BOOK 7: TRIGONOMETRY IN THE COORDINATE PLANE

TABLE OF CONTENTS

HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

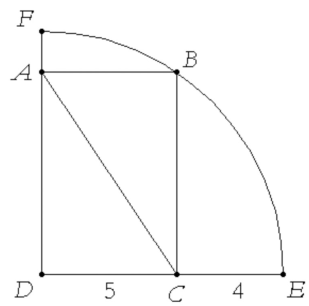
re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

1

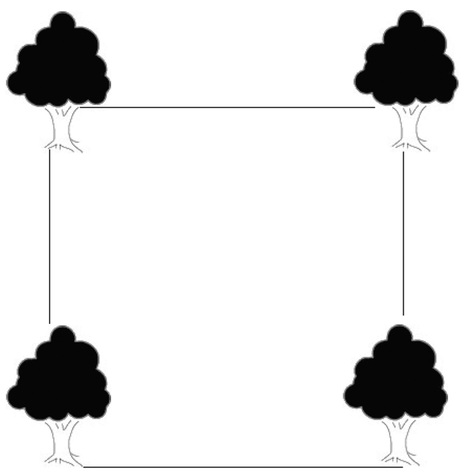
In the following picture, rectangle ABCD is inscribed in a quarter-circle. $DC = 5$, and $CE = 4$. Can you figure out the length of diagonal AC?



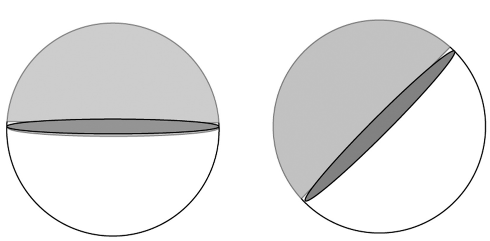
Often a problem that can seem particularly perplexing can be solved by looking at it in a different way. Sometimes the best way to keep track of the different information and variables in a problem is to draw a picture of some sort, to **visualize** the information so that it is in a form that is easier to understand. You saw this last year as you solved a variety of problems, at times by constructing models, and at times by finding clever ways to visually represent things that at first seemed quite nonvisual.

visualize

2 Four trees are planted at each corner of a square park. The city wants to expand the park to twice its current area, but in such a way that the park is still a square, and none of the four trees is in the interior of the park. (The trees cannot be transplanted.) Draw a plan for a new park that meets these criteria.

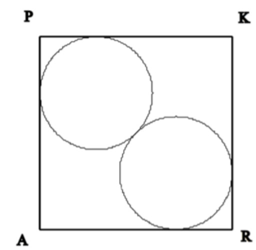


3 The shaded parts of the spheres below are hemispheres.



You throw three darts onto the surface of a globe, each from a randomly chosen direction. What is the probability that all three darts lie in one hemisphere?

4 The circles in the diagram each have radius 1 cm, are tangent to each other and also to the square PARK. Their centers are on the line PR. Find the area of the square PARK.



5 There are many different triangles with side lengths 12 and 13 cm. Which of these triangles has the greatest area?

6 A rectangle and a square are inscribed in congruent circles. The rectangle has a width of 6 and a length of 8. What is the area of the square?

7 Three identical, spherical oranges are placed in a bin as part of a supermarket display. The bin is exactly long and wide enough to have two oranges fit snugly in the bottom, but there's plenty of room to place the third orange on top of these two. If the radius of an orange is 2 inches, find the height of this small stack.

8 A street has parallel curbs 40 ft apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet, and each stripe is 50 feet long. Find the distance between the stripes.

9 Let A and B be any two points in a plane.

a. How many different circles can you draw that go through points A and B? Can you give the radius of the smallest possible circle? Of the biggest?

b. How many different rectangles can you draw with opposite vertices on points A and B?

10 PQ and QR are diagonals of two faces of a cube. Find the measure of $\angle PQR$.

11 A trapezoid is inscribed in a circle of radius 5 cm so that one base is a diameter of the circle, and the other base has length 5 cm. What is the perimeter of the trapezoid?

12 If you start with $\frac{1}{2}$ then add $\frac{1}{4}$, then $\frac{1}{8}$, then $\frac{1}{16}$, and so on, ad infinitum, what do you suppose the answer would be? Draw a diagram that would justify your response.

13 How many sides does a cube have? How about a pyramid?

Can you build a closed 3-dimensional shape out of 4 flat sides? How about out of 3 flat sides? Give examples, or explain why not.

14 Suppose you have a box that has a base of 1 inch by 5 inches and that stands 8 inches tall. How many ½ inch radius spherical balls can you get into this box if you can't let any ball protrude above the top of the box?

15 Craziola, the wacky pizza guy, has decided he wants to cut a pizza into as many pieces as possible, with as few straight cuts as possible. He doesn't care at all if the pieces are of equal size, he just wants to make the most number of distinct pieces. With 1 cut, he produces 2 pieces. With 2 cuts, he creates a maximum of 4 pieces, no matter how crazy the 2 cuts he makes are. How many pieces can he possibly make with 3 cuts? 4 cuts? What about n cuts? Can you find the pattern?

16 Is it possible to arrange six unsharpened pencils so that they all touch each other?

visualize

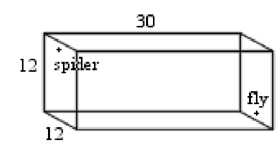
17 Mr. Shimano gives an extremely difficult Japanese test. The highest score was 74% and the lowest 31%. Rather than give a retest, Mr. Shimano decides to raise the 74% to 93% and the 31% to 61%. For a student who scored 57 on the original test, what score do you think Mr. Shimano should give him after the adjustment?

18 A circle is sitting inside an equilateral triangle so that it's tangent to the sides of the triangle in three places. Another equilateral triangle is inside the circle with its three vertices on the circle. If the length of a side of the smaller triangle is 1 cm, find the length of a side of the bigger triangle.

19 Two sides of a triangle measure 6 and 11 cm respectively. If the length of the third side is also an integer, what possible lengths can the third side have?

20 What is the maximum number of acute angles a convex polygon can have?

21 A hungry spider and a fly are in a room 30 feet long, 12 feet wide, and 12 feet high. The spider is on one of the smaller walls, 6 feet from each side and 1 foot from the ceiling. The fly is on the opposite wall, 6 feet from each side and 1 foot from the floor. Assume that the fly does not move (it is paralyzed by fear!) and find the shortest path that the spider can take to eat the fly.



HABITS

LESSON 1: CIRCULAR FUNCTIONS

Introduction

You've seen sine, cosine, and tangent before. In fact, you've even used them to find missing sides in triangles.

1. How would you define the sine of θ ? Cosine of θ ?

2. Use careful construction of a triangle to find $\sin 37^\circ$. Do not use the sine function on your calculator!

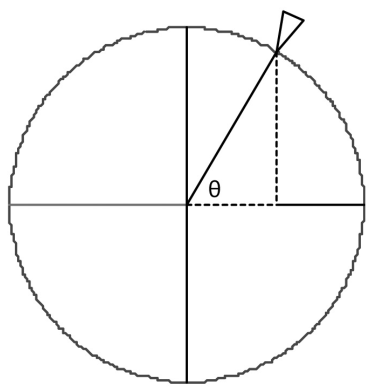
3. Why can't you use a construction to find $\sin 102^\circ$?

Even though we can't construct a triangle to find the sine of 102°, your calculator can tell you an approximate value! In this lesson, we will discover a new way to think about sine and cosine—a way that makes calculating the values of $\sin 102^\circ$, $\cos 1002^\circ$, and $\tan(-10002^\circ)$ possible.

Development

Most of the problems in this section require carefully drawn diagrams.

4 A wheel of radius one foot is placed so that its center is at the origin, and a pin on the rim is at (1, 0). The diagram below shows the wheel after it has been spun an angle θ in a counterclockwise direction.



Now consider the function $P(\theta)$, which outputs the coordinates of the pin after the wheel has been spun an angle θ in a counterclockwise direction. So, for example, $P(0^\circ) = (1, 0)$ and $P(270^\circ) = (0, -1)$. Find $P(\theta)$ when:

a. $\theta = 90^\circ$

b. $\theta = 45^\circ$. Give an exact answer.

c. $\theta = 30^\circ$. Give an exact answer.

d. $\theta = 57^\circ$

e. θ is some measure between 0° and 90°. (Your answer should be expressed in terms of θ.)

f. For values of θ between 0° and 90°, how are $\cos\theta$ and $P(\theta)$ related?

5 Now, let's consider angles greater than 90°.

a. Calculate $P(150^\circ)$, with P being the same position function as in question 4.

b. What is $P(253^\circ)$?

c. Compare these values to the sine and cosine of 150° and 253°.

d. Using the diagram and your calculations, what do you think the tangent of 253° is? The tangent of 150°? Check them on your calculator.

It is the convention for rotations that motion in a counterclockwise direction is considered **positive**, while motion in a clockwise direction is considered **negative**. So if our wheel is spun 57° counterclockwise we would input 57° in our function $P(\theta)$, as we did in problem 1, but if our wheel is spun 57° clockwise we would input -57° in our function $P(\theta)$. So the output for would be the coordinates of the pin after the wheel is spun 57° clockwise.

6 Find:

a. $P(-240^\circ)$. Give an exact answer.

b. $P(-2640^\circ)$. Give an exact answer.

c. $P(-237^\circ)$.

d. $\tan(-237^\circ)$.

e. $P(-\theta)$, where $0^\circ < \theta < 90^\circ$. (Your answer should be expressed in terms of θ.)

7 Now define sine, cosine, and tangent of θ for every value of θ.

8 For angles between 0 and 90 degrees, is using your definition in problem 7 equivalent to using your definition in problem 1? Why or why not?

9 $P(46280^\circ) = (-0.94, -0.34)$. Without using the cosine button on your calculator, find the cosine of 44480°.

10 Let $\cos\theta = -0.4$.

a. For how many angles is that true?

b. How many of these angles are between $-180°$ and $360°$?

c. With the help of your calculator find all the angles in part b.

11 Assuming that $\cos 80° = 0.17$, use the symmetry of the circle to find $\cos 100°$, $\cos(-260°)$, $\cos 260°$, $\cos 280°$, $\sin 190°$, and $\sin(-10°)$.

12 Let's revisit question 4, but with a wheel of radius 7 feet instead of 1 foot. The wheel is still centered at $(0, 0)$, and still with a pin at $(7, 0)$. Let $Q(\theta)$ be the function which outputs the coordinates of the pin after this larger wheel has been spun an angle θ in a counterclockwise direction.

a. What is $Q(48°)$?

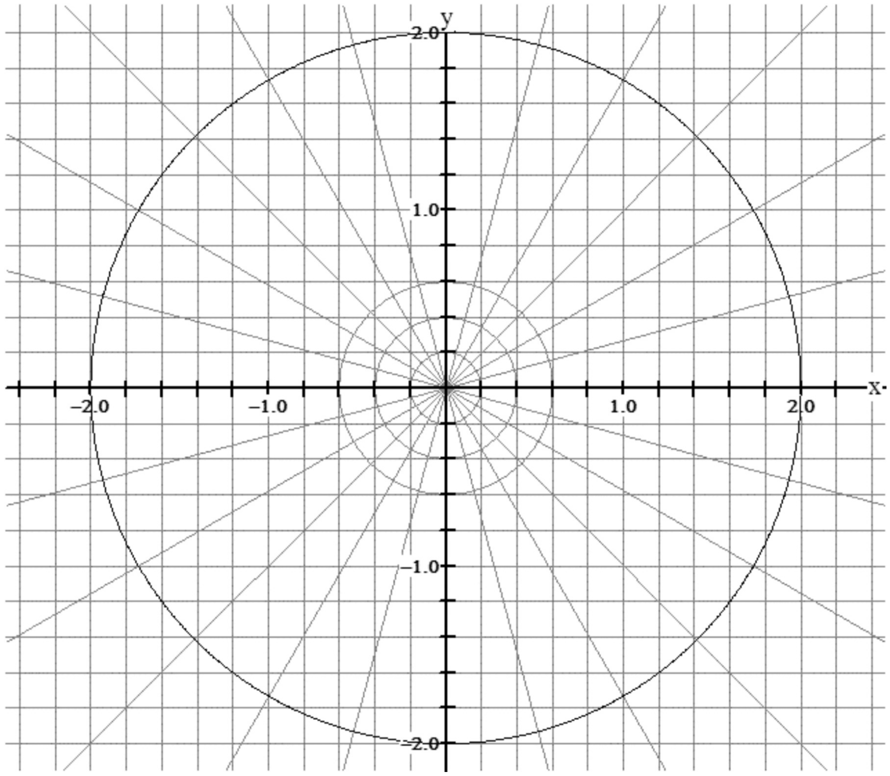
b. What is $Q(109°)$?

c. How would you define sine and cosine of θ using this $Q(\theta)$ function? How would you define tangent?

d. How would you define sine, cosine, and tangent using a circle of radius r?

13

In the diagram below, there is a circle of radius 2 inches, with radii drawn at 15° intervals.



a. Use careful estimates and the conclusion of problem 12 to calculate the sines of 30°, 60°, 90°, 120°, ..., 360°.

b. Using your calculations in part a, sketch a graph with θ on the horizontal axis and $\sin\theta$ on the vertical axis. (Use values of θ from $-360°$ to $360°$.)

Surprise! There are actually three more trigonometric ratios (functions) in addition to the three you already know. Here are their names and definitions:

Cosecant of angle θ, written $\csc\theta$, is defined thus: $\csc\theta = \frac{r}{y}$

Secant of angle θ, written $\sec\theta$, is defined thus: $\sec\theta = \frac{r}{x}$

Cotangent of angle θ, written $\cot\theta$, is defined thus: $\cot\theta = \frac{x}{y}$

14

Use the unit circle to find $\csc\theta$, $\sec\theta$ and $\cot\theta$ for $\theta = 240°$.

Practice

15 Use the symmetry of the circle to complete the following chart. Give exact answers. Copy the chart into your notebooks.

θ	30°	45°	60°	315°	–210°	210°	–315°	150°	240°
$\sin\theta$									
$\cos\theta$									
$\tan\theta$									

16 (For this problem use the circle of problem 13.)

a. Use careful measurements and the conclusion of problem 7 to calculate the cosines of 30°, 60°, 90°, etc.

b. Using your calculations in part a, sketch a graph with θ on the horizontal axis and $\cos\theta$ on the vertical axis. (Use values of θ from -360° to 360°.)

17 Determine, without using your calculator, which of the following expressions are the same as $\sin 27^\circ$.

$\sin(180^\circ - 27^\circ)$, $\sin(180^\circ + 27^\circ)$, $\sin(-27^\circ)$, $\sin(360^\circ + 27^\circ)$, $\sin(-207^\circ)$

18 Find, without using your calculator, two of the following expressions which are the same.

$\sin 27^\circ$, $\cos 27^\circ$, $\sin(-153^\circ)$, $\cos(-153^\circ)$, $\cos 63^\circ$

19 Find at least two values for θ that fit the equation $\sin\theta^\circ = \frac{\sqrt{3}}{2}$. How many such values are there?

Problems

20 At constant speed, a wheel rotates once counterclockwise every 8 seconds. The center of the wheel is $(0, 0)$ and its radius is 1 foot. A pin is initially at $(1, 0)$. Where is it 69 seconds later?

21 A wheel whose radius is 1 is placed so that its center is at $(3, 2)$. A pin on the rim is located at $(4, 2)$. The wheel is spun θ degrees in the counterclockwise direction. Now what are the coordinates of that pin? Does your answer work for 90 degrees? 180 degrees?

22 For the following equations use a circle of radius 2 (and your calculator for part b only) to find all solutions θ between 0° and 360°:

a. $\cos\theta = -\frac{\sqrt{3}}{2}$

b. $\tan\theta = 6.3138$

c. $\sin\theta = -\frac{\sqrt{2}}{2}$

d. $\cos\theta = \cos(251^\circ)$

e. $\sin\theta = \sin 580^\circ$

f. $(\tan\theta)^2 = 3$

23 Find all solutions t between 360° and 720° :

a. $\cos t = \sin t$ (no calculator)

b. $\sin t = -0.9397$

c. $\cos t < \frac{\sqrt{3}}{2}$ (no calculator)

In the next few problems you are asked to come up with conjectures or to examine the validity of various statements. You might want to see if the conclusion reached is even plausible, by looking at specific, easy-to-check cases. A simple check will either give the conclusion credence (and thus makes it worth trying to prove) or disprove it instantly.

24 Asked to simplify the expression $\sin(180^\circ - \theta)$, Alex volunteered the following solution:
$\sin(180^\circ - \theta) = \sin 180^\circ - \sin\theta$, and, because $\sin 180^\circ$ is zero, it follows that $\sin(180^\circ - \theta)$ is the same as $-\sin\theta$.

a. Is this conclusion plausible?

b. If it is plausible, try to prove the result. If it isn't, can you come up with a correct way to express $\sin(180^\circ - \theta)$ in simpler form?

c. Answer the same questions for $\cos(180^\circ - \theta)$.

25 Find simpler, equivalent expressions for the following: $\sin(180^\circ + \theta)$, $\cos(180^\circ + \theta)$, $\tan(180^\circ + \theta)$, $\cos(360^\circ - \theta)$, $\sin(360^\circ - \theta)$, $\tan(360^\circ - \theta)$, $\cos(360^\circ + \theta)$, $\sin(360^\circ + \theta)$, $\tan(180^\circ - \theta)$, $\tan(360^\circ + \theta)$

26 Is it the case that $\sin(90^\circ - \theta) = \cos\theta$? Explain. Then write simpler, equivalent expressions for
$\cos(90^\circ - \theta),\ \sin(90^\circ + \theta),\ \cos(90^\circ + \theta)$.

27 Is $\sin(-\theta)$ always the same as $-\sin(\theta)$? What can be said about $\cos(-\theta)$?

28 Do the following problems without using a calculator. Explain your reasoning.

a. Which is larger: $\cos 311^\circ$ or $\cos 312^\circ$?

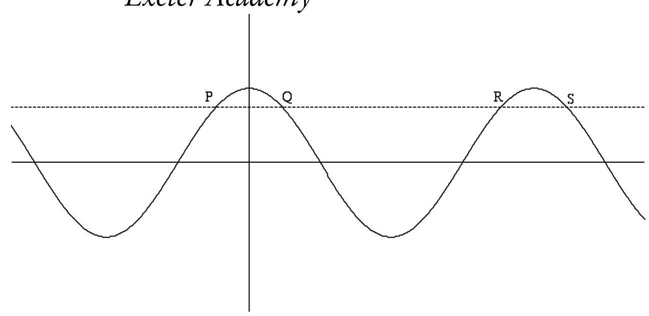
b. Which is larger: $\sin 311^\circ$ or $\sin 312^\circ$?

29 If $\sin A$ is known to be 0.96, then what is $\cos A$? What if it is also known that A is an obtuse angle?

30 Hendrickson is thinking of an angle θ where $\tan\theta > \sin\theta$, and also where $\sin\theta < \cos\theta$. Give two possible values for θ that are in different quadrants.

31 Rodney is running around the circular track $x^2 + y^2 = 10000$, whose radius is 100 meters, at 4 meters per second. Rodney starts at the point $(100, 0)$ and runs in the counterclockwise direction. After 30 minutes of running, what are Rodney's coordinates? *Copyright Phillips Exeter Academy*

32 Below are the graphs of $y = \cos x$ and $y = 0.7431$ (dotted). Given that $Q = (42, 0.7431)$, find coordinates for the intersection points P, R, and S without using a calculator. Use a calculator to check your answers. *Copyright Phillips Exeter Academy*



33 For how many angles in the first quadrant will $\sin\theta$ be rational?

34 One way to find the sine of a 60 degree angle is to draw a circle of radius 2 and draw the related 30°-60°-90° triangle. Find the length of the arc of the circle that is intercepted by this 60 degree angle.

35 Choose an angle θ and calculate $(\cos\theta)^2 + (\sin\theta)^2$. Repeat with several other values of θ. Explain the results. (Note that it is customary to write $\cos^2\theta + \sin^2\theta$ instead of $(\cos\theta)^2 + (\sin\theta)^2$.)

36 For any angle θ, is it true that $1 + \tan^2\theta = \sec^2\theta$? If so, prove it. If not, produce a counterexample.

37 Asked to find an expression that is equivalent to $\cos(\alpha + \beta)$, Laura responded $\cos(\alpha) + \cos(\beta)$. What do you think of Laura's answer, and why?

38 On your calculator, draw a graph of the function $y = \tan x$ with $0^\circ < x < 360^\circ$. Something appears to be wrong here. Say what you think is wrong and explain what's going on.

39 Solve the following without finding the angle θ.

a. Given that $\sin\theta = \frac{12}{13}$, with $0^\circ < \theta < 90^\circ$, find the values of $\cos\theta$ and $\tan\theta$.

b. Given that $\cos\theta = \frac{7}{25}$, with $270^\circ < \theta < 360^\circ$, find $\sin\theta$ and $\tan\theta$.

c. Suppose that $\csc\theta = -\frac{25}{7}$ and $\tan\theta < 0$. Evaluate, in fractional form, the remaining five trigonometric functions of θ.

d. The point $P(-12, 16)$ is on a circle whose center is the origin. Find the cosine of angle θ, the angle between the positive x-axis and the radius to P.

e. The point P is on a circle whose center is the origin, and the x-coordinate of P is five times its y-coordinate. Find the cosine of angle θ, the angle between the positive x-axis and the radius to P.

40 Is it possible for $\sin\theta$ to be exactly twice the size of $\cos\theta$? If so, find such an angle θ. If not, explain why not.

41 Find the three smallest positive solutions to $2\sin\theta = -1.364$.

42 Don't use a calculator for this problem.

a. Reduce: $\dfrac{5ab-10bc}{ab}$

b. Reduce: $\dfrac{3x^2-24}{3x}$

c. Solve for x: $2x^2-3x-2=0$

d. Add: $\dfrac{7}{4}+\dfrac{3a}{b}$

e. Subtract: $\dfrac{x}{4}-\dfrac{x-1}{2}$

43 Find all the solutions x, given $0^\circ \le x < 360^\circ$.

a. $(\cos x - .5)(\sin x + \dfrac{\sqrt{3}}{2}) = 0$

b. $(\cos x)^2 - 5\cos x + 4 = 0$

c. $3(\tan x)^2 + 5\tan x = 0$

d. $2(\sin x)^2 + \sin x = 3$

Exploring in Depth

44 Paul rides a Ferris wheel for five minutes. The diameter of the wheel is 10 meters, and its center is 6 meters above the ground. Each revolution of the wheel takes 30 seconds. Paul's fear of heights kicks in when he is more than 9 meters above the ground. For how many seconds does Paul feel uncomfortable?

45 Jasper's bike has wheels that are 27 inches in diameter. After the front wheel picks up a tack, Jasper rolls another 100 feet and stops. How far above the ground is the tack?

46 You have a circular dartboard. Its target area is defined in an unusual way. Take any point within the dartboard. Draw a straight line that goes through this point and the center of the board. Measure the angle this line makes with the line going from the center to the "easternmost" point on the dart board. If the tangent of this angle is between -0.2 and 0.6, then the point is shaded. Otherwise, it is not shaded. What are your chances of hitting this target area?

47 Given that $\sin\theta = k$, and that $0^\circ < \theta < 90^\circ$, find expressions for $\cos\theta$ and $\tan\theta$.

48 What do the graphs of $y = \sin x$ and $y = \sin 2x$ have in common, and how do they differ? How about the graphs of $y = \cos x$ and $y = \cos mx$, where m is any positive integer?

49 Starting at the same spot on a circular track that is 80 meters in diameter, Andy and Brandon run in opposite directions, at 300 meters per minute and 240 meters per minute, respectively. They run for 50 minutes. What distance separates Andy and Brandon when they finish? Interpret the word *distance* any way you wish.

LESSON 2: LAW OF SINES AND LAW OF COSINES

Introduction

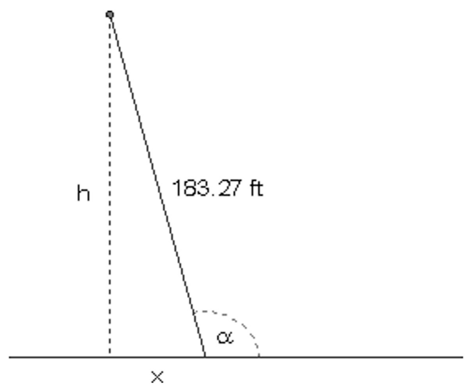
The Leaning Tower of Pisa

The Leaning Tower of Pisa has been leaning almost since the onset of construction in 1173. From ground to top, the tower is 183.27 ft along the lowest side and 186.02 ft along the highest side. The angle of slant of the tower—formed by its shorter slant height and the ground—is 95.5°. An apocryphal tale states that Galileo Galilei (1564-1642), an Italian physicist, mathematician, and philosopher, dropped two cannon balls of different weights from the top of this leaning tower in trying to demonstrate that the descending speed of a falling body is independent of its weight.

1. What is the distance traveled by an object dropped from the top of the Tower of Pisa, on its lowest side, when it hits the ground?

2. What is the horizontal distance from the point where the object hits the ground to the base of the tower?

The Tower of Pisa first acquired a slant after the third floor was built in 1178. More recently, in 1990, it was closed to the public because of safety fears. In fact, the tower was on the verge of collapse, and it was projected that it would have collapsed between 2030 and 2040. However, it has been straightened a bit, but still remains slanted (the tower has been reopened to the public). Thus, the angle of slant of the Tower of Pisa, α, has been changing for centuries. The following is a sketch of the situation.



3 In trying to determine a general expression for the distance, h, traveled by an object dropped from the top of the Tower of Pisa's lowest side when it hits the ground (assuming that the shorter length of the tower has not changed), Gerald found the following expression.

$$h = 183.27 \sin \alpha$$

For the horizontal distance from the point where the object hits the ground to the base of the tower, Gerald found the following expression.

$$x = -183.27 \cos \alpha$$

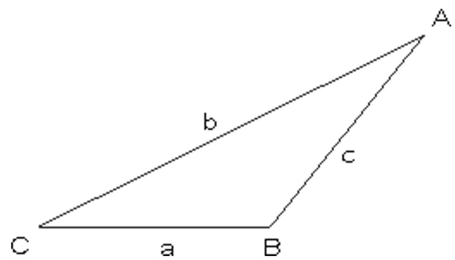
Do you agree with these statements? Explain your answer.

4 You have no more than two seconds after reading the statement of this problem to solve it.

What is the value of $\sin^2\left(\dfrac{\sqrt[7]{123456}}{\pi+13}\right)+\cos^2\left(\dfrac{\sqrt[7]{123456}}{\pi+13}\right)$?

Development

We follow the convention of labeling the angles of a triangle using capital letters, and the lengths of the corresponding opposite sides with the corresponding lower-case letters. For example, we may label A, B, C the angles of the triangle, and a, b, c the corresponding opposite sides, as in the figure below.

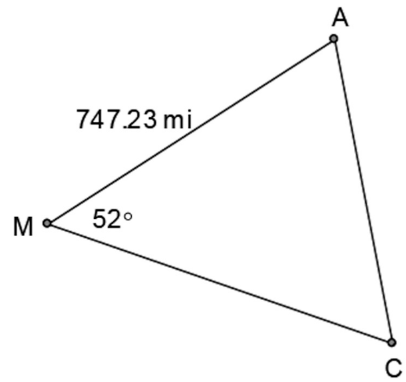


The Bermuda Triangle

The “Bermuda Triangle” or “Devil’s Triangle” is an imaginary area located off the southeastern Atlantic coast of the United States of America, which is noted for a supposedly high incidence of unexplained disappearances of ships and aircraft. The vertices of the triangle are generally believed to be Bermuda, Miami (Florida), and San Juan (Puerto Rico).

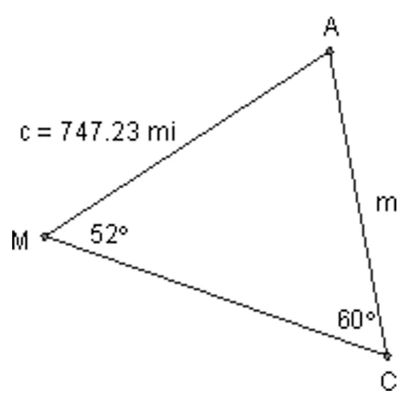
One of the amazing stories from this triangle tells that an aircraft (A in the figure below) was 747.23 miles from the airport of Miami (M in the figure), on the line joining Miami and Bermuda, when its crew received an SOS signal from Cyclops (C in the figure), a ship located at a point on the line joining Miami airport and

San Juan, which was sinking. Flight controllers at the airport were able to estimate the measure of $\angle AMC$ to be 52°.

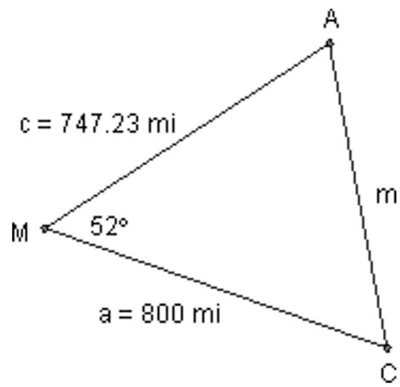


Based on the previous story, Mr. Thomas Howard designed a problem for his mathematics class. The initial situation described in the story, however, contained more than the minimal information required to solve the problem; therefore, he divided the class into two groups, Group I and Group II, and gave each group a problem with different pieces of information, but the same goal to determine how many miles the aircraft had to travel through the purportedly dangerous Bermuda Triangle before reaching Cyclops.

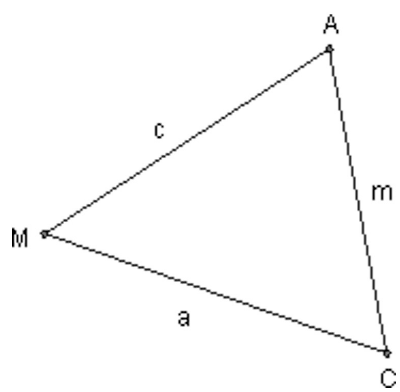
5 Group I: Besides the general information described above, it is known that $m\angle C = 60^\circ$. What is the distance between the aircraft A and the sinking ship Cyclops C? Find a solution to this problem and explain how you arrived at your answer.



6 Group II: Besides the original and general information, it is known that the distance from Cyclops to the airport of Miami is 800 miles, as shown in the figure below. What is the distance between the aircraft A and the sinking ship Cyclops C? Find a solution to this problem and explain how you arrived at your answer.



7 In Problem 5, if the distance given were m and the one to be found were c, the problem could be solved in a similar way. Consider the more general triangle below.



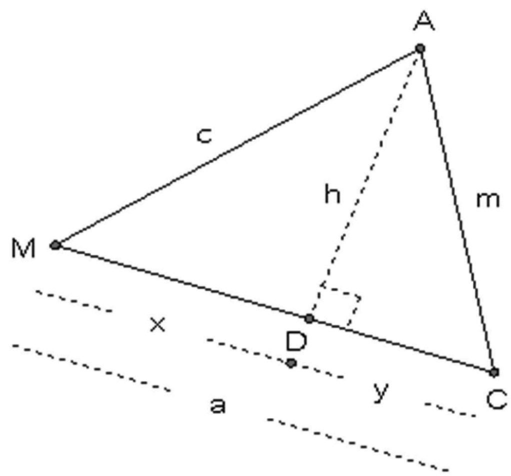
a. Draw the altitude from A to side $\overline{MC}$, and prove that
$$\frac{c}{\sin C} = \frac{m}{\sin M}.$$

Hint: If h is the length of the altitude from A to $\overline{MC}$, find h in ΔADM in terms of c and $\angle M$. Then find $\sin C$ in ΔADC.

b. Now, prove that $\dfrac{m}{\sin M} = \dfrac{a}{\sin A}$

In group II, Rebecca noticed that if the measure of $\angle M$ were 90° rather than 52°, the Pythagorean Theorem would guarantee that $m^2 = a^2 + c^2$. "However," she said, "since $\angle M$ is less than 90°, the Pythagorean relationship among m, a, and c must be adjusted." This adjustment is the result that you are about to find in Part c of the following problem.

8 Consider the triangle in Mr. Howard's problem in a more general form, as shown below, and prove the following.



a. $h = c \sin M$

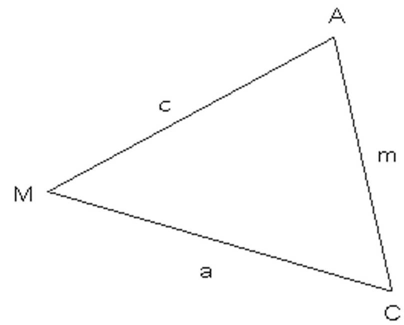
b. $y = a - c \cos M$

c. $m^2 = c^2 \sin^2 M + (a - c \cos M)^2$

9 In part c of the previous problem, expand the square on the right side of the equality. Then use the trigonometric identities that you have learned thus far to find an expression as simple as possible relating m to c, a, and $\angle M$.

As announced above, the expression relating m with c, a, and $\angle M$ that you may have found in the previous problem is the adjustment to the Pythagorean relationship $m^2 = a^2 + c^2$ required when $m\angle M < 90^\circ$. In a similar way, adjustments to the Pythagorean relationship relating a to m and c as well as to that relating c to a and m may be needed when either $m\angle A$ or $m\angle C$ is not 90°, as in the case illustrated above.

10 Consider the general triangle shown below.



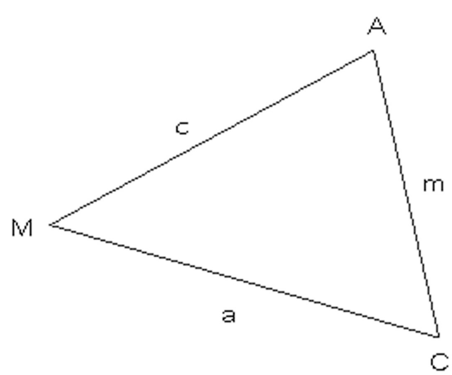
a. If $m\angle A = 90^\circ$, how would a be related to m and c?

b. Conjecture an adjustment to the Pythagorean relationship in part a that may be required when $m\angle A \neq 90^\circ$. Would the proof of your conjecture in this case be quite different from that developed in Problems 8 and 9? Explain.

c. If we had that $m\angle C = 90^\circ$, how would c be related to a and m?

d. Conjecture an adjustment to the Pythagorean relationship in part c that may be required when $m\angle C \neq 90^\circ$. Would the proof of your conjecture in this case be quite different from that required in Part b? Explain.

Summarizing, given a triangle as the one below,



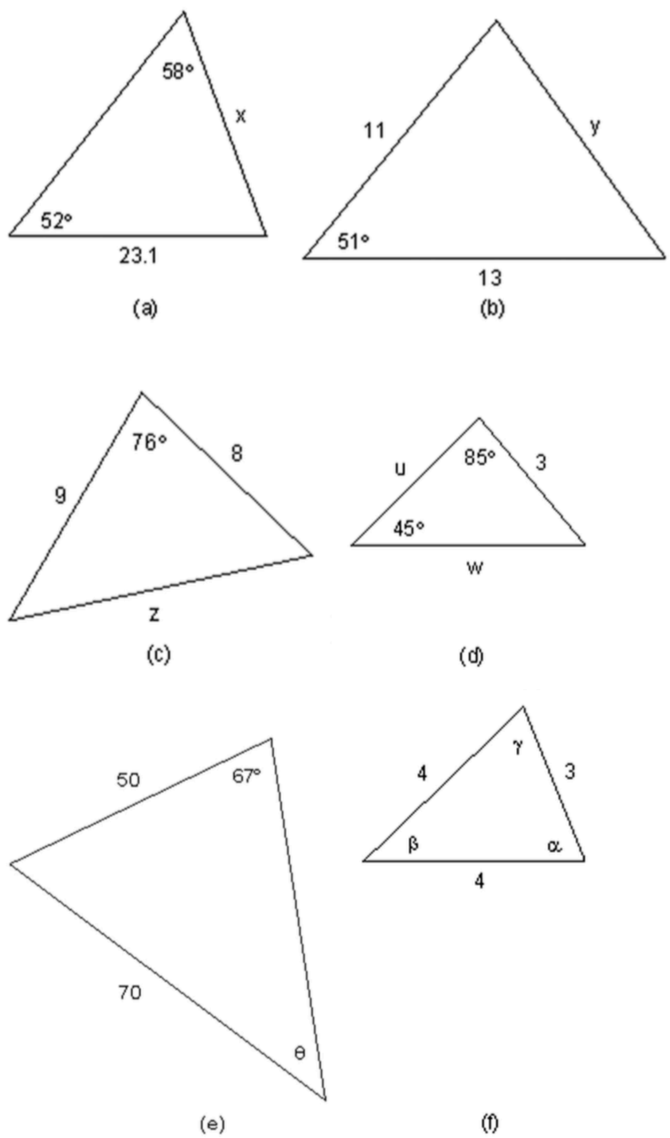
two sets of equalities have been found.

On the one hand, the set of equalities that you may have proven in Problem 7 is known as the **Law of Sines**.

On the other hand, the set of equalities that you may have found in Problems 9, 10b, and 10d is known as the **Law of Cosines**.

Practice

11 Find the indicated side lengths or angle measures in the following figures.

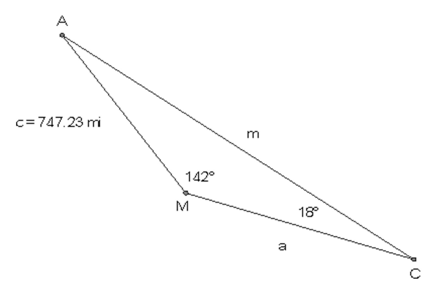


Further Development

When the students in Mr. Howard's class shared their findings (the Law of Sines and the Law of Cosines) the information contained in these laws was considered "the key" to find missing side lengths or angles in any triangle, when basic information about the triangle is known. However, in reality in all the proofs only acute triangles—triangles with each of their three angles being less than 90°—were used.

12

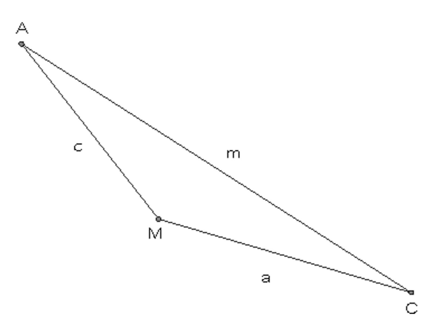
Regarding Mr. Howard's problem, consider the following situation where $m\angle M = 142^\circ$.



Does the proof of the Law of Sines (in Problem 7), or that of the Law of Cosines (in Problems 8 and 9), support the use of either of these laws to find m in this triangle, which is not an acute triangle? Explain.

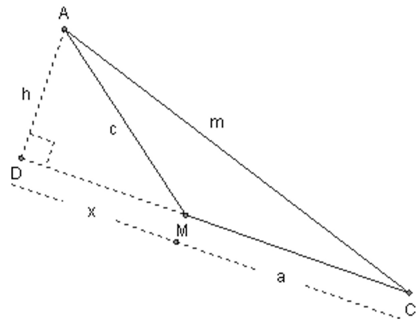
13

Consider the following obtuse ΔAMC (that is, a triangle containing an angle greater than 90°).



Prove that $\dfrac{c}{\sin C} = \dfrac{m}{\sin M} = \dfrac{a}{\sin A}$.

14 Consider again the obtuse ΔAMC. Dashed lines have been added to help you prove the equality stated in Part (a).



a. Prove that $m^2 = a^2 + c^2 - 2ac\cos M$.

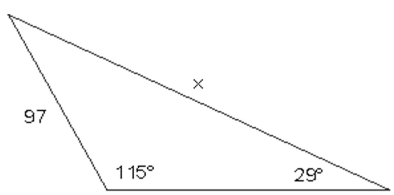
b. Prove that $a^2 = m^2 + c^2 - 2mc\cos A$.

c. Prove that $c^2 = m^2 + a^2 - 2ma\cos C$.

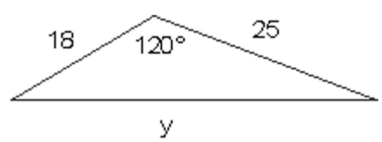
15 Is the Law of Sines or the Law of Cosines worth remembering? Would it be easier always to construct appropriate perpendicular lines and use only trigonometric ratios to solve the problems that may require them?

Practice

16 Find the indicated side lengths or angle measures in the following figures.



(a)



(b)

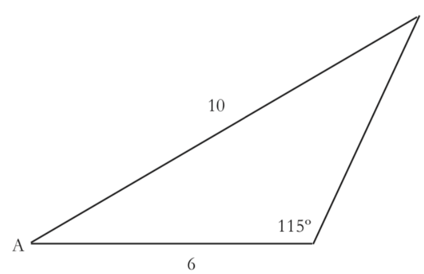
17 Two tracking stations are monitoring the path of a satellite, which has passed to the west of both stations. From station A, the angle of elevation to the satellite is 85.5 degrees. From station B, the angle of elevation to the satellite is 82 degrees. Stations A and B are 65 miles apart.

a. Find the distance from the satellite to tracking station A.

b. Find the height of the satellite above the ground.

18 The distance from Chicago to St. Louis is 440 km, from St. Louis to Atlanta 795 km, and from Atlanta to Chicago 950 km. What are the angles in the triangle with these three cities as vertices?

19 In the figure below, find the measure of angle A.

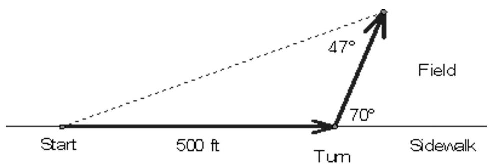


After having done these problems in the Practice section, you may be better prepared to answer question 15, repeated here:

20 Is the Law of Sines or the Law of Cosines worth remembering? Would it be easier always to construct appropriate perpendicular lines and use only trigonometric ratios to solve the problems that may require them?

Problems

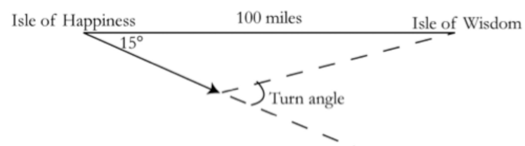
21 Melissa walks along the path shown below: She goes 500 ft along a sidewalk adjacent to a field, then turns 70 degrees, walks a way across the field, and stops. Looking back, she measures a 47 degree angle between her path across the field and her line of sight to her starting point.



a. Find the distance that Melissa walked across the field.

b. How far away is Melissa from her starting point?

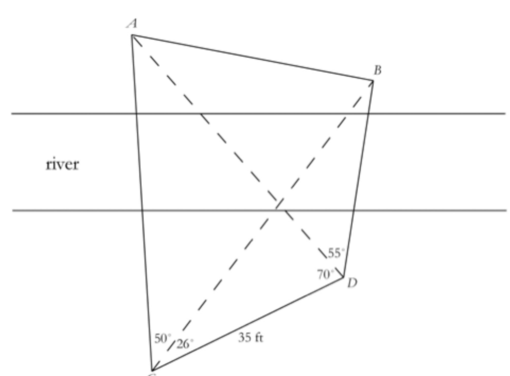
22 A sailboat is attempting to sail between two islands, 100 miles apart. From the very beginning, a wind blows the boat 15 degrees off its course. After the sailboat has been sailing for an hour and a half at 25 mph, it corrects its course so it is sailing straight toward the second island.



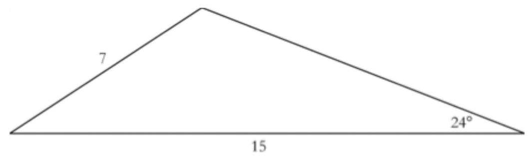
a. By how much does the sailboat need to increase its speed if it wants to arrive at its destination at the same time it would have going 25 mph along the straight path?

b. Find the "turn angle" – the number of degrees the sailboat needed to turn in order to correct its course.

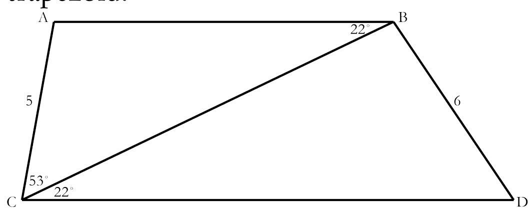
23 Julie needs to find the distance between two trees A and B on the opposite side of a river. On her side of the river, she chooses two points C and D, 35 feet apart. Then she measures the angles shown in the diagram. What distance between the trees will she calculate?



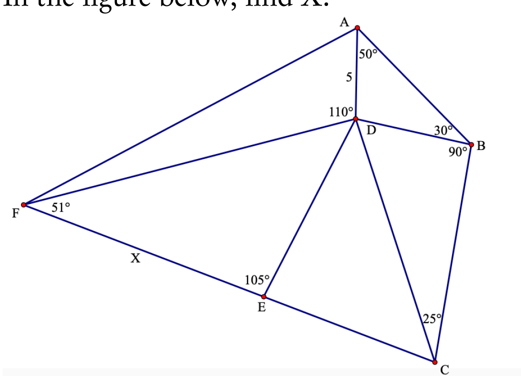
24 Find all three angles of this triangle. Check to be sure that your answer is plausible.



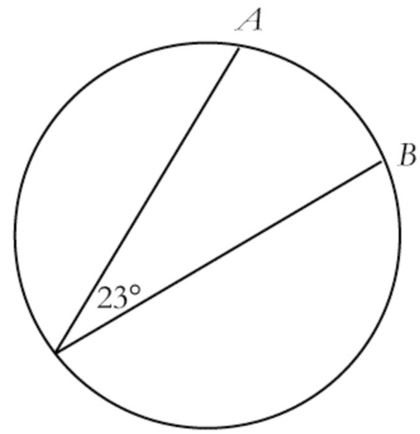
25 Find the measure of all the angles in the trapezoid.



26 In the figure below, find X.



27 In the figure below, the length of the chord drawn from A is 8, and the length of the chord drawn from B is 10. Find the length of arc AB. (Hint: you will need to know the radius of the circle.)



28 The function ThirdSide takes an angle, theta, and outputs the third side of a triangle with sides 3 and 4 and included angle theta.

a. Find $Thirdside(80^\circ)$.

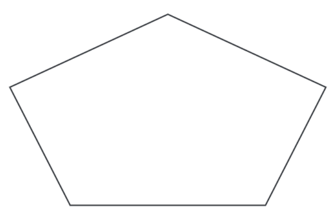
b. What are the minimum and maximum possible values for $Thirdside(\theta)$? Justify your answer in two different ways:

i. by visualizing what different triangles would look like for different values of θ.
ii. by looking at the Law of Cosines formula and seeing how the value of theta affects the value of each term.

Often times, a problem that seems to be hard may be simplified a lot by just drawing a diagram or adding a couple of lines—or even just points—to a diagram already in place. These additions (to the concrete situation given) are implemented to **visualize** and better understand problems which may have been initially confusing. In the following problems, 29 through 33, you will have the chance to use this mathematical habit of mind repeatedly.

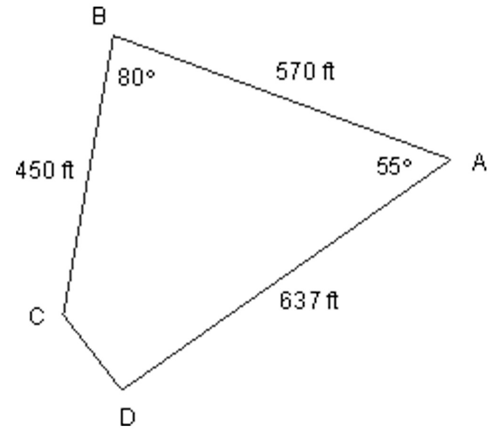
29 A triangle has a 13-inch, a 14-inch, and a 15-inch side. To the nearest tenth of an inch, how long is the median drawn to the 14-inch side? (Recall that a median is a line segment drawn from a vertex of a triangle to the midpoint of the opposite side.)

30 A “half-regular” pentagon isn’t perfectly regular, but it does fold perfectly in half (the left half is the same as the right half – it’s symmetric across a vertical axis). The half-regular pentagon below has a top angle of 160°, a side length of 8 for the top two sides, a side length of 9 for the base, and a total height of 14 (from the top point down to the base).

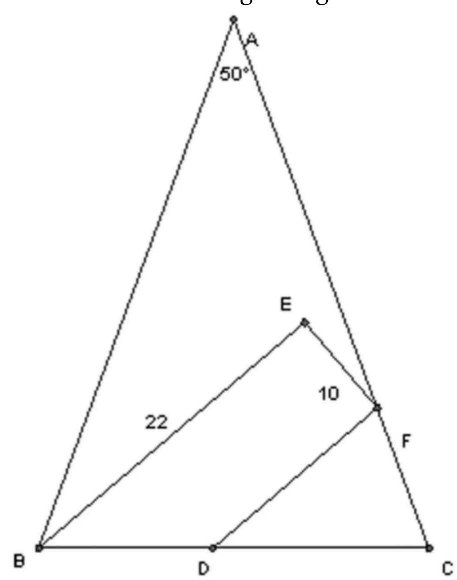


Find the area and perimeter of the pentagon.

31 The diagram below represents a plot of a piece of land. Find the area of the plot.



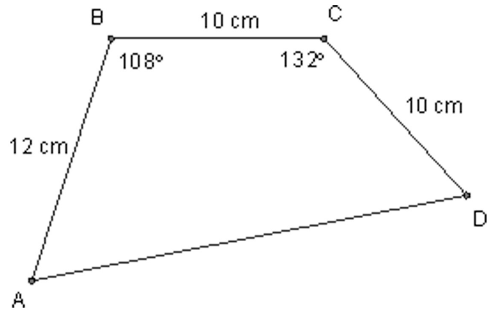
32 In this diagram, ΔABC and ΔCDF are both isosceles. $AB = AC$ and $DF = DC$. $\angle E$ and $\angle EFD$ are right angles.



a. Find all the angles in the diagram.

b. Find AF.

33 The lengths of three sides and the measure of two angles of a quadrilateral are given, as shown in the figure below.

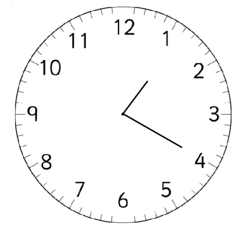


a. Determine the length of the diagonals of this quadrilateral. Round answers to two decimal places.

b. Determine the perimeter of this quadrilateral. Round the answer to two decimal places.

c. Determine the area of this quadrilateral. Round the answer to two decimal places.

34 You're looking at the hour hand and the minute hand on a clock at exactly 1:20. The tips of the hands are 3 inches apart. The hour hand is 2.15 inches long. (Remember – the hour hand is not pointing directly at the 1, since it's after 1:00!)

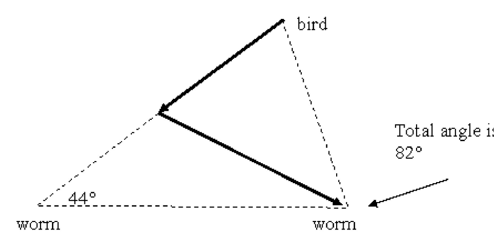


a. What's the angle between the hands?

b. Draw the triangle that this forms, and find the other angles in the triangle.

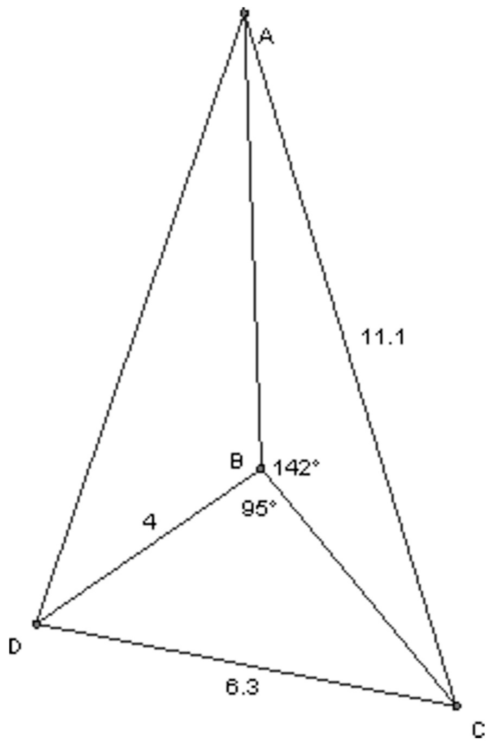
c. Find the length of the minute hand.

35 A bird sees two worms on the ground. The worms are 23 inches apart. The bird flies at the worm on the left, but when it's exactly halfway to the worm, it turns and flies to the worm on the right. The angles are as marked, and the angle marked 82° refers to the entire angle on the right side. Find all the other lengths and angles in the diagram below.

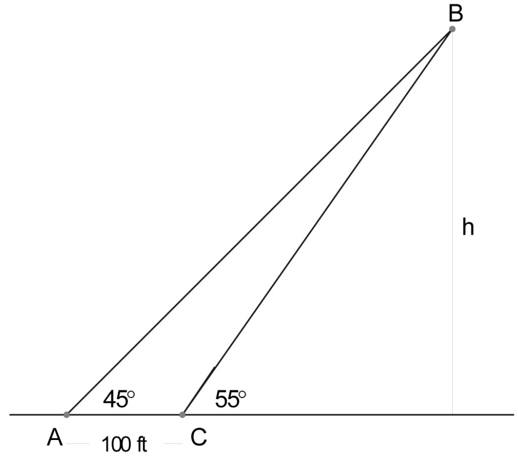


36 Use the law of cosines to determine the angles in a triangle that has sides of lengths 7.3, 23.1, and 15.7. Why would this problem be easier if the sides were 8.1, 23.9, and 16.2?

37 In the figure below, find AB.



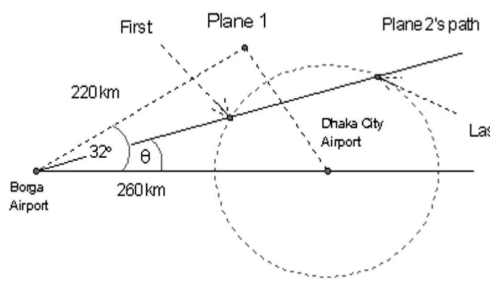
38 A balloon, B, is tethered to the ground by wires $\overline{AB}$ and $\overline{CB}$ as shown in the figure below. How high, h, is the balloon above the ground?



39 In any triangle ABC, prove that $m\angle C = \cos^{-1}\left(\dfrac{a^2+b^2-c^2}{2ab}\right)$.

Exploring in Depth

40 Borga and Dhaka City are two of the main cities in Bangladesh. Borga Airport and Dhaka City Airport are 260 km apart. The ground controllers at Dhaka City monitor planes within a 100-km radius of the airport.



a. Plane 1 is 220 km from Borga Airport at an angle of 32° to the straight line between the airports. Is it within the range of Dhaka City Ground Control?

b. Plane 2 takes off from Borga Airport toward Dhaka City Airport at an angle θ with the line between the airports. If θ is small enough, there is a point when Plane 2 first comes within range of Dhaka City Ground Control, and another point when it is last within range. Is there a value of θ for which Plane 2 is within range of Dhaka City Ground Control at just one point? If so, what is the magnitude of this angle?

c. If $\theta = 15^\circ$, how far will Plane 2 be from Borga Airport when it first comes within range of Dhaka City Ground Control? How far from Borga Airport is it when it is last within range?

41 A triangle has six parts: three sides and three angles.

a. If we know only two out of the six parts of a triangle, is it enough information to precisely describe what triangle it is? Explain.

b. What if we know its three angles? Is it enough information to precisely describe that triangle? Explain.

c. What is the minimal information about the six parts of a triangle needed to precisely describe a triangle?

42 Don't use a calculator for this problem.

a. Reduce: $\dfrac{x^2y + y^2x}{xy}$

b. Reduce: $\dfrac{4x - 20y}{16x + 20y}$

c. Simplify: $\left(\dfrac{84x^5y^2}{14x^{-2}y^4}\right)^{-2}$

d. Rewrite using fractional exponents: $\sqrt{\sqrt[3]{\sqrt{x}}}$

e. If $a = \sqrt{b}$, find a^3 in terms of b.

As you may have explained in Problem 41, in general a triangle is determined by three of its six parts, where at least one of these parts is a side. These are the possibilities. Case SAA: when one side and two angles are known. Case SSA: when two sides and the angle opposite one of those sides are known. Case SAS: when two sides and the included angle are known. Case SSS: when the three sides are known.

43 In the Case SSA described above, what can we say about a triangle for which two sides and the angle opposite one of those sides are known?

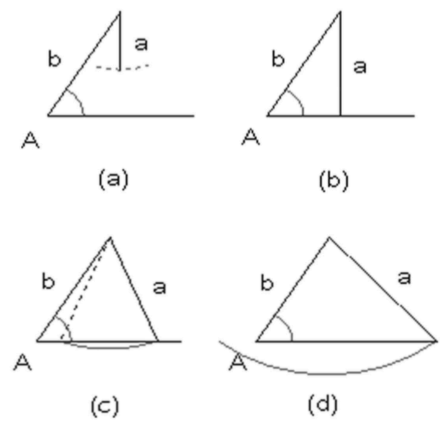
Note: Case SSA is known as the **ambiguous case**. Are there good reasons for this name?

44 Find the side lengths and measures of the angles of $\triangle ABC$ if $m\angle A = 43.1^\circ$, $a = 186.2$, and $b = 248.6$.

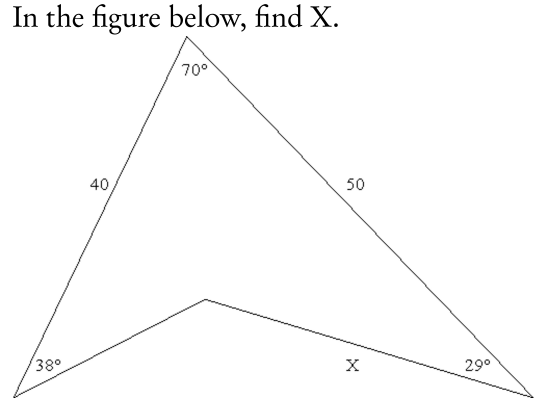
45 Find the side lengths and measures of the angles of the angles of ΔABC if $m\angle A = 42^\circ$, $a = 70$, and $b = 122$.

46 **The Ambiguous Case**

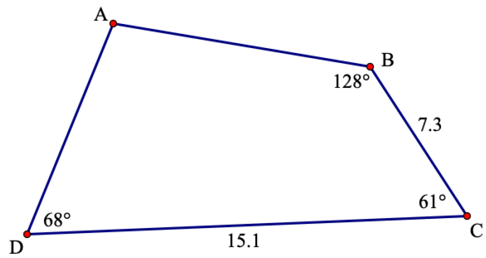
In Case SSA, when two sides and an angle opposite one of those sides are given, it is possible that none, one, or two triangles may exist satisfying the given information. These possibilities are illustrated in the figure below, where $\angle A$, a, and b are the angle and two sides given. For each case, (a) through (d), explain how $\angle A$, a, and b are related.



47 In the figure below, find X.



48 In the figure below, find the lengths of the other two sides of this quadrilateral.



49 **Heron's Formula**

Suppose you have ΔABC.

a. Prove that its area, A, is given by
$A = \frac{1}{2}\, ab \sin C$

b. Prove that
$A^2 = \frac{1}{4}\, a^2 b^2 \left(1 - \cos C\right)\left(1 + \cos C\right)$

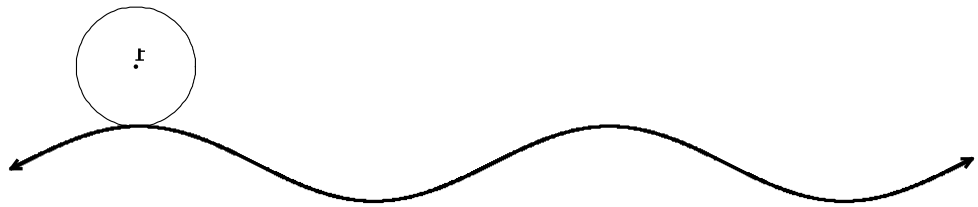
c. Use the Law of Cosines to express $\cos C$ in terms of a, b, and c, and from Part b prove Heron's Formula:
$A = \sqrt{s(s-a)(s-b)(s-c)}$,

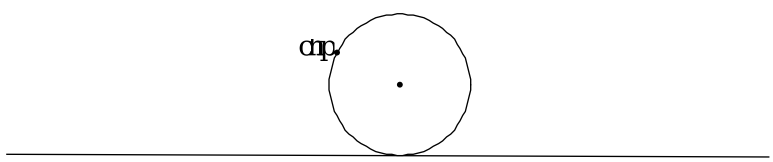
where $s = \frac{1}{2}\left(a + b + c\right)$ is the **semiperimeter** of the triangle.

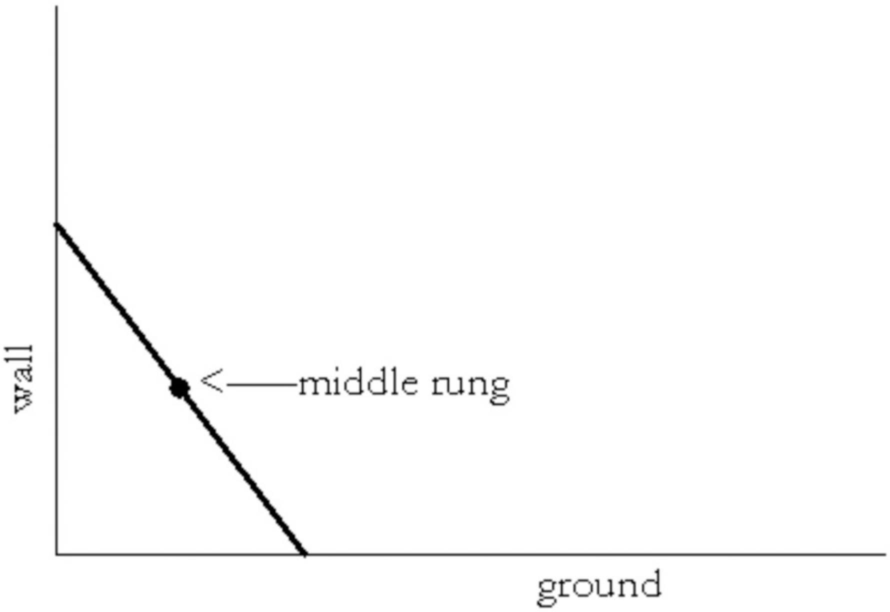
LESSON 3: LOCI

Introduction

Picture your favorite seat on a gently turning Ferris wheel, the hub of the wheel of a bicycle ridden along an undulating country road, that chip on the rim of the wheel of the old trolley as it hits the smooth flat road, or the middle rung of a ladder as the ladder slides down a wall and along the flat ground.







1 Sketch the path of:

a. The Ferris wheel seat.

b. The wheel's hub.

c. The chip on the rim of the wheel of the old trolley. (Experiment with this one a bit.)

d. The ladder's middle rung. (Experiment with this one a lot.)

To check your answer to problem 1d, try the webpage,
http://homepage.mac.com/dscher/ladder.html

These four familiar settings provide a peep into a very interesting concept in mathematics. In the first instance, you might have drawn a circle for the path of your favorite seat. Mathematicians would refer to this path as the locus (plural, loci) of the seat, and would perhaps state it more formally/awkwardly as follows:

"The locus of the seat of the Ferris wheel is a circle whose center is the axis of the Ferris wheel and whose radius is equal to the distance from your seat to the axis."

So, in general, the **locus** of an object or point is the graph of all the possible positions the object or point could occupy under certain restrictions. This is the focus of our lesson. We will try to describe loci in words as well as in equations where possible, as well as to recognize shapes represented by particular equations.

[For this lesson you will need to have a protractor, ruler and compass.]

Development

For each of problems 2 through 7 do the following: On a sheet of graph paper set up coordinate axes. Find a point which satisfies the stated condition, and label it P. Draw as many such points P as you can. Decide whether these points form any particular shape. If so, describe that shape.

2 The x-coordinate of the point is 5.

3 The point is always equidistant from the two points $A\,(3,-5)$ and $B\,(-1,-3)$.

4 The x-coordinate of the point is always less than or equal to the y-coordinate of the point.

5 The distance of the point from the x-axis is always twice its distance from the y-axis.

6 The distance from the point to the origin is always 6.

7 First graph the points $M\,(2,3)$, $A\,(6,3)$ and $C\,(2,7)$. Now the restriction on a point P is that the points P, M, A and C must form a parallelogram. What is the locus of P?

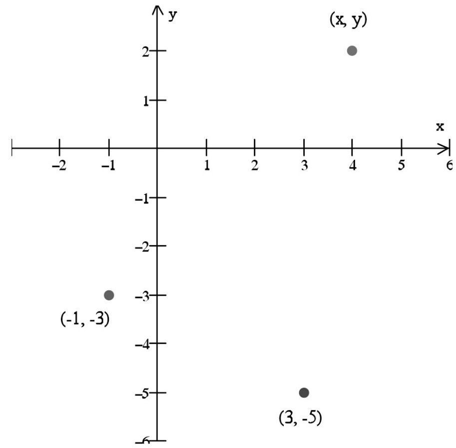
Remember, when you describe something, be sure that your description is complete. It should thoroughly cover every aspect of what's being described.

Your description should also be unambiguous. This means that it should be completely clear, and not open to any different interpretations. You should ask yourself, "Would it be possible for the reader to read my description carefully, but misunderstand what I mean?" If so, you know you need to be more specific.

Finally, your description should be understandable to the intended reader.

One could describe the locus of P in problem 3 as the **perpendicular bisector** of the line segment AB. Describing a locus in words makes it relatively easy to draw the locus, but it is also useful to describe the locus with an equation or equations, when this is possible. An equation allows you to find specific locus points quite quickly.

8 Returning to problem 3, in which P is always equidistant from the two points $A\,(3,-5)$ and $B\,(-1,-3)$, let the coordinates of P be (x, y).



a. The condition for P could be written "The distance from P to A is the same as the distance from P to B." Using the coordinates and diagram above, translate this sentence into symbols.

b. In part a, you should have written an equation. Simplify this equation as much as possible.

c. Check your answer in part b for plausibility in a couple of ways.

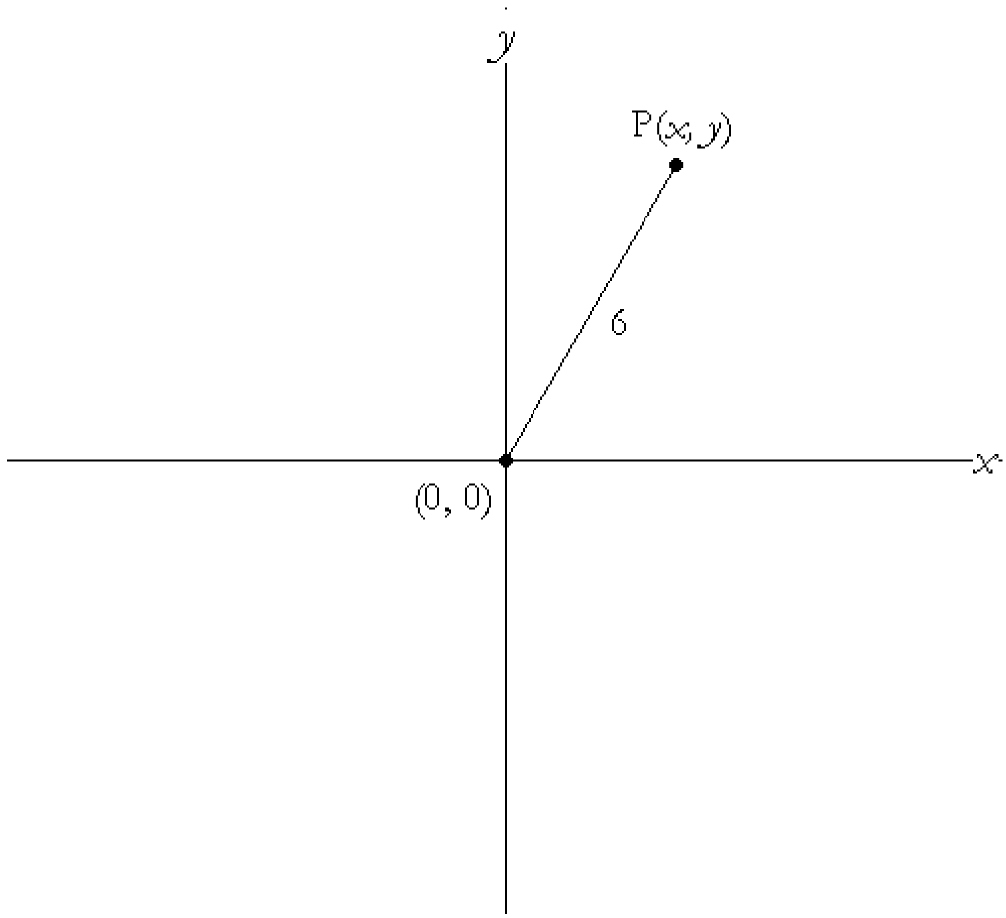
i. First check to see that the point $(1,-4)$ satisfies your equation in part b. Why must this be the case?
ii. Second, choose any other point whose coordinates satisfy your equation and do a calculation to see if the distance from that point to A and B is the same.

9 Returning again to problem 3, let's check to see if your description of the locus was indeed correct.

a. Find the slope and midpoint of $\overline{AB}$.

b. Use your answers for part a to find the equation of the perpendicular bisector of $\overline{AB}$.

10 Looking again at problem 6, with the help of distance formula or Pythagoras' Theorem, write an equation to describe the path of the point P.



11 Write an equation/inequality for the locus in each of the following. Also, in each case check that your answer is plausible by seeing if a couple of easily tested points indeed satisfy your equation.

a. Problem 2 (P's x-coordinate is 5.)

b. Problem 4 (The x-coordinate of P is always less than or equal to the y -coordinate of P.)

c. Problem 5 (P's distance from the x-axis is always twice its distance from the y-axis.)

12 The distance from P to the point is always 6.

a. Describe the locus of P.

b. Write an equation for the locus of P.

13 The distance from P to the point $S\,(h, k)$ is always r.

a. Describe the locus of P.

b. Write an equation for the locus of P.

c. Check that your answer for part b is reasonable when h and k are negative numbers far from zero and r is a small positive number, and then check it again for when h and k are negative numbers close to zero and r is a large positive number.

14 Describe the locus of the point whose equation is:

a. $(x-h)^2+(y-k)^2=r^2$

b. $(x+4)^2+(y+13)^2=100$

15 The equation of the locus of point P is given. Describe or draw the graph of the locus. (Making a chart first could be quite useful.)

a. $x+y=10$

b. $x=|y|$

c. $x^2=y^2$

16 One could argue that in some cases a description of a locus by equation is better than a description in words. What do you think? Give a couple of examples to support your position.

Practice

17 In each case describe the graph and, if possible, write an equation for the locus of P. Make sure to check that the equations you come up with are plausible, by checking a couple of interesting and/or "extreme" points that "should" be part of the locus.

a. The x-coordinate of P is always 7.

b. P's distance from the y-axis is three times its distance from the x-axis.

c. P's distance from the y-axis is always greater than its distance from the x-axis.

d. The distance from P to the origin is always less than 6.

e. Graph the points $M\,(5,3)$ and $N\,(10,3)$. Triangle MPN is always isosceles with base $\overline{MN}$.

f. Graph the points $E\,(-5,3)$ and $D\,(-10,3)$. Triangle EPD is always equilateral.

18 Draw or describe the locus of a point P whose coordinates satisfy the equation $3x + 4y = 12$.

19 Write an equation for the locus of a point which is always 7 units from the point

a. $(2,-7)$.

b. $(10,8)$.

20 Describe the locus of a point whose equation is given by $(x-2)^2 + (y+4)^2 = 25$.

Problems

21

Describe the following loci.

a. P is equidistant from the points $A\,(3,5)$, $B\,(3,8)$ and $C\,(10,8)$.

b. P is equidistant from the points $A\,(3,5)$, $B\,(3,8)$, $C\,(10,8)$, and $D\,(11,5)$.

c. P always forms a triangle with the points $R\,(3,-5)$ and $S\,(-1,4)$.

22

Describe the locus given by each of the following equations:

a. $x^2 - 4x + 4 + y^2 + 8y + 16 = 25$
(Hint: recall that
$x^2 - 4x + 4 = (x-2)^2$ and
$y^2 + 8y + 16 = (y+4)^2$.)

b. $x^2 + 6x + 9 + y^2 - 10y + 25 = 1$

c. $x^2 + 10x + y^2 - 6y = 15$

23

If you wish to plant some trees so that each tree is equidistant from every other tree, how many trees are you able to plant?

24

Graph the points $J\,(2,-6)$ and $L\,(2,10)$. In each case describe the locus of P. (You might want to do some careful drawing.)

a. P is such that $PJ + PL$ is equal to 16 units.

b. P is such that $PJ + PL$ is equal to 18 units.

25

Graph the points $A\,(0,-3)$ and B$(0,7)$. P is always 4 units from the line segment AB. Describe the locus of P.

26

What would the locus of the points a distance 6 from $(0,0)$ be if one could only move an integral amount left/right and up/down?

27

Graph the points $M\,(2,3)$, $A\,(5,7)$ and $C\,(-1,5)$. Point P is such that the points P, M, A and C always form a parallelogram. Describe the locus of P as fully as possible.

With harder problems that have formulas that you are deriving, a plausibility check is a necessity, as it is all too easy to make a conceptual or algebraic mistake when coming up with the equation, and a plausibility check is a quick way to see if you have made some kind of error. For the next few problems be sure to make a plausibility check so that you can be more confident your answer isn't flawed.

28 Graph the points $R(5,0)$ and $S(-5,0)$. Point P is such that the angle RPS is always a right angle.

a. Describe the locus of P. (Carefully plot points on graph paper, using a protractor or some object with a 90^o angle.)

b. Write an equation for the locus of P.

c. What general statement do parts a and b suggest?

29 Point P is such that its distance from the point $(0,2)$ is the same as its distance from the line$y=-2$.

a. Carefully sketch the locus of P.

b. Describe the shape of the locus.

c. Write and simplify an equation for the locus.

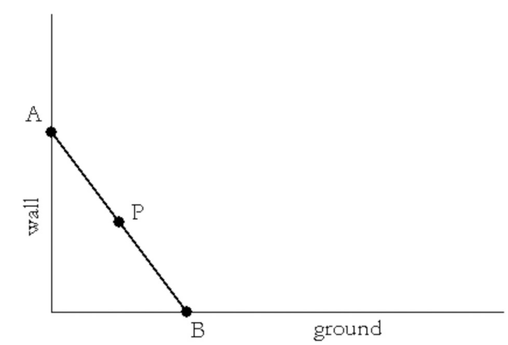
30 Construct a locus problem different from the ones you have encountered in this lesson.

Exploring in Depth

31 Graph the points $M(5,3)$ and $N(10,3)$. Triangle MPN is always isosceles. What is the locus of P?

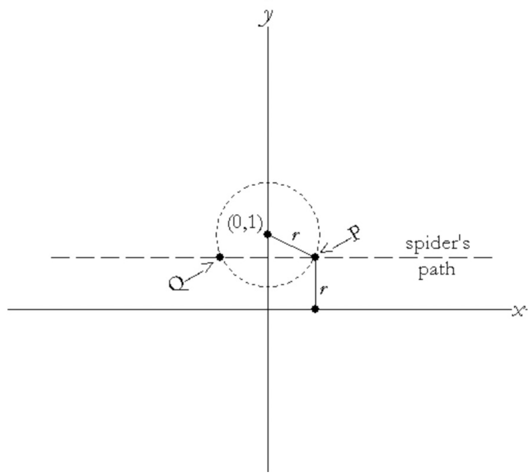
32 Graph the points $R(5,0)$ and . Point P is such that the measure of $\angle RPS$ is always 30°. The locus of P seems to be two pentagons, one at the top of $\overline{RS}$ and one at the bottom. What do you think? Be sure to make a cogent argument.

33 The diagram below models the ladder problem of problem 1. Find the equation of the locus of the midpoint of the ladder $\overline{AB}$ as A slides along the wall and B along the ground. (Hint: besides the length of the ladder another length remains constant throughout the motion.)



34 E is the point $(3,5)$ and T is the point $(3,11)$. Describe and find an equation for the locus of P, given that the area of triangle EPT is always 12 cm^2.

35 In the spiders and flies game, flies fly in circles and spiders crawl on horizontal lines. In this particular game, the flies must fly in circles with center $(0, 1)$. And here is the rule for spiders: Spiders crawling on a line r units from the x-axis can only catch flies that are moving on the circle whose radius is r. So flies will be caught at the points P and Q in the diagram below.



a. Suppose flies were flying on the circle with radius 5. At what point would they be caught by spiders?

b. Would any flies be caught at the point $(2, 2.5)$? How about the points $(1, 1)$, $(3, 3)$ and $(0, 0.5)$?

c. Carefully sketch the locus of the point where a spider catches a fly.

d. Describe the shape of the locus.

e. Write and simplify an equation for the locus.

36 In one spiders and flies game in problem 35 the equation of the locus of the capture is $y = 4x^2$. What are the rules for this particular game?

37 Don't use a calculator for this problem.

a. Reduce: $\frac{a^2-b^2}{a+b}$

b. Separate and reduce: $\frac{\sqrt{x}+x^2}{\sqrt{x^3}}$

c. Simplify: $\sqrt[3]{x\sqrt{x\sqrt[6]{x}}}$

d. Simplify: $\frac{4^{-1}+4^0+4^{\frac{1}{2}}}{4^{-1}-4^{-\frac{1}{2}}}$

e. If $2^x = 7$, what is the value of 2^{2x+1}?

Park School Mathematics

This curriculum was written by members (past and present) of the upper school math department at the Park School of Baltimore. This was possible thanks to Park School's F. Parvin Sharpless Faculty and Curricular Advancement (FACA) program, which supports faculty every summer in major curricular projects. In addition to the support of FACA by the Nathan L. Cohen Faculty Enhancement fund and the Joseph Meyerhoff FACA fund, this project was also funded by grants from the E. E. Ford Foundation, the Benedict Foundation, Josh and Genine Fidler, and an anonymous donor.