Data Structure:

Binary Search Tree

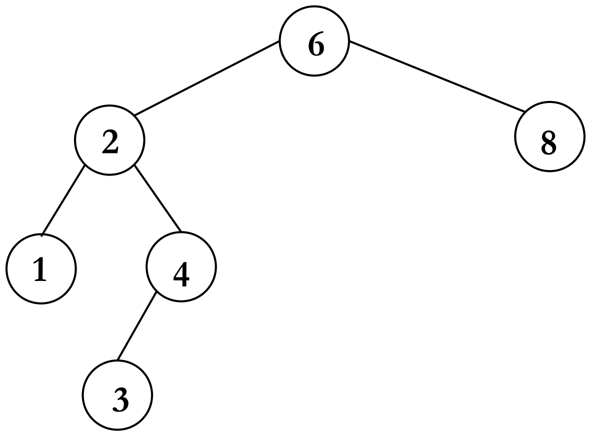
Binary Search Tree

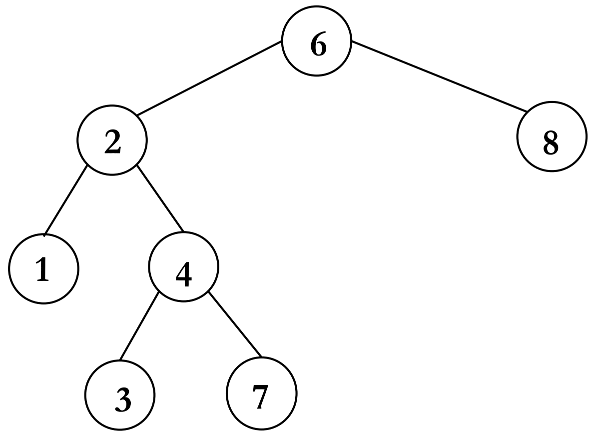
- one of the most fundamental problems in data structure design
  - there is a set of records $R_1, R_2, \ldots, R_n$, which are associated with distinct key values $X_1, X_2, \ldots, X_n$, respectively.
  - given a search key x, find the record if it occurs in the set.

- for every node X in the tree,
  - the values of all the keys in its left subtree are smaller than the key value in X
  - the values of all the keys in its right subtree are larger than the key value in X

Binary Search Tree

- which one can be the binary search tree?





Search in a linear array

- sequential search
  - simply store the keys in a linear array and search sequentially
  - insertion: $O(1)$, searching $O(n)$
- binary search
  - searching: $O(\log n)$, insertion/deletion: $O(n)$

Binary Search Tree

A BST ADT can process the following requests

- insert(x,T):
  - insert x into T
  - if x already exists, do appropriate action (e.g., do nothing, return error message, increment reference count.)

- delete(x,T):
  - delete x from T
  - if x does not exist, issue an error message

- find(x,T): search x in T

  return either True/False or the pointer to the record

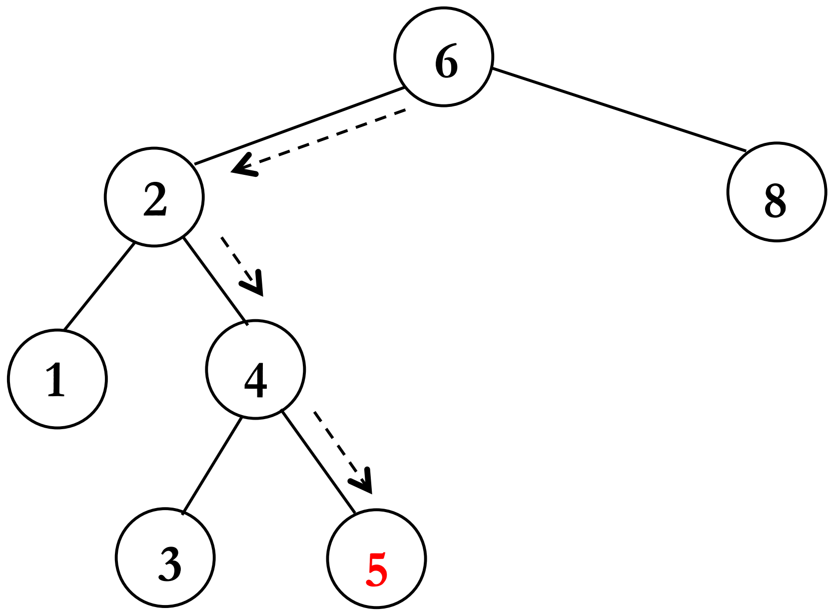
Binary Search Tree: data structure

```
struct TreeNode;
typedef struct TreeNode*   SearchTree;
typedef struct TreeNode*   Node;

struct TreeNode
{
   ElementType Element;
   SearchTree  Left;
   SearchTree  Right;
}
```

Binary Search Tree: Find

```
Node Find( ElementType X, SearchTree T )
{

   if  ( T == NULL )
        return NULL;

   if  ( X < T->Element )
        return Find( X, T->Left );

   else if ( X > T->Element )
        return Find( X, T->Right );

   else    /* X == T->Element */
        return T;
}
```

Find 5



Binary Search Tree: Find

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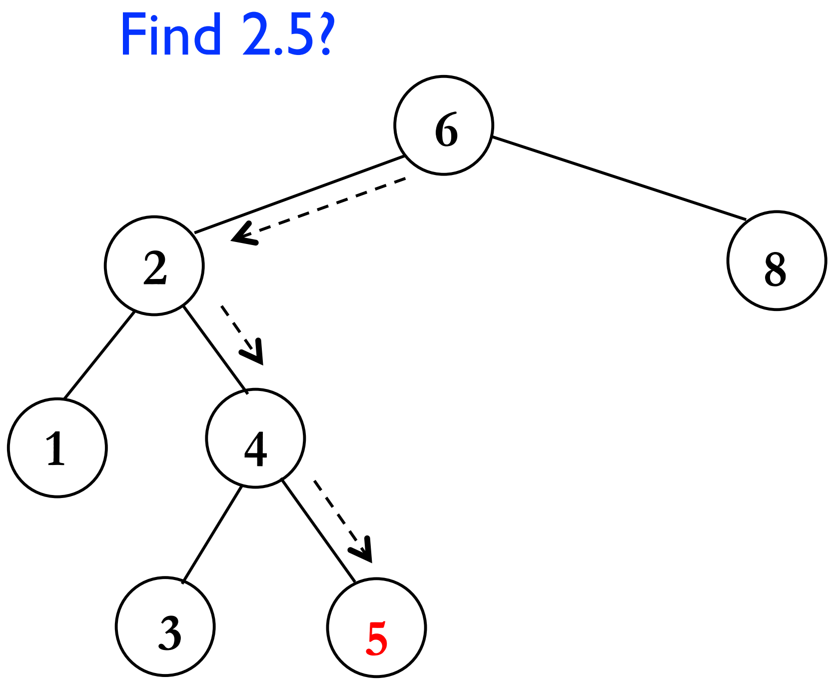
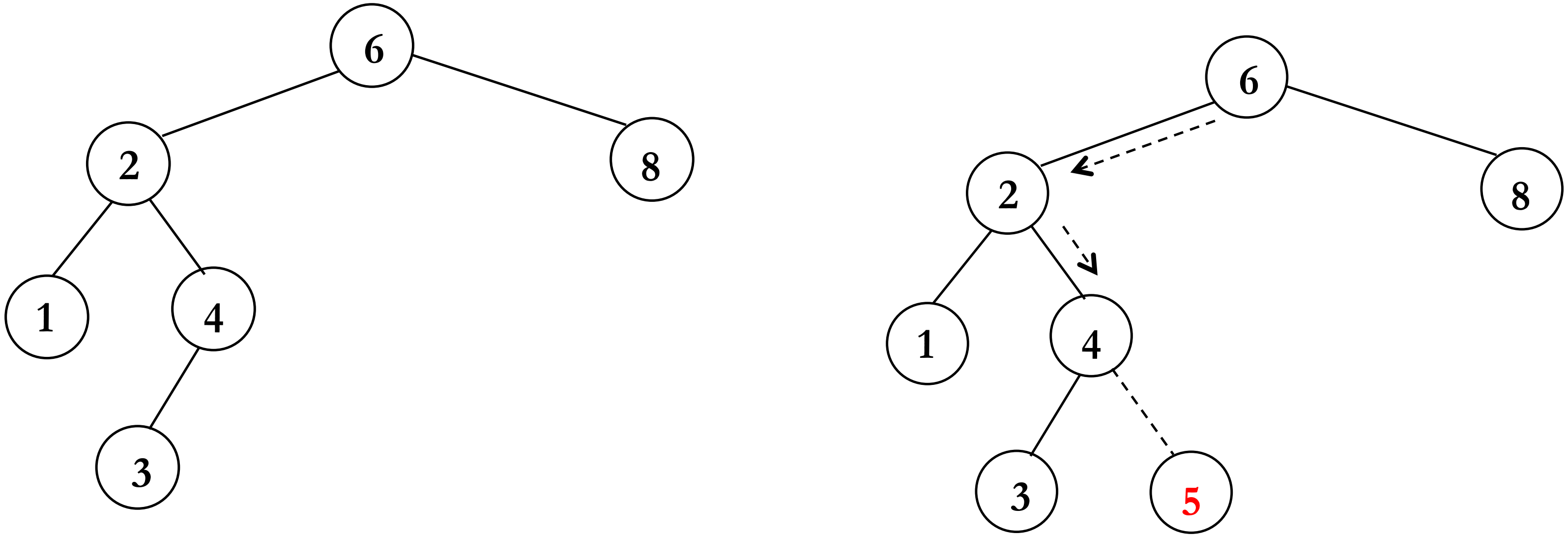


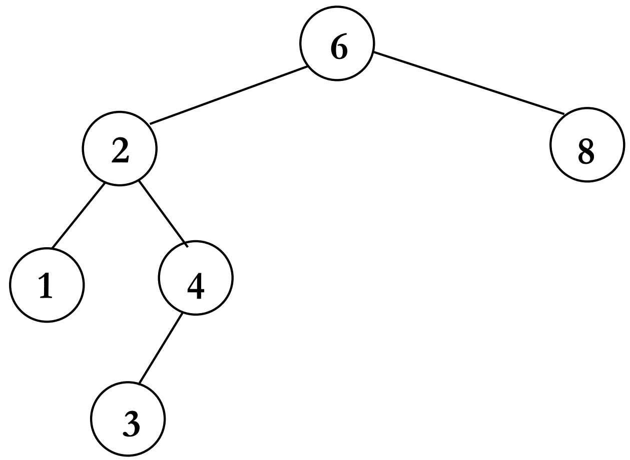
Binary Search Tree: Insert

insertion of 5



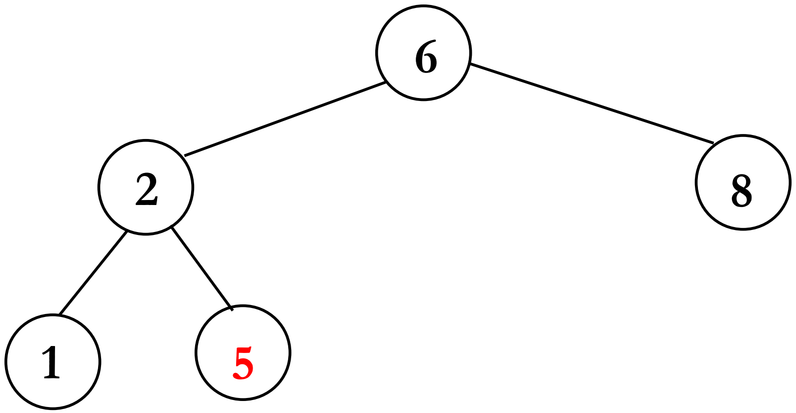
Binary Search Tree: Insert

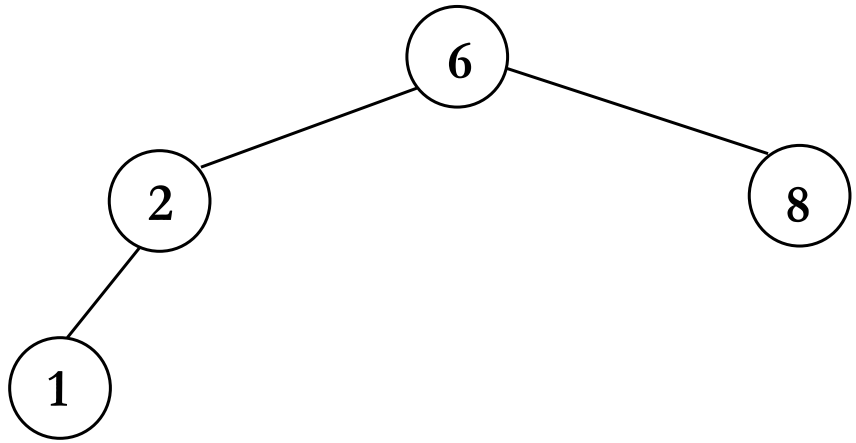
```
SearchTree  Insert ( ElementType X, SearchTree T )
{
    if( T == NULL ) {

        T = malloc( sizeof( struct TreeNode ) );
        if( T == NULL )
            FatalError( “Out of space!!!” );
        else
        {
            T->Element = X;
            T->Left = T->Right = NULL;
        }
    } else if( X < T->Element ) {
        T->Left = Insert( X, T->Left );

    } else if( X > T->Element )
      T->Right = Insert( X, T->Right );

    /* Else X is in the tree already; we’ll do nothing */

    return T;  /* Do not forget this line! */
}
```



Binary Search Tree: Delete

If the node to be deleted is a leaf , just delete it!

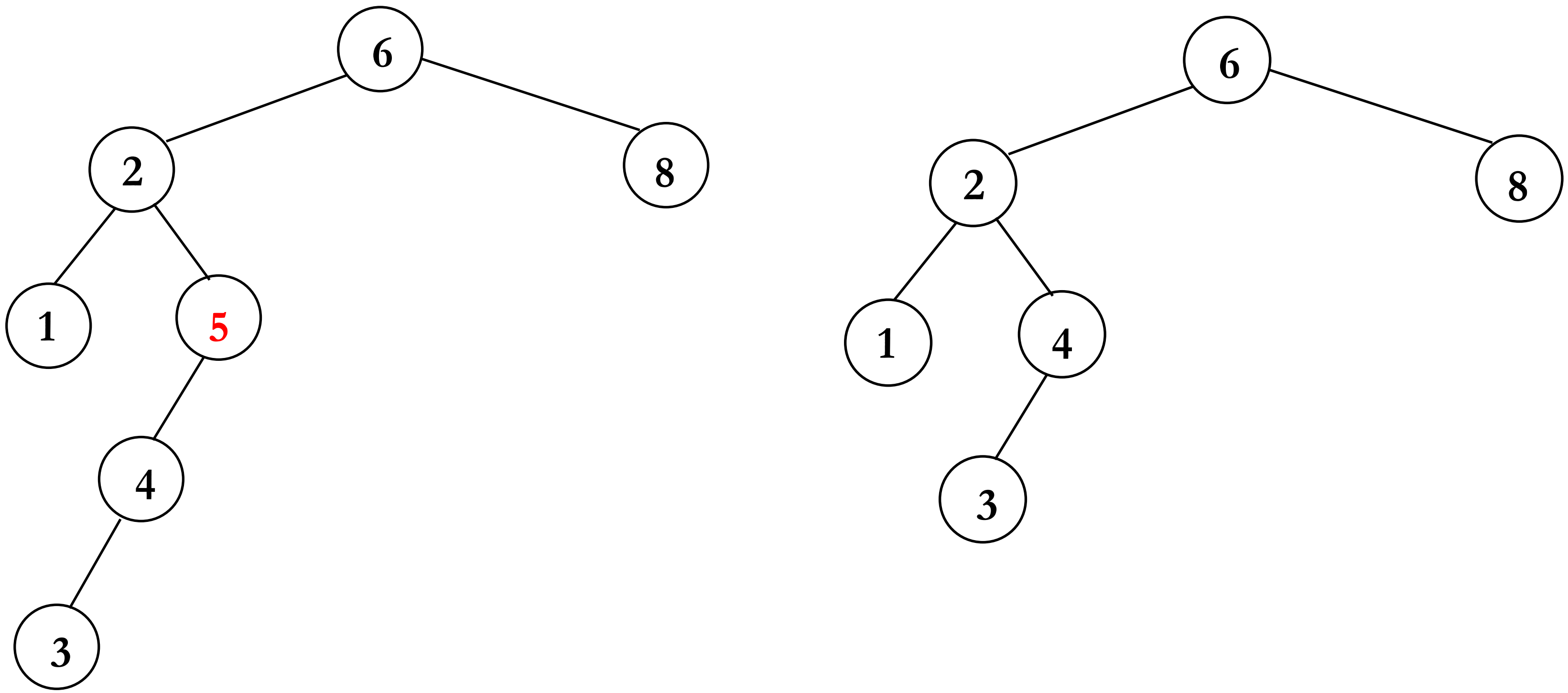




Binary Search Tree: Delete

If the node to be deleted has one child,

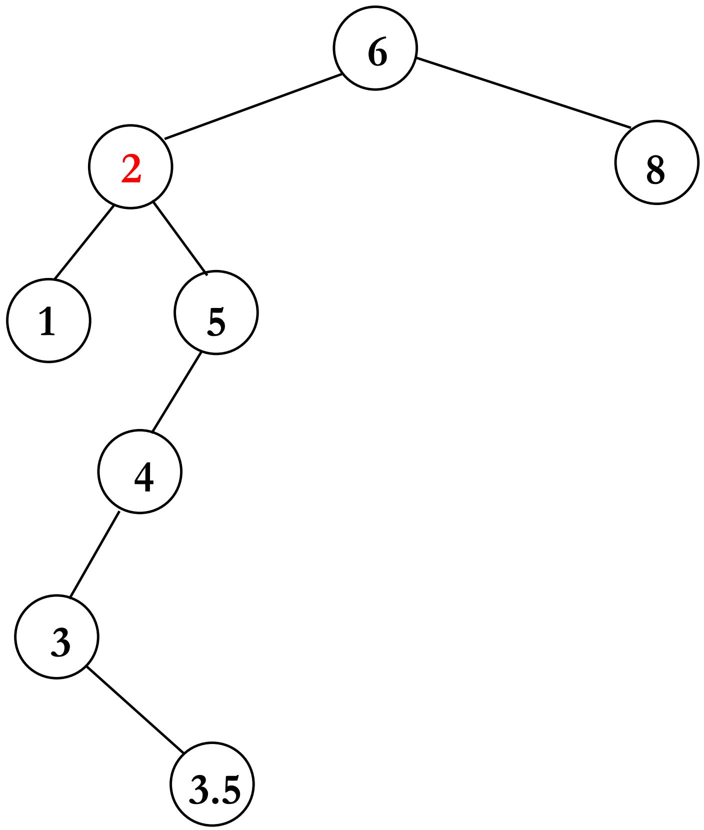
the child of the node is connected to the parent of the node

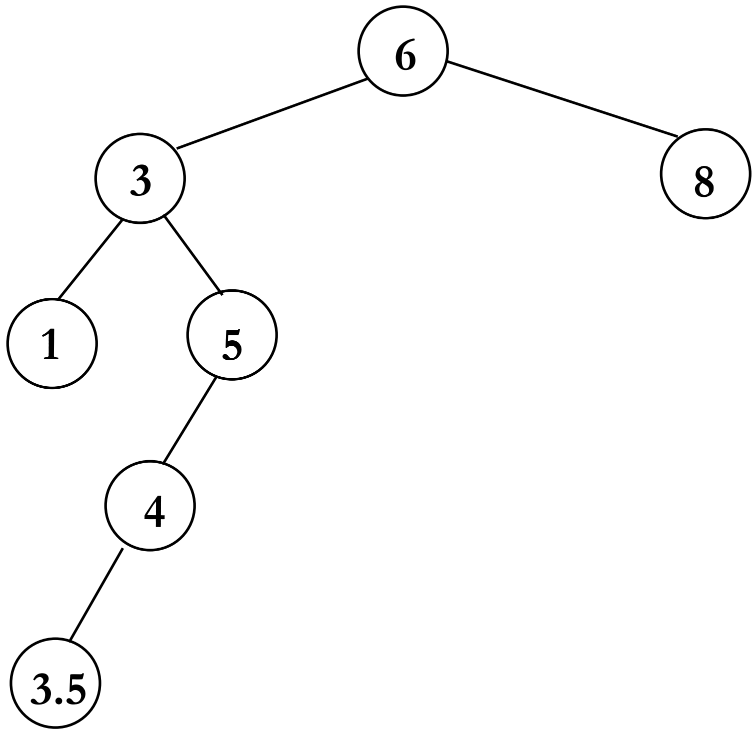


Binary Search Tree: Delete

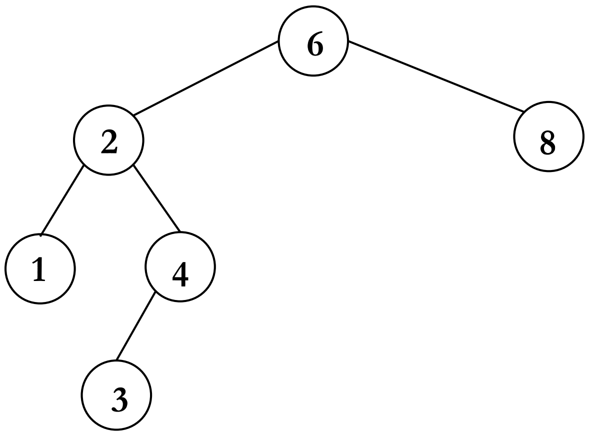
If the node to be deleted has both children,

it is replaced with the smallest node in the right subtree.





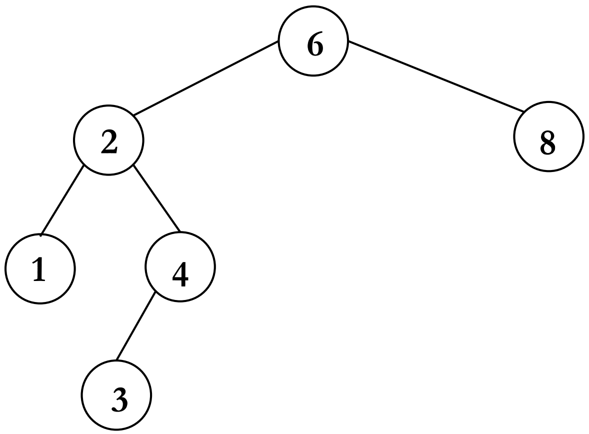
Binary Search Tree: FindMin



recursive implementation

```
Node  FindMin( SearchTree T )
{
    if( T == NULL )
        return NULL;

    else if  ( T->Left == NULL )
        return T;

    else  /* T->Left != NULL */
        return FindMin( T->Left );
}
```

Binary Search Tree: FindMax



nonrecursive implementation

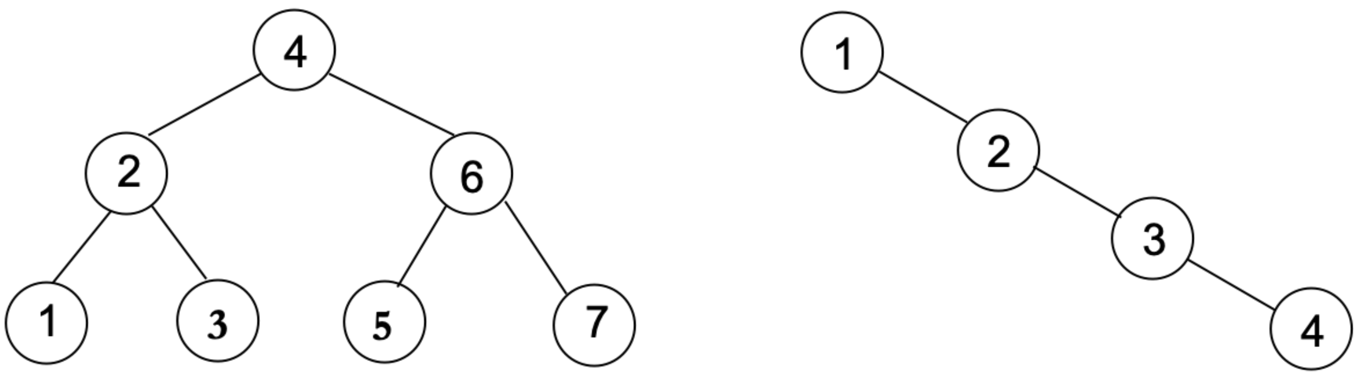
```
Node  FindMax( SearchTree T )
{
  if (T == NULL)
    return NULL

  else
          while( T->Right != NULL )
                T = T->Right;

     return T;
}
```

Binary Search Tree: delete

```
SearchTree  Delete( ElementType X, SearchTree T )
{
      Node TmpCell;

      if   ( T == NULL )
            Error( "Element not found" );

      else if   ( X < T->Element )          /* Go Left */
            T->Left = Delete( X, T->Left );

      else if   ( X > T->Element )          /* Go Right */
            T->Right = Delete( X, T->Right );

      else if   ( T->Left && T->Right )       {    /* found the node to be deleted */

              TmpCell = FindMin( T->Right );
              T->Element = TmpCell->Element;
              T->Right = Delete( T->Element, T->Right );

      }  else {                                    /* 1 or 0 child */
             TmpCell = T;
             if( T->Left == NULL )
                  T = T->Right;
             else if( T->Right == NULL )
                  T = T->Left;
             free(TmpCell);
      }
        return T;
}
```

Analysis of binary search tree

- The runtime of find(), insert(), and delete() is proportional to the height of the tree.
- What is the height of the tree with N nodes?
  - Worst case: Linear tree $O(n)$
  - Best case: Complete binary tree $O(\log n)$



Best Case: Insert 4, 2, 6, 1, 3, 5, 7　　Worst Case: insert 1, 2, 3, 4 ...

- Average case depends on the distribution of insertion/deletion
  - Assumption: insertions only
  - The order in which the keys are inserted is completely random.
    => will average all possible n! insertion orders.

Binary Search Tree: average-case analysis

- Internal Path Length D(N) is the sum of depths of all nodes in a binary search tree T with N nodes, which is O(NlogN)
  - An N-node tree consists of a root, an i-node left subtree and an (N – i – 1)-node right subtree for $0 \leq i < N$
  - let D(i) and D(N – i – 1) are internal path lengths of the left and right subtree w.r.t. their roots, respectively.

D(0) = D(1) = 0

D(N) = D(i) + D(N – i – 1) + N – 1

for the edge from the root to the root of the left and right subtree

The average value of D(i) (or D(N – i – 1)) is $\frac{1}{N}\sum_{j=0}^{N-1} D(j)$;

$$D(N) = \frac{2}{N}\sum_{j=0}^{N-1} D(j) + N - 1$$

Binary Search Tree: average-case analysis

- Prove $D(n) \le 4nlogn, n \ge 1$
  - Base: D(1) = 0
  - Induction: Assume that the theorem is true for $1 \le n \le k-1$

$$D(k) = \frac{2}{k}[\sum_{j=1}^{k-1} D(j)] + k - 1$$

$$\le \frac{2}{k}[\sum_{j=1}^{k-1} 4j\log j] + k - 1$$

$$= \frac{8}{k}[\sum_{j=1}^{k-1} j\log j] + k - 1$$

$$\le \frac{8}{k}[\sum_{j=1}^{\lceil k/2 \rceil - 1} j\log(\frac{k}{2}) + \sum_{j=\lceil k/2 \rceil}^{k-1} j\log k] + k - 1$$

$$\le \frac{8}{k}[\frac{k^2}{2}\log k - \frac{k^2}{8}] + k - 1$$

$$= 4k\log k - 1 \le 4k\log k$$

Binary Search Tree: balanced binary tree

- non-random insertion to BST can produce unbalanced trees
- can we rebalance the tree so that the tree always has $O(\log n)$ height?
- We need *balance* information for each node.