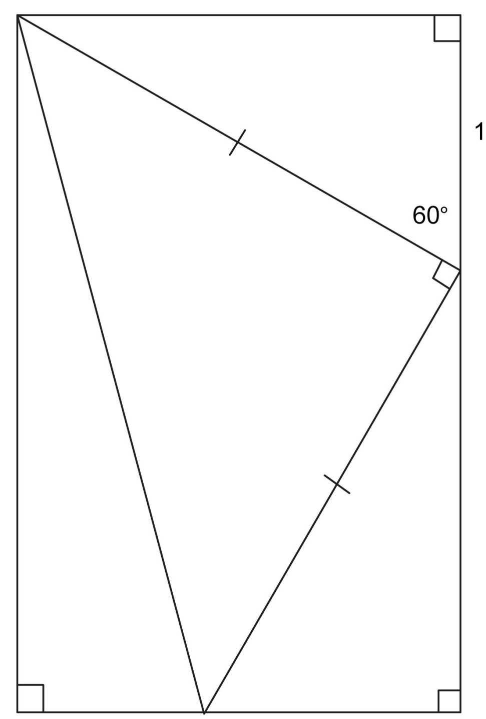
c. Wouldn't it be easier to just define $\text{Oblongness}(a, b)$ simply as $\frac{a}{b}$? Why is the given definition better?

d. Cody has a rectangle, in which he has drawn one diagonal. He finds that the angle between the diagonal and the long side of the rectangle is 12 degrees. What is the oblongness of Cody's rectangle?

40 This diagram shows a rectangle that has been formed by bordering an isosceles right triangle with three other right triangles, one of which has a 60-degree angle as shown.

a. Given that the length marked is 1 unit long, find as many lengths and angles in the picture as you can. (Give answers in exact form.)

b. Use this diagram to find the exact value of the sine of a 75-degree angle.



41 Susan and Emily are both standing on an east-west sidewalk, looking at the top of a 20-foot building. Susan is standing due south of the building, and has to look up at an angle of 50° to see it. Emily, who is standing to Susan's right, has to look up at an angle of 35° to see the building. How far apart are Susan and Emily?

42

Don't use a calculator for this problem.

a. Simplify: $-(7x)(x-5)$

b. Simplify: $3(x-1)-2(x-4)$

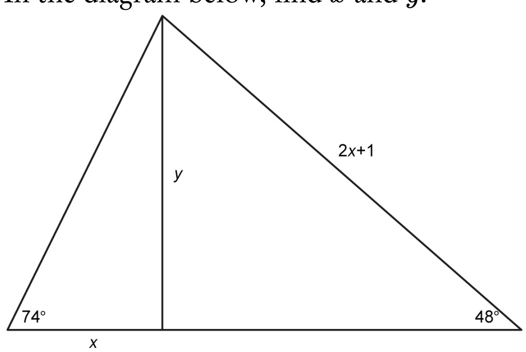
c. Solve the inequality $|x-5|>7$.

d. Factor: $x^2-9x+20$

e. Solve for x: $\frac{2}{x}-5=4$

43

In the diagram below, find x and y.



44

The following questions are related to a famous puzzle called "Buffon's needle," which you may want to look up if you are interested.

a. You're throwing a tiddlywink with a 4 cm radius onto a floor with square tiles measuring 12 cm on a side. What's the probability that the tiddlywink will make it onto the floor without landing on a crack?

b. Now you're playing a different version of the game, where you throw a needle between evenly spaced stripes on the floor. Your needle is 2 inches long, and the stripes are also 2 inches apart. Assuming that the needle will land at an angle of 72 degrees to the stripes, what's the probability that it will land without overlapping any of the stripes?

45 Here are some questions about an "extreme value" of the sine.

a. Ask your calculator to tell you the sine of an 80° angle. Then ask for the sine of an 89° angle. Does this suggest a value for the sine of a 90° angle? Have your calculator confirm your guess.

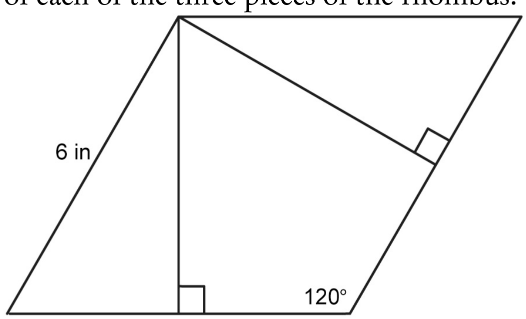
b. Using a protractor, draw a right triangle with an 80° angle. Then measure the hypotenuse and the side opposite the 80° angle. How do they compare? How do you think those two sides would compare in a right triangle with an 89° angle?

c. Have a conversation with a classmate (or friend, guardian, spiritual advisor) about how someone might have decided what the sine of a 90° angle should be. Remember that our definition of sine as $\frac{\text{opposite}}{\text{hypotenuse}}$ in a right triangle won't quite work.

Exploring in Depth

(Problems 46-49 copyright Phillips Exeter Academy.)

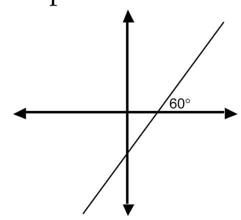
46 A rhombus has four 6-inch sides and two 120° angles. Two altitudes are drawn in from one of the 120° angles. Find the area of each of the three pieces of the rhombus.



47 A right triangle has a 24 cm perimeter, and its hypotenuse is twice as long as its shorter leg. Find the lengths of all three sides of this triangle.

48 A triangle has a 60-degree angle and a 45-degree angle, and the side opposite the 45-degree angle is 240 mm long. To the nearest mm, how long is the side opposite the 60-degree angle?

49 A line of positive slope is drawn so that it makes a 60 degree angle with the x-axis. What is the slope of this line?



50 Play the "Buffon's needle" game: take a needle and draw parallel lines on a sheet of paper, spaced the needle's length apart. Toss the needle so that it lands on the paper, and record whether the needle crosses a line or not. Pool your data with others who have played the game, and use the data to compute the probability that a needle thrown on the paper will cross a line. Once you have the probability, try dividing by two and then taking the reciprocal of your answer. This process should result in something close to a number with which you're familiar.

51 Pick an angle, any angle. Use your calculator to find its sine and cosine. Now square each of these numbers and add them together. Try it again with another angle. Notice anything? Write down your conjecture in as concise a form as you can.

52 Let's try to prove your conjecture in Problem 51. Draw a right triangle ABC, with the right angle at C, and use a, b and c to represent the lengths of the sides opposite angles A, B, and C respectively. Write ratios for sin A and cos A, and with the help of the Pythagorean Theorem, prove your conjecture.

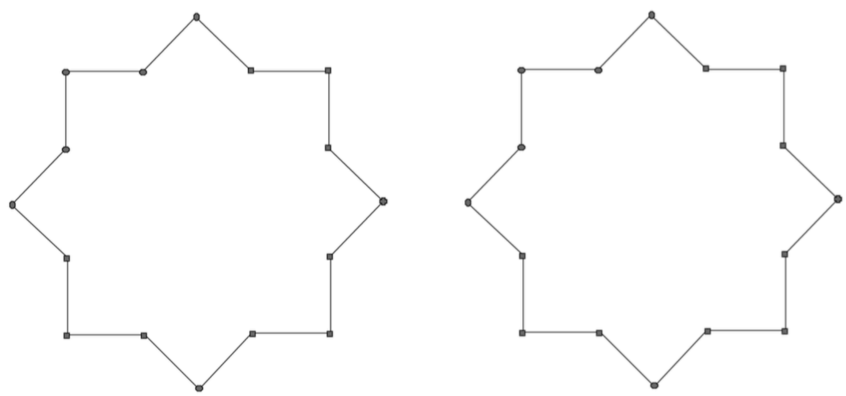
LESSON 3: TRIG AND SHAPES

Introduction

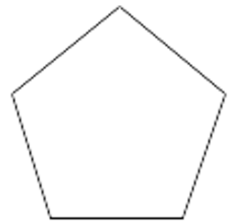
So far, the trigonometry that you've learned applies specifically to right triangles. Even with shapes you see around you that don't look like right triangles, with a little creativity you can still analyze these shapes using right triangles.

An "octastar" is a perfectly regular eight-pointed star.

1 Find two different ways to split up the octastar into right triangles.

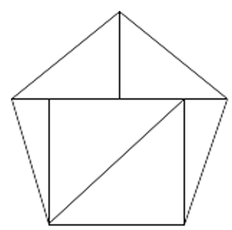


2 Here's a regular pentagon with each side measuring 2 inches.



a. Find the measure of each angle of the pentagon by first drawing a circle around the pentagon. (You might want to use the center of the circle.)

b. Let's split this pentagon into six right triangles. (You can assume that the lines that look vertical and horizontal are vertical and horizontal.) Fill in all the angles that you can figure out in the new diagram.



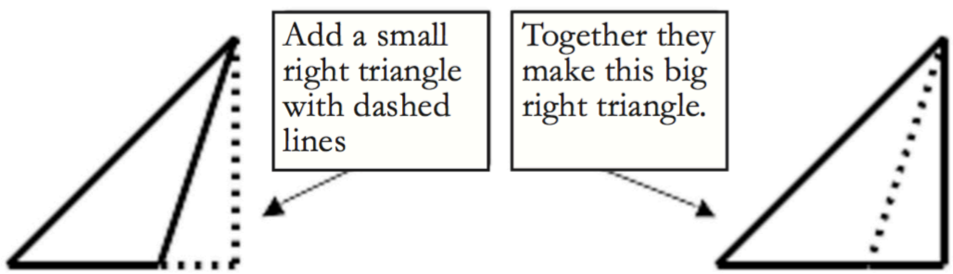
c. Now that you have all the angles, what more do you need to find in order to calculate the area of the pentagon? (Saying what you need to find is enough... you don't need to actually calculate the area.)

Development

In the diagrams so far, every drawing has been split up by turning it into a collection of right triangles pieced together – in other words, you add up the triangles to get the picture you want. Here's a slightly different way to use that strategy. Suppose you're investigating the following triangle:

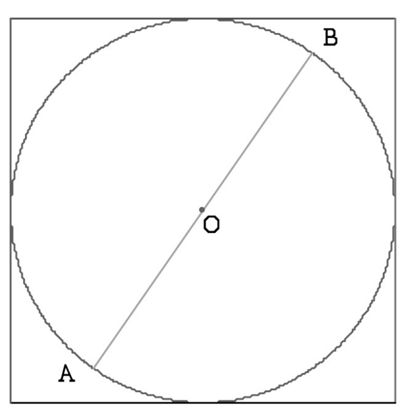
You could split it into two right triangles by adding the following line:

But suppose for some reason you didn't want to split up the long side of the triangle. Instead, you might notice that if you add a small right triangle on to your diagram, you get a big right triangle:

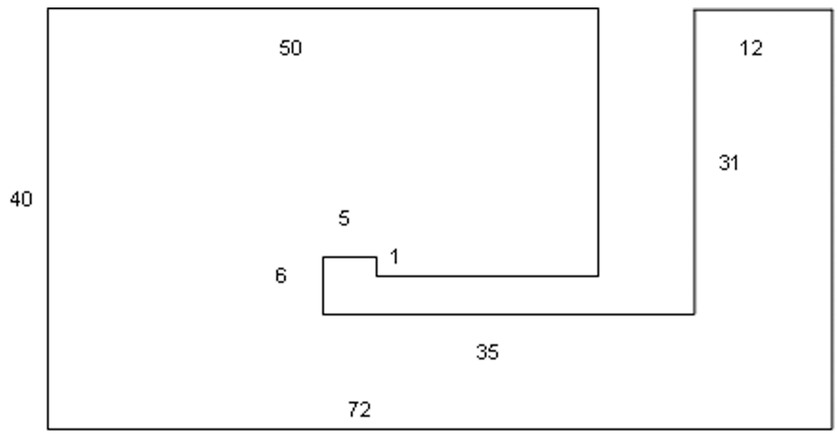


One strategy to think about when you **visualize** a problem is that you can alter the diagram to see it in a new way. Finding a new way to see a picture, or figuring out an interesting way to split a diagram into pieces, can give you insight into the problem.

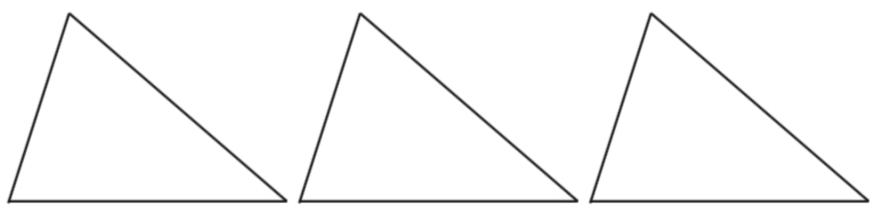
3 In the diagram below a circle is fitted exactly into a square. The center of the circle is O and the length of diameter AB is 3.5 cm. What is the area of the square?



4 Find the area of this shape:



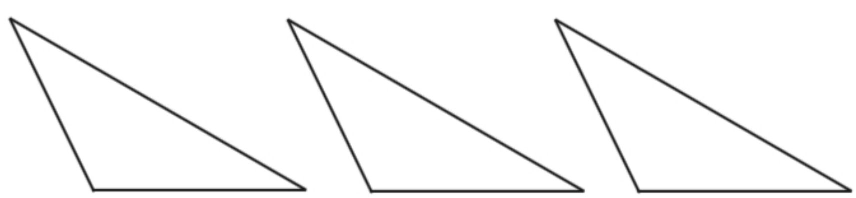
5 Split each of these three identical triangles into two right triangles, each in a different way.



6 The line segments you used to split the triangles in problem 5 are called **altitudes**.

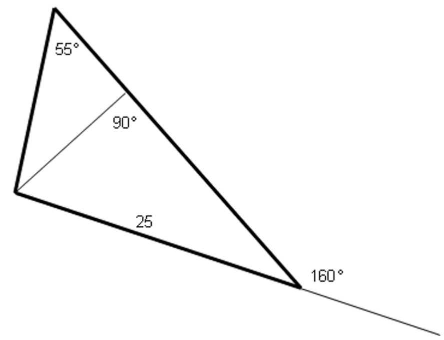
a. Why do you think they are so named? Write a clear definition of an altitude.

b. Draw altitudes for the three identical triangles below, each in a different way.

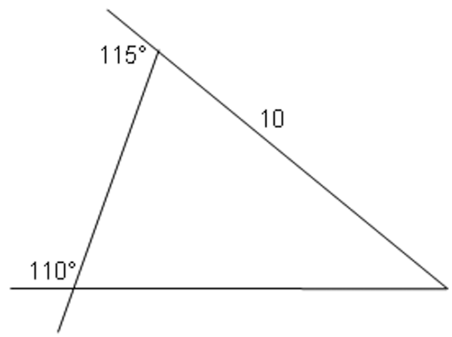


For the rest of this lesson, your numerical answers should either be exact or have 2-decimal place accuracy, unless you are asked to estimate an answer.

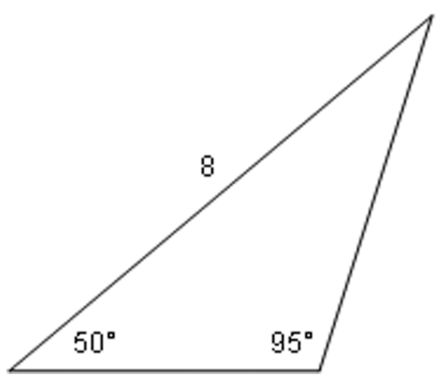
7 Use trigonometry to find the lengths of all the sides of the bold triangle.



8 Use trigonometry to find the lengths of all the sides of the triangle.

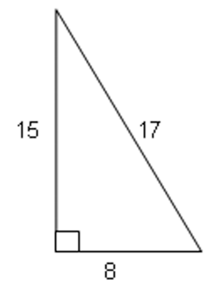


9 Use trigonometry to find the lengths of all the sides of the triangle.



Inverse Trigonometry

So far, given the right information, you have been able to find the length of sides of a triangle, and might well wonder if there might be a way to find the angles when the appropriate information has been given. For example, could you find the angle at the top of the right triangle below? You could take a guess, check your result, and refine your guess.



10 Make a guess. Think about how you could test your guess for accuracy, given what you know about sines (or cosines). Test it and adjust your guess until you get the correct answer to the nearest degree.

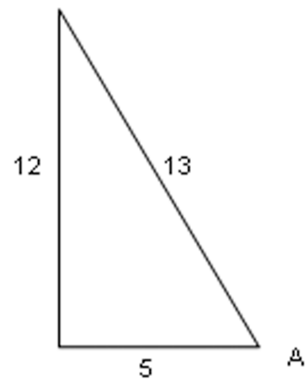
There are ways to figure out the trigonometry of every angle exactly, but when you type "sin 67", for example, into your calculator, it is basically looking the answer up on a table: sin 67° is about .9205.

Now suppose you know that an angle has a sine of .9205, but you don't remember the degree measure of the angle. You can try to figure out the angle, as you did in problem 10 above, using reasoning. Or, you can use your calculator to look at the sine table, backwards – this is like asking "The sine of what angle would give me .9205?"

To do this, first make sure your calculator is set in degree mode. Then, type "$\sin^{-1}$ .9205" into your calculator. This asks your calculator to give you an angle whose sine is .9205. You will get an answer of about 66.999°. (Notice that the answer we got is a tiny bit different from the angle we started with – this is because .9205 is rounded.) Your calculator has "$\cos^{-1}$" and "$\tan^{-1}$" buttons as well.

Practice

11

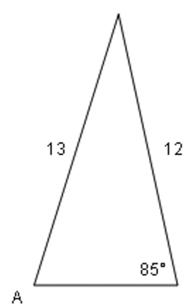


a. In this right triangle, what is the sine of angle A? Use your calculator (inverse trig) to find A.

b. What is the tangent of angle A? Use inverse trig with this number to find angle A. (Your answer should be the same as that of part a!)

12

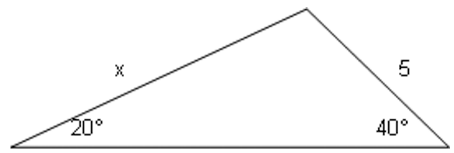
Find the measure of angle A in the picture below.



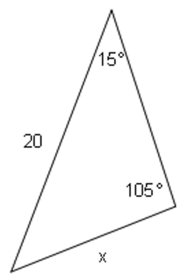
13

Find x in each problem.

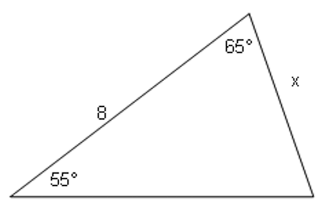
a.



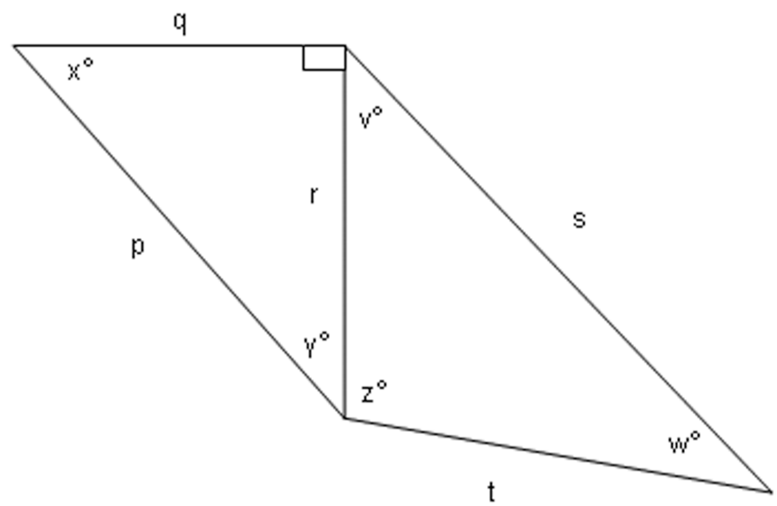
b.



c.



14 For the picture below, decide if each statement is true or false. If you think a statement is false, say why.



a. $p^2 + q^2 = r^2$

b. $\sin w^{\circ} = \frac{r}{s}$

c. $\sin y^{\circ} = \frac{r}{p}$

d. $\tan x^{\circ} = \frac{r}{p}$

e. $r^2 + t^2 = s^2$

15 Solve for x in each equation.

a. $\sin x = .57$

b. $1.3 = \tan x$

c. $6 \cos x = 5$

d. $\frac{\sin x}{6} = .12$

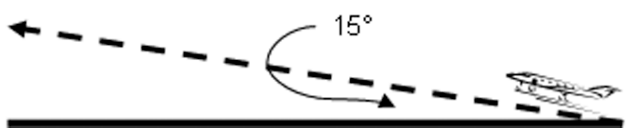
e. $16 = \frac{10}{\tan x}$

Problems

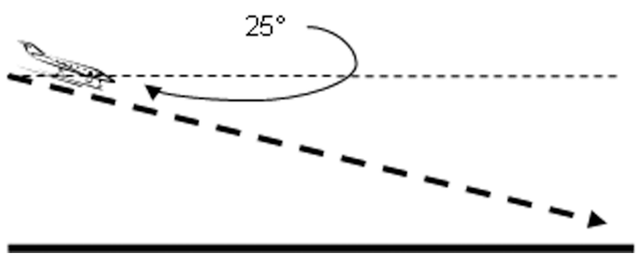
Angles of Descent and Elevation

In real-world problems, angles are often measured relative to the ground, or to the horizontal.

In this diagram, the plane is taking off at an **angle of elevation** of 15°.

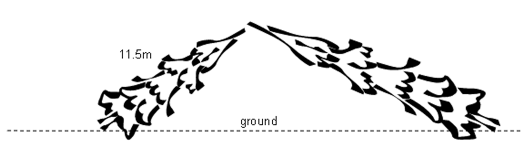


In this diagram, the plane is landing with an **angle of descent** of 25°. Another name for this angle is the **angle of depression**.



16 The descending plane in the picture above is 100m above the ground. The angle of descent is 25° as shown. How far in the air does the plane fly from where it starts in the picture to where it lands?

17 Two trees are leaning on each other. The tree on the left is 11.5m long, at an angle of elevation of 26°. The tree on the right has an angle of elevation of 11°. Find the length of the second tree.



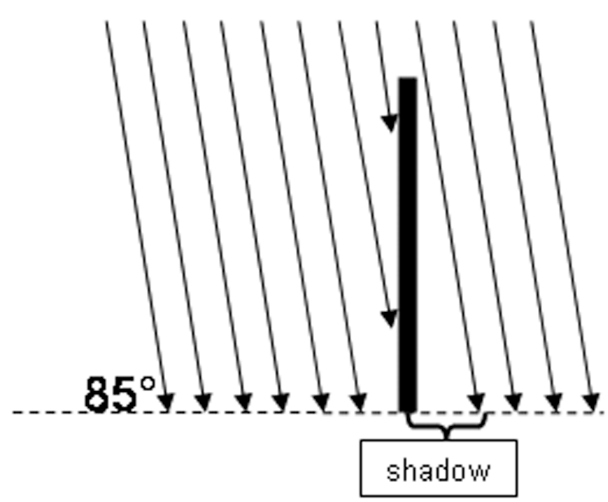
18 A bird is sitting on top of a pole, looking at a bug on the ground. The bug is 8 ft from the base of the pole, and the bird's line of sight down to the bug has an angle of depression of 55°. Draw a diagram to represent the situation, and determine the height of the pole.

19 A plane takes off from Chicago O'Hare Airport, flying up at an angle of elevation of 2°, for 410 miles. Then it tilts down and starts to fly directly towards BWI Airport, at an angle of descent of 3°.

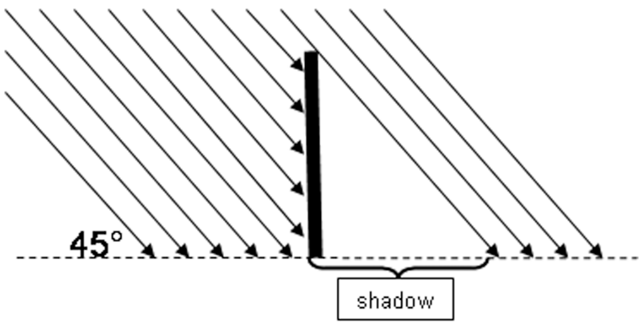
a. Draw a picture. How far does the plane fly on the downward part of its path?

b. How far is BWI from O'Hare (along the ground)?

In problems 20 through 22, let's define **angle of the sun** to mean the angle between the sun's light rays and the ground. (Note that since the sun is extremely far away, all the light rays are essentially parallel). So, around noon, the angle of the sun is close to 90° and you have a very short shadow:



Late in the afternoon, on the other hand, the angle of the sun is much smaller, and you have a longer shadow:



20 Just before noon, the angle of the sun is 89°.

a. How long is the shadow of a 40 ft tall house?

b. How tall is a building that has a 25 ft shadow?

21 At 5pm, a 100-foot building has a 40 ft shadow.

a. Estimate, without using your calculator, the angle of the sun.

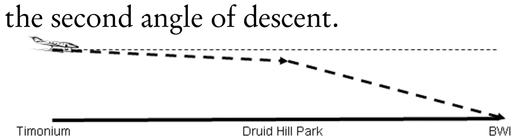
b. Now calculate the angle of the sun.

22 What's the angle of the sun if everyone has a shadow that's 5 times their height?

23 Give the lengths of the 3 sides of a right triangle, where exactly two of the sides are irrational and one of the angles has a sine of $\frac{1}{2}$.

24 A dartboard is shaped like a regular hexagon with 10-inch sides. Its vertices are (in this order) A,B,C,D,E,F. You get a point if you hit the target region, rectangle ACDF. What are your chances of getting a point?

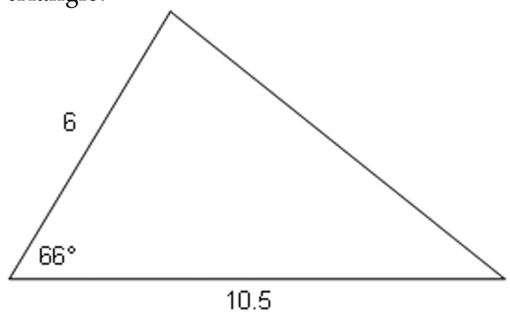
25 A plane is landing at BWI airport. You start observing the plane when it's directly above Timonium, which is 15 miles away from the airport. At this time the plane is .55 miles above the ground. It's flying at an angle of descent of 1°, until it's above Druid Hill Park, which is halfway between Timonium and BWI. Then the plane changes its angle of descent so that it's heading directly towards the airport. Find the second angle of descent.



26 You are kicking a ball off the ground at a wall. You win a point if the ball hits the wall between 4 and 5 feet off the ground. If you're standing 10 feet from the wall, what's the range of angles you could kick the ball at, to get the point?

27 You fold a piece of wire in half, and stand it up on the table on its two ends. The angle of the fold is 48°, and the two ends are 6 inches apart. How long was the wire? (Draw a picture).

28 Find the missing side length in this triangle.



29 Your house is 10 miles west of Baltimore, and your friend's house is 11 miles north-west of Baltimore.

a. Estimate, without using your calculator, the distance between the two houses. (Hint: what's the angle between west and north-west?)

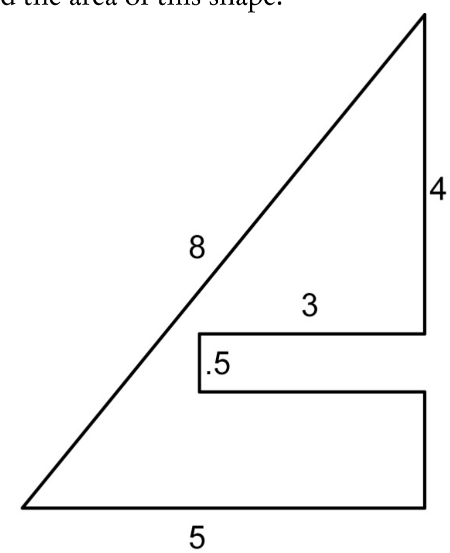
b. How far apart are the two houses?

30 Starting at your front door, you walk 25 feet forward, then make a slight, 10° turn to the right, and walk 5 more feet. Draw a picture, and find how far you now are from your front door.

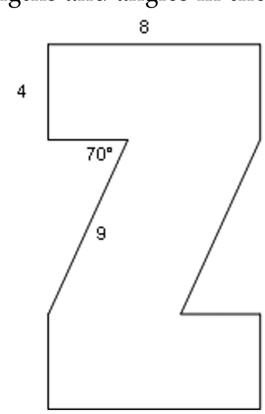
31 Two ants on the ground are 15 feet apart. They are looking up at a balloon hovering up in the air between them. If the first ant looks up at the balloon, the angle of elevation of the ant's line of sight is 84°. For the second ant, it's 52°. Draw a picture, and then find the distance of the balloon from each of the two ants.

For the next few problems – **visualize** - find a new way to see a picture, or figure out an interesting way to split a diagram into pieces, giving you more insight into the problem.

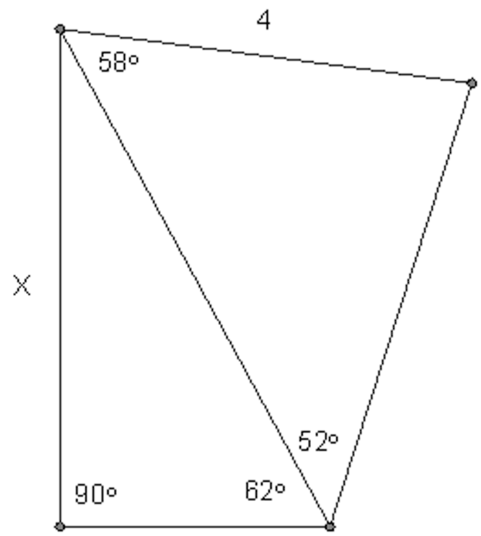
32 Find the area of this shape:



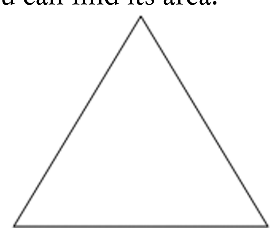
33 This letter Z, composed of two rectangles and a parallelogram, is symmetrical (the top part is the same as the bottom part). Find all lengths and angles in the picture.



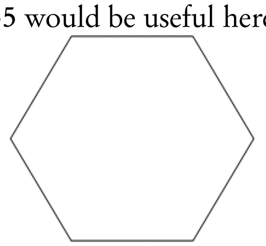
34 Find X.



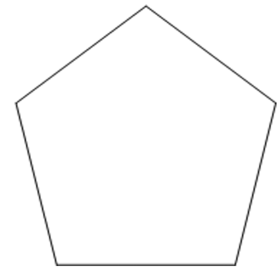
35 This is an equilateral triangle of side length 2 inches. Find a way to split up the triangle so that you can find its area.



36 This is a regular hexagon of side length 2 inches. Find a way to split up the picture so that you can find its area. (The result in problem 35 would be useful here.)



37 Back to our regular pentagon of side length 2 inches.



a. What's its area?

b. What's its height?

38 Two cars start next to each other. The red car drives 8 miles west, and the black car drives 13 miles north-east. It might be helpful to make this diagram on a grid.

a. Draw the paths of the two cars. What's the angle between them?

b. What's the distance between the two cars, after they're done driving?

39 You are standing near the edge of a cliff (at the top). Your feet are 4.5 feet away from the edge. If you look at the edge of the cliff, your line of sight is at an angle of depression of 53°.

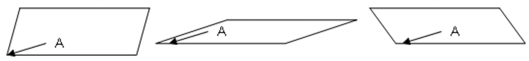
a. Estimate your height (Pretend your eyes are at the top of your head).

b. Now determine your height using trigonometry.

c. In the same line of sight as the edge of the cliff, you see the bottom of a cactus. If the cliff is 95 feet tall, how far away from the base of the cliff is the cactus?

40

You have a four-sided shape with side lengths 10, 4, 10, and 4 (in inches), and with hinges at all the corners – you can bend it like this:



a. Let A be the angle indicated in the picture above.a. What kind of quadrilateral is always formed?

b. What kind of quadrilateral is always formed?

c. If A=71°, find the area of the quadrilateral.

d. If A=100°, find the area of the quadrilateral.

e. What angle A would make the shape have the biggest area? Explain your answer

41

Line n is described by $y = 2x + 3$. Point P has coordinates $(4, 7)$. Determine the measure of the angle formed by line n and the line containing point P and the y-intercept of line n.

42

Don't use a calculator for this problem.

a. Divide: $8\frac{2}{3} \div 4\frac{1}{3}$

b. Simplify: $4 - 2(-1)^5 - (-2)^3$

c. Find an equation for the line that goes through the points $(-10, 9)$ and $(5, 3)$.

d. For which values of x does $\frac{x+1}{x+2} = \frac{x+2}{x+4}$?

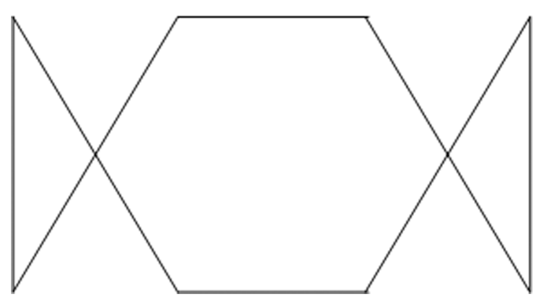
e. Evaluate the expression $\frac{-|x|}{|-x|}$ if $x = 3$, and then if $x = -2$.
Can you find a value of x that does not fit the pattern you see?

43

Find the measure of the angle formed by a diagonal of a cube and a diagonal of one of the faces of the cube.

44

A piece of candy looks like a regular hexagon with two isosceles triangles. The piece of candy is 1 inch tall (the top and bottom of the hexagon line up with the top and bottom points of the triangle).

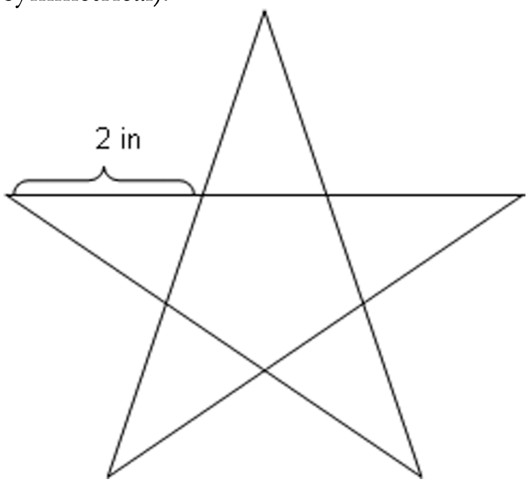


a. Find all the angles in the picture.

b. Tinker with the diagram as necessary so that you can find all the lengths in the picture.

c. Find the exact area of the candy.

45 This is a perfect star (everything is symmetrical).

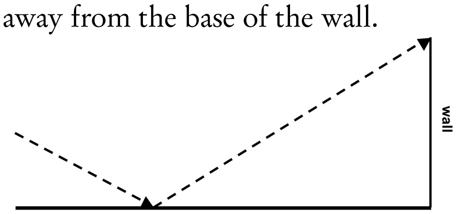


a. Find all the angles in the picture.

b. Find all the lengths in the picture.

c. Find the area of the star.

46 You throw a tennis ball from a height of 5.5 ft off the ground, and it makes an angle of 35° with the ground when it hits and when it bounces back up. After bouncing up it hits a wall. You are standing 26 feet away from the base of the wall.



a. How far away from your feet did the tennis ball bounce?

b. How high up on the wall did the tennis ball hit?

c. How far did the tennis ball travel?

47 A ladder is propped against a wall. The top of the ladder is 19 feet above ground, and the ladder makes an angle of 75° with the ground.

a. Estimate, without using your calculator, the length of the ladder.

b. How long is the ladder?

c. The ladder slides 4 feet down the wall. How far away from the wall is the base of the ladder, after the ladder's slide?

48 You have a 20 foot ladder propped against a wall. It makes an angle of 55 degrees with the floor. Then it slides down a little bit. After the slide it makes an angle of 25 degrees with the floor.

a. How far along the wall did it slide?

b. How far along the ground did it slide?

49 Jill is 5.7 feet tall, and Elizabeth is shorter. They stand facing each other. Jill looks down at Elizabeth's feet, and her line of sight has an angle of depression of 76 degrees. (You can assume people's eyes are at the top of their heads).

a. Draw a diagram of the situation with all the information filled in. How far apart are they standing?

b. When Elizabeth looks up to the top of Jill's head, her line of sight is at an angle of elevation of 19 degrees. How tall is Elizabeth?

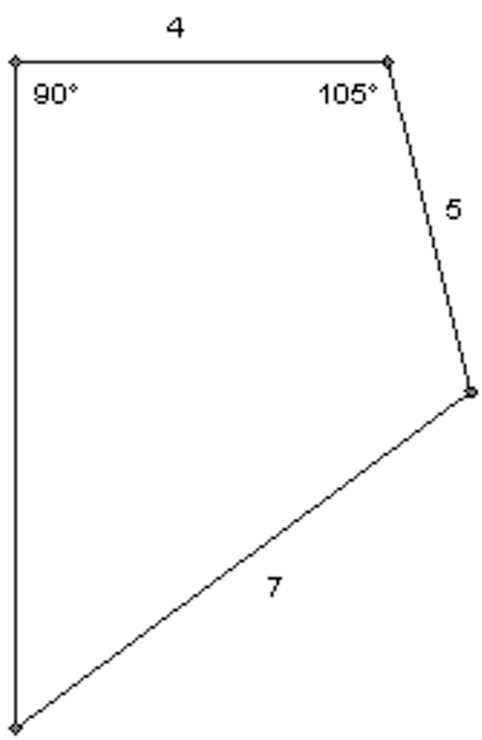
50 Point Q has coordinates $(5, 2)$.

a. Find the image of Q after it's been rotated 30 degrees counterclockwise around $(0, 0)$.

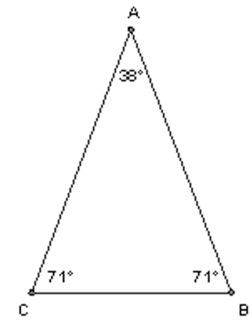
b. Find the image of Q after it's been rotated 30 degrees counterclockwise around $(1, 3)$.

Exploring in Depth

51 Find the missing side length.

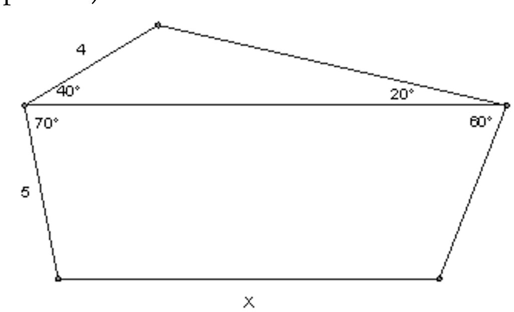


52 The perimeter of the triangle below is 100. Find the lengths of the sides. (Write equations!)



53 You leave your house, and walk 10 miles north. Then you make a 30 degree turn to the right and walk 8 more miles. Then you make a sharp turn to the right (more than a 90 degree turn) and walk some more, ending up exactly 15 miles due east from your house. How long was this last piece of your walk?

54 Find X. (You can assume that the two lines that appear to be parallel are indeed parallel).



55 You are X inches tall. One afternoon, you notice that your shadow got 4.5 inches longer between when you left school (when the angle of the sun was 72°) and when you got home (when the angle of the sun was 68°).

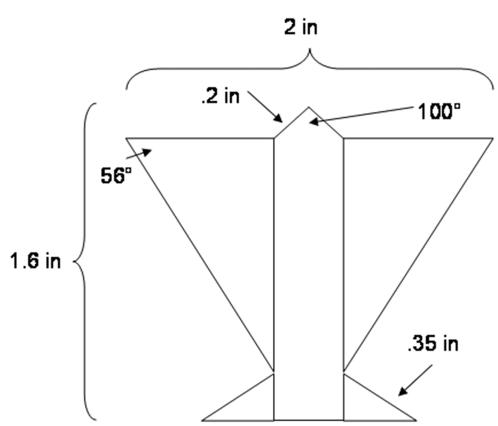
a. In terms of X, how long is your shadow when you leave school?

b. In terms of X, how long is your shadow when you get home?

c. Find X.

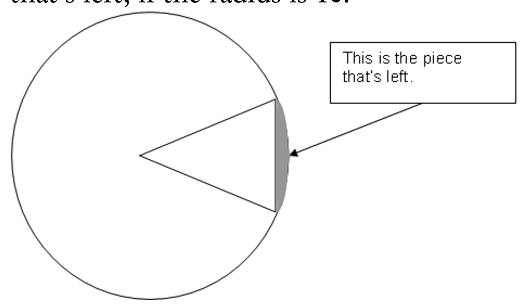
56 A plane is flying east, with an angle of descent of 10°. After flying 43 miles, it turns around quickly and starts flying due west, at an angle of descent of 14°, for X miles – pretty far past where it started. It ends up exactly 18.5 miles away from where it started. Draw a picture, and then find X.

57 The strangely pointy butterfly below is perfectly symmetrical across the middle, and all the angles that look like right angles are perfect right angles. He is 2 inches across at the widest point, and 1.6 inches tall. Find all the lengths in the picture.

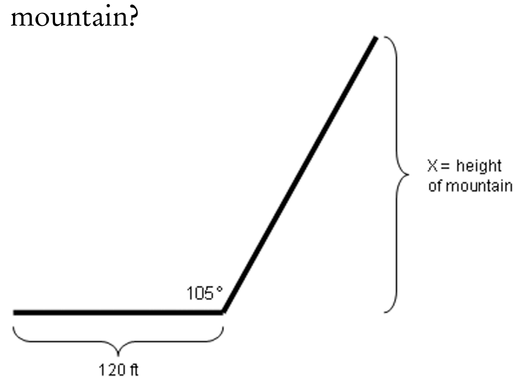


58 Let line n and point P be as in problem 41: $y = 2x + 3$ and $(4, 7)$. Determine the coordinates of the reflection of P over line n.

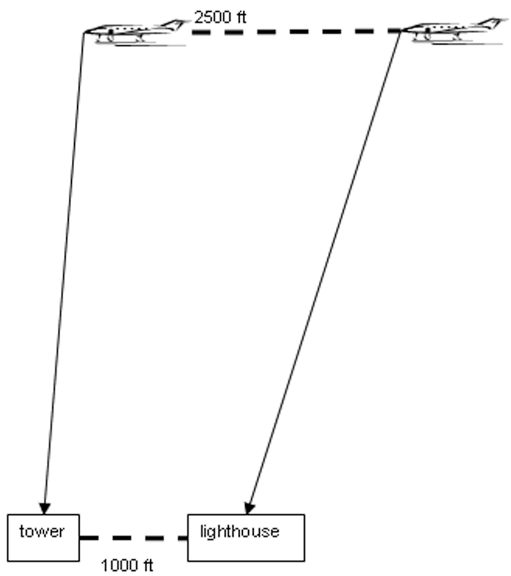
59 You take a 40 degree sector of a circle (a sector is like a slice of a cake), and cut off the triangle. What's the area of the piece that's left, if the radius is 10?



60 At 5pm one day, you are attempting to measure the height of a mountain. First, you observe that while you are 6 ft tall, your shadow is only 2.5 feet long. Then, you observe that the mountain's shadow extends 120 ft past the foot of the mountain. Also, the angle at the base of the mountain is 105°. How tall is the mountain?

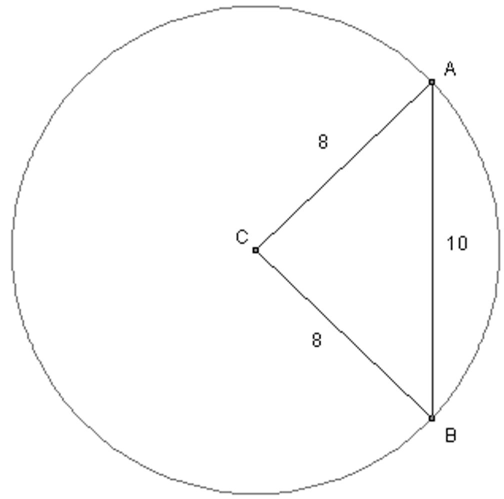


61 A plane is flying west at a constant altitude. The pilot looks down and sees a lighthouse, and his line of sight is at an angle of depression of 85° - the lighthouse is almost directly below the plane. After the plane flies 2500 more feet forward, the pilot looks down again, this time at an angle of 89°, and sees a tower that he knows to be 1000 feet west of the lighthouse. What is the altitude of the plane? (Note – the picture is not to scale!)



62 You see the top of a palm tree 14 yards away from you, at an angle of elevation of 40 degrees. Behind the palm, you can tell that there's a short pine tree. The top of the pine is 20 yards away from you, at an angle of elevation of 10 degrees. How far apart are the tops of the two trees?

63 This circle has a radius of 8 inches, and the chord AB has a length of 10 inches. Find the arc length of arc AB.



64 On a hiking trip, you start at point A and hike 5 miles in a direction that's 4° north of due east. Then, you turn your path slightly, by 10° to your right, and walk until you are directly east of point A where you started.

a. Draw a diagram of the situation. How long do you have to hike in this new direction before you make it back onto your intended trail?

b. How much shorter would your path have been, if you just walked straight east instead of up and then down?

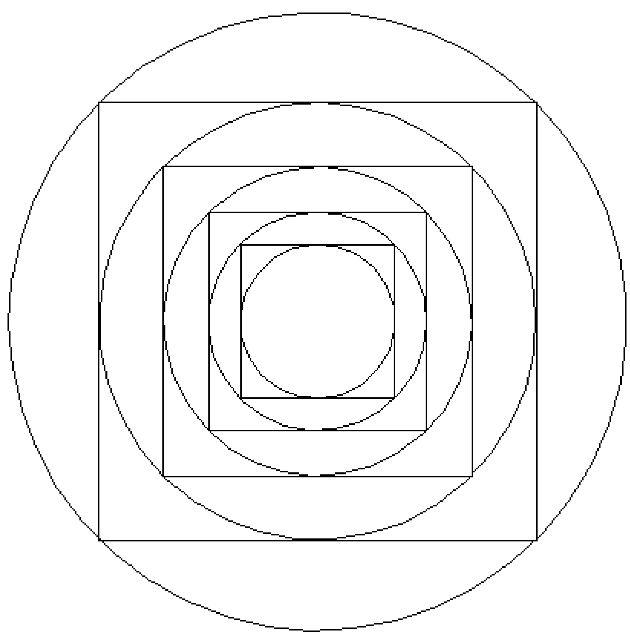
65 Two motion-detector cameras sense an unidentified object hovering in the air somewhere above them. The first motion detector is on the ground, and it registers the object's location as 8.6 miles away, at an angle of elevation of 80°. The second motion detector is on a tower 2 miles above the ground, and registers the object's location as 10 miles away at an angle of elevation of 40°.

Draw a diagram of the situation, and find the distance between the two motion detectors.

66 The area of a regular quadrilateral (a square) where each side is length 1 will of course be 1^2, which is an integer. Do any of the other regular polygons with sides of length 1 have integral area? How about rational area?

When you have a diagram, it is sometimes helpful to **Fill in all the Information you Can**, even if you think it has nothing to do with the problem you have to solve. This is another example of **tinkering**. Doing this will help you to see patterns and to notice things about the diagram that you might not have noticed otherwise. It is a wonderful thing to do when you are stuck on a problem.

1 The radius of the smallest circle is one unit. What is the ratio of the area of the largest circle to the area of the smallest circle?

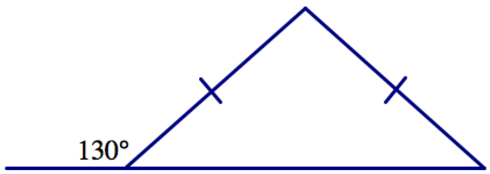


geometric tinkering

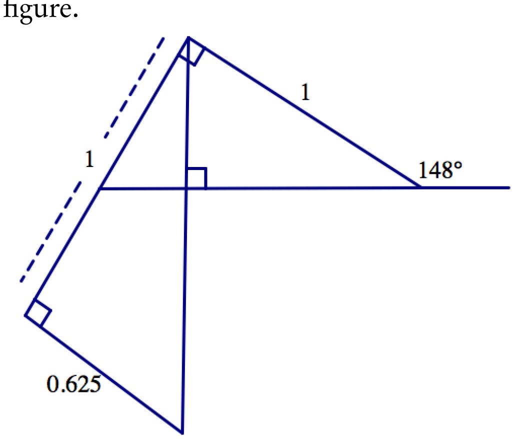
In the problem on the previous page, rather than trying to figure out the ratio of the two areas immediately, it makes more sense to figure out some smaller, manageable thing about the figure, and see where that leads us. A clear place to start is with the smallest circle. Once we find the area of that circle, we can then find the square that surrounds it. Can we then find the area of the second largest circle? By looking at the smallest parts, we can see the pattern which allows us to solve the original problem.

In your previous experience with geometry, you probably learned some basic facts about angles, lines, and triangles. For instance, two angles that form a line add up to 180 degrees, and so do the three angles of a triangle. You may also know that, in an isosceles triangle, the angles opposite the sides that are the same length are also the same. These facts will help you with many of the problems that follow.

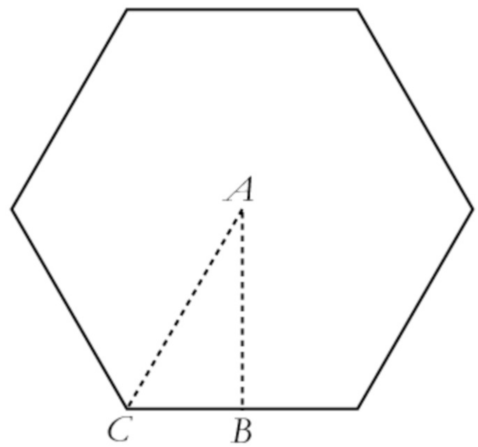
2 The two marked lengths are congruent. Find the measures of all the angles in the figure.



3 Find the measures of all the angles in the figure.

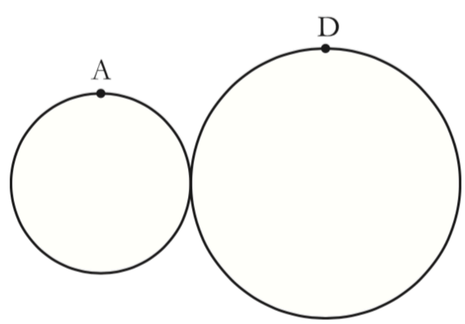


4 The area of this hexagon is 60. Find the product of the lengths of the segments AB and BC.

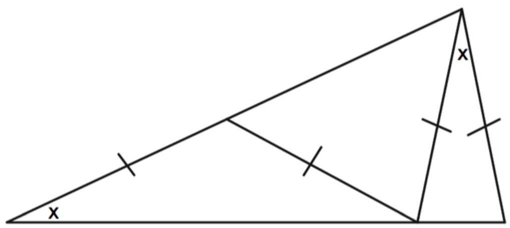


5 A smaller circle has radius 2 and a larger circle has radius 3. *A* and *D* are points on the tops of these circles. The circles are touching in the middle, as shown.

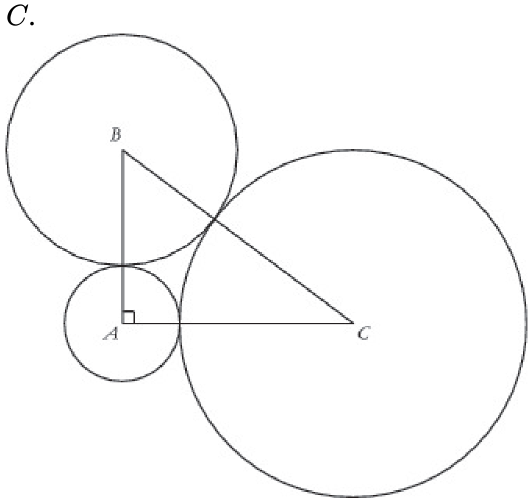
Find the distance from *A* to *D*.



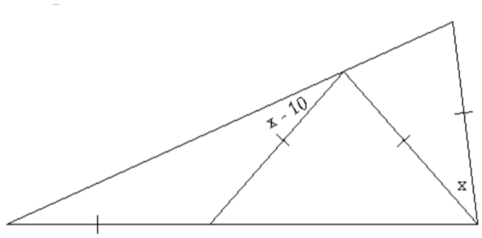
6 Given that the four marked lengths are congruent, find x.



7 In the following diagram, $BC = 5$, $AC = 4$, and the circle B has four times the area of circle A. Find the area of circle C.



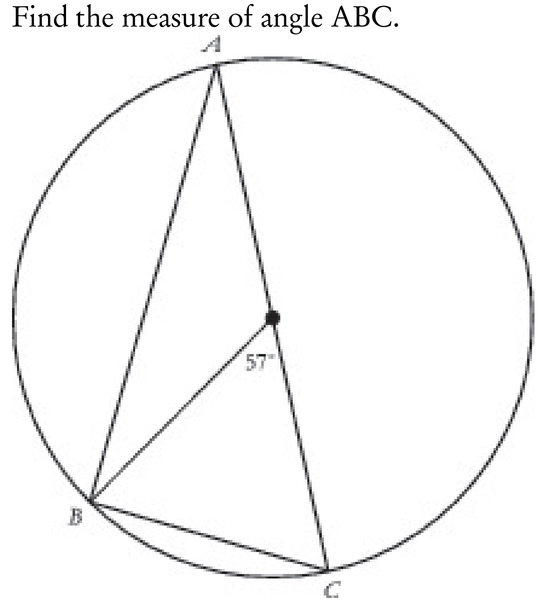
8 Find x.



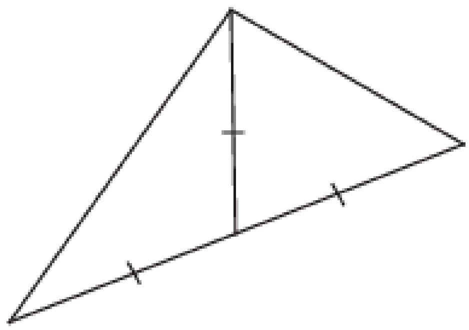
9 Draw a square with vertices A, B, C, and D, in that order. Then draw an equilateral triangle with vertices A, B, and E, where E is a point outside the square. Connect points E and C with a line segment. Find the measure of angle ECB.

10 Repeat the previous problem, but this time place point E inside the square. Again, find the measure of angle ECB.

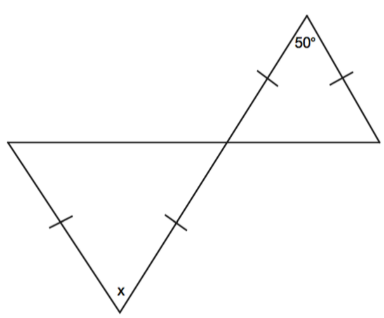
11 Find the measure of angle ABC.



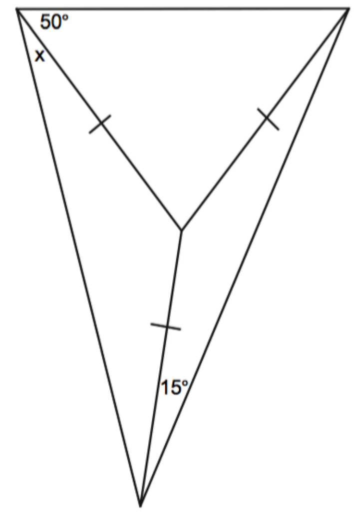
12 Prove that the large triangle below is a right triangle. It may help to label some angles x and other angles y (give angles the same letter if you know they are the same!)



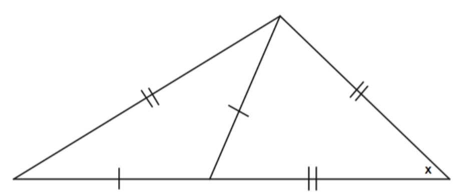
13 Find x.



14 Find x.



15 Find x.



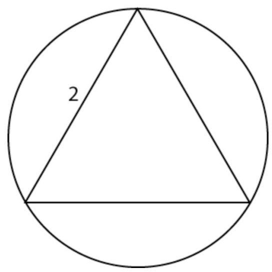
16 A rectangular solid always has 12 edges (count the edges of a tissue box), and 6 faces (count the 2D rectangles that form the tissue box.) Each face of a rectangular solid has two diagonals, formed by connecting opposite corners.

Find the dimensions of a rectangular solid where the 12 edges and the diagonals on all 6 faces are integers.

geometric tinkering

17

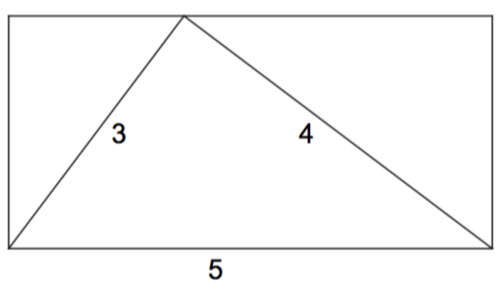
The triangle below is equilateral. Find the area of the circle.



18

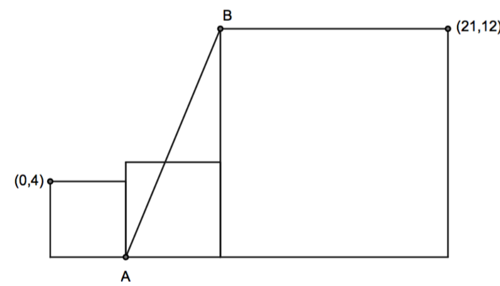
The triangle below is the famous “3-4-5” right triangle; famous because it is one of the simplest triangles that satisfies the Pythagorean theorem: $3^2 + 4^2 = 5^2$.

As you can see, the 3-4-5 right triangle has been placed in a rectangle. What is the height of the rectangle?



19

Three squares are lined up along the x-axis as shown, and the points with coordinates (0,4) and (21, 12) are labeled accordingly. Find AB. *Copyright mathleague.com*.

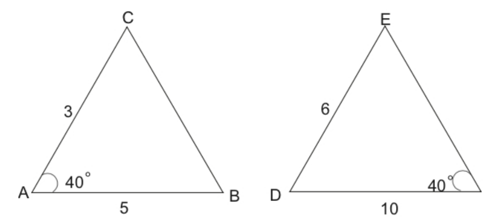


HABITS

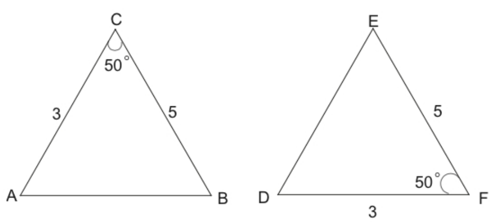
SUMMARY AND REVIEW

1 For each pair of triangles shown, say if they must be similar, must not be similar, or if there is not sufficient information to tell. If the triangles are similar, write the similarity statement (with a "∼"), paying attention to the correct order of the letters.

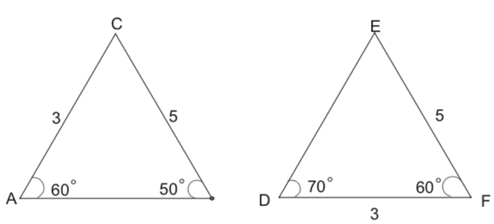
a.



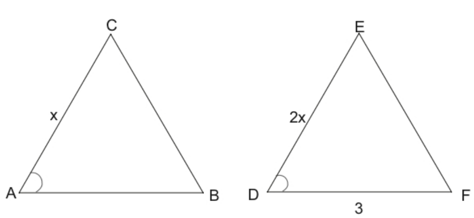
b.



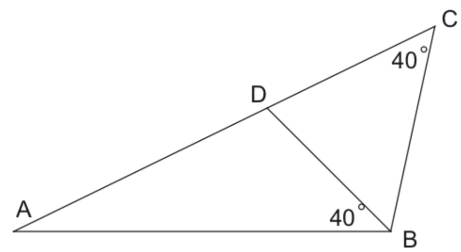
c.



d.

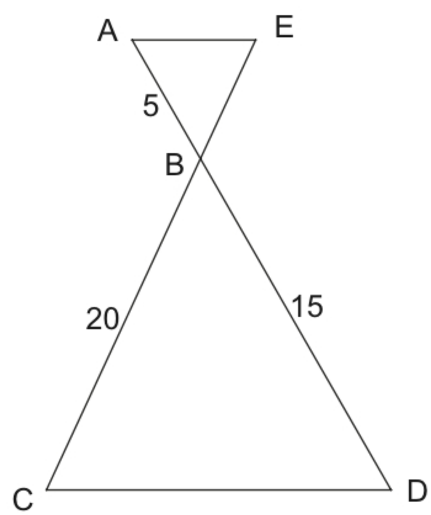


e.

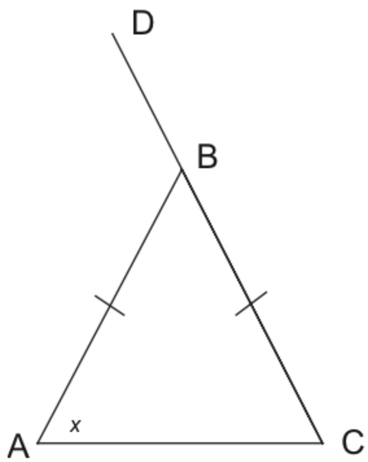


2 a. In the figure below, AE is parallel to CD. Why are the triangles similar?

b. Find all the missing side lengths.

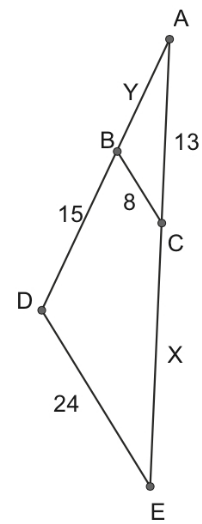


3 Below is an isosceles triangle with one of its sides extended. If angle A is x degrees, find the measure of angle ABD in terms of x.

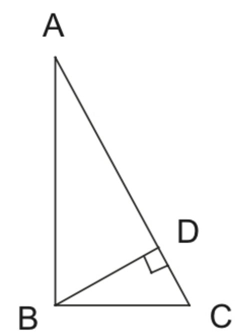


4 In triangle ABC, it is given that angle A is 59 degrees and angle B is 53 degrees The altitude from B to line AC is extended until it intersects the line through A that is parallel to segment BC; they meet at K. Calculate the size of angle AKB.

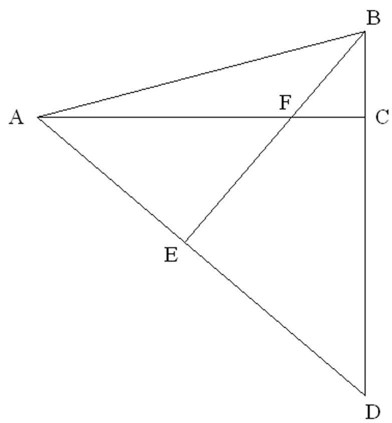
5 In the picture below, $BC || DE$. Find X and Y.



6 ABC is a right triangle. The altitude to its hypotenuse is drawn, hitting the hypotenuse at point D. $AD = 9$ and $DC = 4$. Find x, the length of the altitude.



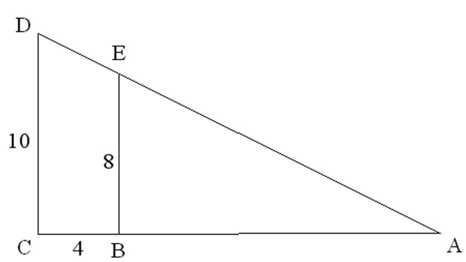
7 In the figure below, $\angle ACD$ and $\angle BED$ are right angles. Name all the triangles in the figure that are similar to each other, using correct similarity notation.



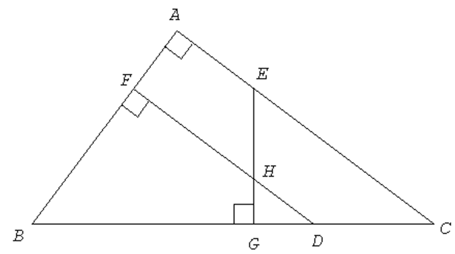
8 In the figure below, $\overline{BE} \parallel \overline{CD}$, and $\angle ABE$ and $\angle C$ are right angles. Find each of the following.

a. AB

b. ED



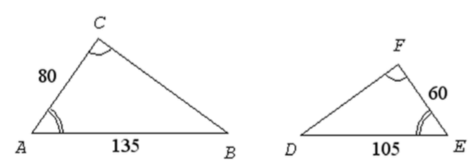
9 Look carefully at the diagram below before answering the following questions.



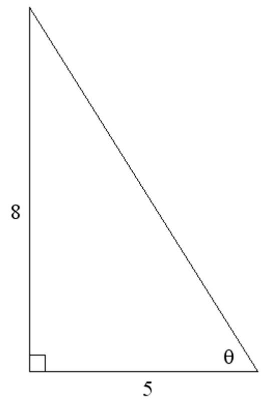
a. Name as many pairs of similar triangles as you can. Use correct similarity notation.

b. If $EC = 5.6$, $CD = 3$, $GD = 1.48$, and $FH = 3.75$, calculate the length of $\overline{BF}$. You must show your work clearly.

10 What's wrong with the picture drawn below?



11 Use the triangle below to determine the values of the following expressions.



a. The exact length of the hypotenuse. This means don't approximate its length.

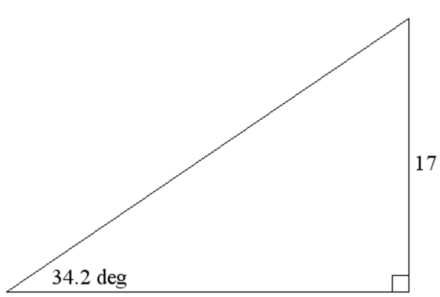
b. $\sin\theta$

c. $\cos\theta$

d. $\tan\theta$

e. $\sin(90^\circ - \theta)$

12 Use trigonometry to calculate the lengths of the unknown sides of the right triangle given below. Show your work.



13 Recall that the convention for labeling sides and angles of triangles is to put side a opposite angle A, etc.
Find the missing information in each triangle:

a. In triangle ABC, $\angle A = 90^\circ, \angle B = 25^\circ$, and $a = 18$. Find b and c.

b. In triangle XYZ, $\angle X = 90^\circ, \angle Y = 37^\circ$, and $z = 25$. Find x and y.

14 Why does your calculator give you an error when you try to do $\sin^{-1}(2)$, but not when you do $\tan^{-1}(2)$?

15 The CN tower in Toronto is the highest tower in the world. You are standing 100 m away from the base of a tower. You can just see the top of the tower when you look up at a 79.7° angle. How tall is the tower?

16 A student looks out of a second-story school window and sees the top of the school flagpole by angling his line of sight 22° above the horizontal. The student is 18 ft above the ground and 50 ft from the flagpole. Find the height of the flagpole.

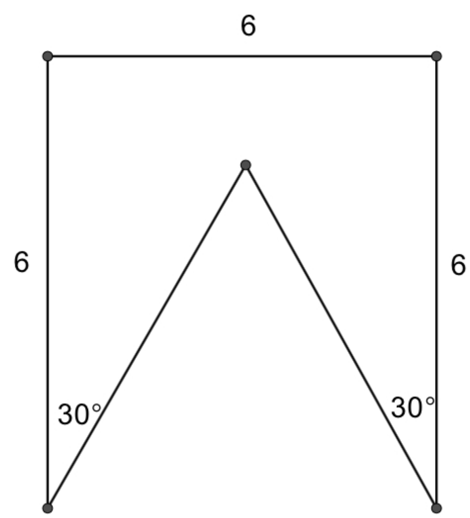
17 Squidward is in a plane that is 160 miles north and 85 miles east of SpongeBob airport.

a. How far away is the airport?

b. What direction should the plane fly to go directly to the airport? (Your answer should be very specific and include an angle)

18 An observer in a lighthouse 350 feet above sea level observes two ships approaching. The angle of depression to the first ship is 4°, and to the second is 7°. How far apart are the ships?

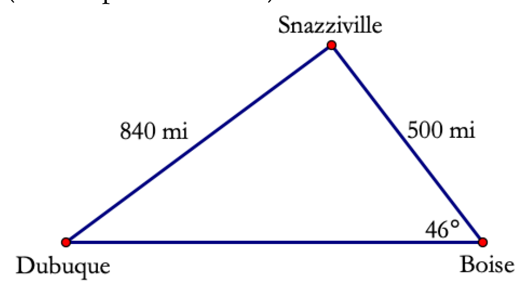
19 What is the perimeter and area of the figure below?



20 A triangle has sides of length g, g, and $\frac{7g}{5}$. Is it a right triangle? Explain.

21 $\triangle ABC$ is a right triangle with right angle B. A line segment is drawn from angle B perpendicular to $\overline{AC}$. Where it intersects this side mark the point D. Let $BC = 12$ and $DC = 11.07692$. Find the area of $\triangle ABC$. Show your work.

22 A plane flies 500 miles from Boise to Snazziville, then changes direction at Snazziville and flies 840 miles to Dubuque. (see the picture below).

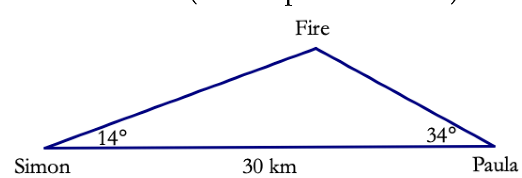


a. What is the measure of the angle with vertex Dubuque?

b. How far is it from Boise to Dubuque?

c. If the plane always flies at 600 miles per hour, how much time was wasted going by way of Snazziville instead of going directly to Dubuque?

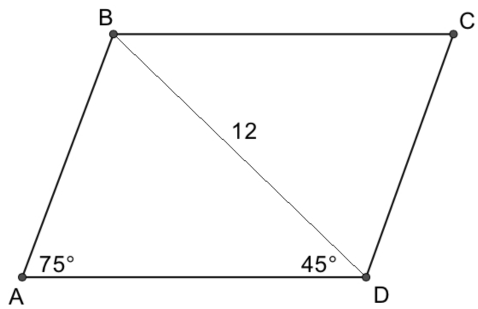
23 Simon and Paula are 30 kilometers apart, and they each spot a religious idol on fire in the distance (see the picture below).



a. How far are Simon and Paula from the fire?

b. What is the area of the triangle?

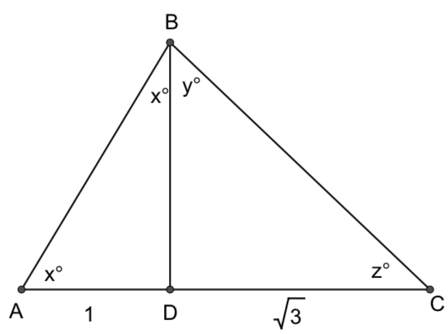
24 In parallelogram ABCD below, $BD = 12$, $\angle A = 75^\circ$, and $\angle BDA = 45^\circ$.



a. Find AB.

b. Find the area of the parallelogram.

25 In the figure below, find x, y, z, AB and BC.



26 The altitude (or height) of an equilateral triangle is 6. Find the triangle's perimeter.

27 The perimeter of a regular octagon is 72.

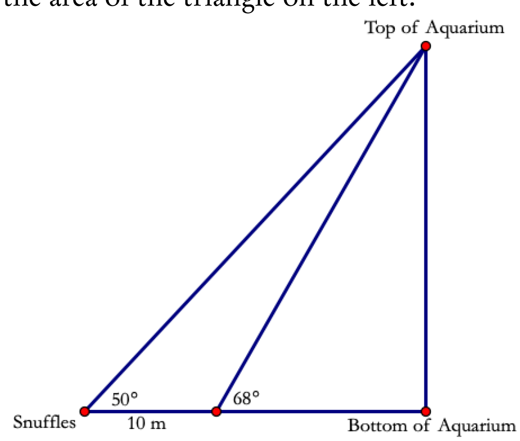
a. What is the length of each side?

b. What is the sum of its angles?

c. Find the area of the octagon.

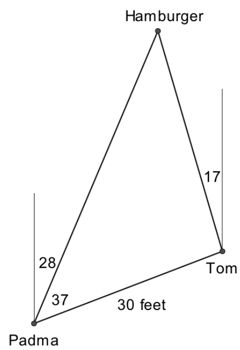
28 A right triangle has a perimeter of 24, and its hypotenuse is three times as long as its shorter leg. Find the lengths of all three sides of the triangle.

29 When Snuffles looks up to the top of the Aquarium, he is looking up at an angle of 68°. When he steps back 10 meters, now when he looks up to the top of the aquarium, he is looking up at only 50°. Determine the height of the aquarium, and the area of the triangle on the left.



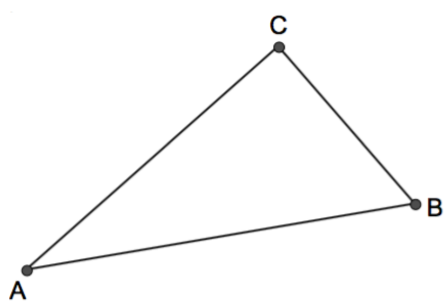
30

Tom Colicchio and Padma Lakshmi haven't eaten in 5 days as the food that has been cooked for them hasn't been fancy enough for them to deign to eat it. By this point, however, they are both starving. As in the picture below, a hamburger appears suddenly. Assuming Padma runs $1\frac{2}{3}$ times as fast as Tom, who gets to the burger first?



31

In right triangle ACB (where C is the right angle):

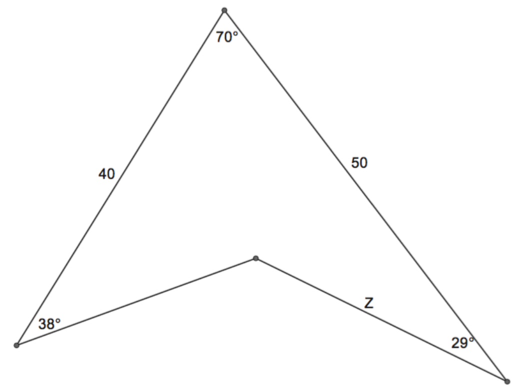


a. If a = 12 and $\angle B = 71^\circ$, determine the rest of the angles and sides of the triangle.

b. If, instead, c = 13 and b = 6, determine the rest of the angles and sides of this other triangle.

32

In the diagram below, find Z.



33

Socrates, the Ancient Greek philosopher, was known for being a trickster. One day, Socrates claimed to have a triangle whose lengths were 5, 8, and 10. He also said that two of the angles in the triangle were 50° and 27°.

Explain why Socrates' triangle can't possibly exist.

34

Andrew decides to measure the height of a tree by using an idea he learned in class. He takes two poles—one 10 feet long and the other 14 feet long—and walks up to a very tall tree. He places the 14-ft tall pole to the side, walks with the 10-ft pole straight out 36 feet from the tree and plants the pole so that it is parallel to the tree. He then walks along the same line father away from the tree until the line-of-sight from the ground to the top of the 10-ft pole intersects with the top of the tree. He marks this spot on the ground and measures from it to the base of the 10-ft pole. This distance is 6 feet. He then takes the other pole—the 14-ft one—and walks in the opposite direction from the 10-ft pole until he's 40 feet from the tree. He plants the 14-ft pole so that it is parallel to the tree. He then walks along the same line farther away from the tree until the line-of-sight from the ground to the top of the 14-ft pole lines up with the top of the tree. He marks this spot on the ground and measures from it to the base of the 14-ft pole. This distance is *d* feet. (You'll notice that I'm not telling you what this distance is. Heh, heh, heh!) Assume the ground is level throughout the entire area that Andrew is measuring.

a. Draw a diagram that shows the situation and all of the key measurements.

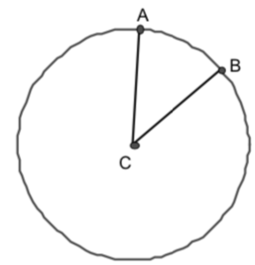
b. Determine the height of the tree. Show your work.

c. Determine what *d* is. Show your work.

d. Did Andrew need both poles to measure the height of the tree? Explain.

35

The circle below has a circumference of 48π. The length of the arc AB is 12.

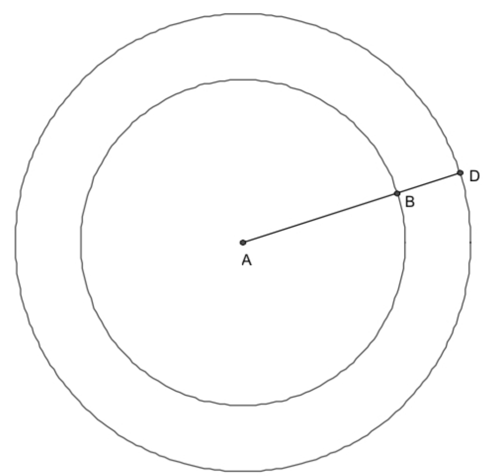


a. What is the radius of the circle?

b. What is $\angle ACB$?

c. If Josephine removed sector ACB (that is, the "piece of pie") from the circle, what would the area be of what was left?

36



a. If the area between the circles above is 51π, and $AD = 10$, find BD.

b. If, alternatively, the area between the circles is 24π and $BD = 2$, find AB.

Park School Mathematics

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