확률과 통계

Class 3

Chapter 4
Continuous Distributions

4.1 Probability density

Probability density

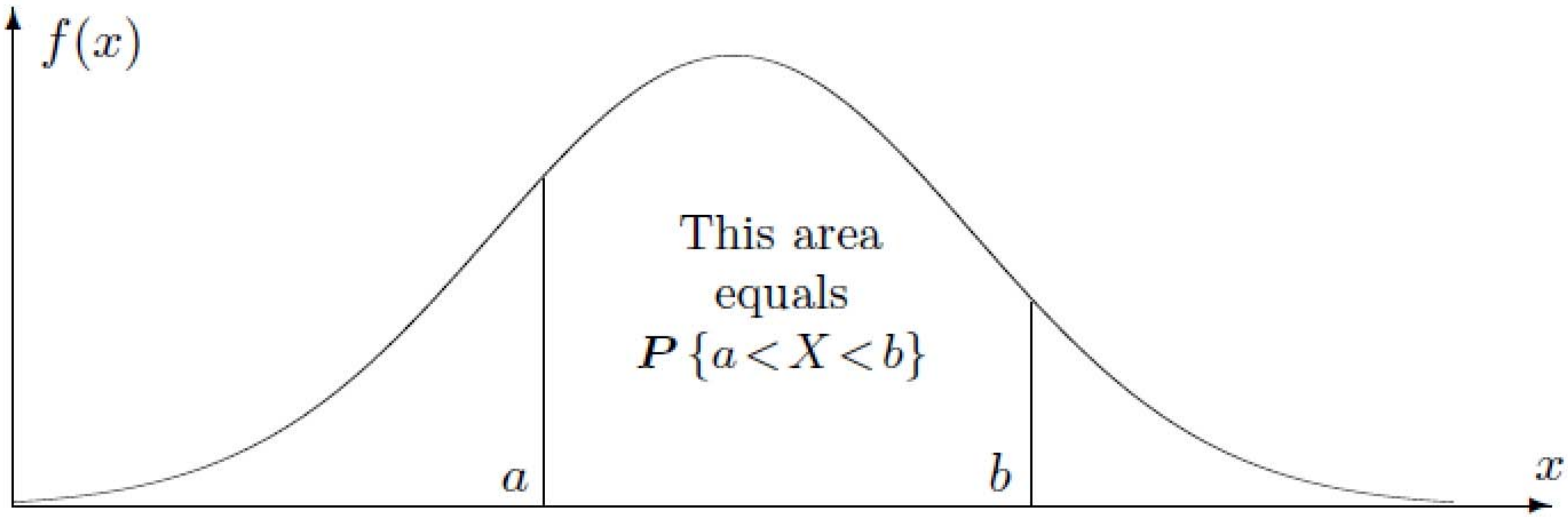


FIGURE 4.1: *Probabilities are areas under the density curve.*

DEFINITION 4.1

Probability density function (pdf, density) is the derivative of the cdf, $f(x) = F'(x)$. The distribution is called **continuous** if it has a density.

$$\int_a^b f(x)dx = F(b) - F(a) = \boldsymbol{P}\{a < X < b\}$$

Probability density function

$$\boxed{\begin{gathered} f(x) = F'(x) \\ \boldsymbol{P}\{a < X < b\} = \int_a^b f(x)dx \end{gathered}}$$

$$\int_{-\infty}^{b} f(x)dx = \boldsymbol{P}\{-\infty < X < b\} = F(b)$$

$$\int_{-\infty}^{+\infty} f(x)dx = \boldsymbol{P}\{-\infty < X < +\infty\} = 1.$$

$$P(x) = \boldsymbol{P}\{x \leq X \leq x\} = \int_{x}^{x} f = 0.$$

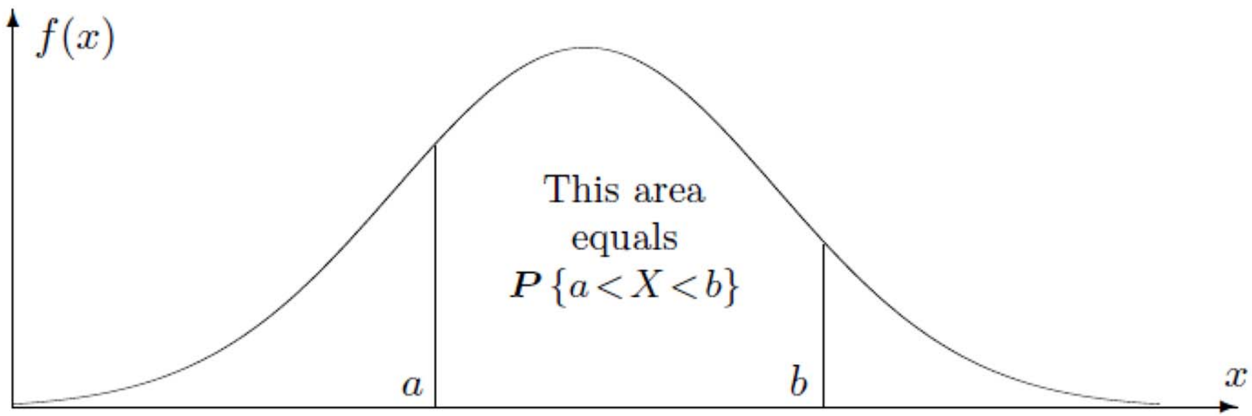


FIGURE 4.1: *Probabilities are areas under the density curve.*

Example 4.1. The lifetime, in years, of some electronic component is a continuous random variable with the density

$$f(x) = \begin{cases} \dfrac{k}{x^3} & \text{for} \quad x \geq 1 \\ 0 & \text{for} \quad x < 1. \end{cases}$$

Find k, draw a graph of the cdf $F(x)$, and compute the probability for the lifetime to exceed 5 years.

<u>Solution</u>. Find k from the condition $\int f(x)dx = 1$:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{1}^{+\infty} \frac{k}{x^3}dx = -\left.\frac{k}{2x^2}\right|_{x=1}^{+\infty} = \frac{k}{2} = 1.$$

$$\frac{d}{dx}x^c = cx^{c-1}$$

Hence, $k = 2$. Integrating the density, we get the cdf,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{1}^{x} \frac{2}{y^3}dy = -\left.\frac{1}{y^2}\right|_{y=1}^{x} = 1 - \frac{1}{x^2}$$

for $x > 1$. Its graph is shown in Figure 4.2.

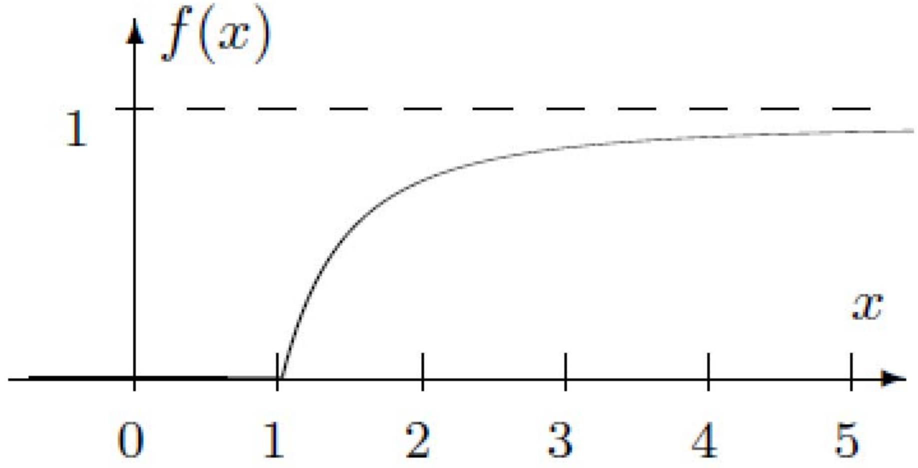


FIGURE 4.2: *Cdf for Example 4.1.*

$$\boldsymbol{P}\{X > 5\} = \int_5^{+\infty} f(x)dx = \int_5^{+\infty} \frac{2}{x^3}dx = -\left.\frac{1}{x^2}\right|_{x=5}^{+\infty} = \frac{1}{25} = 0.04.$$

Distribution	Discrete	Continuous
Definition	$P(x) = P\{X = x\}$ (pmf)	$f(x) = F'(x)$ (pdf)
Computing probabilities	$\boldsymbol{P}\{X \in A\} = \sum_{x \in A} P(x)$	$\boldsymbol{P}\{X \in A\} = \int_A f(x)dx$
Cumulative distribution function	$F(x) = \boldsymbol{P}\{X \leq x\} = \sum_{y \leq x} P(y)$	$F(x) = \boldsymbol{P}\{X \leq x\} = \int_{-\infty}^{x} f(y)dy$
Total probability	$\sum_{x} P(x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$

TABLE 4.1: *Pmf* $P(x)$ *versus pdf* $f(x)$.

Joint and marginal densities

DEFINITION 4.2

For a vector of random variables, the **joint cumulative distribution function** is defined as

$$F_{(X,Y)}(x,y) = \boldsymbol{P}\{X \leq x \cap Y \leq y\}.$$

The **joint density** is the *mixed derivative* of the joint cdf,

$$f_{(X,Y)}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{(X,Y)}(x,y).$$

Distribution	**Discrete**	**Continuous**
Marginal distributions	$P(x) = \sum_y P(x,y)$ $P(y) = \sum_x P(x,y)$	$f(x) = \int f(x,y)dy$ $f(y) = \int f(x,y)dx$
Independence	$P(x,y) = P(x)P(y)$	$f(x,y) = f(x)f(y)$
Computing probabilities	$\boldsymbol{P}\{(X,Y) \in A\}$ $= \sum_{(x,y)\in A} \sum P(x,y)$	$\boldsymbol{P}\{(X,Y) \in A\}$ $= \iint_{(x,y)\in A} f(x,y)\,dx\,dy$

TABLE 4.2: *Joint and marginal distributions in discrete and continuous cases.*

Expectation and variance

$$\mu = \mathbf{E}(X) = \int x f(x) dx$$

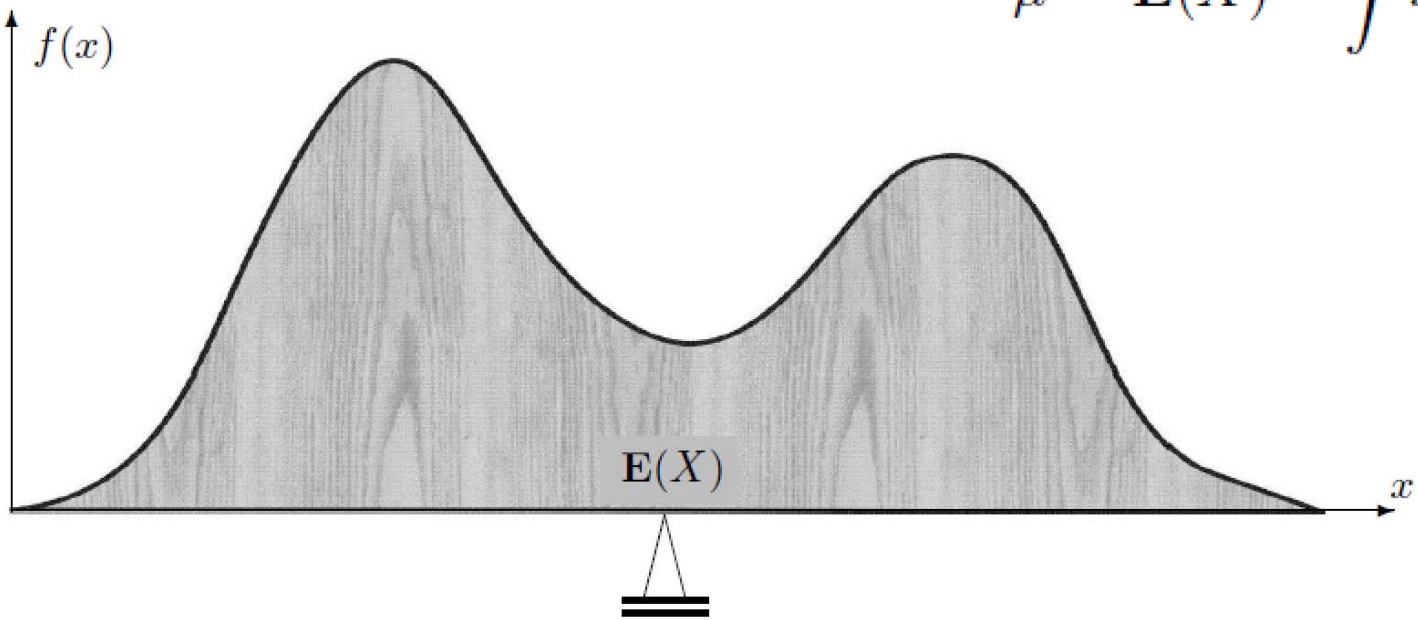


FIGURE 4.3: *Expectation of a continuous variable as a center of gravity.*

Example 4.2. A random variable X in Example 4.1 has density

$$f(x) = 2x^{-3} \text{ for } x \geq 1.$$

Its expectation equals

$$\mu = \mathbf{E}(X) = \int x\, f(x) dx = \int_1^{\infty} 2x^{-2} dx = -2x^{-1}\Big|_1^{\infty} = 2.$$

$$\sigma^2 = \mathrm{Var}(X) = \int x^2 f(x) dx - \mu^2 = \int_1^{\infty} 2x^{-1} dx - 4 = 2\ln x\Big|_1^{\infty} - 4 = +\infty.$$

Discrete	Continuous
$\mathbf{E}(X) = \sum_x x P(x)$	$\mathbf{E}(X) = \int x f(x) dx$
$\mathrm{Var}(X) = \mathbf{E}(X-\mu)^2$ $= \sum_x (x-\mu)^2 P(x)$ $= \sum_x x^2 P(x) - \mu^2$	$\mathrm{Var}(X) = \mathbf{E}(X-\mu)^2$ $= \int (x-\mu)^2 f(x) dx$ $= \int x^2 f(x) dx - \mu^2$
$\mathrm{Cov}(X,Y) = \mathbf{E}(X-\mu_X)(Y-\mu_Y)$ $= \sum_x \sum_y (x-\mu_X)(y-\mu_Y) P(x,y)$ $= \sum_x \sum_y (xy) P(x,y) - \mu_x \mu_y$	$\mathrm{Cov}(X,Y) = \mathbf{E}(X-\mu_X)(Y-\mu_Y)$ $= \iint (x-\mu_X)(y-\mu_Y) f(x,y)\, dx\, dy$ $= \iint (xy) f(x,y)\, dx\, dy - \mu_x \mu_y$

TABLE 4.3: *Moments for discrete and continuous distributions.*

4.2 Families of continuous distributions

4.2.1 Uniform distribution

$$f(x) = \frac{1}{b-a}, \quad a < x < b,$$

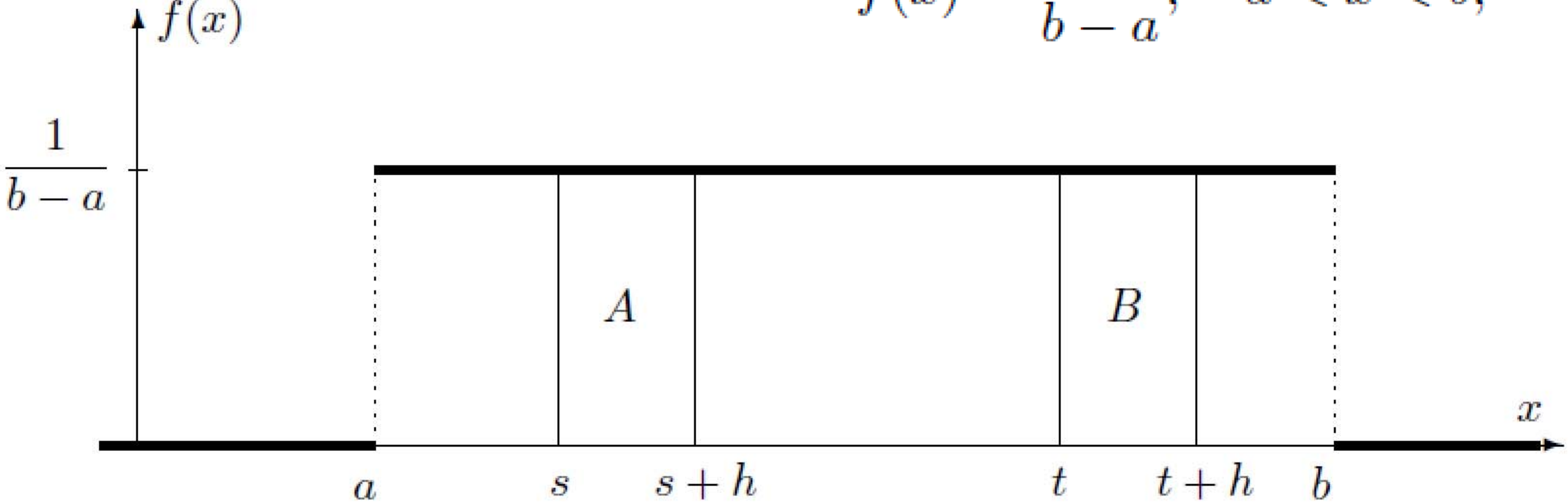


FIGURE 4.4: *The Uniform density and the Uniform property.*

The uniform property

For any $h > 0$ and $t \in [a, b-h]$, the probability

$$\boldsymbol{P}\{\, t < X < t+h \,\} = \int_t^{t+h} \frac{1}{b-a}\, dx = \frac{h}{b-a}$$

Example 4.3. In Figure 4.4, rectangles A and B have the same area, showing that $\boldsymbol{P}\{s < X < s+h\} = \boldsymbol{P}\{t < X < t+h\}$. $\Diamond$

Example 4.4. If a flight scheduled to arrive at 5 pm actually arrives at a Uniformly distributed time between 4:50 and 5:10, then it is equally likely to arrive before 5 pm and after 5 pm, equally likely before 4:55 and after 5:05, etc. $\Diamond$

Standard uniform distribution

- b=1, a=0

$$Y = \frac{X - a}{b - a} \qquad\qquad X = a + (b - a)Y$$

Expectation and variance

$$\mathbf{E}(Y) = \int y f(y) dy = \int_0^1 y dy = \frac{1}{2}$$

$$\mathrm{Var}(Y) = \mathbf{E}(Y^2) - \mathbf{E}^2(Y) = \int_0^1 y^2 dy - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$X = a + (b - a)Y$$

$$\mathbf{E}(X) = \mathbf{E}\{a + (b-a)Y\} = a + (b-a)\,\mathbf{E}(Y) = a + \frac{b-a}{2} = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \mathrm{Var}\{a + (b-a)Y\} = (b-a)^2\,\mathrm{Var}(Y) = \frac{(b-a)^2}{12}.$$

Uniform distribution

$$
\begin{aligned}
(a,b) &= \text{range of values}\\
f(x) &= \frac{1}{b-a}, \quad a<x<b\\
\mathbf{E}(X) &= \frac{a+b}{2}\\
\mathrm{Var}(X) &= \frac{(b-a)^2}{12}
\end{aligned}
$$

4.2.2 Exponential distribution

- Exponential distribution is often used to model *time*:
  waiting time, interarrival time, hardware lifetime, failure time, time between telephone calls, etc.
- As we shall see below, in a sequence of rare events, when the number of events is **Poisson**, the time between events is **Exponential**.

- Exponential distribution has density

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0. \tag{4.1}$$

- With this density, we compute the Exponential cdf, mean, and variance as

$$\begin{aligned}
F(x) &= \int_0^x f(t)dt = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x} \quad (x > 0), && (4.2)\\
\mathbf{E}(X) &= \int t f(t)dt = \int_0^\infty t\lambda e^{-\lambda t} dt = \frac{1}{\lambda} \quad \left(\begin{array}{c} \textit{integrating} \\ \textit{by parts} \end{array} \right), && (4.3)\\
\mathrm{Var}(X) &= \int t^2 f(t)dt - \mathbf{E}^2(X) \\
&= \int_0^\infty t^2 \lambda e^{-\lambda t} dt - \left(\frac{1}{\lambda}\right)^2 \quad (\textit{by parts twice}) && (4.4)\\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.
\end{aligned}$$

- The quantity λ is a parameter of Exponential distribution, and its meaning is clear from $E(X) = 1/\lambda$. If X is time, measured in minutes, then λ is a frequency, measured in min^{-1}.
- For example, if arrivals occur every half a minute, on the average, then $E(X) = 0.5$ and $\lambda = 2$, saying that they occur with a frequency (arrival rate) of 2 arrivals per minute.
- This λ has the same meaning as the parameter of Poisson distribution in **Section 3.4.5**.

Poisson distribution	
	λ = frequency, average number of events
	$P(x) = e^{-\lambda}\dfrac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$
	$\mathbf{E}(X) = \lambda$
	$\mathrm{Var}(X) = \lambda$

Partial integration

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx$$

Times between rare events are Exponential

- What makes Exponential distribution a good model for interarrival times?
- Apparently, this is not only experimental, but also a mathematical fact.
- As in **Section 3.4.5**, consider the sequence of rare events, where the number of occurrences during time t has Poisson distribution with a parameter proportional to t.
- This process is rigorously defined in **Section 6.3.2** where we call it Poisson process.
- Event "the time T until the next event is greater than t" can be rephrased as "zero events occur by the time t," and further, as "$X = 0$," where X is the number of events during the time interval $[0, t]$.
- This X has Poisson distribution with parameter λt.
- It equals 0 with probability
$$P_X(0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}.$$
- Then we can compute the cdf of T as
$$F_T(t) = 1 - \boldsymbol{P}\{T > t\} = 1 - \boldsymbol{P}\{X = 0\} = 1 - e^{-\lambda t}, \qquad (4.5)$$
and here we recognize the Exponential cdf.
- Therefore, the time until the next arrival has Exponential distribution.

Example 4.5

- Jobs are sent to a printer at an average rate of 3 jobs per hour.

a. What is the expected time between jobs?
   - Job arrivals represent rare events, thus the time T between them is Exponential with the given parameter $\lambda = 3$ hrs^{-1} (jobs per hour).
   - $E(T) = 1/\lambda = 1/3$ hours or 20 minutes between jobs;

b. What is the probability that the next job is sent within 5 minutes?
   - Convert to the same measurement unit: 5 min = (1/12) hrs.
   - Then,
     $\boldsymbol{P}\{T < 1/12 \text{ hrs}\} = F(1/12)$
     $= 1 - e^{-\lambda(1/12)} = 1 - e^{-1/4} = 0.2212.$

Memoryless property

$$\boldsymbol{P}\{T > t + x \mid T > t\} = \boldsymbol{P}\{T > x\} \qquad \text{for } t, x > 0.$$

PROOF: From (4.2), $\boldsymbol{P}\{T > x\} = e^{-\lambda x}$. Also, by the formula (2.7) for conditional probability,

$$\boldsymbol{P}\{T > t + x \mid T > t\} = \frac{\boldsymbol{P}\{T > t + x \cap T > t\}}{\boldsymbol{P}\{T > t\}} = \frac{\boldsymbol{P}\{T > t + x\}}{\boldsymbol{P}\{T > t\}} = \frac{e^{-\lambda(t + x)}}{e^{-\lambda t}} = e^{-\lambda x}.$$

$\square$

Exponential distribution

$$\begin{aligned}
\lambda &= \text{frequency parameter, the number of events} \\
&\ \text{per time unit} \\
f(x) &= \lambda e^{-\lambda x}, \quad x > 0 \\
\mathbf{E}(X) &= \frac{1}{\lambda} \\
\mathrm{Var}(X) &= \frac{1}{\lambda^2}
\end{aligned}$$

4.2.3 Gamma distribution

- When a certain procedure consists of α independent steps, and each step takes Exponential(λ) amount of time, then the total time has **Gamma distribution** with parameters α and λ.
- Gamma distribution can be widely used for the total time of a multistage scheme, for example, related to downloading or installing a number of files.
- In a process of rare events, with Exponential times between any two consecutive events, the time of the α-th event has Gamma distribution because it consists of α independent Exponential times.

Example 4.6 (Internet promotions)

Property 1:	$\Gamma(n+1) = n\,\Gamma(n)$
Property 2:	$\Gamma(\frac{1}{2}) = \sqrt{\pi}$
Property 3:	$\Gamma(\frac{n}{2}) = \frac{2^{(1-n)}(n-1)!\sqrt{\pi})}{(\frac{n-1}{2})!}$
Property 4:	$\lvert\Gamma(ni)\rvert = \sqrt{\frac{\pi}{n \sinh(\pi n)}}$
Euler's reflection formula:	$\Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin(\pi n)}$
Duplication formula:	$\Gamma(n)\Gamma(n+\frac{1}{2}) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$

- Users visit a certain internet site at the average rate of 12 hits per minute.
- Every sixth visitor receives some promotion that comes in a form of a flashing banner.
- Then the time between consecutive promotions has Gamma distribution with parameters $\alpha = 6$ and $\lambda = 12$.

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0. \quad (4.7)$$

$$\Gamma(z) = \int_0^\infty \frac{t^{z-1}\,dt}{\exp t} \qquad (\mathrm{Re}\, z > 0)$$

$$\boxed{\begin{array}{rcl} \text{Gamma}(1, \lambda) & = & \text{Exponential}(\lambda) \\ \text{Gamma}(\alpha, 1/2) & = & \text{Chi-square}(2\alpha) \end{array}}$$

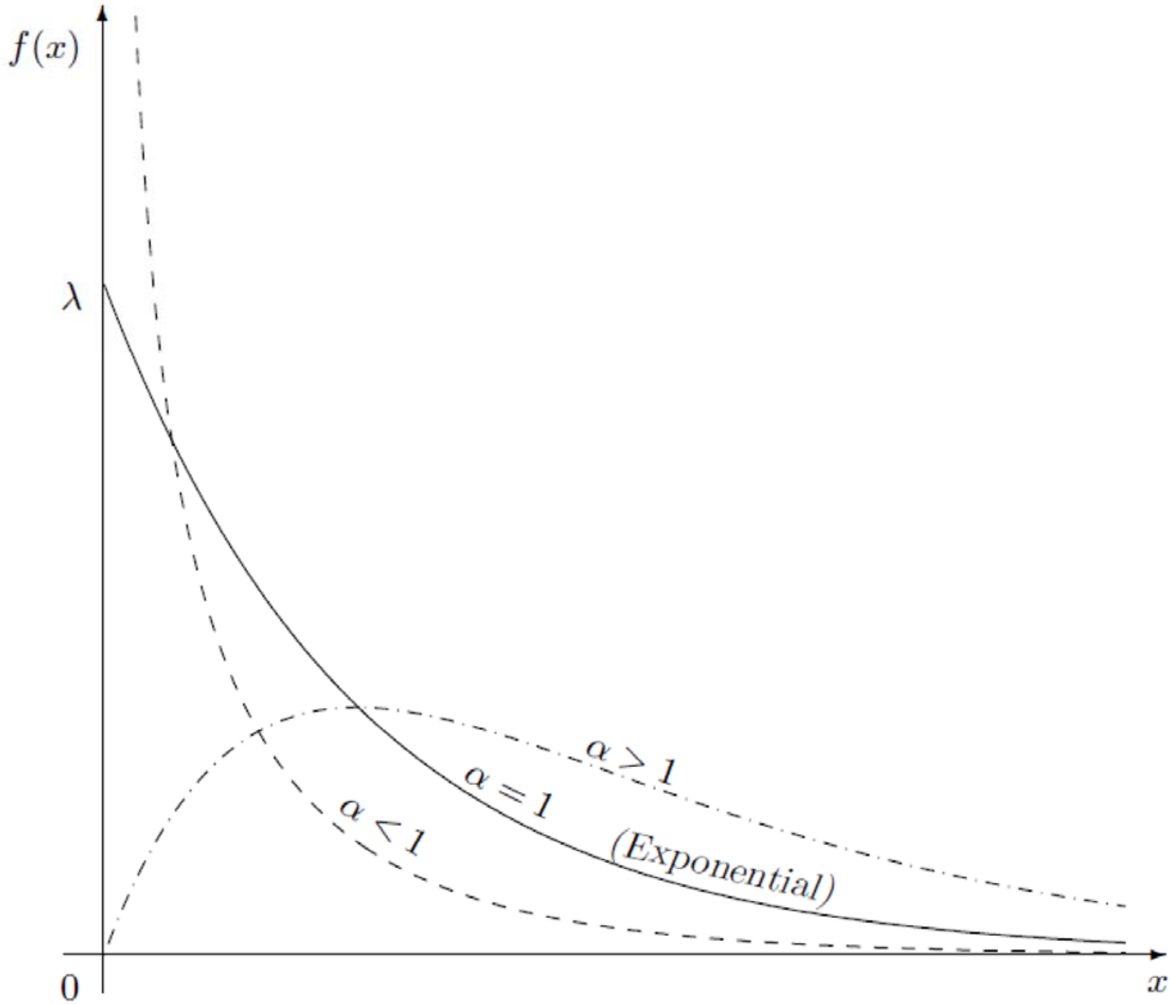


FIGURE 4.5: *Gamma densities with different shape parameters* α.

Gamma cdf has the form

$$F(t) = \int_0^t f(x)dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-\lambda x} dx.$$

First, let us notice that $\int_0^\infty f(x)dx = 1$ for Gamma and all the other densities. Then, integrating (4.7) from 0 to ∞, we obtain that

$$\boxed{\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha} \qquad \text{for any } \alpha > 0 \text{ and } \lambda > 0} \tag{4.9}$$

Substituting $\alpha + 1$ and $\alpha + 2$ in place of α, we get for a Gamma variable X,

$$\mathbf{E}(X) = \int x f(x) dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}$$

(using the equality $\Gamma(t+1) = t\Gamma(t)$ that holds for all $t > 0$),

$$\mathbf{E}(X^2) = \int x^2 f(x) dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}} = \frac{(\alpha+1)\alpha}{\lambda^2},$$

and therefore,

$$\operatorname{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}^2(X) = \frac{(\alpha+1)\alpha - \alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}. \tag{4.11}$$

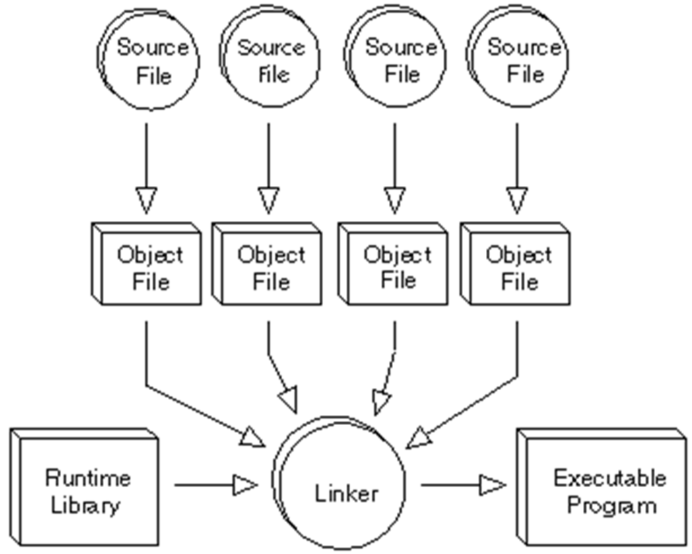
Gamma distribution

$$
\begin{aligned}
\alpha &= \text{shape parameter} \\
\lambda &= \text{frequency parameter} \\
f(x) &= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0 \\
\mathbf{E}(X) &= \frac{\alpha}{\lambda} \\
\mathrm{Var}(X) &= \frac{\alpha}{\lambda^2}
\end{aligned}
$$

Example 4.7 (Total compilation time)

- Compilation of a computer program consists of 3 blocks that are processed sequentially, one after another.
- Each block takes Exponential time with the mean of 5 minutes, independently of other blocks.

(a) Compute the expectation and variance of the total compilation time.
  - The total time T is a sum of three independent Exponential times, therefore, it has Gamma distribution with $\alpha = 3$.



- The frequency parameter λ equals (1/5) min^{-1} because the Exponential compilation time of each block has expectation $1/\lambda = 5$ min.
- For a Gamma random variable T with $\alpha = 3$ and $\lambda = 1/5$,

$$\mathbf{E}(T) = \frac{3}{1/5} = 15 \text{ (min)} \quad \text{and} \quad \mathrm{Var}(T) = \frac{3}{(1/5)^2} = 75 \text{ (min}^2\text{)}$$

(b) Compute the probability for the entire program to be compiled in less than 12 minutes.

- A direct solution involves two rounds of integration by parts,

$$
\begin{aligned}
\boldsymbol{P}\{T < 12\} &= \int_0^{12} f(t)dt = \frac{(1/5)^3}{\Gamma(3)} \int_0^{12} t^2 e^{-t/5} dt \\
&= \frac{(1/5)^3}{2!} \left(-5t^2 e^{-t/5} \Big|_{t=0}^{t=12} + \int_0^{12} 10te^{-t/5} dt \right) \\
&= \frac{1/125}{2} \left(-5t^2 e^{-t/5} - 50te^{-t/5} \Big|_{t=0}^{t=12} + \int_0^{12} 50e^{-t/5} dt \right) \\
&= \frac{1}{250} \left(-5t^2 e^{-t/5} - 50te^{-t/5} - 250e^{-t/5} \right) \Big|_{t=0}^{t=12} \\
&= 1 - e^{-2.4} - 2.4e^{-2.4} - 2.88e^{-2.4} = \underline{0.4303}.
\end{aligned}
\tag{4.13}
$$

Gamma-Poisson formula

- Computation of Gamma probabilities can be significantly simplified by thinking of a Gamma variable as the time between some rare events.
- In particular, one can avoid lengthy integration by parts, as in **Example 4.7**, and use Poisson distribution instead.
- Indeed, let T be a Gamma variable with an integer parameter α and some positive λ.
- This is a distribution of the time of the α-th rare event.
- Then, the event $\{T > t\}$ means that the α-th rare event occurs after the moment t, and therefore, fewer than α rare events occur before the time t.
- We see that
$$\{T > t\} = \{X < \alpha\},$$
where X is the number of events that occur before the time t.
- This number of rare events X has Poisson distribution with parameter (λt); therefore, the probability
$$\boldsymbol{P}\{T > t\} = \boldsymbol{P}\{X < \alpha\}$$
and the probability of a complement
$$\boldsymbol{P}\{T \le t\} = \boldsymbol{P}\{X \ge \alpha\}$$
can both be computed using the Poisson distribution of X.

Gamma-Poisson formula

For a Gamma(α, λ) variable T and a Poisson(λt) variable X,

$$P\{T > t\} = P\{X < \alpha\}$$
$$P\{T \le t\} = P\{X \ge \alpha\}$$

Example 4.8
(Total compilation time, continued)

- Here is an alternative solution to **Example 4.7(b)**.
- According to the Gamma-Poisson formula
  with $\alpha = 3$, $\lambda = 1/5$, and $t = 12$,
  $$\boldsymbol{P}\{T < 12\} = \boldsymbol{P}\{X \geq 3\} = 1 - F(2) = 1 - 0.5697 = 0.430$$
  from **Table A3** for the Poisson distribution of X
  with parameter $\lambda t = 2.4$.
- Furthermore, we notice that the four-term mathematical expression
  that we obtained in **(4.13)**
  after integrating by parts represents precisely
  $$\boldsymbol{P}\{X \geq 3\} = 1 - P(0) - P(1) - P(2).$$

Example 4.9

- Lifetimes of computer memory chips have Gamma distribution with expectation $\mu = 12$ years and standard deviation $\sigma = 4$ years.
- What is the probability that such a chip has a lifetime between 8 and 10 years?
- Solution.
  - Step 1, parameters.
    - From the given data,
      compute parameters
      of this Gamma distribution.
    - Using **(4.12)**,
      obtain a system of two equations
      and solve them for α and λ,
      $$\begin{cases} \mu = \alpha/\lambda \\ \sigma^2 = \alpha/\lambda^2 \end{cases} \Rightarrow \begin{cases} \alpha = \mu^2/\sigma^2 = (12/4)^2 = 9, \\ \lambda = \mu/\sigma^2 = 12/4^2 = 0.75. \end{cases}$$
- Step 2, probability.
  - We can now compute the probability,
    $\boldsymbol{P}\{8 < T < 10\} = F_T(10) - F_T(8)$. (4.15)
  - For each term in **(4.15)**,
    we use the Gamma-Poisson formula
    with $\alpha = 9$, $\lambda = 0.75$, and $t = 8, 10$,
    $F_T(10) = \boldsymbol{P}\{T \leq 10\} = \boldsymbol{P}\{X \geq 9\}$
    $= 1 - F_X(8) = 1 - 0.662 = 0.338$
    from **Table A3** for a Poisson variable X
    with parameter $\lambda t = (0.75)(10) = 7.5$;
    $F_T(8) = \boldsymbol{P}\{T \leq 8\} = \boldsymbol{P}\{X \geq 9\}$
    $= 1 - F_X(8) = 1 - 0.847 = 0.153$
    from the same table,
    this time with parameter $\lambda t = (0.75)(8) = 6$.
  - Then,
    $\boldsymbol{P}\{8 < T < 10\} = 0.338 - 0.153 = 0.185$.

- Enter a value in BOTH of the first two text boxes.
- Click the **Calculate** button.
- The Calculator will compute the Poisson and Cumulative Probabilities.

Poisson random variable (x)	8
Average rate of success	7.5
Poisson Probability: P(X = 8)	0.137328592653878
Cumulative Probability: P(X < 8)	0.524638526487606
Cumulative Probability: P(X ≤ 8)	0.661967119141484
Cumulative Probability: P(X > 8)	0.338032880858516
Cumulative Probability: P(X ≥ 8)	0.475361473512394

- Enter a value in BOTH of the first two text boxes.
- Click the **Calculate** button.
- The Calculator will compute the Poisson and Cumulative Probabilities.

Poisson random variable (x)	8
Average rate of success	6
Poisson Probability: $P(X = 8)$	0.103257733530844
Cumulative Probability: $P(X < 8)$	0.743979760453717
Cumulative Probability: $P(X \leq 8)$	0.847237493984561
Cumulative Probability: $P(X > 8)$	0.152762506015439
Cumulative Probability: $P(X \geq 8)$	0.256020239546283

4.2.4 Normal distribution

Normal distribution

$$
\begin{aligned}
\mu &= \text{expectation, location parameter} \\
\sigma &= \text{standard deviation, scale parameter} \\
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty \\
\mathbf{E}(X) &= \mu \\
\mathrm{Var}(X) &= \sigma^2
\end{aligned}
$$

DEFINITION 4.3

Normal distribution with "standard parameters" $\mu = 0$ and $\sigma = 1$ is called **Standard Normal distribution.**

<u>NOTATION</u>

$$\begin{aligned} Z &= \text{Standard Normal random variable} \\ \phi(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ Standard Normal pdf} \\ \Phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \text{ Standard Normal cdf} \end{aligned}$$

$$Z = \frac{X - \mu}{\sigma}. \qquad X = \mu + \sigma Z.$$

Example 4.10
(Computing Standard Normal probabilities)

- For a Standard Normal random variable Z,

$$\boldsymbol{P}\{Z < 1.35\} = (1.35) = 0.9115$$
$$\boldsymbol{P}\{Z > 1.35\} = 1 - (1.35) = 0.0885$$
$$\boldsymbol{P}\{-0.77 < Z < 1.35\} = (1.35) - (-0.77)$$
$$= 0.9115 - 0.2206 = 0.6909$$

  according to **Table A4**.

- Notice that

$$\boldsymbol{P}\{Z < -1.35\} = 0.0885 = \boldsymbol{P}\{Z > 1.35\},$$

  which is explained by the symmetry
  of the Standard Normal density in **Figure 4.6**.

- Due to this symmetry,
  "the left tail," or the area to the left of (−1.35)
  equals "the right tail,"
  or the area to the right of 1.35.

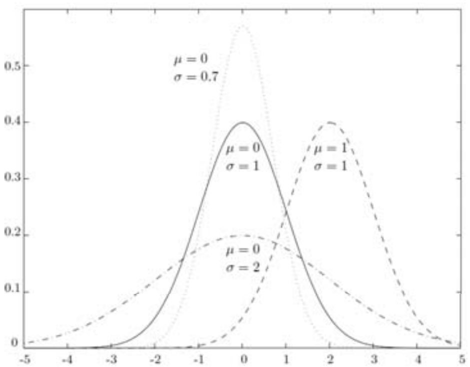
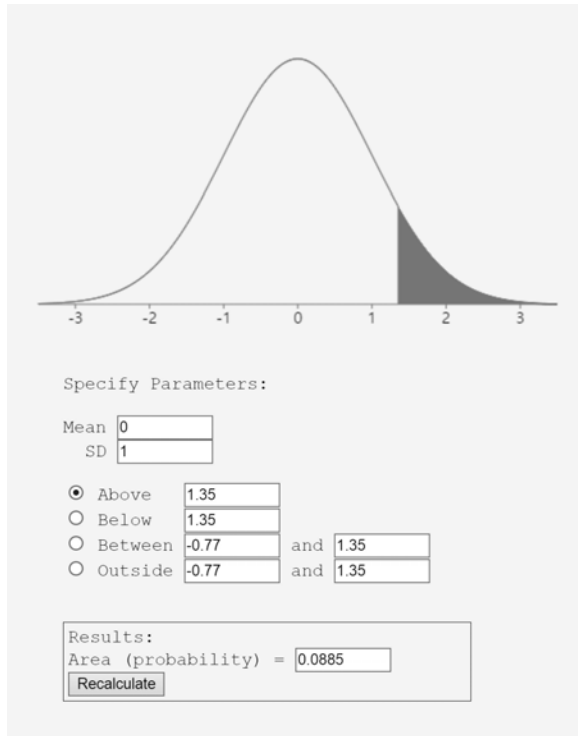


FIGURE 4.6: *Normal densities with different location and scale parameters.*



Example 4.11
(Computing non-standard Normal probabilities)

- Suppose that the average household income in some country is 900 coins, and the standard deviation is 200 coins.
- Assuming the Normal distribution of incomes, compute the proportion of "the middle class," whose income is between 600 and 1200 coins.

- Solution.
  - Standardize and use **Table A4**.
  - For a Normal($\mu = 900$, $\sigma = 200$) variable X,

$$\boldsymbol{P}\{600 < X < 1200\}$$
$$= \boldsymbol{P}\left\{\frac{600-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{1200-\mu}{\sigma}\right\}$$
$$= \boldsymbol{P}\left\{\frac{600-900}{200} < Z < \frac{1200-900}{200}\right\}$$
$$= \boldsymbol{P}\{-1.5 < Z < 1.5\}$$
$$= (1.5) - (-1.5) = 0.9332 - 0.0668$$
$$= 0.8664.$$

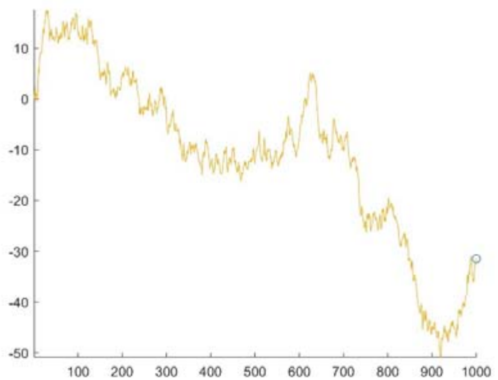
Example 4.12 (Inverse problem)

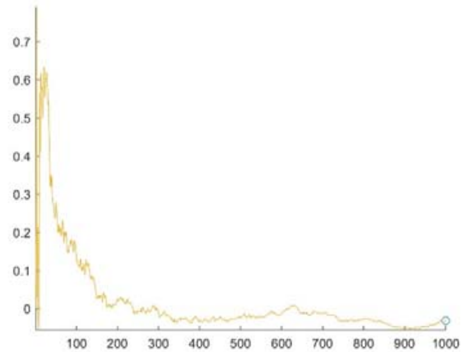
- The government of the country in **Example 4.11** decides to issue food stamps to the poorest 3% of households.
- Below what income will families receive food stamps?

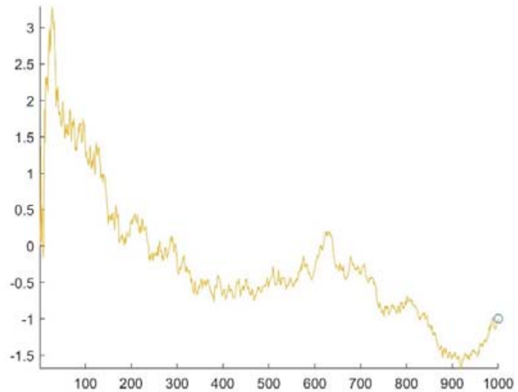
- Solution.
  - We need to find such income x that $\boldsymbol{P}\{X < x\} = 3\% = 0.03$.
  - This is an equation that can be solved in terms of x.
  - Again, we standardize first, then use the table:
    $$\boldsymbol{P}\{X < x\} = \boldsymbol{P}\left\{Z < \frac{X-\mu}{\sigma}\right\} = 0.03,$$
    from where $x = \mu + \sigma^{-1}(0.03)$.
  - In **Table A4**, we have to find the probability, the *table entry* of 0.03.
  - We see that $\Phi(-1.88) \approx 0.03$.
  - Therefore, $\Phi^{-1}(0.03) = -1.88$, and $x = \mu + \sigma(-1.88) = 900 + (200)(-1.88) = 524$ (coins) is the answer.
  - In the literature, the value $\Phi^{-1}(\alpha)$ is often denoted by $z_{1-\alpha}$.

4.3 Central Limit Theorem

- We now turn our attention to sums of random variables,
  $S_n = X_1 + \ldots + X_n$,
  that appear in many applications.
- Let $\mu = \mathbf{E}(X_i)$ and $\sigma = \mathrm{Std}(X_i)$ for all $i = 1, \ldots, n$.
- How does S_n behave for large n?
- Matlab Demo.
  - The following MATLAB code is a good illustration to the behavior of partial sums S_n.
  - `S(1)=0;`
  - `for n=2:1000; S(n)=S(n-1)+randn; end;`
  - `n=1:1000; comet(n,S); pause(3); pause(3);`
  - `comet(n,S./n); pause(3);`
  - `comet(n,S./sqrt(n)); pause(3);`







Theorem 1 (Central Limit Theorem)

- Let $X_1, X_2, \ldots$ be independent random variables with the same expectation $\mu = \mathbf{E}(X_i)$ and the same standard deviation $\sigma = \mathrm{Std}(X_i)$, and let
$$S_n = \sum_{i=1}^{n} X_i = X_1 + \ldots + X_n.$$
- As $n \to \infty$, the standardized sum
$$Z_n = \frac{S_n - \mathbf{E}(S_n)}{\mathrm{Std}(S_n)} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$
converges in distribution to a Standard Normal random variable, that is,
$$F_{Z_n}(z) = \boldsymbol{P}\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq z\right\} \to \Phi(z) \quad (4.18)$$
for all z.

- This theorem is very powerful because it can be applied to random variables $X_1, X_2, \ldots$ having virtually any thinkable distribution with finite expectation and variance.
- As long as n is large (the rule of thumb is $n > 30$), one can use Normal distribution to compute probabilities about S_n.
- **Theorem 1** is only one basic version of the Central Limit Theorem.
- Over the last two centuries, it has been extended to large classes of dependent variables and vectors, stochastic processes, and so on.

Example 4.13 (Allocation of disk space)

- A disk has free space of 330 megabytes.
- Is it likely to be sufficient for 300 independent images,
  if each image has expected size of 1 megabyte with a standard deviation of 0.5 megabytes?
- Solution.
  - We have $n = 300$, $\mu = 1$, $\sigma = 0.5$.
  - The number of images n is large,
    so the Central Limit Theorem applies to their total size S_n.
  - Then,
    $$\boldsymbol{P}\{\text{sufficient space}\} = \boldsymbol{P}\{S_n \leq 330\} = \boldsymbol{P}\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \frac{330 - (300)(1)}{0.5\sqrt{300}}\right\} \approx (3.46) = 0.9997.$$
  - This probability is very high, hence,
    the available disk space is very likely to be sufficient.

Example 4.14 (Elevator)

- You wait for an elevator, whose capacity is 2000 pounds.
- The elevator comes with 10 adult passengers.
- Suppose your own weight is 150 lbs,
  and you heard that human weights are normally distributed with the mean of 165 lbs
  and the standard deviation of 20 lbs.
- Would you board this elevator or wait for the next one?
- Solution.
  - In other words, is overload likely?
  - The probability of an overload equals
    $$\boldsymbol{P}\{S_{10} + 150 > 2000\} = \boldsymbol{P}\left\{\frac{S_{10} - (10)(165)}{20\sqrt{10}} > \frac{2000 - 150 - (10)(165)}{20\sqrt{10}}\right\} = 1 - (3.16) = 0.0008.$$
  - So, with probability 0.9992 it is safe to take this elevator.
  - It is now for you to decide.

- Binomial variable
  = sum of independent Bernoulli variables
- Negative Binomial variable
  = sum of independent Geometric variables
- Gamma variable
  = sum of independent Exponential variables

Normal approximation to Binomial distribution

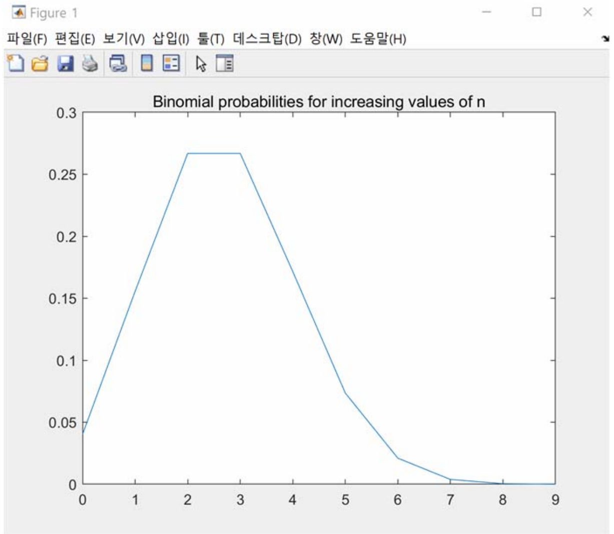
- Binomial variables represent a special case of $S_n = X_1 + ... + X_n$, where all X_i have Bernoulli distribution with some parameter p.
- We know from **Section 3.4.5** that small p allows to approximate Binomial distribution with Poisson, and large p allows such an approximation for the number of failures.
- For the moderate values of p (say, $0.05 \le p \le 0.95$) and for large n, we can use **Theorem 1**:

$$\text{Binomial}(n, p) \approx \text{Normal}\left(\mu = np, \sigma = \sqrt{np(1-p)}\right) \quad (4.19)$$

Matlab Demo

- To visualize how Binomial distribution gradually takes the shape of Normal distribution when $n \to \infty$, you may execute the following MATLAB code that graphs Binomial(n, p) pmf for increasing values of n.

```
for n=1:2:100; x=0:n;
p=gamma(n+1)./gamma(x+1)./gamma(n-x+1).*(0.3).bx.*(0.7).b(n-x);
plot(x,p);
title('Binomial probabilities for increasing values of n');
pause(1); end;
```



Continuity correction

$$P_X(x) = \boldsymbol{P}\{X = x\} = \boldsymbol{P}\{x - 0.5 < X < x + 0.5\}$$

Example 4.15

- A new computer virus attacks a folder consisting of 200 files.
- Each file gets damaged with probability 0.2 independently of other files.
- What is the probability that fewer than 50 files get damaged?
- Solution.
  - The number X of damaged files has Binomial distribution with $n = 200$, $p = 0.2$, $\mu = np = 40$, and $\sigma = \sqrt{\mathrm{np}(1-\mathrm{p})} = 5.657$.
  - Applying the Central Limit Theorem with the continuity correction,
    $\boldsymbol{P}\{X < 50\} = \boldsymbol{P}\{X < 49.5\}$
    $$= \boldsymbol{P}\left\{\frac{X-40}{5.657} < \frac{49.5-40}{5.657}\right\} = (1.68) = 0.9535.$$

- Notice that the properly applied continuity correction replaces 50 with 49.5, not 50.5.
- Indeed, we are interested in the event that X is strictly less than 50.
- This includes all values up to 49 and corresponds to the interval [0, 49] that we expand to [0, 49.5].
- In other words, events $\{X < 50\}$ and $\{X < 49.5\}$ are the same; they include the same possible values of X.
- Events $\{X < 50\}$ and $\{X < 50.5\}$ are different because the former includes $X = 50$, and the latter does not.
- Replacing $\{X < 50\}$ with $\{X < 50.5\}$ would have changed its probability and would have given a wrong answer.

Q&A