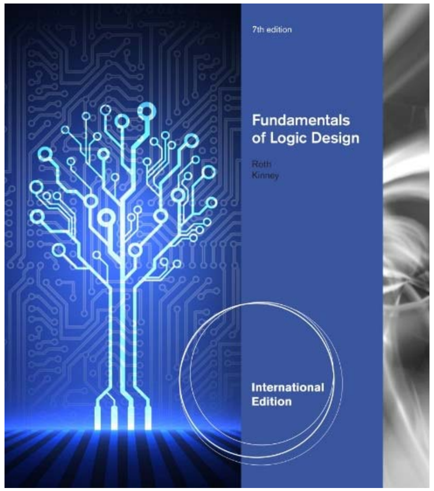
CHAPTER 3

BOOLEAN ALGEBRA

(continued)



This chapter in the book includes:

Objectives

Topics introduced in this chapter:

- Apply Boolean laws and theorems to manipulation of expression
  - Simplifying
  - Finding the complement
  - Multiplying out and factoring
- Exclusive-OR and Equivalence operation(Exclusive-NOR)
- Consensus theorem

3.1 Multiplying Out and Factoring Expressions

To obtain a sum-of-product form ➔ Multiplying out using distributive laws

$$X(Y+Z) = XY + XZ$$

$$(X+Y)(X+Z) = X + YZ$$

Theorem for multiplying out:

$$(X+Y)(X'+Z) = XZ + X'Y \qquad (3\text{-}3)$$

If $X = 0$, (3-3) reduces to $Y(1+Z) = 0 + 1*Y$ or $Y = Y$.

If $X = 1$, (3-3) reduces to $(1+Y)Z = Z + 0*Y$ or $Z = Z$.

because the equation is valid for both $X = 0$ and $X = 1$, it is always valid.

The following example illustrates the use of Theorem (3-3) for factoring:

Theorem for factoring:

$$AB + A'C = (A+C)(A'+B)$$

3.1 Multiplying Out and Factoring Expressions

i) Multiplying out using Theorem :

$$(Q+AB')(C'D+Q')=QC'D+Q'AB'$$

ii) Multiplying out using distributive laws :

$$(Q+AB')(C'D+Q')=QC'D+\boxed{QQ'+AB'C'D}+AB'Q'$$

Redundant terms

multiplying out: (1) distributive laws (2) theorem(3-3)

$$(A+B+C')(A+B+D)(A+B+E)(A+D'+E)(A'+C)$$

$$=(A+B+C'D)(A+B+E)[AC+A'(D'+E)]$$

$$=(A+B+C'DE)(AC+A'D'+A'E)$$

$$=AC+\cancel{ABC}+A'BD'+A'BE+A'C'DE \qquad (3\text{-}4)$$

What theorem was applied to eliminate *ABC* ?

3.1 Multiplying Out and Factoring Expressions

To obtain a product-of-sum form ➔ Factoring using distributive laws

Theorem for factoring:

$$\underbrace{A\overbrace{B+A'}C} = (A+C)(A'+B)$$

Example of factoring:

$$AC + A'BD' + A'BE + A'C'DE$$

$$= \underbrace{AC}_{XZ} + \underbrace{A'}_{X'}(\underbrace{BD' + BE + C'DE}_{Y})$$

$$= (A + BD' + BE + C'DE)(A' + C)$$

$$= [\underbrace{A + C'DE}_{X} + \underbrace{B}_{Y}(\underbrace{D' + E}_{Z})](A' + C)$$

$$= (A + B + C'DE)(A + \cancel{C'DE} + D' + E)(A' + C)$$

$$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C) \qquad \text{(3-5)}$$

3.2 Exclusive-OR and Equivalence Operations

Exclusive-OR

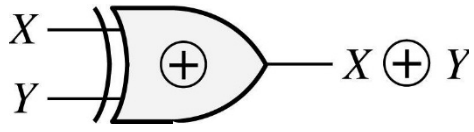
$$0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$$

$$1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$$

Truth Table

XY	$X \oplus Y$
0 0	0
0 1	1
1 0	1
1 1	0

Symbol



$$\boxed{X \oplus Y = X'Y + XY'} \quad (3\text{-}6)$$

3.2 Exclusive-OR and Equivalence Operations

Theorems for Exclusive-OR:

$$X \oplus 0 = X$$

$$X \oplus 1 = X'$$

$$X \oplus X = 0$$

$$X \oplus X' = 1$$

$$X \oplus Y = Y \oplus X \text{ (commutative law)}$$

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z \text{ (associative law)}$$

$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive law)}$$

$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y'$$

3.2 Exclusive-OR and Equivalence Operations

Equivalence operation (Exclusive-NOR)

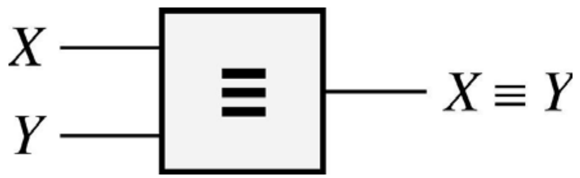
$$(0 \equiv 0) = 1 \qquad (0 \equiv 1) = 0$$

$$(1 \equiv 0) = 0 \qquad (1 \equiv 1) = 1$$

Truth Table

XY	$X \equiv Y$
0 0	1
0 1	0
1 0	0
1 1	1

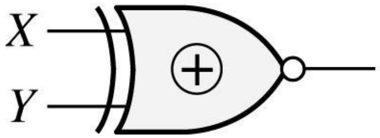
Symbol



$$\boxed{(X \equiv Y) = XY + X'Y'} \qquad (3\text{-}17)$$

3.2 Exclusive-OR and Equivalence Operations

Exclusive-NOR



$(X \oplus Y)' = (X \equiv Y)$

Example of EX-OR and Equivalence: $F = (A'B \equiv C) + (B \oplus AC')$

$$
\begin{aligned}
F &= [(A'B)C + (A'B)'C'] + [B'(AC') + B(AC')'] \\
&= A'BC + (A+B')C' + AB'C' + B(A'+C) \\
&= B(A'C + A' + C) + C'(A + B' + AB') = B(A'+C) + C'(A+B')
\end{aligned}
$$

Useful theorem: $\boxed{(XY' + X'Y)' = XY + X'Y'}$ (3-19)

$$
\begin{aligned}
A' \oplus B \oplus C &= [A'B' + (A')'B] \oplus C \\
&= (A'B' + AB)C' + (A'B' + AB)'C && \text{(by (3-6))} \\
&= (A'B' + AB)C' + (A'B + AB')C && \text{(by (3-19))} \\
&= A'B'C' + ABC' + A'BC + AB'C
\end{aligned}
$$

3.3 The Consensus Theorem

Consensus Theorem $XY + X'Z + YZ = XY + X'Z$

Proof :

$$XY + X'Z + YZ = XY + X'Z + (X + X')YZ$$
$$= (XY + XYZ) + (X'Z + X'YZ)$$
$$= XY(1+Z) + X'Z(1+Y) = XY + X'Z$$

Example:

consensus

$$a'b' + ac + bc' + b'c + ab = a'b' + ac + bc'$$

consensus

Dual form of consensus theorem $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$

Example:

$$(a + b + c')(a + b + d')(b + c + d') = (a + b + c')(b + c + d')$$

3.3 The Consensus Theorem

Example: eliminate *BCD*

$$A'C'D + A'BD + \cancel{BCD} + ABC + ACD'$$

Example: eliminate *A'BD, ABC*

$$A'C'D + \cancel{A'BD} + BCD + \cancel{ABC} + ACD'$$

Example: Reducing an expression by adding a term and eliminate.

$$F = ABCD + B'CDE + A'B' + BCE'$$

$$F = ABCD + B'CDE + A'B' + BCE' + \boxed{ACDE}$$

Consensus

Term added

From ABCD & B'CDE

Final expression

$$F = A'B' + BCE' + ACDE$$

3.4 Algebraic Simplification of Switching Expressions

1. Combining terms $XY + XY' = X$

Example: $abc'd' + abcd' = abd'$ $[X = abd', Y = c]$

Adding terms using $X + X = X$

$$ab'c + abc + a'bc = ab'c + abc + abc + a'bc = ac + bc$$

Example: $(a+bc)(d+e') + a'(b'+c')(d+e') = d + e'$

$[X = d + e', Y = a + bc, Y' = a'(b'+c')]$

2. Eliminating terms $X + XY = X$ $XY + X'Z + YZ = XY + X'Z$

Example: $a'b + a'bc = a'b$ $[X = a'b]$

$a'bc' + bcd + a'bd = a'bc' + bcd$ $[X = c, Y = bd, Z = a'b]$

3.4 Algebraic Simplification of Switching Expressions

3. Eliminating literals $X + X'Y = X + Y$

Example:

$$\begin{aligned} A'B + A'B'C'D' + ABCD' &= A'(B + B'C'D') + ABCD' \\ &= A'(B + C'D') + ABCD' \\ &= B(A' + ACD') + A'C'D' \\ &= B(A' + CD') + A'C'D' \\ &= A'B + BCD' + A'C'D' \end{aligned}$$

4. Adding redundant terms (Adding *xx'*, multiplying *(x+x')*, adding *yz* to *xy+x'z*, adding *xy* to *x*, etc…)

Example:

$$\begin{aligned} & WX + XY + X'Z' + WY'Z' && \text{(add } WZ' \text{ by consensus theorem)} \\ &= WX + XY + X'Z' + WY'Z' + WZ' && \text{(eliminate } WY'Z'\text{)} \\ &= WX + XY + X'Z' + WZ' && \text{(eliminate } WZ'\text{)} \\ &= WX + XY + X'Z' && \text{(3-27)} \end{aligned}$$

3.5 Proving Validity of an Equation

Proving an equation valid

1. Construct a truth table and evaluate both sides – tedious, not elegant method
2. Manipulate one side by applying theorems until it is the same as the other side
3. Reduce both sides of the equation independently
4. Apply same operation in both sides (complement both sides, add 1 or 0)

3.5 Proving Validity of an Equation

Prove : $A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC$

$$A'BD' + BCD + ABC' + AB'D$$

$$= A'BD' + BCD + ABC' + AB'D + BC'D' + A'BC + ABD$$

(add consensus of *A'BD'* and *ABC'*)

(add consensus of *A'BD'* and *BCD*)

(add consensus of *BCD* and *ABC'*)

$$= AD + A'BD' + BCD + ABC' + BC'D' + A'BC = BC'D' + AD + A'BC$$

(eliminate consensus of *BC'D'* and *AD*)

(eliminate consensus of *AD* and *A'BC*)

(eliminate consensus of *BC'D'* and *A'BC*)

3.5 Proving Validity of an Equation

Some of Boolean Algebra are not true for ordinary algebra

Example: If $x + y = x + z$, then $y = z$ — **True in ordinary algebra**

$1 + 0 = 1 + 1$ but $0 \neq 1$ — **Not True in Boolean algebra**

Example: If $xy = xz$, then $y = z$ — **True in ordinary algebra** / **Not True in Boolean algebra**

But the converses are True

Example: If $y = z$, then $x + y = x + z$ — **True in ordinary algebra**

If $y = z$, then $xy = xz$ — **True in Boolean algebra**

%Reason: Subtraction and Division is not defined in Boolean Algebra