Multivariate logistic distribution

July 2, 2019

1 Latent variable notation

We are interested in modeling multivariate binomial outcomes. We will write the observed outcome as $\mathbf{y}_i = (y_{i1}, y_{i2}, \ldots, y_{ip})'$, a vector of binary variables. To keep our notations consistent, we will model $\mathbf{y}_i$ using a latent variable approach:

$$\begin{aligned} \mathbf{y}_i &= 1(\mathbf{z}_i > 0) \\ \mathbf{z}_i &\sim \text{distrib}(\mu, \Sigma^2). \end{aligned} \tag{1}$$

where $\mathbf{z}_i$ is a vector of continuous variable following a probability distribution with mean μ and covariance Σ^2. This approach generalizes the univariate logistic regression.

2 Models

2.1 Logistic regression

First, consider the univariate logistic regression (with intercept only):

$$\text{logit } \Pr(y_i = 1) = \mu. \tag{2}$$

This is equivalent to writing

$$\begin{aligned} \mathbf{y}_i &= 1(\mathbf{z}_i > 0) \\ \mathbf{z}_i &\sim \text{logistic}(\mu, 1), \end{aligned} \tag{3}$$

where $\text{logistic}(\cdot|\mu, s)$ is a logistic *distribution* with location parameter μ and a scale parameter s.

One way to extend this model to account for multivariate observation is to use a mixed model approach:

$$\begin{aligned} \text{logit } \Pr(\mathbf{Y}_i = y_i) &= \boldsymbol{\mu} + \mathbf{r}_i \\ \mathbf{r}_i &\sim \mathcal{N}(0, \sigma^2 R), \end{aligned} \tag{4}$$

where $\mathcal{N}(0, \sigma^2 R)$ is a multivariate normal distribution with covariance $\sigma^2 R$ (R is the correlation matrix). For convenience, we assume that random effects variance σ^2 is constant among response variables. Equivalently, we can write this as

$$\begin{aligned}
\mathbf{y}_i &= 1(\mathbf{z}_i > 0) \\
\mathbf{z}_i &\sim \text{logistic}(\boldsymbol{\mu} + \mathbf{r}_i, 1) \\
\mathbf{r}_i &\sim \mathcal{N}(0, \sigma^2 R),
\end{aligned} \tag{5}$$

which can be further expanded as

$$\begin{aligned}
\mathbf{y}_i &= 1(\mathbf{z}_i > 0) \\
\mathbf{z}_i &= \boldsymbol{\mu} + \mathbf{r}_i + \boldsymbol{\epsilon}_i \\
\mathbf{r}_i &\sim \mathcal{N}(0, \sigma^2 R) \\
\epsilon_{ij} &\sim \text{logistic}(0, 1)
\end{aligned} \tag{6}$$

Essentially, we have a continuous latent variable $\mathbf{z}_i$ which has a mean $\boldsymbol{\mu}$ and two "error" terms: $\mathbf{r}_i$, which follows a multivariate normal with covariance $\sigma^2 R$, and $\boldsymbol{\epsilon}_i$, which follows an independent logistic (each marginal distribution is an i.i.d. logistic distribution). Due to two levels of uncertainties, it becomes much harder to estimate the correlation structure R. I don't have a good analytical argument for this but I hope this is somewhat intuitive... I'll compare this with other models later; this might make things slightly clearer.

Identifiability of the random effects variance

It doesn't seem like Jonathan is completely convinced that σ^2 is not identifiable; he says that it is "practically" unidentifiable. Let's try to do some math. Consider a univariate logistic regression with an underlying normal random effects on the mean:

$$\begin{aligned}
y_i &= 1(z_i > 0) \\
z_i &\sim \text{logistic}(\mu + r_i, 1) \\
r_i &\sim \mathcal{N}(0, \sigma^2)
\end{aligned} \tag{7}$$

Then, the marginal likelihood of this model can be written as

$$\prod_{i=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ 1(z_i > 0)^{y_i} 1(z_i \leq 0)^{1-y_i} \right\} f(z_i|\mu + r_i, 1) dz_i g(r_i|0, \sigma^2) dr_i, \tag{8}$$

where f is the pdf of the standard logistic distribution and g is the pdf of the standard normal. This is ugly. When $y_i = 0$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left\{ 1(z_i > 0)^{y_i} 1(z_i \leq 0)^{1-y_i} \right\} f(z_i|\mu + r_i, 1) dz_i \\
&= \int_{-\infty}^{0} f(z_i|\mu + r_i, 1) dz_i \\
&= \frac{1}{1 + \exp(\mu + r_i)}
\end{aligned} \tag{9}$$

Then,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{1(z_i>0)^{y_i}1(z_i\le 0)^{1-y_i}\right\}f(z_i|\mu+r_i,1)dz_i g(r_i|0,\sigma^2)dr_i \\
&=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\frac{1}{1+\exp(\mu+r_i)}\exp\left(-\frac{r_i^2}{2\sigma^2}\right)dr_i
\end{aligned}
\tag{10}
$$

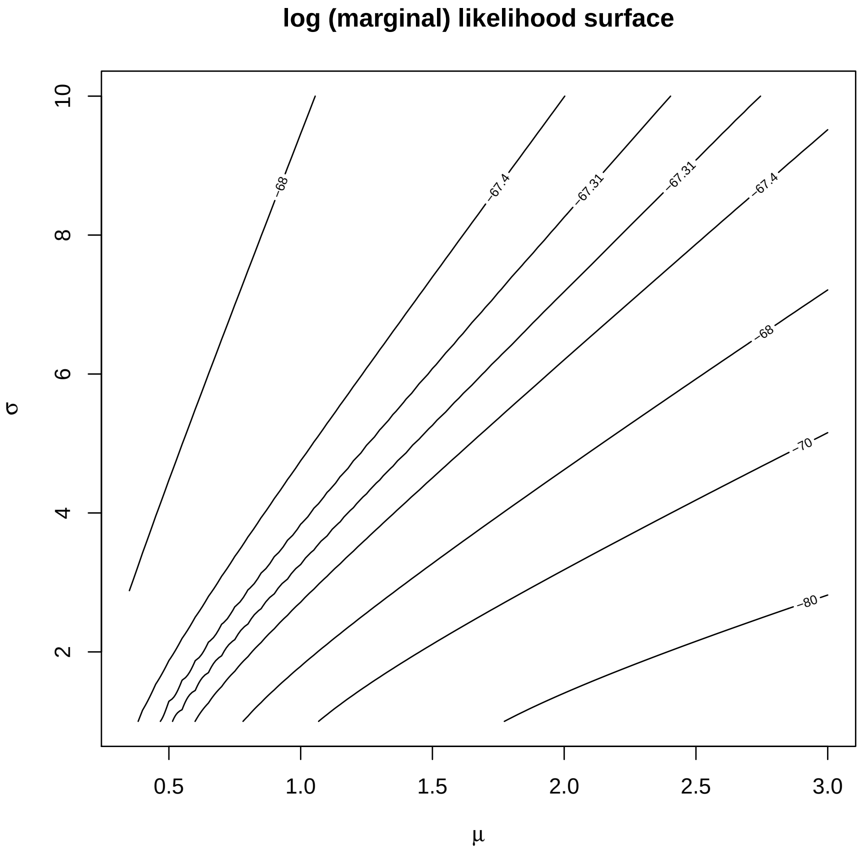
I can't evaluate this integral analytically but eventually the marginal likelihood can be written as

$$
\begin{aligned}
&\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n\left(\int_{-\infty}^{\infty}\frac{1}{1+\exp(\mu+r_i)}\exp\left(-\frac{r_i^2}{2\sigma^2}\right)dr_i\right)^{n_0} \\
&\times\left(\int_{-\infty}^{\infty}\left(1-\frac{1}{1+\exp(\mu+r_i)}\right)\exp\left(-\frac{r_i^2}{2\sigma^2}\right)dr_i\right)^{n_1},
\end{aligned}
\tag{11}
$$

where n_0 is the number of 0's and n_1 is the number of 1's.

We can work out a numerical example. Assume $n_1 = 60$ and $n_0 = 40$. Then, the MLE of μ of the logistic regression without random effects is approximately 0.4 (plogis(0.4) is approximately 0.6). As we increase σ, we see that our estimate of μ increases. Here, we show the log marginal likelihood surface:

```
library(emdbook)
ifun1 <- function(r, mu, sigma)
        1/(1 + exp(mu + r)) * exp(-r^2/(2 * sigma^2))
ifun2 <- function(r, mu, sigma)
        (1 - 1/(1 + exp(mu + r))) * exp(-r^2/(2 * sigma^2))

llfun <- function(mu, sigma, n1=60, n0=40, log=TRUE) {
        first <- integrate(ifun1, -200, 200, mu=mu, sigma=sigma)
        second <- integrate(ifun2, -200, 200, mu=mu, sigma=sigma)

        ll <- (n0 + n1) * log(1/sqrt(2 * pi * sigma^2)) +
                n0 * log(first[[1]]) + n1 * log(second[[1]])

        if (!log) ll <- exp(ll)

        ll
}

muvec <- seq(0.35, 3, by=0.05)
sigmavec <- exp(seq(log(1), log(10), length.out=100))

contour(muvec, sigmavec, apply2d(llfun, muvec, sigmavec),
                xlab=expression(mu),
                ylab=expression(sigma),
                main="log (marginal) likelihood surface",
                levels=c(-67.31, -67.4, -68, -70, -80))
```



We can see that there's a relatively flat region starting from $\sigma \approx 0$ and $\mu \approx 0.5$ (see line -67.31).

2.2 Probit regression

Probit regression provides a natural way of defining the underlying correlation for multivariate binomial response:

$$\begin{aligned} \mathbf{y}_i &= 1(\mathbf{z}_i > 0) \\ \mathbf{z}_i &\sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{R}) \end{aligned} \tag{12}$$

In other words,

$$\begin{aligned} \mathbf{y}_i &= 1(\mathbf{z}_i > 0) \\ \mathbf{z}_i &\sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{R}) \end{aligned} \tag{13}$$

which can be rewritten as

$$\begin{aligned}\mathbf{y}_i &= 1(\mathbf{z}_i > 0)\\ \mathbf{z}_i &= \boldsymbol{\mu} + \boldsymbol{\epsilon}_i\\ \boldsymbol{\epsilon}_i &\sim \mathcal{N}(0, \mathbf{R}).\end{aligned} \tag{14}$$

This model tries to capture the correlation among the latent "residuals". Compare this expression with the mixed model approach:

$$\begin{aligned}\mathbf{y}_i &= 1(\mathbf{z}_i > 0)\\ \mathbf{z}_i &= \boldsymbol{\mu} + \mathbf{r}_i + \boldsymbol{\epsilon}_i\\ \mathbf{r}_i &\sim \mathcal{N}(0, \sigma^2 R)\\ \epsilon_{ij} &\sim \text{logistic}(0, 1)\end{aligned} \tag{15}$$

The mixed model approach seeks to decompose the latent residual into two terms. We're going to have less power to detect the correlation structure.

2.3 Multivariate logistic distribution

Multivariate logistic distribution suggested by O'Brien allows us to model residual correlations while preserving the marginal logistic distribution:

$$\begin{aligned}\mathbf{y}_i &= 1(\mathbf{z}_i > 0)\\ \mathbf{z}_i &= \boldsymbol{\mu} + \log\left(\frac{F(\mathbf{e}_i)}{1 - F(\mathbf{e}_i)}\right)\end{aligned} \tag{16}$$

where $\mathbf{e}_i$ comes from a multivariate distribution with mean 0 and some correlation structure and F is the univariate cumulative distribution function of $\mathbf{e}_i$. Then, the resulting distribution of $\mathbf{z}_i$ also has a very similar correlation structure (see code below) as $\mathbf{e}_i$ and each z_{ij} follows a logistic distribution.

```
## very similar correlations
library(mvtnorm)

## Warning:  package 'mvtnorm' was built under R version 3.5.2
##
## Attaching package:  'mvtnorm'
## The following object is masked from 'package:emdbook':
##
##     dmvnorm

corr <- matrix(
        c(1, .1, .3,
          .1, 1, -.3,
          .3, -.3, 1),
        3, 3
)
```

```r
set.seed(101)
rr <- rmvnorm(10000, sigma=corr)
cor(rr)

##             [,1]        [,2]       [,3]
## [1,] 1.00000000  0.09329252  0.2929604
## [2,] 0.09329252  1.00000000 -0.3104384
## [3,] 0.29296044 -0.31043837  1.0000000

cor(log(pnorm(rr)/(1 - pnorm(rr))))

##            [,1]       [,2]       [,3]
## [1,] 1.0000000  0.0926942  0.2898303
## [2,] 0.0926942  1.0000000 -0.3088695
## [3,] 0.2898303 -0.3088695  1.0000000
```

Regardless of what model one decides to use, there should be a way to convert the estimate into a single, consistent scale...? Predict probability from posterior and convert that into odds ratio...? Not sure yet.