

# Using moment equations to understand heterogeneity

## Definition

We want to write  $Y_o(v; D)$  for **Type**, **order**, **variable**, and **Domain**.

We define total as follows:  $T_o(v; D) = \int v(a)^o D(a) da$ . Since we're mainly interested with susceptibility for now,  $T_o$  represents  $T_o(.; S)$ .

We define  $M_i = T_i/T_0$ . Then,  $M_1$  is the mean susceptibility and  $\kappa = \frac{M_2 M_0}{M_1^2} - 1$  is the squared coefficient of variance (CV).

## SI example

We have  $\dot{S}(a) = -\Lambda \sigma(a) S(a)$ . Integrating gives us  $\dot{T}_0 = -\Lambda T_1$ . More generally, we have  $\dot{T}_i = -\Lambda T_{i+1}$ . Given that  $M_1$  is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using  $M$  defined above, we also have the following equations:  $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$ .

Given that  $M_2 = (1 + \kappa)M^2$  and assuming that  $\kappa$  stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M} S^\kappa$ , where  $\hat{M}$  is the mean susceptibility of the susceptible population at a disease free equilibrium.

We can define  $\kappa = \kappa_2$  and extend this idea to  $\kappa_i = \frac{M_i M_{i-2}}{M_{i-1}^2} - 1$ . We can derive an equation for  $\kappa_i$  as well:

$$\begin{aligned} \dot{\kappa}_i &= \frac{M_{i-2} M_{i-1} \dot{M}_i + M_{i-1} M_i \dot{M}_{i-2} - 2 M_{i-2} M_i \dot{M}_{i-1}}{M_{i-1}^3} \\ &= -\Lambda \frac{M_{i-2} M_{i-1} (M_{i+1} - M_i M_1) + M_{i-1} M_i (M_{i-1} - M_{i-2} M_1) - 2 M_{i-2} M_i (M_i - M_{i-1} M_1)}{M_{i-1}^3} \\ &= -\Lambda \frac{M_{i-2} M_{i-1} M_{i+1} + M_{i-1}^2 M_i - 2 M_{i-2} M_i^2}{M_{i-1}^3} \\ &= -\Lambda \frac{M_{i-2} (\kappa_{i+1} + 1) M_i^2 + M_{i-1}^2 M_i - 2 M_{i-2} M_i^2}{M_{i-1}^3} \\ &= -\Lambda \frac{(\kappa_{i+1} - 1) M_{i-2} M_i^2 + M_{i-1}^2 M_i}{M_{i-1}^3} \\ &= -\Lambda \frac{(\kappa_{i+1} - 1) (\kappa_i + 1) M_i M_{i-1}^2 + M_{i-1}^2 M_i}{M_{i-1}^3} \\ &= -\Lambda \frac{(\kappa_{i+1} \kappa_i - \kappa_i + \kappa_{i+1}) M_i}{M_{i-1}} \end{aligned}$$

Using chain rule, we can also do this:

$$\begin{aligned}
\frac{d\kappa_i}{dt} &= \frac{d\kappa_i}{dT_{i-2}} \frac{dT_{i-2}}{dt} \\
-\Lambda \frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}} &= -\Lambda M_{i-1} T_0 \frac{d\kappa_i}{dT_{i-2}} \\
\frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}^2} &= T_0 \frac{d\kappa_i}{dT_{i-2}} \\
\frac{(\kappa_i + 1)(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})}{M_{i-2}} &= T_0 \frac{d\kappa_i}{dT_{i-2}}
\end{aligned}$$

If we let  $i = 2$ , we have:

$$(\kappa_2 + 1)(\kappa_3\kappa_2 - \kappa_2 + \kappa_3) = T_0 \frac{d\kappa_2}{dT_0}$$

This doesn't work for some reason... It seems like we can't integrate it this way. Assuming that  $\kappa_3$  stays constant might be a problem.

If we assume that  $\kappa_3$  stays constant, we get the following separable differential equation:

$$\begin{aligned}
\int \frac{1}{T_0} dT_0 &= \int \frac{1}{(\kappa_2 + 1)(\kappa_3\kappa_2 - \kappa_2 + \kappa_3)} d\kappa_2 \\
\int \frac{1}{T_0} dT_0 &= \int \left( \frac{1}{\kappa_2 + 1} + \frac{1 - \kappa_3}{\kappa_3\kappa_2 - \kappa_2 + \kappa_3} \right) d\kappa_2 \\
\ln(T_0) &= \ln(\kappa_2 + 1) - \ln(\kappa_3\kappa_2 - \kappa_2 + \kappa_3) + C
\end{aligned}$$

If we set  $(\hat{T}_0, \hat{\kappa}_2) = (1, \hat{\kappa}_2)$ , we have  $C = \ln(\kappa_3\hat{\kappa}_2 - \hat{\kappa}_2 + \kappa_3) - \ln(\hat{\kappa}_2 + 1)$ . Solving the differential equation, we find that

$$\kappa = \frac{\kappa_3 S - e^C}{(1 - \kappa_3)S + e^C}$$

, which doesn't work....

## SIS example