

Using moment equations to understand heterogeneity

Definition

We want to write $Y_o(v; D)$ for **Type**, **order**, **variable**, and **Domain**.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a) da$. Since we're mainly interested with susceptibility for now, T_o represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, M_1 is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

SI example

We have $\dot{S}(a) = -\Lambda \sigma(a) S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that M_1 is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using M defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1 + \kappa)M^2$ and assuming that κ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M} S^\kappa$, where \hat{M} is the mean susceptibility of the susceptible population at a disease free equilibrium.

We can define $\kappa = \kappa_2$ and extend this idea to $\kappa_i = \frac{M_i M_{i-2}}{M_{i-1}^2} - 1$. This also We can derive an equation for κ_i as well:

$$\begin{aligned} \kappa_i &= \frac{M_{i-2} M_{i-1} \dot{M}_i + M_{i-1} M_i \dot{M}_{i-2} - 2 M_{i-2} M_i \dot{M}_{i-1}}{M_{i-1}^3} \\ &= -\Lambda \frac{M_{i-2} M_{i-1} (M_{i+1} - M_i M_1) + M_{i-1} M_i (M_{i-1} - M_{i-2} M_1) - 2 M_{i-2} M_i (M_i - M_{i-1} M_1)}{M_{i-1}^3} \\ &= -\Lambda \frac{M_{i-2} M_{i-1} M_{i+1} + M_{i-1}^2 M_i - 2 M_{i-2} M_i^2}{M_{i-1}^3} \\ &= -\Lambda \frac{M_{i-2} (\kappa_{i+1} + 1) M_i^2 + M_{i-1}^2 M_i - 2 M_{i-2} M_i^2}{M_{i-1}^3} \\ &= -\Lambda \frac{(\kappa_{i+1} - 1) M_{i-2} M_i^2 + M_{i-1}^2 M_i}{M_{i-1}^3} \\ &= -\Lambda \frac{(\kappa_{i+1} - 1)(\kappa_i + 1) M_i M_{i-1}^2 + M_{i-1}^2 M_i}{M_{i-1}^3} \\ &= -\Lambda \frac{(\kappa_{i+1} \kappa_i - \kappa_i + \kappa_{i+1}) M_i}{M_{i-1}} \end{aligned}$$