Using moment equations to understand heterogeneity

Definition

We want to write $Y_o(v; D)$ for **T**y**p**e, **o**rder, **v**ariable, and **D**omain.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a)da$. Since we're mainly interested with susceptibility for now, T_o is represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, M_1 is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

SI model

We have $\dot{S}(a) = -\Lambda\sigma(a)S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that M_1 is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using M defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1+\kappa)M^2$ and assuming that κ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M}S^\kappa$, where $\hat{M}$ is the mean susceptibility of the susceptible population at a disease free equilibrium.

Definition - $\kappa_i = \frac{M_i}{M_{i-1}M_1} - 1$

$$\begin{aligned}
\dot{\kappa}_i &= \frac{M_{i-1}M_1\dot{M}_i - M_i M_1 \dot{M}_{i-1} - M_i M_{i-1}\dot{M}_1}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{M_{i-1}M_1(M_{i+1} - M_i M_1) - M_i M_1(M_i - M_{i-1}M_1) - M_i M_{i-1}(M_2 - M_1^2)}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{M_{i-1}M_1 M_{i+1} - M_i M_1 M_i - M_i M_{i-1} M_2 + M_i M_{i-1} M_1^2}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{(\kappa_{i+1}+1)M_i M_{i-1} M_1^2 - M_i M_1 M_i - (\kappa_2+1)M_i M_{i-1} M_1^2 + M_i M_{i-1} M_1^2}{M_{i-1}^2 M_1^2} \\
&= -\Lambda \frac{(\kappa_{i+1}+1)M_i M_{i-1} M_1 - (\kappa_i+1)M_{i-1} M_i M_1 - \kappa_2 M_i M_{i-1} M_1}{M_{i-1}^2 M_1} \\
&= -\Lambda \frac{(\kappa_{i+1}+1)M_i - (\kappa_i+1)M_i - \kappa_2 M_i}{M_{i-1}} \\
&= -\Lambda M_i \frac{\kappa_{i+1} - (\kappa_2 + \kappa_i)}{M_{i-1}} \\
&= -\Lambda M_1 (\kappa_i + 1)\{\kappa_{i+1} - (\kappa_2 + \kappa_i)\}
\end{aligned}$$

When $i = 2$, we have $\dot{\kappa} = -\Lambda M(\kappa+1)(\kappa_3 - 2\kappa)$, where $\kappa = \kappa_2$. For gamma distribution, $\kappa_3 = 2\kappa$. Let's assume that $r = \kappa_3/\kappa$ stays constant. Then, we have $\dot{\kappa} = -\Lambda M(r-2)(\kappa+1)(\kappa)$. We can do this:

$$
\begin{aligned}
\frac{d\kappa}{dt} &= \frac{d\kappa}{dS}\frac{dS}{dt} \\
-\Lambda M(r-2)(\kappa+1)(\kappa) &= -\Lambda M S\frac{d\kappa}{dS} \\
(r-2)(\kappa+1)(\kappa) &= S\frac{d\kappa}{dS} \\
\frac{(r-2)}{S} &= \frac{1}{\kappa(\kappa+1)}\frac{d\kappa}{dS} \\
\int\frac{(r-2)}{S}dS &= \int\frac{1}{\kappa(\kappa+1)}d\kappa \\
(r-2)\log(S) &= \log(\kappa) - \log(\kappa+1) + C
\end{aligned}
$$

We let the initial values $(S(0), \kappa(0)) = (1, \hat{\kappa})$, then we have $C = \log(\frac{\hat{\kappa}+1}{\hat{\kappa}})$. We can continue with the derivative:

$$
\begin{aligned}
\log(S^{r-2}) &= \log(e^C\frac{\kappa}{\kappa+1}) \\
S^{r-2} &= e^C\frac{\kappa}{\kappa+1} \\
S^{r-2}(\kappa+1) &= e^C\kappa \\
S^{r-2} &= (e^C - S^{r-2})\kappa \\
\kappa &= \frac{S^{r-2}}{e^C - S^{r-2}}
\end{aligned}
$$

How accurate is this?

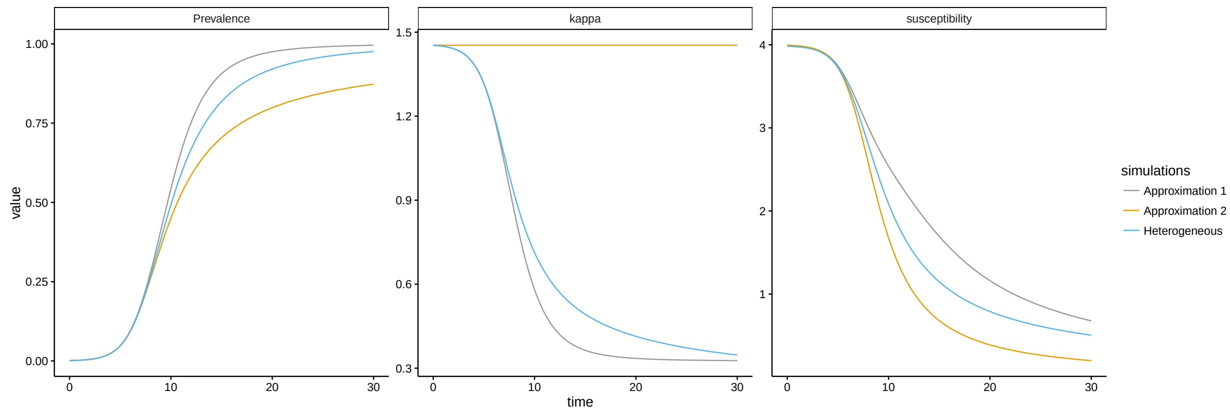
Ex 1 - lognormal distribution

Comparison of 3 simulations Linear equation:

$$r \approx (\hat{r}-2)S + 2$$

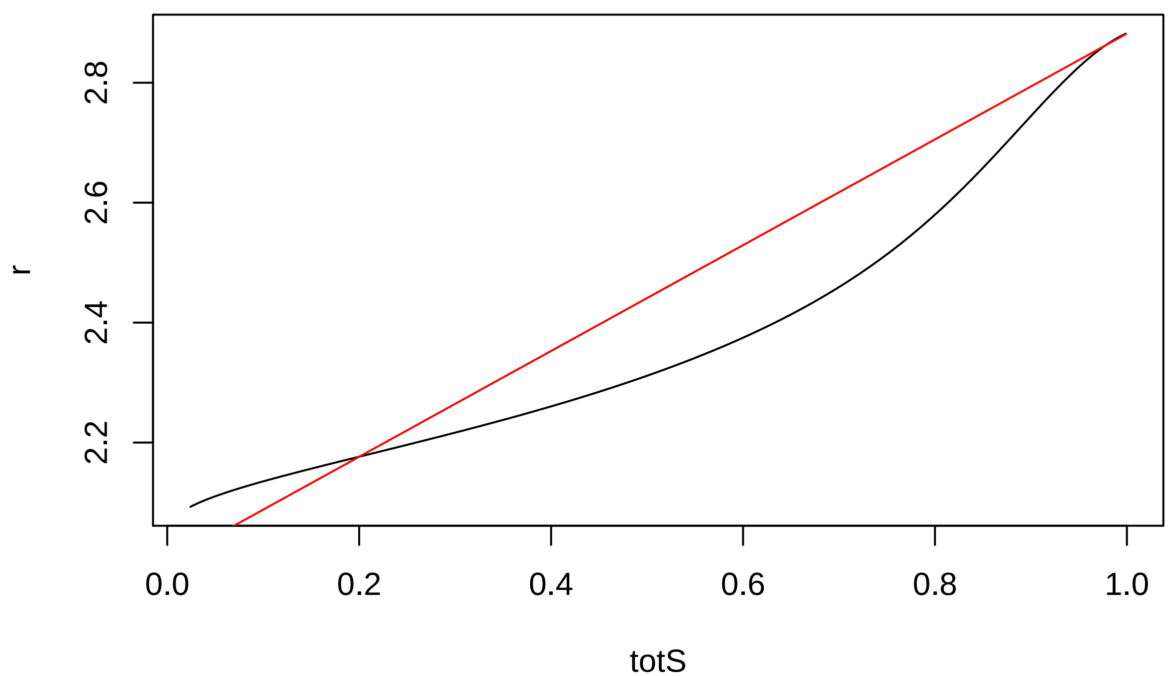
- Heterogeneous model
- Approximated model: linear equation for κ_3/κ_2 coupled with approximated equation for κ
- Approximated model: constant κ

```r
source("plotFunctions.R")
baseSim <- list(r.lnorm.het, r.lnorm.app1, r.lnorm.app2)
name <- c("Heterogeneous", "Approximation 1", "Approximation 2")
plotSim(baseSim, name)
```



Linear extrapolation of the κ_3/κ_2 ratio approximates the prevalence trajectory better sice it matches the initial κ trajectory but it doesn't approximate κ so well near the equilibrium. . . . How does r vs S look like?

```
plot(cbind(totS, r= k3/k2), type = "l")
lines(cbind(totS,( k3[1]/k2[1] - 2)*totS + 2), col = 2)
```



SIS model

Here is a simple SIS model: $\dot{S}(a) = \rho(N(a) - S(a)) - \Lambda\sigma(a)S(a)$ and $\dot{I} = \int \Lambda\sigma S(a)da - I$, where $N(a)$ is the initial distribution of the susceptible individuals in a disease free equilibrium. We are going to define

$\bar{Y}_o = Y_o(v;N)$. For this model, we have $\dot{T}_i = \rho(\bar{T}_i - T_i) - \Lambda T_{i+1}$. Using chain rule, we can also get an equation for $M = M_1$:

$$\begin{aligned}
\dot{M}_i &= \frac{\dot{T}_i}{T_0} - \frac{\dot{T}_0 T_i}{T_0^2} \\
&= (\rho(\bar{T}_i/T_0 - M_i) - \Lambda M_{i+1}) - \rho(M_i \bar{T}_0/T_0 - M_i) + \Lambda M_i M_1 \\
&= \rho(\bar{M}_i - M_i)\frac{\bar{T}_0}{T_0} - \Lambda(M_{i+1} - M_i M_1) \\
&= \rho(\bar{M}_i - M_i)/T_0 - \Lambda \kappa_{i+1} M_i M_1
\end{aligned}$$

When $i = 1$, $\dot{M} = \rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa M_1^2$.

Derivative of κ

New variable: $\varphi_i = \kappa_i + 1$.

$$\begin{aligned}
\dot{\varphi} &= \varphi(\frac{\dot{M}_2}{M_2} - 2\frac{\dot{M}_1}{M_1}) \\
&= \varphi(\frac{\rho(\bar{M}_2 - M_2)/T_0 - \Lambda\kappa_3 M_2 M_1}{M_2} - 2\frac{\rho(\bar{M}_1 - M_1)/T_0 - \Lambda\kappa_2 M_1^2}{M_1}) \\
&= \varphi\{\frac{\rho}{T_0}(\frac{(\bar{M}_2 - M_2)}{M_2} - 2\frac{(\bar{M}_1 - M_1)}{M_1}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{M_1\bar{M}_2 + M_1 M_2 - 2M_2\bar{M}_1}{M_2 M_1}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{\bar{M}_2}{M_2} - 2\frac{\bar{M}_1}{M_1} + 1) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{\bar{\varphi}}{\varphi}\frac{\bar{M}_1^2}{M_1^2} - 2\frac{\bar{M}_1}{M_1} + 1) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{\bar{\varphi}\bar{M}_1^2 - 2\varphi\bar{M}_1 M_1 + \varphi M_1^2}{\varphi M_1^2}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\} \\
&= \varphi\{\frac{\rho}{T_0}(\frac{(\bar{\varphi} - \varphi)\bar{M}_1^2 + \varphi(\bar{M}_1 - M_1)^2}{\varphi M_1^2}) - \Lambda M_1(\varphi_3 - 2\varphi + 1)\}
\end{aligned}$$