

Using moment equations to understand heterogeneity

Definition

We want to write $Y_o(v; D)$ for **Type**, **order**, **variable**, and **Domain**.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a) da$. Since we're mainly interested with susceptibility for now, T_o is represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, M_1 is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

SI model

We have $\dot{S}(a) = -\Lambda \sigma(a) S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that M_1 is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using M defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1 + \kappa)M_1^2$ and assuming that κ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M} S^\kappa$, where \hat{M} is the mean susceptibility of the susceptible population at a disease free equilibrium.

Definition - $\kappa_i = \frac{M_i}{M_{i-1} M_1} - 1$

$$\begin{aligned} \dot{\kappa}_i &= \frac{M_{i-1} M_1 \dot{M}_i - M_i M_1 \dot{M}_{i-1} - M_i M_{i-1} \dot{M}_1}{M_{i-1}^2 M_1^2} \\ &= -\Lambda \frac{M_{i-1} M_1 (M_{i+1} - M_i M_1) - M_i M_1 (M_i - M_{i-1} M_1) - M_i M_{i-1} (M_2 - M_1^2)}{M_{i-1}^2 M_1^2} \\ &= -\Lambda \frac{M_{i-1} M_1 M_{i+1} - M_i M_1 M_i - M_i M_{i-1} M_2 + M_i M_{i-1} M_1^2}{M_{i-1}^2 M_1^2} \\ &= -\Lambda \frac{(\kappa_{i+1} + 1) M_i M_{i-1} M_1^2 - M_i M_1 M_i - (\kappa_2 + 1) M_i M_{i-1} M_1^2 + M_i M_{i-1} M_1^2}{M_{i-1}^2 M_1^2} \\ &= -\Lambda \frac{(\kappa_{i+1} + 1) M_i M_{i-1} M_1 - (\kappa_i + 1) M_{i-1} M_i M_1 - \kappa_2 M_i M_{i-1} M_1}{M_{i-1}^2 M_1} \\ &= -\Lambda \frac{(\kappa_{i+1} + 1) M_i - (\kappa_i + 1) M_i - \kappa_2 M_i}{M_{i-1}} \\ &= -\Lambda M_i \frac{\kappa_{i+1} - (\kappa_2 + \kappa_i)}{M_{i-1}} \\ &= -\Lambda M_1 (\kappa_i + 1) \{ \kappa_{i+1} - (\kappa_2 + \kappa_i) \} \end{aligned}$$

How accurate is this?

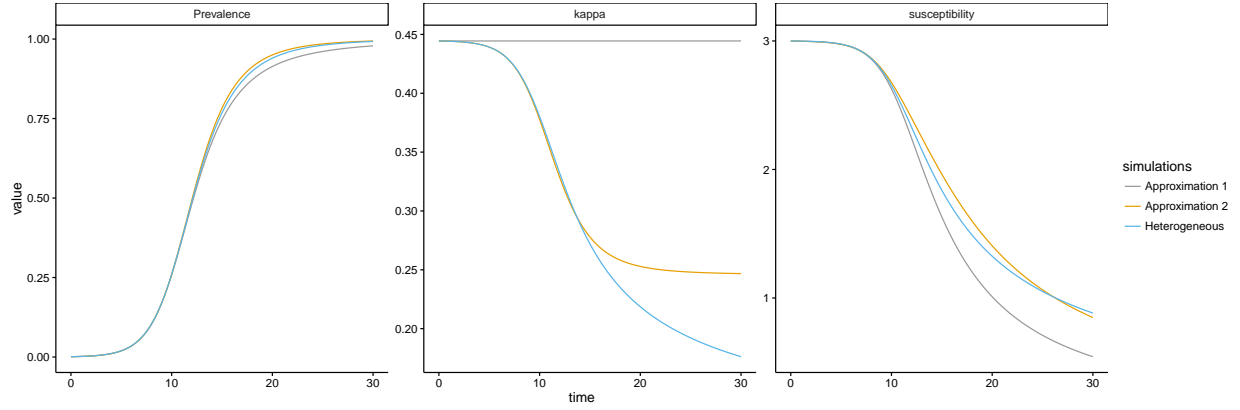
Ex 1 - lognormal distribution

Linear equation:

$$r \approx (\hat{r} - 2)S + 2$$

- Heterogeneous model
- Approximated model: linear equation for κ_3/κ_2 coupled with approximated equation for κ
- Approximated model: constant κ

```
source("plotFunctions.R")
name <- c("Heterogeneous", "Approximation 1", "Approximation 2")
plotSim(SI.list.lnorm, name)
```



SIS model

Here is a simple SIS model: $\dot{S}(a) = \rho(N(a) - S(a)) - \Lambda\sigma(a)S(a)$ and $\dot{I} = \int \Lambda\sigma S(a)da - I$, where $N(a)$ is the initial distribution of the susceptible individuals in a disease free equilibrium. We are going to define $\bar{Y}_o = Y_o(v; N)$. For this model, we have $\dot{T}_i = \rho(\bar{T}_i - T_i) - \Lambda T_{i+1}$. Using chain rule, we can also get an equation for $M = M_1$:

$$\begin{aligned} \dot{M}_i &= \frac{\dot{T}_i}{T_0} - \frac{\dot{T}_0 T_i}{T_0^2} \\ &= (\rho(\bar{T}_i/T_0 - M_i) - \Lambda M_{i+1}) - \rho(M_i \bar{T}_0/T_0 - M_i) + \Lambda M_i M_1 \\ &= \rho(\bar{M}_i - M_i) \frac{\bar{T}_0}{T_0} - \Lambda(M_{i+1} - M_i M_1) \\ &= \rho(\bar{M}_i - M_i)/T_0 - \Lambda \kappa_{i+1} M_i M_1 \end{aligned}$$

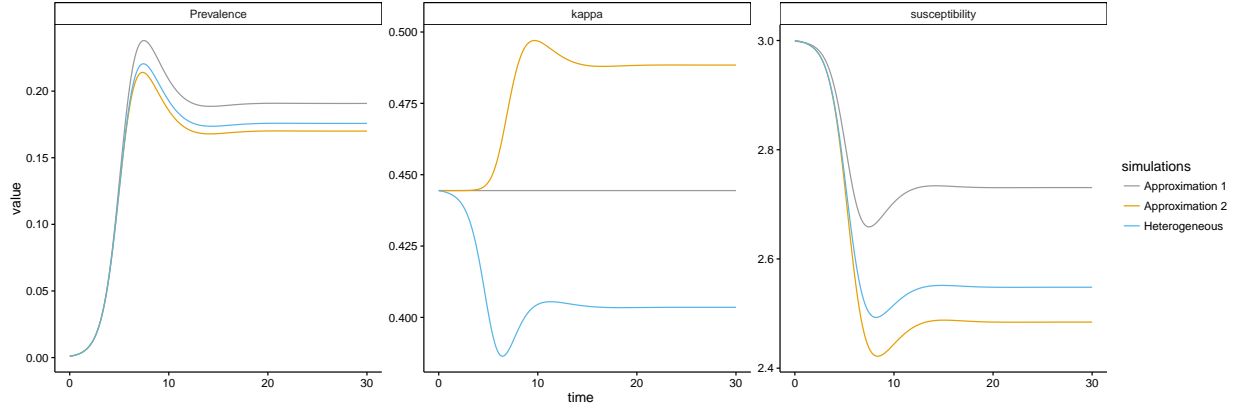
When $i = 1$, $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda \kappa M_1^2$.

Derivative of κ

New variable: $\varphi_i = \kappa_i + 1$.

$$\begin{aligned}
\dot{\varphi} &= \varphi \left(\frac{\dot{M}_2}{M_2} - 2 \frac{\dot{M}_1}{M_1} \right) \\
&= \varphi \left(\frac{\rho(\bar{M}_2 - M_2)/T_0 - \Lambda \kappa_3 M_2 M_1}{M_2} - 2 \frac{\rho(\bar{M}_1 - M_1)/T_0 - \Lambda \kappa_2 M_1^2}{M_1} \right) \\
&= \varphi \left\{ \frac{\rho}{T_0} \left(\frac{\bar{M}_2 - M_2}{M_2} - 2 \frac{(\bar{M}_1 - M_1)}{M_1} \right) - \Lambda M_1 (\varphi_3 - 2\varphi + 1) \right\} \\
&= \varphi \left\{ \frac{\rho}{T_0} \left(\frac{M_1 \bar{M}_2 + M_1 M_2 - 2 M_2 \bar{M}_1}{M_2 M_1} \right) - \Lambda M_1 (\varphi_3 - 2\varphi + 1) \right\} \\
&= \varphi \left\{ \frac{\rho}{T_0} \left(\frac{\bar{M}_2}{M_2} - 2 \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda M_1 (\varphi_3 - 2\varphi + 1) \right\} \\
&= \varphi \left\{ \frac{\rho}{T_0} \left(\frac{\bar{\varphi}}{\varphi} \frac{\bar{M}_1^2}{M_1^2} - 2 \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda M_1 (\varphi_3 - 2\varphi + 1) \right\} \\
&= \varphi \left\{ \frac{\rho}{T_0} \left(\frac{\bar{\varphi} \bar{M}_1^2 - 2\varphi \bar{M}_1 M_1 + \varphi M_1^2}{\varphi M_1^2} \right) - \Lambda M_1 (\varphi_3 - 2\varphi + 1) \right\} \\
&= \varphi \left\{ \frac{\rho}{T_0} \left(\frac{(\bar{\varphi} - \varphi) \bar{M}_1^2 + \varphi (\bar{M}_1 - M_1)^2}{\varphi M_1^2} \right) - \Lambda M_1 (\varphi_3 - 2\varphi + 1) \right\}
\end{aligned}$$

```
load("SIS_sim.rda")
plotSim(SIS.list.lnorm, name)
```



Approximation 1:

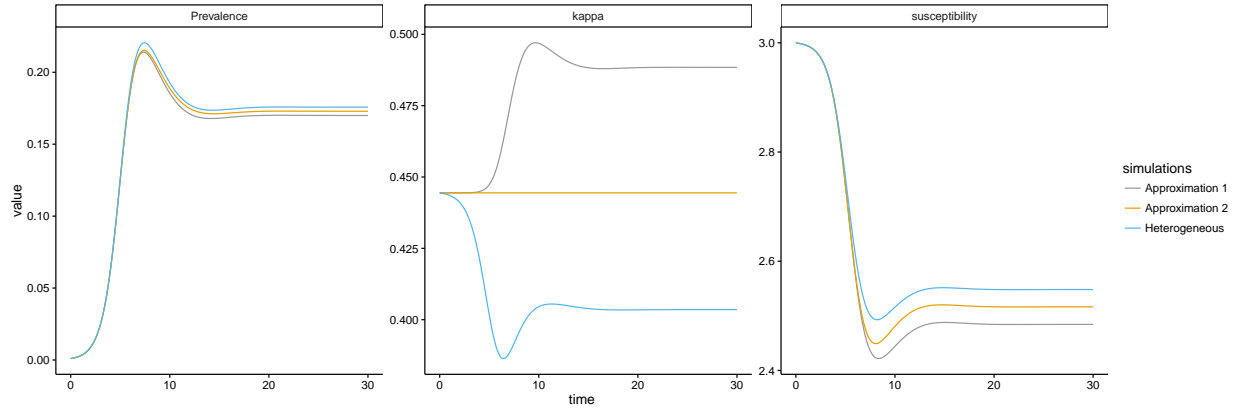
- Constant κ
- $M = \hat{M} S^\kappa$

Approximation 2:

- $\dot{\varphi} = \varphi \left\{ \frac{\rho}{T_0} \left(\frac{(\bar{\varphi} - \varphi) \bar{M}_1^2 + \varphi (\bar{M}_1 - M_1)^2}{\varphi M_1^2} \right) \right\}$
- $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda \kappa M_1^2$

Another set of simulations that is similar to the one above:

```
load("SIS_sim2.rda")
plotSim(SIS.list.lnorm2, name)
```



Approximation 1:

- Constant κ
- $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$

Approximation 2:

- $\dot{\varphi} = \varphi \left\{ \frac{\rho}{T_0} \left(\frac{(\bar{\varphi} - \varphi)\bar{M}_1^2 + \varphi(\bar{M}_1 - M_1)^2}{\varphi M_1^2} \right) \right\}$
- $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$

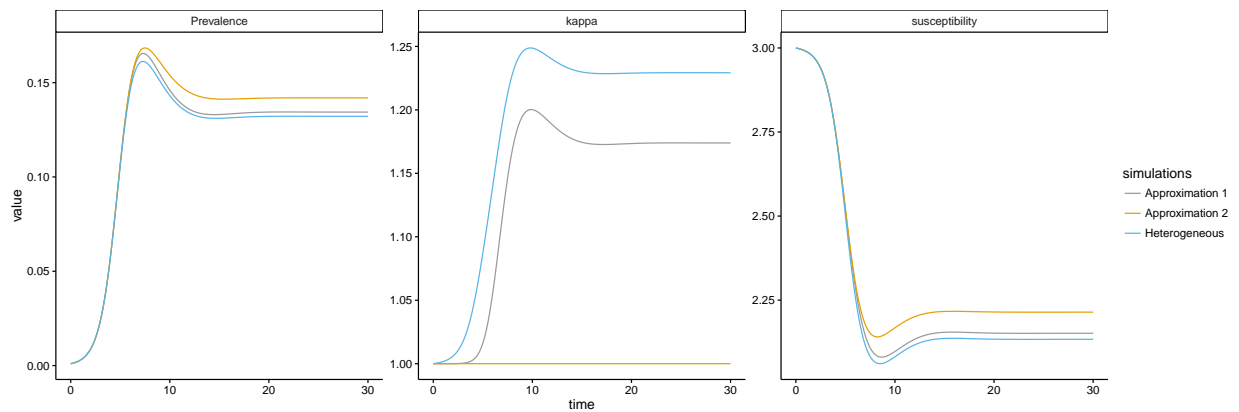
Holding κ_3 constant does a horrible job so I got rid of it

More distributions

I'm going to make same figure as the one above with difference distributions

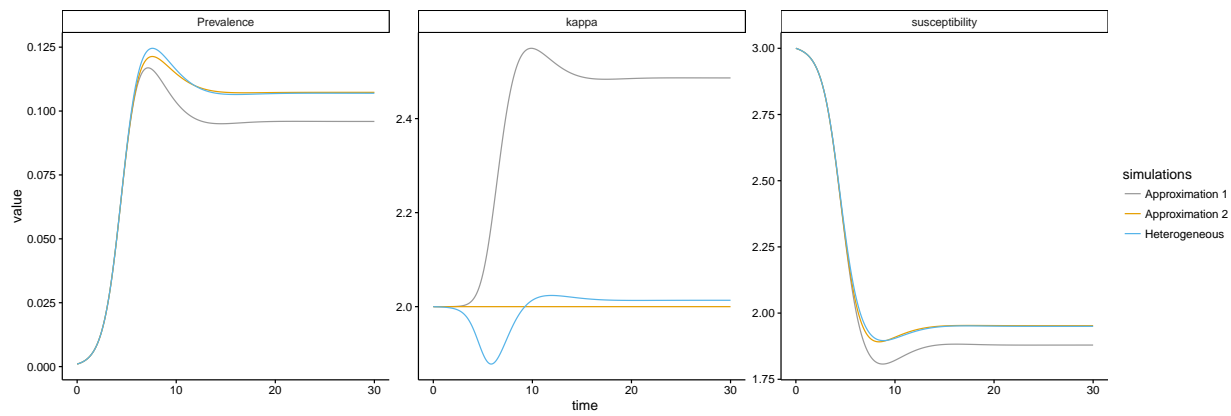
- 1) $f(x) = 1/(1+x)^4$ and $\kappa = 1$

```
load("SIS_simCV1.rda")
plotSim(SIS.list.CV1, name)
```



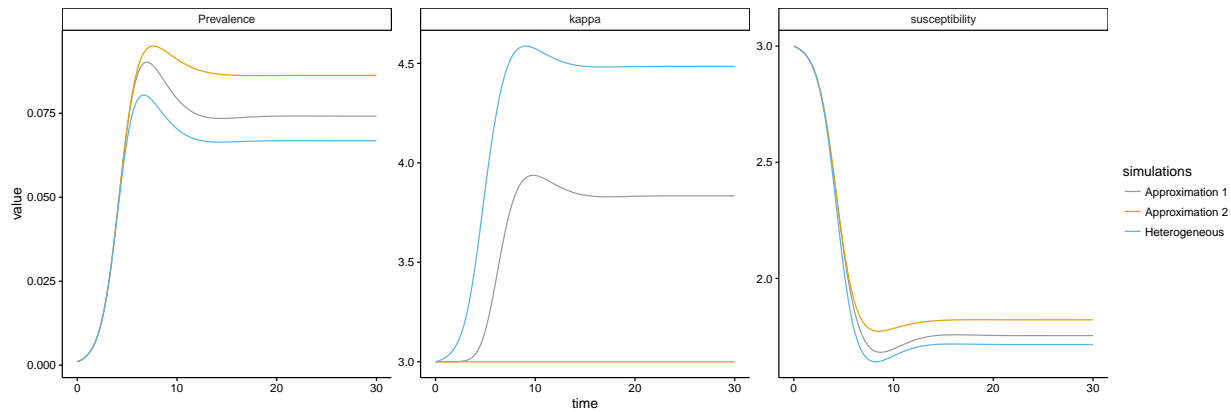
- 2) $f(x) = 1/(1+x^2)$ and $\kappa = 2$

```
load("SIS_simCV2.rda")
plotSim(SIS.list.CV2, name)
```



3) $f(x) = 1/(1+x)$ and $\kappa = 3$

```
load("SIS_simCV3.rda")
plotSim(SIS.list.CV3, name)
```



Why can't we keep φ_3 constant?

Let's try to get some understanding for φ_3 .

$$\varphi_i = \frac{M_i}{M_{i-1}M_1}$$

$$\begin{aligned}
\log \varphi_3 &= \log \frac{M_3}{M_2 M_1} \\
\dot{\varphi}_3 &= \varphi_3 \left(\frac{\dot{M}_3}{M_3} - \frac{\dot{M}_2}{M_2} - \frac{\dot{M}_1}{M_1} \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{M}_3 - M_3}{M_3} - \frac{\bar{M}_2 - M_2}{M_2} - \frac{\bar{M}_1 - M_1}{M_1} \right) - \Lambda \kappa_4 M_1 + \Lambda \kappa_3 M_1 + \Lambda \kappa_2 M_1 \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{M}_3 - M_3}{M_3} - \frac{\bar{M}_2 - M_2}{M_2} - \frac{\bar{M}_1 - M_1}{M_1} \right) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1)M_1 \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{M}_3}{M_3} - \frac{\bar{M}_2}{M_2} - \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1)M_1 \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{M}_3}{M_3} - \frac{\bar{M}_2}{M_2} - \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1)M_1 \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{\varphi}_3 \bar{M}_2 \bar{M}_1}{\varphi_3 M_2 M_1} - \frac{\bar{\varphi}_2 \bar{M}_1^2}{\varphi_2 M_1^2} - \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1)M_1 \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{\varphi}_3 \bar{\varphi}_2 \bar{M}_1^3}{\varphi_3 \varphi_2 M_1^3} - \frac{\bar{\varphi}_2 \bar{M}_1^2}{\varphi_2 M_1^2} - \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1)M_1 \right) \\
&= \varphi_3 \left(\frac{\rho}{T_0} \left(\frac{\bar{\varphi}_3 \bar{\varphi}_2 \bar{M}_1^3 - \bar{\varphi}_2 \varphi_3 \bar{M}_1^2 M_1 - \varphi_3 \varphi_2 M_1^2 + \varphi_3 \varphi_2 M_1^3}{\varphi_3 \varphi_2 M_1^3} \right) - \Lambda(\varphi_4 - \varphi_3 - \varphi_2 + 1)M_1 \right)
\end{aligned}$$

Can we go back to the old definition? $\varphi_i = \frac{M_i M_{i-2}}{M_{i-1}^2}$

First, we need a general equation for understanding M_i :

$$\begin{aligned}
\dot{M}_i &= \frac{\dot{T}_i}{T_0} - \frac{\dot{T}_0 T_i}{T_0^2} \\
&= (\rho(\bar{T}_i/T_0 - M_i) - \Lambda M_{i+1}) - \rho(M_i \bar{T}_0/T_0 - M_i) + \Lambda M_i M_1 \\
&= \rho(\bar{M}_i - M_i) \frac{1}{T_0} - \Lambda(M_{i+1} - M_i M_1)
\end{aligned}$$

Probably can't do much with this...? Equation for φ ...

$$\begin{aligned}
\dot{\varphi} &= \varphi \left(\frac{\dot{M}_2}{M_2} - 2 \frac{\dot{M}_1}{M_1} \right) \\
&= \varphi \left(\frac{\frac{\rho}{T_0} (\bar{M}_2 - M_2) - \Lambda(M_3 - M_2 M_1)}{M_2} - 2 \frac{\frac{\rho}{T_0} (\bar{M}_1 - M_1) - \Lambda(M_2 - M_1^2)}{M_1} \right) \\
&= \varphi \left(\frac{\rho}{T_0} \left(\frac{\bar{M}_2}{M_2} - 2 \frac{\bar{M}_1}{M_1} + 1 \right) - \Lambda \frac{(M_3 M_1 - M_2 M_1^2) - 2(M_2^2 - M_2 M_1^2)}{M_1 M_2} \right) \\
&= \varphi \left(\frac{\rho}{T_0} \left(\frac{\bar{M}_2}{M_2} - 2 \frac{\bar{M}}{M} + 1 \right) - \Lambda(\varphi_3 \varphi - 2\varphi + 2)M \right) \\
&= \varphi \left(\frac{\rho}{T_0} \left(\frac{\bar{\varphi} \bar{M}^2}{\varphi M^2} - 2 \frac{\bar{M}}{M} + 1 \right) - \Lambda(\varphi_3 \varphi - 2\varphi + 2)M \right) \\
&= \varphi \left(\frac{\rho}{T_0} \left(\frac{(\bar{\varphi} - \varphi) \bar{M}_1^2 + \varphi(\bar{M}_1 - M_1)^2}{\varphi M_1^2} \right) - \Lambda(\varphi_3 \varphi - 2\varphi + 2)M \right)
\end{aligned}$$