Using moment equations to understand heterogeneity

Definition

We want to write $Y_o(v; D)$ for **T**y**p**e, **o**rder, **v**ariable, and **D**omain.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a) da$. Since we're mainly interested with susceptibility for now, T_o is represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, M_1 is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

SI example

We have $\dot{S}(a) = -\Lambda\sigma(a)S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that M_1 is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using M defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1+\kappa)M^2$ and assuming that κ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M} S^\kappa$, where $\hat{M}$ is the mean susceptibility of the susceptible population at a disease free equilibrium.

Idea 1 - $\kappa_i = \frac{M_i M_{i-2}}{M_{i-1}^2} - 1$

$$\begin{aligned}
\dot{\kappa_i} &= \frac{M_{i-2}M_{i-1}\dot{M}_i + M_{i-1}M_i\dot{M}_{i-2} - 2M_{i-2}M_i\dot{M}_{i-1}}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}M_{i-1}(M_{i+1} - M_iM_1) + M_{i-1}M_i(M_{i-1} - M_{i-2}M_1) - 2M_{i-2}M_i(M_i - M_{i-1}M_1)}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}M_{i-1}M_{i+1} + M_{i-1}^2M_i - 2M_{i-2}M_i^2}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}(\kappa_{i+1}+1)M_i^2 + M_{i-1}^2M_i - 2M_{i-2}M_i^2}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}-1)M_{i-2}M_i^2 + M_{i-1}^2M_i}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}-1)(\kappa_i+1)M_iM_{i-1}^2 + M_{i-1}^2M_i}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}}
\end{aligned}$$

Idea 2 - $\kappa_i = \frac{M_i}{M_{i-1}M_1} - 1$

$$\begin{aligned}
\dot{\kappa}_i &= \frac{M_{i-1}M_1\dot{M}_i - M_iM_1\dot{M}_{i-1} - M_iM_{i-1}\dot{M}_1}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{M_{i-1}M_1(M_{i+1} - M_iM_1) - M_iM_1(M_i - M_{i-1}M_1) - M_iM_{i-1}(M_2 - M_1^2)}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{M_{i-1}M_1M_{i+1} - M_iM_1M_i - M_iM_{i-1}M_2 + M_iM_{i-1}M_1^2}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{(\kappa_{i+1}+1)M_iM_{i-1}M_1^2 - M_iM_1M_i - (\kappa_2+1)M_iM_{i-1}M_1^2 + M_iM_{i-1}M_1^2}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{(\kappa_{i+1}+1)M_iM_{i-1}M_1 - (\kappa_i+1)M_{i-1}M_iM_1 - \kappa_2M_iM_{i-1}M_1}{M_{i-1}^2M_1} \\
&= -\Lambda\frac{(\kappa_{i+1}+1)M_i - (\kappa_i+1)M_i - \kappa_2M_i}{M_{i-1}} \\
&= -\Lambda M_i\frac{\kappa_{i+1} - (\kappa_2 + \kappa_i)}{M_{i-1}} \\
&= -\Lambda M_1(\kappa_i+1)\{\kappa_{i+1} - (\kappa_2+\kappa_i)\}
\end{aligned}$$

When $i = 2$, we have $\dot{\kappa} = -\Lambda M(\kappa+1)(\kappa_3 - 2\kappa)$, where $\kappa = \kappa_2$. For gamma distribution, $\kappa_3 = 2\kappa$. Let's assume that $r = \kappa_3/\kappa$ stays constant. Then, we have $\dot{\kappa} = -\Lambda M(r-2)(\kappa+1)(\kappa)$. We can do this:

$$\begin{aligned}
\frac{d\kappa}{dt} &= \frac{d\kappa}{dS}\frac{dS}{dt} \\
-\Lambda M(r-2)(\kappa+1)(\kappa) &= -\Lambda MS\frac{d\kappa}{dS} \\
(r-2)(\kappa+1)(\kappa) &= S\frac{d\kappa}{dS} \\
\frac{(r-2)}{S} &= \frac{1}{\kappa(\kappa+1)}\frac{d\kappa}{dS} \\
\int\frac{(r-2)}{S}dS &= \int\frac{1}{\kappa(\kappa+1)}d\kappa \\
(r-2)\log(S) &= \log(\kappa) - \log(\kappa+1) + C
\end{aligned}$$

We let the initial values $(S(0), \kappa(0)) = (1, \hat{\kappa})$, then we have $C = \log(\frac{\hat{\kappa}+1}{\hat{\kappa}})$. We can continue with the derivative:

$$\begin{aligned}
\log(S^{r-2}) &= \log(e^C\frac{\kappa}{\kappa+1}) \\
S^{r-2} &= e^C\frac{\kappa}{\kappa+1} \\
S^{r-2}(\kappa+1) &= e^C\kappa \\
S^{r-2} &= (e^C - S^{r-2})\kappa \\
\kappa &= \frac{S^{r-2}}{e^C - S^{r-2}}
\end{aligned}$$

How accurate is this?

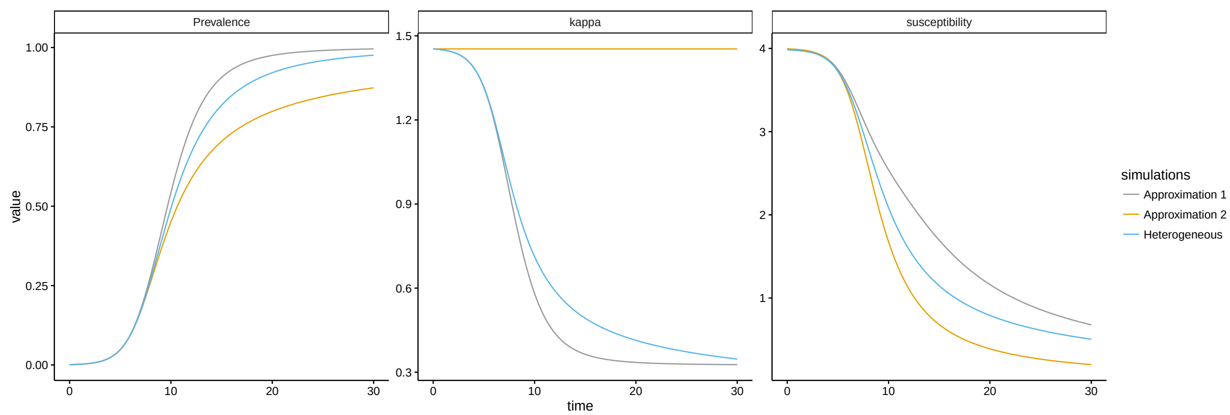
Ex 1 - lognormal distribution

Comparison of 3 simulations Linear equation:

$$r \approx (\hat{r} - 2)S + 2$$

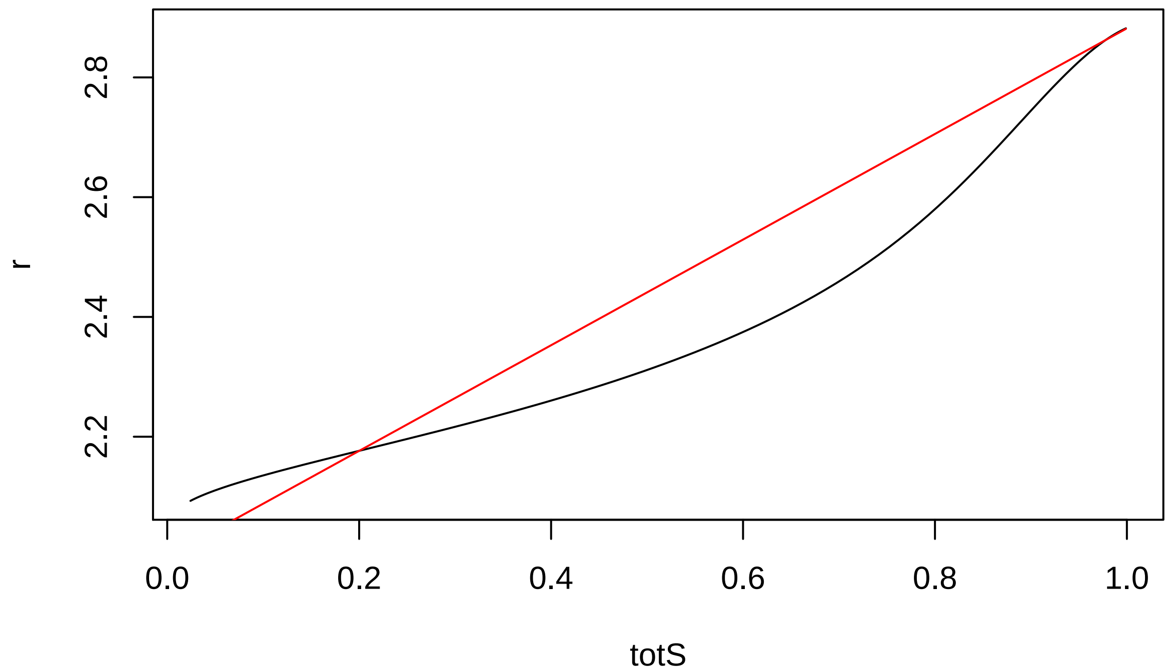
- Heterogeneous model
- Approximated model: linear equation for κ_3/κ_2 coupled with approximated equation for κ
- Approximated model: constant κ

```
source("plotFunctions.R")
baseSim <- list(r.lnorm.het, r.lnorm.app1, r.lnorm.app2)
name <- c("Heterogeneous", "Approximation 1", "Approximation 2")
plotSim(baseSim, name)
```



Linear extrapolation of the κ_3/κ_2 ratio approximates the prevalence trajectory better sice it matches the initial κ trajectory but it doesn't approximate κ so well near the equilibrium.... How does r vs S look like?

```
plot(cbind(totS, r= k3/k2), type = "l")
lines(cbind(totS,( k3[1]/k2[1] - 2)*totS + 2), col = 2)
```



Can we get some understanding for κ_3/κ_2? Equation for $\dot{\kappa}$ shows us that the equilibrium value of $\kappa^* = \kappa_3/2$ but this might give us something else... Given that $\dot{\kappa}_3 = -\Lambda M(\kappa_3 + 1)(\kappa_4 - (\kappa_2 + \kappa_3))$, we can find the equation for the derivative of r:

$$\begin{aligned}
\dot{r} &= \frac{\dot{\kappa}_3\kappa - \dot{\kappa}\kappa_3}{\kappa^2} \\
&= -\Lambda M \frac{(\kappa_3+1)(\kappa_4 - (\kappa+\kappa_3))\kappa - (\kappa+1)(\kappa_3 - 2\kappa)\kappa_3}{\kappa^2} \\
&= -\Lambda M \frac{(\kappa_3\kappa_4 - \kappa\kappa_3 - \kappa_3^2 + \kappa_4 - \kappa - \kappa_3)\kappa - (\kappa\kappa_3 - 2\kappa^2 + \kappa_3 - 2\kappa)\kappa_3}{\kappa^2} \\
&= -\Lambda M \frac{(\kappa\kappa_3\kappa_4 - \kappa^2\kappa_3 - \kappa\kappa_3^2 + \kappa\kappa_4 - \kappa^2 - \kappa\kappa_3) - (\kappa\kappa_3^2 - 2\kappa^2\kappa_3 + \kappa_3^2 - 2\kappa\kappa_3)}{\kappa^2} \\
&= -\Lambda M \frac{\kappa\kappa_3\kappa_4 - \kappa^2\kappa_3 - \kappa\kappa_3^2 + \kappa\kappa_4 - \kappa^2 - \kappa\kappa_3 - \kappa\kappa_3^2 + 2\kappa^2\kappa_3 - \kappa_3^2 + 2\kappa\kappa_3}{\kappa^2} \\
&= -\Lambda M \frac{\kappa\kappa_3\kappa_4 + \kappa^2\kappa_3 - 2\kappa\kappa_3^2 + \kappa\kappa_4 - \kappa^2 + \kappa\kappa_3 - \kappa_3^2}{\kappa^2} \\
&= -\Lambda M(\kappa_4 r + \kappa_3 - 2\kappa_3 r + \frac{\kappa_4}{\kappa_3} r - 1 + r - r^2) \\
&= -\Lambda M(\frac{\kappa_4}{\kappa_3}\kappa r^2 + \kappa r - 2\kappa r^2 + \frac{\kappa_4}{\kappa_3} r - 1 + r - r^2) \\
&= -\Lambda M((\frac{\kappa_4}{\kappa_3}\kappa - 2\kappa - 1)r^2 + (\frac{\kappa_4}{\kappa_3} + \kappa + 1)r - 1)
\end{aligned}$$

or... we can do this...

$$
\begin{aligned}
\dot{r} &= \frac{\dot{\kappa}_3}{\kappa} - \frac{\dot{\kappa}\kappa_3}{\kappa^2} \\
&= -\Lambda M(\kappa_3+1)(\frac{\kappa_4}{\kappa_3}r - (1+r)) + \Lambda M(\kappa+1)(r-2)r
\end{aligned}
$$

SIS example

Here is a simple SIS model: $\dot{S}(a) = \mu(N(a) - S(a)) - \Lambda\sigma(a)S(a)$, where $N(a)$ is the initial distribution of the susceptible individuals in a disease free equilibrium. We are going to define $T_o(N) = T_o(v; N)$. For this model, we have $\dot{T}_i = \mu(T_i(N) - T_i) - \Lambda T_{i+1}$. Using chain rule, we can also get an equation for $M = M_1$:

$$
\begin{aligned}
\dot{M}_i &= \frac{\dot{T}_i}{T_0} - \frac{\dot{T}_0 T_i}{T_0^2} \\
&= (\mu(T_i(N)/T_0 - M_i) - \Lambda M_{i+1}) - \mu(M_i T_0(N)/T_0 - M_i) + \Lambda M_i M_1 \\
&= \mu(M_i(N) - M_i)\frac{T_0(N)}{T_0} - \Lambda(M_{i+1} - M_i M_1) \\
&= \mu(M_i(N) - M_i)/T_0 - \Lambda\kappa_{i+1}M_i M_1
\end{aligned}
$$

When $i = 1$, $\dot{M} = \mu(M_1(N) - M_1)/T_0 - \Lambda\kappa M_1^2 \ldots$ What can we do with this?

Idea 1

I want to know if we can approximate $M_i(N)/M_i$. This term appears in the κ equation and maybe we can do something about it:

$$
\begin{aligned}
\frac{d}{dt}(\frac{M_i(N)}{M_i}) &= -\frac{M_i(N)}{M_i^2}\dot{M}_i \\
&= -\frac{M_i(N)}{M_i^2}\{\mu(M_i(N) - M_i)/T_0 - \Lambda\kappa_{i+1}M_i M_1\} \\
&= -\frac{M_i(N)}{M_i}\{\mu(\frac{M_i(N)}{M_i} - 1)/T_0 - \Lambda\kappa_{i+1}M_1\}
\end{aligned}
$$

If we let $i = 1$ and $R = M(N)/M$ and, we have the following equations:

$$
\dot{R} = -R\{\mu(R-1)/T_0 - \Lambda\kappa M\}
$$

We can substitute $M = M(N)/R$:

$$
\begin{aligned}
\dot{R} &= -R\{\mu(R-1)/T_0 - \Lambda\kappa M(N)/R\} \\
&= \Lambda\kappa M(N) - R\mu(R-1)/T_0
\end{aligned}
$$

We also know that since $1/T_0 = \frac{M_1(N)}{RT_1}$. Substituting this gives us

$$
\begin{aligned}
\dot{R} &= \Lambda\kappa M(N) - \mu(R-1)\frac{M_1(N)}{T_1} \\
&= \Lambda\kappa M(N) - \mu(R-1)\frac{M(N)}{T_1} \\
&= M(N)(\Lambda\kappa - \mu(R-1)/T_1)
\end{aligned}
$$

Can we relate this to $\dot{T}_0 = \mu(1 - T_0) - \Lambda T_1$?

Can we get some understanding for κ?

$$\begin{aligned}
\dot{\kappa}_i &= \frac{M_{i-1}M_1\dot{M}_i - M_iM_1\dot{M}_{i-1} - M_iM_{i-1}\dot{M}_1}{M_{i-1}^2M_1^2} \\
&= \mu\frac{M_{i-1}M_1(M_i(N) - M_i) - M_iM_1(M_{i-1}(N) - M_{i-1}) - M_iM_{i-1}(M_1(N) - M_1)}{M_{i-1}^2M_1^2T_0} \\
&- \Lambda M_1(\kappa_i + 1)\{\kappa_{i+1} - (\kappa + \kappa_i)\} \\
&= \mu\frac{M_{i-1}M_1M_i(N) - M_iM_1M_{i-1}(N) - M_iM_{i-1}M_1(N) + M_1M_{i-1}M_i}{M_{i-1}^2M_1^2T_0} \\
&- \Lambda M_1(\kappa_i + 1)\{\kappa_{i+1} - (\kappa + \kappa_i)\} \\
&= \mu\frac{M_{i-1}M_1M_i(N) - M_iM_1M_{i-1}(N) - M_iM_{i-1}M_1(N)}{M_{i-1}^2M_1^2T_0} \\
&+ \mu\frac{\kappa_i + 1}{T_0} - \Lambda M_1(\kappa_i + 1)\{\kappa_{i+1} - (\kappa + \kappa_i)\} \\
&= \mu\frac{(\kappa_i(N) + 1)M_{i-1}(N)M_1(N) - (\kappa_i + 1)M_1M_{i-1}(N) - (\kappa_i + 1)M_{i-1}M_1(N)}{M_{i-1}M_1T_0} \\
&+ \mu\frac{\kappa_i + 1}{T_0} - \Lambda M_1(\kappa_i + 1)\{\kappa_{i+1} - (\kappa + \kappa_i)\} \\
&= \mu\frac{(\kappa_i(N) + 1)M_{i-1}(N)M_1(N)}{M_{i-1}M_1T_0} - \mu\frac{(\kappa_i + 1)M_{i-1}(N)}{M_{i-1}T_0} - \mu\frac{(\kappa_i + 1)M_1(N)}{M_1T_0} \\
&+ \mu\frac{\kappa_i + 1}{T_0} - \Lambda M_1(\kappa_i + 1)\{\kappa_{i+1} - (\kappa + \kappa_i)\} \\
&= \frac{\mu}{T_0}(\frac{(\kappa_i(N) + 1)M_{i-1}(N)M_1(N)}{M_{i-1}M_1} - \frac{(\kappa_i + 1)M_{i-1}(N)}{M_{i-1}} - \frac{(\kappa_i + 1)M_1(N)}{M_1} + \kappa_i + 1) \\
&- \Lambda M_1(\kappa_i + 1)\{\kappa_{i+1} - (\kappa + \kappa_i)\}
\end{aligned}$$

When $i = 2$, we have

$$\begin{aligned}
\dot{\kappa} &= \frac{\mu}{T_0}(\frac{(\kappa(N) + 1)M_1(N)^2}{M_1^2} - 2\frac{(\kappa + 1)M_1(N)}{M_1} + \kappa + 1) - \Lambda M_1(\kappa + 1)(\kappa_3 - 2\kappa) \\
&= \frac{\mu}{T_0}(\frac{(\kappa(N) + 1)M_1(N)^2}{M_1^2} - 2\frac{(\kappa + 1)M_1(N)}{M_1} + \kappa + 1) - \Lambda M_1(\kappa + 1)(\kappa_3 - 2\kappa)
\end{aligned}$$

Using R defined above, we have:

$$\dot{\kappa} = \frac{\mu}{T_0}((\kappa(N) + 1)R^2 - 2(\kappa + 1)R + \kappa + 1) - \Lambda M_1(\kappa + 1)(\kappa_3 - 2\kappa)$$

It doesn't seem like this is useful...

Idea 2

Can we see how accurate $M = \hat{M}T_0^\kappa$ is for SIS model?

$$
\begin{aligned}
\dot{M} &= \frac{dM}{dT_0}\frac{dT_0}{dt} \\
&= \kappa \hat{M} T_0^{\kappa-1}\{\mu(1-T_0) - \Lambda T_1\} \\
&= \mu\kappa \hat{M} T_0^{\kappa-1} - \mu\kappa \hat{M} T_0^{\kappa} - \Lambda\kappa \hat{M} T_0^{\kappa-1} T_1 \\
&= \mu\kappa \hat{M} T_0^{\kappa}/T_0 - \mu\kappa M - \Lambda\kappa \hat{M} T_0^{\kappa-1} M T_0 \\
&= \mu\kappa M/T_0 - \mu\kappa M - \Lambda\kappa M^2 \\
&= \mu(\kappa M - \kappa M T_0)/T_0 - \Lambda\kappa M^2
\end{aligned}
$$

Idea 3

$$
\begin{aligned}
\dot{T} &= \mu(1-T) - \Lambda T_1 \\
\dot{M} &= \mu(M(N) - M)/T - \Lambda\kappa M^2
\end{aligned}
$$

Can we do this?

$$
\begin{aligned}
\frac{dM}{dt} &= \frac{dM}{dT}\frac{dT}{dt} \\
\mu(M(N) - M)/T - \Lambda\kappa M^2 &= (\mu(1-T) - \Lambda T_1)\frac{dM}{dT}
\end{aligned}
$$

Random notes:

$$
\begin{aligned}
M &= T_1/T_0 \\
\dot{T}_1 &= \mu(T_1(N) - T_1) - \Lambda T_2 \\
\dot{T} &= \mu(1-T) - \Lambda T_1
\end{aligned}
$$

Suppose κ_3 stays constant. How do we define κ_3?

$$
\kappa_3 + 1 = M_3 M_1 / M_2^2
$$