Digital Systems and Binary Numbers

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K-S. Sohn

Contents

Digital age and information ages

- Digital computers
  - general purposes
  - many scientific, industrial and commercial applications
- Digital systems
  - telephone switching exchanges
  - digital camera
  - electronic calculators, PDA's
  - digital TV
- Discrete information-processing systems
  - manipulate discrete elements of information

Signal

- An information variable represented by physical quantity
- For digital systems, the variable takes on discrete values
  - Two level, or binary values are the most prevalent values
- Binary values are represented abstractly by:
  - digits 0 and 1
  - words (symbols) False (F) and True (T)
  - words (symbols) Low (L) and High (H)
  - and words On and Off.
- Binary values are represented by values or ranges of values of physical quantities

Digital System

- Digital systems = {Digital modules = {Digital circuits}}
  - A digital system is an interconnection of digital modules.
  - To understand the operation of each digital module, it is necessary to have a basic knowledge of digital circuits and their logical function.

Number System

- Digits of base-r number
  - $0,1,\ldots,d_{r-2}, d_{r-1}$,
    - where r is positive integer.
- Ex. of digits
  - Decimal digits : 0,1,2,3,4,5,6,7,8,9
  - Binary digits(bits) : 0,1
  - Hexadecimal digits : 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E
    - The six letters (in addition to the 10 integers) in hexadecimal represent: 10, 11, 12, 13, 14, and 15, respectively.

Number Systems

- Decimal number

Base

$$a_j$$

Power

$$\cdots a_5 a_4 a_3 a_2 a_1 . a_{-1} a_{-2} a_{-3} \cdots$$

Decimal point

$$\Rightarrow \cdots + 10^3 a_3 + 10^2 a_2 + 10^1 a_1 + 10^0 a_0 + 10^{-1} a_{-1} + 10^{-2} a_{-2} + 10^{-3} a_{-3} + \cdots$$

  - ex.) $7.329 = 7\times 10^0 + 3\times 10^{-1} + 2\times 10^{-2} + 9\times 10^{-3}$

- General forma of base-r system

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \cdots + a_{-m} r^{-m}$$

  - Coefficient: $0 \le a_j \le r-1$

Examples of number representation

Example: Base-2 number (Binary number)

$$(11010.11)_2 = (26.75)_{10}$$
$$= 1\times 2^4 + 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 0\times 2^0 + 1\times 2^{-1} + 1\times 2^{-2}$$

Example: Base-5 number (Pentadecimal number)

$$(4021.2)_5$$
$$= 4\times 5^3 + 0\times 5^2 + 2\times 5^1 + 1\times 5^0 + 2\times 5^{-1} = (511.5)_{10}$$

Example: Base-8 number (Octal number)

$$(127.4)_8$$
$$= 1\times 8^3 + 2\times 8^2 + 1\times 8^1 + 7\times 8^0 + 4\times 8^{-1} = (87.5)_{10}$$

Example: Base-16 number (Hexadecimal number)

$$(\mathrm{B65F})_{16} = 11\times 16^3 + 6\times 16^2 + 5\times 16^1 + 15\times 16^0 = (46,687)_{10}$$

Powers of 2

- Special powers of 2
  - 2^{10} (1024) is Kilo, denoted "K"
  - 2^{20} (1,048,576) is Mega, denoted "M"
  - 2^{30} (1,073,741,824) is Giga, denoted "G"
  - 2^{40} (1,099,511,627,776) is Tera, denoted "T"
- Typical powers of 2

n	2^n	n	2^n	n	2^n
0	1	8	256	16	65,536
1	2	9	512	17	131,072
2	4	10	1,024 (1K)	18	262,144
3	8	11	2,048	19	524,288
4	16	12	4,096 (4K)	20	1,048,576 (1M)
5	32	13	8,192	21	2,097,152
6	64	14	16,384	22	4,194,304
7	128	15	32,768	23	8,388,608

Binary Arithmetic

- General arithmetic operation
  - Arithmetic operations with numbers in base-r follow the same rules as decimal numbers.
- Binary arithmetic operations
  - Single Bit Addition with Carry
  - Multiple Bit Addition
  - Single Bit Subtraction with Borrow
  - Multiple Bit Subtraction
  - Multiplication
  - BCD Addition

Binary Arithmetic

- **Addition**

Augend:	**101101**
Addend:	**+100111**
Sum:	**1010100**

- **Subtraction**

Minuend:	**101101**
Subtrahend:	**–100111**
Difference:	**000110**

- **Multiplication**

Multiplicand	**1011**
Multiplier	**× 101**
Partial Products	**1011**
	0000 -
	1011 - -
Product	**110111**

Number-base Conversions

- Ex. 1.1
  - Convert decimal 41 to binary.
  - The process is continued until the integer quotient becomes 0.

	Integer quotient		remainder	Coefficient
41/2	20	+	1	$a_0 = 1$
20/2	10	+	0	$a_1 = 0$
10/2	5	+	0	$a_2 = 0$
5/2	2	+	1	$a_3 = 1$
2/2	1	+	0	$a_4 = 0$
1/2	0	+	1	$a_5 = 1$

$\Rightarrow (41)_{10} = (a5a4a3a2a1a0)_2 = (101001)_2$

Number-base Conversions

- **Example 1.2**

Convert decimal 153 to octal. The required base *r* is 8.

Integer	Remainder
153	
19	1
2	3
0	2

$= (231)_8$

Number-base Conversion

- Example 1.3
  - Convert $(0.6875)_{10}$ to binary.
  - The process is continued until the fraction becomes 0 or until the number of digits has sufficient accuracy.

	Integer		Fraction	Coefficient
$0.6875 \times 2 =$	1	+	0.3750	$a_{-1} = 1$
$0.3750 \times 2 =$	0	+	0.7500	$a_{-2} = 0$
$0.7500 \times 2 =$	1	+	0.5000	$a_{-3} = 1$
$0.5000 \times 2 =$	1	+	0.0000	$a_{-4} = 1$

$$\Rightarrow \boxed{(0.6875)_{10} = (0.a_{-1}a_{-2}a_{-3}a_{-4})_2 = (0.1011)_2}$$

- To convert a decimal fraction to a number expressed in base r, a similar procedure is used. However, multiplication is by r instead of 2, and the coefficients found from the integers may range in value from 0 to r -1 instead of 0 and 1.

Number-base Conversions

- Example 1.4
  - Convert $(0.513)_{10}$ to octal.

$$0.513 \times 8 = 4.104$$
$$0.104 \times 8 = 0.832$$
$$0.832 \times 8 = 6.656$$
$$0.656 \times 8 = 5.248$$
$$0.248 \times 8 = 1.984$$
$$0.984 \times 8 = 7.872$$

$$\Rightarrow \boxed{(0.513)_{10} = (0.406517\ldots)_8}$$

- **From Examples 1.1 and 1.3:** $\mathbf{(41.6875)_{10} = (101001.1011)_2}$
- **From Examples 1.2 and 1.4:** $\mathbf{(153.513)_{10} = (231.406517)_8}$

Number of Different Bases

Decimal (base 10)	Binary (base 2)	Octal (base 8)	Hexadecimal (base 16)
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Octal and Hexadecimal Numbers

- **Conversion from binary to octal**
  - Conversion from binary to octal can be done by positioning the binary number into groups of three digits each, starting from the binary point and proceeding to the left and to the right.

(10	110	001	101	011	·	111	100	000	110)$_2$	= $(26153.7406)_8$
2	6	1	5	3		7	4	0	6	

- **Conversion from binary to hexadecimal**
  - Conversion from binary to hexadecimal is similar, except that the binary number is divided into groups of four digits:

(10	1100	0110	1011	·	1111	0010)$_2$	= $(2C6B.F2)_{16}$
2	C	6	B		F	2	

Octal and Hexadecimal Numbers

- **Conversion from octal/hexadecimal to binary**
  - Conversion from octal or hexadecimal to binary is done by reversing the preceding procedure.

$(673.124)_8$ = (110	111	011	·	001	010	100)$_2$
6	7	3		1	2	4

$(306.D)_{16}$ = (0011	0000	0110	·	1101)$_2$
3	0	6		D

Complements of Numbers

Given n-digit base-r number N

- **Diminished(or reduced) radix complement**
  - (r-1)'s complement of N = $(r^n - 1)$- N
- **Radix complement**
  - r's complement of N = r^n - N (for N≠0) ⇒ (r^n-1)-N+1
    = 0 (for N=1)
- **Examples**
  - Diminished radix complements
    - Base-10: 012398, 246700 ⇒ 987601, 753299, respectively
    - Base-2: 1101100, 0110111 ⇒ 0010011, 1001000, respectivel
  - Radix complements
    - Base-10: 012398, 246700 is 987602, 753300, respectively
    - Base-2: 1101100, 0110111 ⇒ 0010100, 1001001, respectively

Complements of Numbers

- **Complement with radix point (c.f. decimal point)**
  - Step-1. remove the radix point from the number
  - Step-2. obtain r's or (r-1)'s complement
  - Step-3. Put the radix point at the same position

  - Ex. 1's complement with radix point
    - 1101.011 ⇒ 1101011 ⇒ 0010100 ⇒ 0010.100
  - Ex. 2's complement with radix point
    - 1101.011 ⇒ 1101011 ⇒ 0010101 ⇒ 0010.101

Subtraction with Complements

- The subtraction of two n-digit unsigned numbers M-N in base r can be done as follows:
  1. Add the minuend *M* to the r's complement of the subtrahend *N*. $M + (r^n - N) = M - N + r^n$
  2. If $M \geq N$, the sum will produce end end carry r^n, which can be discarded; what is left is the result $M - N$
  3. If $M < N$, the sum does not produce an end carry and is equal to $r^n - (N - M)$, which is r's complement of $(N - M)$. To obtain the answer in a familiar form, take the r's complement of the sum and place a genative sign in front.

Subtraction with Complements

- **Example 1.5**

  Using 10's complement, subtract 72532 – 3250.

$$\begin{array}{rr} M = & 72532 \\ \text{10's complement of} \quad N = & \underline{+96750} \\ \text{Sum} = & 169282 \\ \text{Discard end carry } 10^5 = & \underline{-100000} \\ \text{Answer} = & 69282 \end{array}$$

- **Example 1.6**

  Using 10's complement, subtract 3250 – 72532

$$\begin{array}{rr} \text{M} = & 03250 \\ \text{10's complement of} \quad \text{N} = & \underline{+27468} \\ \text{Sum} = & 30718 \end{array}$$

There is no end carry.

Therefore, the answer is – (10's complement of 30718) = – 69282.

Subtraction with Complements

- **Example 1.7**

Given the two binary numbers *X* = 1010100 and *Y* = 1000011, perform the subtraction (a) X – Y and (b) *Y* – *X* by using 2's complement.

$$\begin{array}{lrr} \text{(a)} & \text{X} = & 1010100 \\ & \text{2's complement of Y} = & \underline{+0111101} \\ & \text{Sum} = & 10010001 \\ & \text{Discard end carry } 2^7 = & \underline{-10000000} \\ & \text{Answer. X} - \text{Y} = & 0010001 \end{array}$$

$$\begin{array}{lrr} \text{(b)} & \text{Y} = & 1000011 \\ & \text{2's complement of X} = & \underline{+0101100} \\ & \text{Sum} = & 1101111 \end{array}$$

There is no end carry. Therefore, the answer is *Y* – *X* = – (2's complement of 1101111) = – 0010001.

Subtraction with Complements

- Subtraction of unsigned numbers
  - Subtraction of unsigned numbers can also be done by means of the (r -1)'s complement. Remember that the (r -1) 's complement is one less then the r's complement.
- Example 1.8
  - Repeat Example 1.7, but this time using 1's complement.

(a) $X - Y = 1010100 - 1000011$

$$\begin{array}{rr} X = & 1010100 \\ \text{1's complement of } Y = & \underline{+\ 0111100} \\ \text{Sum} = & 10010000 \\ \text{End-around carry} = & \underline{+\qquad 1} \\ \text{Answer. } X - Y = & 0010001 \end{array}$$

Subtraction with Complements

- Example 1.8

$$(b) Y - X = 1000011 - 1010100$$

$$\begin{array}{rr} Y = & 1000011 \\ \text{1's complement of } X = & \underline{+0101011} \\ \text{Sum} = & 1101110 \end{array}$$

  - There is no end carry, therefore, the answer is:
    X - Y = -(1's complement of 1101110) = -0010001

Sighed Binary Numbers

- Needs
  - To represent negative integers, we need a notation for negative values.
  - It is customary to represent the sign with a bit placed in the leftmost position of the number.
- Sign bit
  - The convention is to make the sign bit 0 for positive and 1 for negative
  - Example:

Signed-magnitude representation:	10001001
Signed-1's-complement representation:	11110110
Signed-2's-complement representation:	11110111

Signed Binary Numbers

- Sign representation with the sign bit

Decimal	Signed-2's Complement	Signed-1's Complement	Signed Magnitude
+7	0111	0111	0111
+6	0110	0110	0110
+5	0101	0101	0101
+4	0100	0100	0100
+3	0011	0011	0011
+2	0010	0010	0010
+1	0001	0001	0001
+0	0000	0000	0000
−0	—	1111	1000
−1	1111	1110	1001
−2	1110	1101	1010
−3	1101	1100	1011
−4	1100	1011	1100
−5	1011	1010	1101
−6	1010	1001	1110
−7	1001	1000	1111
−8	1000	—	—

Signed Binary Numbers

- Arithmetic addition
  - The addition of two numbers in the signed-magnitude system follows the rules of ordinary arithmetic. If the signs are the same, we add the two magnitudes and give the sum the common sign. If the signs are different, we subtract the smaller magnitude from the larger and give the difference the sign if the larger magnitude.
  - The addition of two signed binary numbers with negative numbers represented in signed-2's-complement form is obtained from the addition of the two numbers, including their sign bits.
  - A carry out of the sign-bit position is discarded.

Signed Binary Numbers

- Arithmetic addition
  - Example

+ 6	00000110	− 6	11111010
+13	00001101	+13	00001101
+ 19	00010011	+ 7	00000111

+ 6	00000110	− 6	11111010
−13	11110011	−13	11110011
− 7	11111001	− 19	11101101

Signed Binary Numbers

- Arithmetic subtraction
  - In 2's complement form:

1. 1. Take the 2's complement of the subtrahend (including the sign bit) and add it to the minuend (including sign bit).
2. 2. A carry out of sign-bit position is discarded.

$$(\pm A) - (+B) = (\pm A) + (-B)$$
$$(\pm A) - (-B) = (\pm A) + (+B)$$

- Example:

(– 6) – (– 13) ⇒ (11111010 – 11110011)

⇒ (11111010 + 00001101)

⇒ 00000111 (+ 7)

BCD Decimal Code

- BCD (Binary-Coded Decimal)

Decimal Symbol	BCD Digit
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

A number with k decimal digits will require 4k bits in BCD. Decimal 396 is represented in BCD with 12bits as 0011 1001 0110, with each group of 4 bits representing one decimal digit. A decimal number in BCD is the same as its equivalent binary number only when the number is between 0 and 9. A BCD number greater than 10 looks different from its equivalent binary number, even though both contain 1's and 0's. Moreover, the binary combinations 1010 through 1111 are not used and have no meaning in BCD.

BCD Decimal Code

- Example
  - Consider decimal 185 and its corresponding value in BCD and binary:

⇒ $\boxed{(185)_{10} = (0001\ 1000\ 0101)_{BCD} = (10111001)_2}$

- BCD addition
  - Example:

4	0100	4	0100	8	1000
+5	+0101	+8	+1000	+9	+1001
9	1001	12	1100	17	10001
			+0110		+0110
			10010		10111

BCD Decimal Code

- Example of BCD addition:
  - Consider the addition of 184 + 576 = 760 in BCD

BCD	1	1		
	0001	1000	0100	184
	+ 0101	0111	0110	+576
Binary sum	0111	10000	1010	
Add 6		0110	0110	
BCD sum	0111	0110	0000	760

Binary Codes for the Decimal Digits

Decimal Digit	BCD 8421	2421	Excess-3	8, 4, −2, −1
0	0000	0000	0011	0000
1	0001	0001	0100	0111
2	0010	0010	0101	0110
3	0011	0011	0110	0101
4	0100	0100	0111	0100
5	0101	1011	1000	1011
6	0110	1100	1001	1010
7	0111	1101	1010	1001
8	1000	1110	1011	1000
9	1001	1111	1100	1111
	1010	0101	0000	0001
Unused bit combi- nations	1011	0110	0001	0010
	1100	0111	0010	0011
	1101	1000	1101	1100
	1110	1001	1110	1101
	1111	1010	1111	1110

Gray Code

- Gray code

Gray Code	Decimal Equivalent
0000	0
0001	1
0011	2
0010	3
0110	4
0111	5
0101	6
0100	7
1100	8
1101	9
1111	10
1110	11
1010	12
1011	13
1001	14
1000	15

ASCII Character Code

- ASCII(Americal Standard Code for Information Interexchange) code

	$b_7b_6b_5$							
$b_4b_3b_2b_1$	000	001	010	011	100	101	110	111
0000	NUL	DLE	SP	0	@	P	‘	p
0001	SOH	DC1	!	1	A	Q	a	q
0010	STX	DC2	“	2	B	R	b	r
0011	ETX	DC3	#	3	C	S	c	s
0100	EOT	DC4	$	4	D	T	d	t
0101	ENQ	NAK	%	5	E	U	e	u
0110	ACK	SYN	&	6	F	V	f	v
0111	BEL	ETB	‘	7	G	W	g	w
1000	BS	CAN	(	8	H	X	h	x
1001	HT	EM	)	9	I	Y	i	y
1010	LF	SUB	*	:	J	Z	j	z
1011	VT	ESC	+	;	K	[	k	{
1100	FF	FS	,	<	L		l	|
1101	CR	GS	–	=	M	]	m	}
1110	SO	RS	.	>	N	^	n	~
1111	SI	US	/	?	O	–	o	DEL

ASCII Character Code

- Features
  - A popular code used to represent information sent as character-based data.
  - It uses 7-bits to represent:
    - 94 Graphic printing characters.
    - 34 Non-printing characters
  - Some non-printing characters are used for text format (e.g. BS = Backspace, CR = carriage return)
  - Other non-printing characters are used for record marking and flow control (e.g. STX and ETX start and end text areas).

Error Detecting Code

- Needs
  - To detect errors in data communication and processing, an eighth bit is sometimes added to the ASCII character to indicate its parity.
- Parity bit
  - A parity bit is an extra bit included with a message to make the total number of 1's either even or odd.
- Example:
  - Consider the following two characters and their even and odd parity:

	With even parity	With odd parity
ASCII A = 1000001	01000001	11000001
ASCII T = 1010100	11010100	01010100

Error Detecting Code

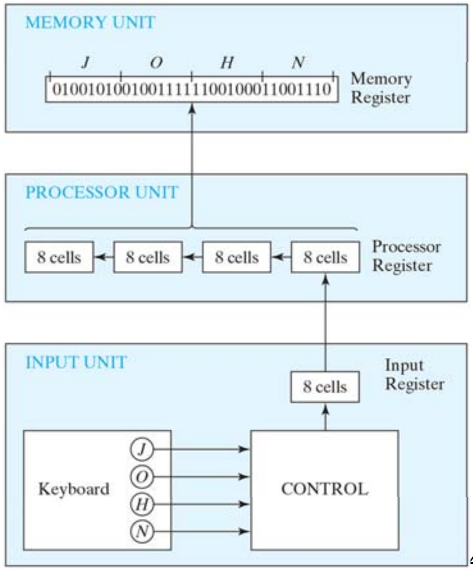
- Features
  - Redundancy (e.g. extra information), in the form of extra bits, can be incorporated into binary code words to detect and correct errors.
  - A simple form of redundancy is parity, an extra bit appended onto the code word to make the number of 1's odd or even. Parity can detect all single-bit errors and some multiple-bit errors.
  - A code word has even parity if the number of 1's in the code word is even.
  - A code word has odd parity if the number of 1's in the code word is odd.

Binary Storage and Registers

- Registers
  - A binary cell is a device that possesses two stable states and is capable of storing one of the two states.
  - A register is a group of binary cells. A register with n cells can store any discrete quantity of information that contains n bits. n cells 2n possible states
- A binary cell
  - two stable state
  - store one bit of information
  - examples: flip-flop circuits, ferrite cores, capacitor
- A register
  - a group of binary cells
  - AX in x86 CPU

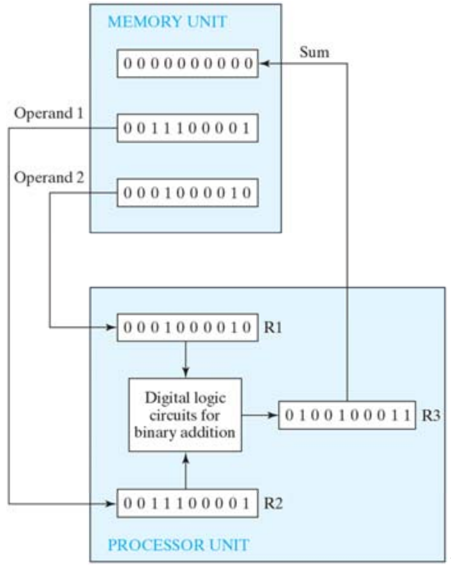
Binary Storage and Registers

- Register Transfer
  - a transfer of the information stored in one register to another
  - one of the major operations in digital system
  - an example ⇒



Binary Storage and Registers

- Binary information processing



Binary Logic

- Definition of binary logic
  - Binary logic consists of binary variables and a set of logical operations.
  - The variables are designated by letters of the alphabet, such as A, B, C, x, y, z, etc, with each variable having two and only two distinct possible values: 1 and 0,
  - There are three basic logical operations: AND, OR, and NOT.

Binary Logic

- Basic logical operations: AND, OR, and NOT

AND			**OR**			**NOT**	
x	y	$x \cdot y$	x	y	$x + y$	x	x'
0	0	0	0	0	0	0	1
0	1	0	0	1	1	1	0
1	0	0	1	0	1		
1	1	1	1	1	1		

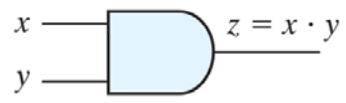
Binary Logic

- Basic logical operations:

1. AND: This operation is represented by a dot or by the absence of an operator. For example, $x \cdot y = z$ or $xy = z$ is read "x AND y is equal to z," The logical operation AND is interpreted to mean that $z = 1$ if only $x = 1$ and $y = 1$; otherwise $z = 0$. (Remember that x, y, and z are binary variables and can be equal either to 1 or 0, and nothing else.)
2. OR: This operation is represented by a plus sign. For example, $x + y = z$ is read "x OR y is equal to z," meaning that $z = 1$ if $x = 1$ or $y = 1$ or if both $x = 1$ and $y = 1$. If both $x = 0$ and $y = 0$, then $z = 0$.
3. NOT: This operation is represented by a prime (sometimes by an overbar). For example, $x' = z$ (or $\bar{x} = z$) is read "not x is equal to z," meaning that z is what z is not. In other words, if $x = 1$, then $z = 0$, but if $x = 0$, then $z = 1$, The NOT operation is also referred to as the complement operation, since it changes a 1 to 0 and a 0 to 1.

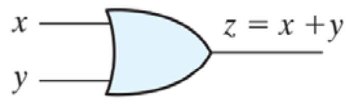
Binary Logic

- Logic Gates
  - Graphic symbols of AND gate



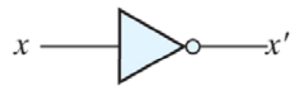
(a) Two-input AND gate

  - Graphic symbols of OR gate



(b) Two-input OR gate

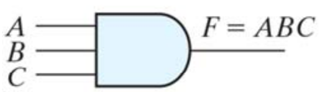
  - Graphic symbols of NOT gate



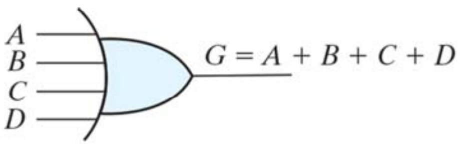
(c) NOT gate or inverter

Binary Logic

- Example: logic gate (logical) operation
  - Gate with multiple inputs



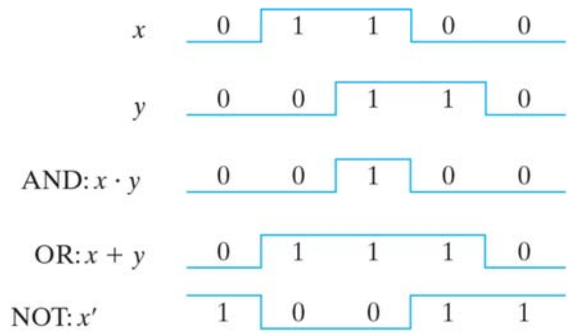
(a) Three-input AND gate



(b) Four-input OR gate

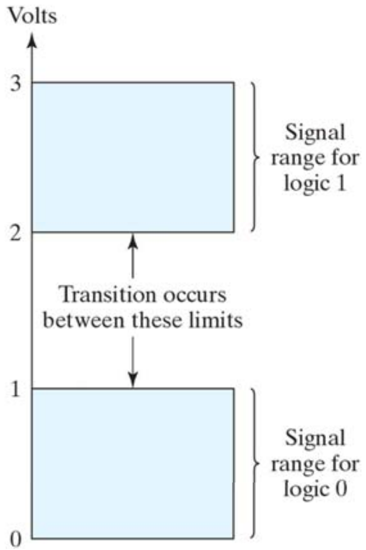
Binary Logic

- Input-output signals for gates



Binary Logic

- Digital signal level



Discussion~~~