Using moment equations to understand heterogeneity

Definition

We want to write $Y_o(v; D)$ for **T**y**pe**, **o**rder, **v**ariable, and **D**omain.

We define total as follows: $T_o(v; D) = \int v(a)^o D(a)da$. Since we're mainly interested with susceptibility for now, T_o is represents $T_o(.; S)$.

We define $M_i = T_i/T_0$. Then, M_1 is the mean susceptibility and $\kappa = \frac{M_2 M_0}{M_1^2} - 1$ is the squared coefficient of variance (CV).

SI example

We have $\dot{S}(a) = -\Lambda\sigma(a)S(a)$. Integrating gives us $\dot{T}_0 = -\Lambda T_1$. More generally, we have $\dot{T}_i = -\Lambda T_{i+1}$. Given that M_1 is the mean susceptibility, we can also write:

$$\dot{S} = -\Lambda M_1 S$$

Using M defined above, we also have the following equations: $\dot{M}_i = -\Lambda(M_{i+1} - M_i M_1)$.

Given that $M_2 = (1+\kappa)M^2$ and assuming that κ stays constant, we can integrate the equation above to obtain the following equation:

$M = \hat{M}S^\kappa$, where $\hat{M}$ is the mean susceptibility of the susceptible population at a disease free equilibrium.

Idea 1 - $\kappa_i = \frac{M_i M_{i-2}}{M_{i-1}^2} - 1$

$$\begin{aligned}
\dot{\kappa_i} &= \frac{M_{i-2}M_{i-1}\dot{M}_i + M_{i-1}M_i\dot{M}_{i-2} - 2M_{i-2}M_i\dot{M}_{i-1}}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}M_{i-1}(M_{i+1} - M_iM_1) + M_{i-1}M_i(M_{i-1} - M_{i-2}M_1) - 2M_{i-2}M_i(M_i - M_{i-1}M_1)}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}M_{i-1}M_{i+1} + M_{i-1}^2M_i - 2M_{i-2}M_i^2}{M_{i-1}^3} \\
&= -\Lambda\frac{M_{i-2}(\kappa_{i+1}+1)M_i^2 + M_{i-1}^2M_i - 2M_{i-2}M_i^2}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}-1)M_{i-2}M_i^2 + M_{i-1}^2M_i}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}-1)(\kappa_i+1)M_iM_{i-1}^2 + M_{i-1}^2M_i}{M_{i-1}^3} \\
&= -\Lambda\frac{(\kappa_{i+1}\kappa_i - \kappa_i + \kappa_{i+1})M_i}{M_{i-1}}
\end{aligned}$$

Idea 2 - $\kappa_i = \frac{M_i}{M_{i-1}M_1} - 1$

$$
\begin{aligned}
\dot{\kappa}_i &= \frac{M_{i-1}M_1\dot{M}_i - M_iM_1\dot{M}_{i-1} - M_iM_{i-1}\dot{M}_1}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{M_{i-1}M_1(M_{i+1} - M_iM_1) - M_iM_1(M_i - M_{i-1}M_1) - M_iM_{i-1}(M_2 - M_1^2)}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{M_{i-1}M_1M_{i+1} - M_iM_1M_i - M_iM_{i-1}M_2 + M_iM_{i-1}M_1^2}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{(\kappa_{i+1}+1)M_iM_{i-1}M_1^2 - M_iM_1M_i - (\kappa_2+1)M_iM_{i-1}M_1^2 + M_iM_{i-1}M_1^2}{M_{i-1}^2M_1^2} \\
&= -\Lambda\frac{(\kappa_{i+1}+1)M_iM_{i-1}M_1 - (\kappa_i+1)M_{i-1}M_iM_1 - \kappa_2M_iM_{i-1}M_1}{M_{i-1}^2M_1} \\
&= -\Lambda\frac{(\kappa_{i+1}+1)M_i - (\kappa_i+1)M_i - \kappa_2M_i}{M_{i-1}} \\
&= -\Lambda M_i\frac{\kappa_{i+1} - (\kappa_2+\kappa_i)}{M_{i-1}} \\
&= -\Lambda M_1(\kappa_i+1)\{\kappa_{i+1} - (\kappa_2+\kappa_i)\}
\end{aligned}
$$

When $i = 2$, we have $\dot{\kappa} = -\Lambda M(\kappa+1)(\kappa_3 - 2\kappa)$, where $\kappa = \kappa_2$. For gamma distribution, $\kappa_3 = 2\kappa$. Let's assume that $r = \kappa_3/kappa$ stays constant. Then, we have $\dot{\kappa} = -\Lambda M(r-2)(\kappa+1)(\kappa)$. We can do this:

$$
\begin{aligned}
\frac{d\kappa}{dt} &= \frac{d\kappa}{dS}\frac{dS}{dt} \\
-\Lambda M(r-2)(\kappa+1)(\kappa) &= -\Lambda MS\frac{d\kappa}{dS} \\
(r-2)(\kappa+1)(\kappa) &= S\frac{d\kappa}{dS} \\
\frac{(r-2)}{S} &= \frac{1}{\kappa(\kappa+1)}\frac{d\kappa}{dS} \\
\int\frac{(r-2)}{S}dS &= \int\frac{1}{\kappa(\kappa+1)}d\kappa \\
(r-2)\log(S) &= \log(\kappa) - \log(\kappa+1) + C
\end{aligned}
$$

We let the initial values $(S(0), \kappa(0)) = (1, \hat{\kappa})$, then we have $C = \log(\frac{\hat{\kappa}+1}{\hat{\kappa}})$. We can continue with the derivative:

$$
\begin{aligned}
\log(S^{r-2}) &= \log(e^C\frac{\kappa}{\kappa+1}) \\
S^{r-2} &= e^C\frac{\kappa}{\kappa+1} \\
S^{r-2}(\kappa+1) &= e^C\kappa \\
S^{r-2} &= (e^C - S^{r-2})\kappa \\
\kappa &= \frac{S^{r-2}}{e^C - S^{r-2}}
\end{aligned}
$$

SIS example

Here is a simple SIS model: $\dot{S}(a) = \mu(N(a) - S(a)) - \Lambda\sigma(a)S(a)$, where $N(a)$ is the initial distribution of the susceptible individuals in a disease free equilibrium. We are going to define $N_o = T_o(v; N)$. For this model, we have $\dot{T}_i = \mu(N_i - T_i) - \Lambda T_{i+1}$. Using chain rule, we can also get an equation for $M = M_1$:

$$\dot{M} = \mu(N_1 - MN_0)/T_0 - \Lambda\kappa M^2$$