# Using moment equations to understand heterogeneity

## Definition

We want to write $Y_o(v; D)$ for **T**y**p**e, **o**rder, **v**ariable, and **D**omain.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a) da$. Since we're mainly interested with susceptibility for now, $T_o$ is represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, $M_1$ is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

## SI model

We have $\dot{S}(a) = -\Lambda\sigma(a)S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that $M_1$ is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using $M$ defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1 + \kappa)M^2$ and assuming that $\kappa$ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M} S^\kappa$, where $\hat{M}$ is the mean susceptibility of the susceptible population at a disease free equilibrium.

**Definition -** $\kappa_i = \frac{M_i}{M_{i-1} M_1} - 1$

$$\begin{aligned}
\dot{\kappa}_i &= \frac{M_{i-1} M_1 \dot{M}_i - M_i M_1 \dot{M}_{i-1} - M_i M_{i-1} \dot{M}_1}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{M_{i-1} M_1 (M_{i+1} - M_i M_1) - M_i M_1 (M_i - M_{i-1} M_1) - M_i M_{i-1} (M_2 - M_1^2)}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{M_{i-1} M_1 M_{i+1} - M_i M_1 M_i - M_i M_{i-1} M_2 + M_i M_{i-1} M_1^2}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{(\kappa_{i+1} + 1) M_i M_{i-1} M_1^2 - M_i M_1 M_i - (\kappa_2 + 1) M_i M_{i-1} M_1^2 + M_i M_{i-1} M_1^2}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{(\kappa_{i+1} + 1) M_i M_{i-1} M_1 - (\kappa_i + 1) M_{i-1} M_i M_1 - \kappa_2 M_i M_{i-1} M_1}{M_{i-1}^2 M_1} \\
&= -\Lambda \frac{(\kappa_{i+1} + 1) M_i - (\kappa_i + 1) M_i - \kappa_2 M_i}{M_{i-1}} \\
&= -\Lambda M_i \frac{\kappa_{i+1} - (\kappa_2 + \kappa_i)}{M_{i-1}} \\
&= -\Lambda M_1 (\kappa_i + 1) \{\kappa_{i+1} - (\kappa_2 + \kappa_i)\}
\end{aligned}$$

How accurate is this?

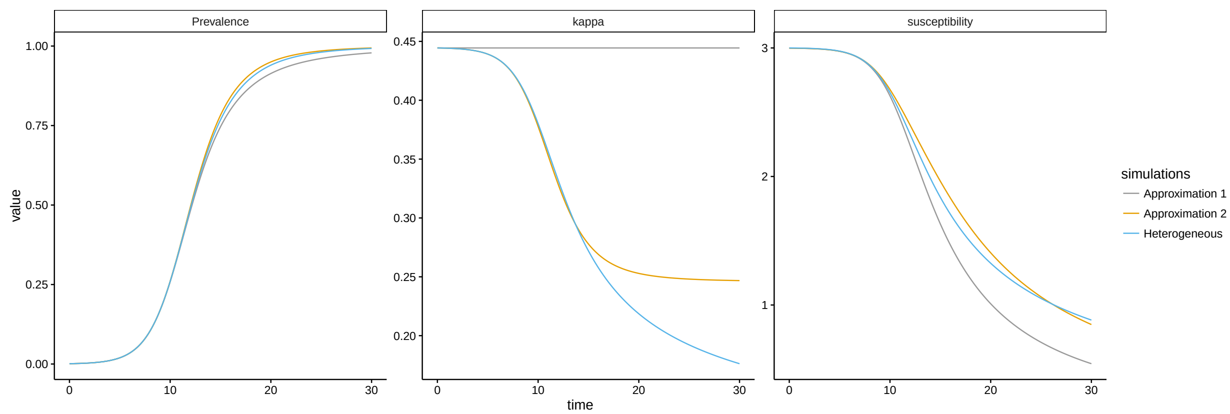
### Ex 1 - lognormal distribution

Linear equation:

$$r \approx (\hat{r} - 2)S + 2$$

- Heterogeneous model
- Approximated model: linear equation for $\kappa_3/\kappa_2$ coupled with approximated equation for $\kappa$
- Approximated model: constant $\kappa$

```
source("plotFunctions.R")
name <- c("Heterogeneous", "Approximation 1", "Approximation 2")
plotSim(SI.list.lnorm, name)
```




## SIS model

Here is a simple SIS model: $\dot{S}(a) = \rho(N(a) - S(a)) - \Lambda\sigma(a)S(a)$ and $\dot{I} = \int \Lambda\sigma S(a)da - I$, where $N(a)$ is the initial distribution of the susceptible individuals in a disease free equilibrium. We are going to define $\bar{Y}_o = Y_o(v; N)$. For this model, we have $\dot{T}_i = \rho(\bar{T}_i - T_i) - \Lambda T_{i+1}$. Using chain rule, we can also get an equation for $M = M_1$:

$$\begin{aligned}
\dot{M}_i &= \frac{\dot{T}_i}{T_0} - \frac{\dot{T}_0 T_i}{T_0^2} \\
&= (\rho(\bar{T}_i/T_0 - M_i) - \Lambda M_{i+1}) - \rho(M_i\bar{T}_0/T_0 - M_i) + \Lambda M_i M_1 \\
&= \rho(\bar{M}_i - M_i)\frac{\bar{T}_0}{T_0} - \Lambda(M_{i+1} - M_i M_1) \\
&= \rho(\bar{M}_i - M_i)/T_0 - \Lambda\kappa_{i+1} M_i M_1
\end{aligned}$$

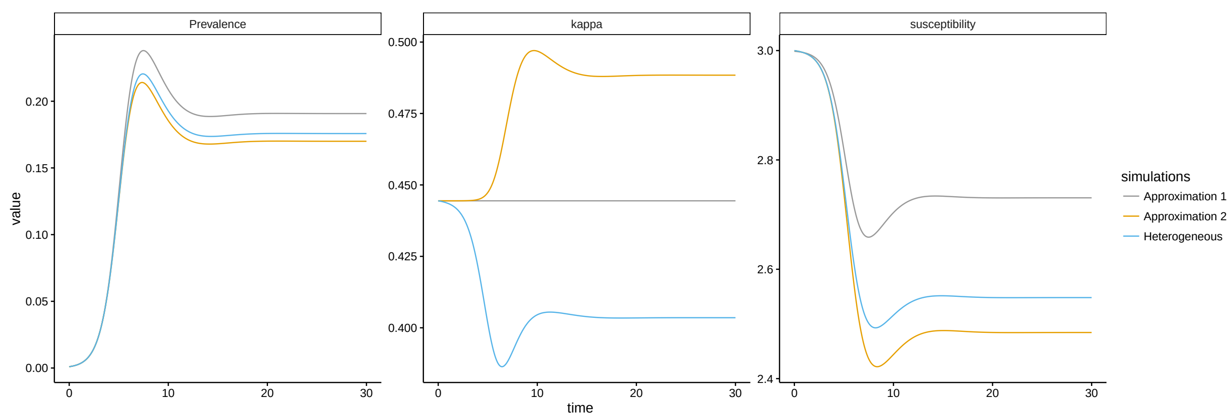
When $i = 1$, $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$.

### Derivative of $\kappa$

New variable: $\varphi_i = \kappa_i + 1$.

$$
\begin{aligned}
\dot{\varphi} &= \varphi(\frac{\dot{M}_2}{M_2} - 2\frac{\dot{M}_1}{M_1}) \\
&= \varphi(\frac{\rho(\bar{M}_2 - M_2)/T_0 - \Lambda\kappa_3 M_2 M_1}{M_2} - 2\frac{\rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa_2 M_1^2}{M_1}) \\
&= \varphi\{\frac{\rho}{T_0}(\frac{(\bar{M}_2 - M_2)}{M_2} - 2\frac{(\bar{M}_1 - M_1)}{M_1}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{M_1\bar{M}_2 + M_1 M_2 - 2M_2\bar{M}_1}{M_2 M_1}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{\bar{M}_2}{M_2} - 2\frac{\bar{M}_1}{M_1} + 1) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{\bar{\varphi}}{\varphi}\frac{\bar{M}_1^2}{M_1^2} - 2\frac{\bar{M}_1}{M_1} + 1) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{\bar{\varphi}\bar{M}_1^2 - 2\varphi\bar{M}_1 M_1 + \varphi M_1^2}{\varphi M_1^2}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{(\bar{\varphi} - \varphi)\bar{M}_1^2 + \varphi(\bar{M}_1 - M_1)^2}{\varphi M_1^2}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\}
\end{aligned}
$$

```r
load("SIS_sim.rda")
plotSim(SIS.list.lnorm, name)
```




Approximation 1:

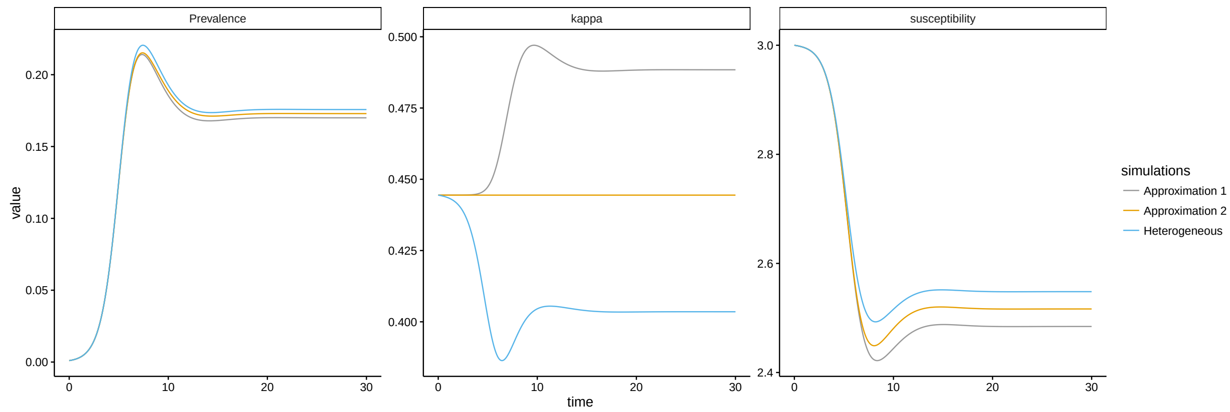
- Constant $\kappa$
- $M = \hat{M}S^{\kappa}$

Approximation 2:

- $\dot{\varphi} = \varphi\{\frac{\rho}{T_0}(\frac{(\bar{\varphi}-\varphi)\bar{M}_1^2+\varphi(\bar{M}_1-M_1)^2}{\varphi M_1^2})$
- $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$

Another set of simulations that is similar to the one above:

```r
load("SIS_sim2.rda")
plotSim(SIS.list.lnorm2, name)
```




Approximation 1:

- Constant $\kappa$
- $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$

Approximation 2:

- $\dot{\varphi} = \varphi\{\frac{\rho}{T_0}(\frac{(\bar{\varphi}-\varphi)\bar{M}_1^2+\varphi(\bar{M}_1-M_1)^2}{\varphi M_1^2})$
- $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$

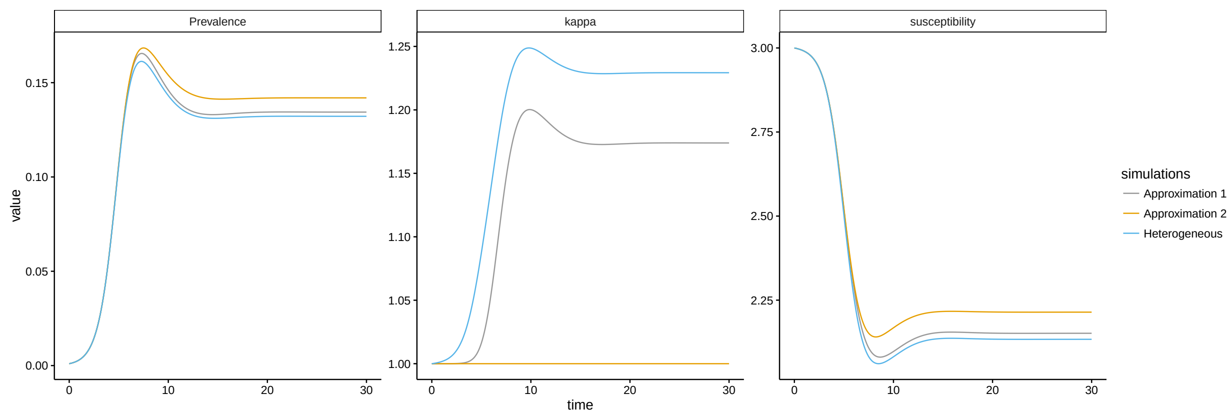
Holding $\kappa_3$ constant does a horrible job so I got rid of it

### More distributions

I'm going to make same figure as the one above with difference distributions

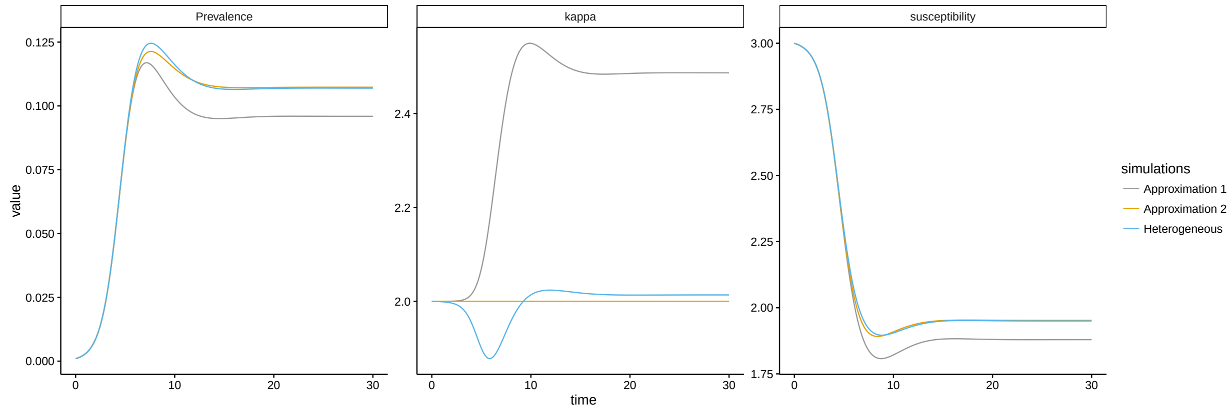
1) $f(x) = 1/(1+x)^4$ and $\kappa = 1$

```r
load("SIS_simCV1.rda")
plotSim(SIS.list.CV1, name)
```



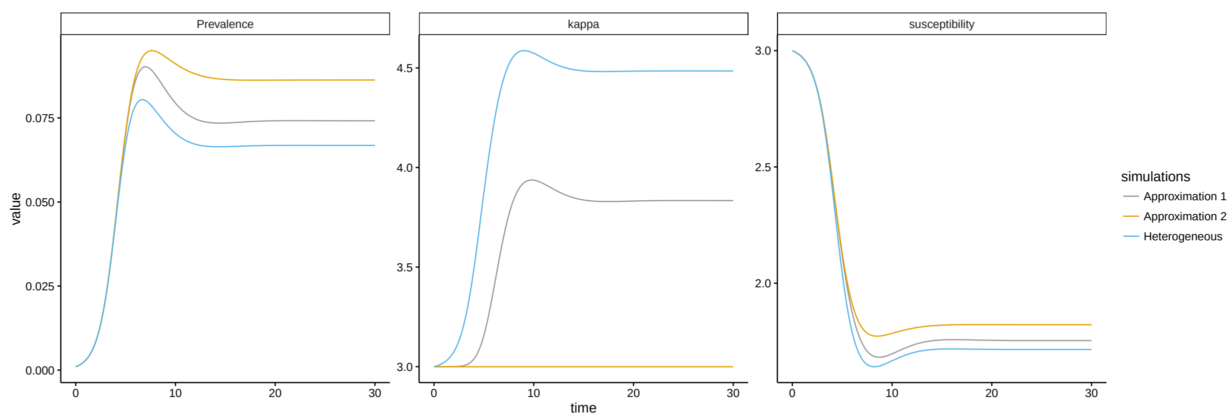

2) $f(x) = 1/(1+x^2)$ and $\kappa = 2$

```r
load("SIS_simCV2.rda")
plotSim(SIS.list.CV2, name)
```




3) $f(x) = 1/(1+x)$ and $\kappa = 3$

```r
load("SIS_simCV3.rda")
plotSim(SIS.list.CV3, name)
```




Why can't we keep $\varphi_3$ constant?

Let's try to get some understanding for $\varphi_3$.

$\varphi_i = \frac{M_i}{M_{i-1} M_1}$

$$
\begin{aligned}
\log \varphi_3 &= \log \frac{M_3}{M_2 M_1} \\
\dot{\varphi}_3 &= \varphi_3 (\frac{\dot{M}_3}{M_3} - \frac{\dot{M}_2}{M_2} - \frac{\dot{M}_1}{M_1}) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{M}_3 - M_3}{M_3} - \frac{\bar{M}_2 - M_2}{M_2} - \frac{\bar{M}_1 - M_1}{M_1}) - \Lambda\kappa_4 M_1 + \Lambda\kappa_3 M_1 + \Lambda\kappa_2 M_1) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{M}_3 - M_3}{M_3} - \frac{\bar{M}_2 - M_2}{M_2} - \frac{\bar{M}_1 - M_1}{M_1}) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1) M_1) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{M}_3}{M_3} - \frac{\bar{M}_2}{M_2} - \frac{\bar{M}_1}{M_1} + 1) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1) M_1) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{M}_3}{M_3} - \frac{\bar{M}_2}{M_2} - \frac{\bar{M}_1}{M_1} + 1) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1) M_1) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{\varphi}_3 \bar{M}_2 \bar{M}_1}{\varphi_3 M_2 M_1} - \frac{\bar{\varphi}_2 \bar{M}_1^2}{\varphi_2 M_1^2} - \frac{\bar{M}_1}{M_1} + 1) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1) M_1) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{\varphi}_3 \bar{\varphi}_2 \bar{M}_1^3}{\varphi_3 \varphi_2 M_1^3} - \frac{\bar{\varphi}_2 \bar{M}_1^2}{\varphi_2 M_1^2} - \frac{\bar{M}_1}{M_1} + 1) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1) M_1) \\
&= \varphi_3 (\frac{\rho}{T_0} (\frac{\bar{\varphi}_3 \bar{\varphi}_2 \bar{M}_1^3 - \bar{\varphi}_2 \varphi_3 \bar{M}_1^2 M_1 - \varphi_3 \varphi_2 M_1^2 + \varphi_3 \varphi_2 M_1^3}{\varphi_3 \varphi_2 M_1^3}) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1) M_1)
\end{aligned}
$$