Using moment equations to understand heterogeneity

Definition

We want to write $Y_o(v; D)$ for **T**y**p**e, **o**rder, **v**ariable, and **D**omain.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a) da$. Since we're mainly interested with susceptibility for now, T_o is represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, M_1 is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

SI example

We have $\dot{S}(a) = -\Lambda\sigma(a)S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that M_1 is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using M defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1+\kappa)M^2$ and assuming that κ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M} S^\kappa$, where $\hat{M}$ is the mean susceptibility of the susceptible population at a disease free equilibrium.

We can define $\kappa = \kappa_2$ and extend this idea to $\kappa_i = \frac{M_i M_{i-2}}{M_{i-1}^2} - 1$. We can derive an equation for κ_i as well:

$$\begin{aligned}
\dot{\kappa_i} &= \frac{M_{i-2}M_{i-1}\dot{M}_i + M_{i-1}M_i\dot{M}_{i-2} - 2M_{i-2}M_i\dot{M}_{i-1}}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}M_{i-1}(M_{i+1} - M_i M_1) + M_{i-1}M_i(M_{i-1} - M_{i-2}M_1) - 2M_{i-2}M_i(M_i - M_{i-1}M_1)}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}M_{i-1}M_{i+1} + M_{i-1}^2 M_i - 2M_{i-2}M_i^2}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}(\kappa_{i+1}+1)M_i^2 + M_{i-1}^2 M_i - 2M_{i-2}M_i^2}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}-1)M_{i-2}M_i^2 + M_{i-1}^2 M_i}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}-1)(\kappa_i+1)M_i M_{i-1}^2 + M_{i-1}^2 M_i}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}}
\end{aligned}$$

Using chain rule, we can also do this:

$$\begin{aligned}
\frac{d\kappa_i}{dt} &= \frac{d\kappa_i}{dT_{i-2}}\frac{dT_{i-2}}{dt} \\
-\Lambda\frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}} &= -\Lambda M_{i-1}T_0\frac{d\kappa_i}{dT_{i-2}} \\
\frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}^2} &= T_0\frac{d\kappa_i}{dT_{i-2}} \\
\frac{(\kappa_i + 1)(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})}{M_{i-2}} &= T_0\frac{d\kappa_i}{dT_{i-2}}
\end{aligned}$$

If we let $i = 2$, we have:

$$(\kappa_2 + 1)(\kappa_3\kappa_2 - \kappa_2 + \kappa_3) = T_0\frac{d\kappa_2}{dT_0}$$

If we assume that κ_3 stays constant, we get the following separable differential equation:

$$\begin{aligned}
\int \frac{1}{T_0}dT_0 &= \int \frac{1}{(\kappa_2 + 1)(\kappa_3\kappa_2 - \kappa_2 + \kappa_3)}d\kappa_2 \\
\int \frac{1}{T_0}dT_0 &= \int (\frac{1}{\kappa_2 + 1} + \frac{1 - \kappa_3}{\kappa_3\kappa_2 - \kappa_2 + \kappa_3})d\kappa_2 \\
\ln(T_0) &= \ln(\kappa_2 + 1) - \ln(\kappa_3\kappa_2 - \kappa_2 + \kappa_3) + C
\end{aligned}$$

If we set $(\hat{T_0}, \hat{\kappa_2}) = (1, \hat{\kappa_2})$, we have $C = \ln(\kappa_3\hat{\kappa}_2 - \hat{\kappa}_2 + \kappa_3) - \ln(\hat{\kappa}_2 + 1)$. Solving the differential equation, we find that

$$\kappa = \frac{\kappa_3 S - e^C}{(1 - \kappa_3)S + e^C}$$

SIS example