Math 3T03 - Topology

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Contents

1 Introduction to topology

1.1 What is topology?

Definition 1.1 (Topology)**.** *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *The union of any collections of elements in τ is also in τ.*
- *THe intersection of any finite collection oi elements in τ is also in τ.*

Definition 1.2. (X, τ) *is a topological space.*

Definition 1.3. *The elements of τ are called the open sets.*

Example 1.1. Consider $X = \{a, b, c\}$. Is $\tau_1 = \{\varnothing, X, \{a\}\}$ a topology?

Proof. Yes, τ_1 is a topology. Clearly, $\varnothing \in \tau_1$ and $X \in \tau_1$ so it suffices to verify the arbitrary unions and finite intersection axioms.

Let $\{V_\alpha\}_{\alpha \in A}$ be a subcollection of τ_1. Then, we want to show

$$\bigcup_{\alpha \in A} V_\alpha \in \tau_1.$$

If $V_\alpha = \varnothing$ for any $\alpha \in A$, then this does not contribute to the union. Hence, φ can be omittied:

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\substack{\alpha \in A \\ V_\alpha \neq \varnothing}} V_\alpha.$$

If $V_\alpha = X$ for some $\alpha \in A$, then

$$\bigcup_{\alpha \in A} V_\alpha = X \in \tau_1$$

and we are done. So we may assume $V_\alpha \neq X$ for all $\alpha \in A$. Similarly, if $V_\alpha = V_\beta$ for $\alpha \neq \beta$, then we can omit V_β from the union. Since τ_1 only contains $\varnothing$, X and $\{a\}$, we must have

$$\bigcup_{\alpha \in A} V_\alpha = \{a\} \in \tau_1.$$

Similarly, if any element of $\{V_\alpha\}_{\alpha \in A}$ is empty, then

$$\bigcap_{\alpha \in A} V_\alpha = \varnothing \in \tau_1.$$

We can therefore assume $V_\alpha \neq \varnothing$ for all $\alpha \in A$. Again, repetition can be ignored and any V_α that eqauls X can be ignored. Then,

$$\bigcap_{\alpha \in A} V_\alpha = \begin{cases} \{a\} \in \tau_1 \\ X \in \tau_1 \end{cases}$$

□

Example 1.2. Consider $X = \{a, b, c\}$. Is $\tau_2 = \{\varnothing, X, \{a\}, \{b\}\}$ a topology?

Proof. No. Note that $\{a\} \in \tau_2$ and $\{b\} \in \tau_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_2$. □

Example 1.3. Consider $X = \{a, b, c\}$. Then, $\tau_3 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ is a topology.

1.2 Set theory

Definition 1.4. *If X is a set and a is an element, we write $a \in X$.*

Definition 1.5. *If Y is a subset of X, we write $Y \subset X$ or $Y \subseteq X$.*

Example 1.4. Suppose X is a set. Then, *power set* of X is the set $P(X)$ whose elements are all subsets of X. In other words,

$$Y \in P(X) \iff Y \subseteq X$$

Note that $P(X)$ is closed under arbitrary unions and intersections. Hence, $P(X)$ is a topology on X.

Definition 1.6. *$P(X)$ is the discrete topology on X.*

Definition 1.7. *The indiscrete (trivial) topology on X is $\{\varnothing, X\}$.*

Definition 1.8. *Suppose U, V are sets. Then, their union is*

$$U \cup V = \{x | x \in U \text{ or } x \in V\}.$$

Note that if $U_1, U_2, \ldots, U_10$ are sets, their union can be written as follows:

- $U_1 \cup U_2 \cup \cdots \cup U_{10}$
- $\bigcup_{k=1}^{10} U_k$
- $\bigcup_{k=\{1,2,\ldots,10\}} U_k$

Definition 1.9. *Let A be any set. Suppose $\forall \alpha \in A$, I have a set U_α. The union of the U_α over $\alpha \in A$ is*

$$\bigcup_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

Similarly, the intersection is

$$\bigcap_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

2 Functions

Definition 2.1. *A function is a rule that assigns to element of a given set A, an element in another set B.*

Often, we use a formula to describe a function.

Example 2.1. $f(x) = \sin(5x)$

Example 2.2. $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Definition 2.2. *A* rule of assignment *is a subset R of the Cartesian product $C \times D$ of two sets with the property that each element of C appears as the first coordinate of at most one ordered pair belonging to R. In other words, a subset R of $C \times D$ is a rule of assignment if it satisfies*

$$\textit{If } (c, d) \in R \textit{ and } (c, d') \in R, d = d'.$$

Definition 2.3. *Given a rule of assignment R, we define the doamin of R to be the subset of C consisting of all first coordinates of elements of R:*

$$\text{domain}(R) \equiv \{c \mid \exists d \in D \, s.t. \, (c, d) \in R\}$$

Definition 2.4. *The image set of R is defined to be the subset of D consisting of all second coordinates.*

$$\text{image}(R) \equiv \{d \mid \exists c \in C \, s.t. \, (c, d) \in R\}$$

Definition 2.5. *A function is a rule of assignment R, together with a set that contains the image set of R. The domain of the rule of assignment is also called the domain of f, and the image of f is defined to be the image set of the rule of assignment.*

Definition 2.6. *The set B is often called the range of f. This is also referred to as the codomain.*

If $A = \text{domain}(f)$, then we write

$$f : A \to B$$

to indicate that f is a function with domain A and codomain B.

Definition 2.7. *Given $a \in A$, we write $f(a) \in B$ for the unique element in B associated to a by the rule of assignment.*

Definition 2.8. *If $S \subseteq A$ is a subset of A, let*

$$f(S)\{f(a) \mid a \in S\} \subseteq B.$$

Definition 2.9. *Given $A_0 \subseteq A$, we can restrict the domain of f to A_0. The restriction is denoted*

$$f|A_0 \equiv \{(a, f(a)) \mid a \in A_0\} \subseteq A \times B.$$

Definition 2.10. *If $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$ is defined to be*

$$\{(a, c) \mid \exists b \in B \text{ s.t. } f(a) = b \text{ and } g(b) = c\} \subseteq A \times C.$$

Definition 2.11. *A function $f : A \to B$ is called injective (or one-to-one) if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$.*

Definition 2.12. *A function $f : A \to B$ is called surjective (or onto) if the image of f equals B, i.e. $f(A) = B$., i.e. if, for every $b \in B$, there exists $a \in A$ with $f(a) = b$.*

Definition 2.13. *$f : A \to B$ is a bijection if it is one-to-one and onto.*

Definition 2.14. *If $B_0 \subseteq B$ is a subset and $f : A \to B$ is a function, then the preimage of B_0 under f is the subset of A given by*

$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}$$

Example 2.3. If B_0 is disjoint from $f(A)$ then $f^{-1}(B_0) = \varnothing$.

3 Relation

Definition 3.1. *A relation on a set A is a subset R of $A \times A$.*

Given a relation R on A, we will write xRy "x is related to y" to mean $(x, y) \in R$.

Definition 3.2. *An equivalence relation on a set is a relation with the following properties:*

- *Reflexivity. xRx holds $\forall x \in A$.*
- *Symmetric. if xRy then yRx holds $\forall x, y \in A$.*
- *Transitive. if xRy and yRz then xRz holds $\forall x, y, z \in A$.*

Definition 3.3. *Given $a \in A$, let $E(a) = \{x \mid x \sim a\}$ denote the equivalence class of a.*

Remark. $E(a) \subseteq A$ and it is nonempty because $a \in E(a)$.

Proposition 3.1. *If $E(a) \cap E(b) \neq \varnothing$ then $E(a) = E(b)$.*

Proof. IAssume the hypothesis, i.e., suppose $x \in E(a) \cap E(b)$. So $x \sim a$ and $x \sim b$. By symmetry, $a \sim x$ and by transitivity $a \sim b$.

Now suppose that $y \in E(a)$. Then $y \sim a$ but we just saw that $a \sim b$ so $y \sim b$ and $y \in E(b)$. Hence, $E(a) \subseteq E(b)$. Likewise, we can show that $E(b) \subseteq E(a)$. □

Definition 3.4. *A partition of a set is a collection of pairwise disjoint subsets of A whose union is all of A.*

Example 3.1. Consider $A = \{1, 2, 3, 4, 5\}$. Then, $\{1, 2\}$ and $\{3, 4, 5\}$ is a partition of A.

Proposition 3.2. *An equivalence relation in a set determines a partition of A, namely the one with equivalence classes as subsets. Conversely, a partition[1] $\{Q_\alpha \mid \alpha \in J\}$ of a set A determines an equivalence relation on A by: $x \sim y$ if and only if $\exists \alpha \in J$ s.t. $x, y \in Q_\alpha$. The equivalence classes of this equivalence relation are precisely the subsets Q_α.*

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[1] Note that J is an index set: $Q_\alpha \subseteq A$ for each $\alpha \in J$ and $Q_\alpha \cap Q_\beta = \varnothing$ if $\alpha \neq \beta$. Furthermore, $\cup_{\alpha \in J} = A$.

4 Finite and infinite sets

Definition 4.1. *A nonempty set A is finite if there is a bijection from A to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{Z}^+$.*

Remark. Consider $n, m \in \mathbb{Z}^+$ with $n \neq m$. Then, there is no bijection from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, m\}$.

Definition 4.2. *Cardinality of a finite set A is defined as follows:*

1. $\mathrm{card}(A) = 0$ *if* $A = \varnothing$.
2. $\mathrm{card}(A) = n$ *if there is a bijection form A to* $\{1, 2, \dots, n\}$.

Note that $\mathbb{Z}^+$ is not finite. How would you prove this? A sneaky approach is that there is a bijection from $\mathbb{Z}^+$ to a proper subset of $\mathbb{Z}^+$.

Consider the shift map:

$$S : \mathbb{Z}^+ \to \mathbb{Z}^+$$

where $S(k) = k + 1$. So S is a bijection from $\mathbb{Z}^+$ to $\{2, 3, \dots\} \subset \mathbb{Z}^+$, a proper subset of $\mathbb{Z}^+$. On the other hand, any proper sbset B of a finite set A with $\mathrm{card}(A) = n$ has $\mathrm{card}(B) < n$. In particular, there is no bijection from A to B. Therefore, $\mathbb{Z}^+$ is not finite.

Theorem 4.1. *A non-empty set is finite if and only if one of the following holds:*

1. $\exists$ *bijection* $f : A \to \{1, 2, \dots, n\}$ *for some* $n \in \mathbb{Z}^+$.
2. $\exists$ *injection* $f : A \to \{1, 2, \dots, N\}$ *for some* $N \in \mathbb{Z}^+$
3. $\exists$ *surjection* $g : \{1, \dots, N\} \to A$ *for some* $N \in \mathbb{Z}^+$

Remark. Finite unions of finite sets are finite. Finite products of finite sets are also finite.

Recall that if X is a set, then

$$X^w = \pi_{n \in \mathbb{Z}^+} X = \{(x_n) \mid x_n \in X \forall n \in \mathbb{Z}^+\}$$

the set of infinite sequences in X.

If $X = \{a\}$ is a singleton, then X^w is also a singleton. Otherwise, if $\mathrm{card}(X) > 1$, then X^w is an infinite set.

Example 4.1. Consider $X = \{0, 1\}$. Then, X^w is a set of binary sequences.

Definition 4.3. *If there exists a bijection from a set X to $\mathbb{Z}^+$, then X is said to be countably infinite.*

Definition 4.4. *Let $\mathcal{A}$ be a nonempty collection of sets. An indexing family is a set J together with a surjection $f : J \to \mathcal{A}$. We use A_α to denote $f(a) \subseteq \mathcal{A}$. So*

$$\mathcal{A} = \{A_\alpha \mid \alpha \in J\}$$

Most of the time, we will be able to use $\mathcal{Z}^+$ for the index set J.

Definition 4.5 (Cartesian product)**.** *Let $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Finite cartesian product is defined as*

$$A_1 \times \cdots \times A_n = \{(x_1, \dots, x_n) \mid x_i \in A_i\}$$

It is a vector space of n tuples of real numbers.

Definition 4.6 (Infinite Cartesian product)**.** *Infinite Cartesian product $\mathbb{R}^\omega$ is an ω-tuple $x : \mathbb{Z}^+ \to \mathbb{R}$, i.e., a sequence*

$$\begin{aligned} x &= (x_i)_{i=1}^\infty = (x_1, x_2, \dots, x_n, \dots) \\ &= (x_i)_{i \in \mathbb{Z}^+} \end{aligned}$$

More generally, if $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Then,

$$\prod_{i \in \mathbb{Z}^+} A_i = \{(a_i)_{i \in \mathbb{Z}^+} \mid a_i \in A_i\}.$$

Definition 4.7. *A set A is countable if A is finite or it is countably infinite.*

Theorem 4.2 (Criterion for countability)**.** *Suppose A is a non-empty set. Then, the following are equivalent.*

1. *A is countable*
2. *There is an injection $f : A \to \mathbb{Z}^+$*
3. *THere is a surjection $g : \mathbb{Z}^+ \to A$*

Theorem 4.3. *If $C \subseteq \mathbb{Z}^+$ is a subset, then C is countable.*

Proof. Either C is finite or infinite. If C is finite, it follows C is countable.

Assume that C is finite. We will define a bijection $h : \mathbb{Z}^+ \to C$ by induction. Let $h(1)$ be the smallest element in C[2]. Suppose

$$h(1), \dots, h(n-1) \in C$$

and are defined. Let $h(n)$ be the smallest element in $C \setminus \{h(1), \dots h(n-1)\}$. (Note that this set is nonempty.)

We want to show that h is inejctive. Let $n, m \in \mathbb{Z}$ where $n \neq m$. Suppose $n < m$. So $h(n) \in \{h(1), \dots, h(n)\}$ and

$$h(n) \in C \setminus \{h(1), h(2), \dots, h(m-1)\}.$$

So $h(n) \neq h(m)$.

We want to show that h is surjective. Suppose $c \in C$. We will show that $c = h(n)$ for some $n \in \mathbb{Z}^+$. First, notice that $h(\mathbb{Z}^+)$ is not contained in $\{1, 2, \dots, c\}$

[2] $C \neq \varnothing$ and every nonempty subset of $\mathbb{Z}^+$ has a smallest elemnt

because $h(\mathbb{Z}^+)$ is an infinite set. Therefore, there must exists $n \in \mathbb{Z}^+$ such that $h(n) > c$. So let m be the smallest element with $h(m) \geq c$. Then for all $i < m$, $h(i) < c$. Then,

$$c \notin \{h(1), \dots, h(m-1)\}.$$

By definition, $h(m)$ is the smallest element in

$$C \setminus \{h(1), \dots, h(m-1)\}.$$

Then, $h(m) \leq c$. Therefore, $h(m) = c$. □

Corollary 4.1. *Any subset of a countable set is countable.*

Proof. Suppose $A \subset B$ with B countable. Then, there exists an injection $f : B \to \mathbb{Z}^+$. The restriction

$$f|_A : A \to \mathbb{Z}^+$$

is also injective. Therefore, A is countable by the criterion. □

Theorem 4.4. *Any countable union of countable is countable. In other words, if J is countable and A_α is countable for all $\alpha \in J$, then $\cup_{\alpha \in J}$ is countable.*

Theorem 4.5. *A finite product of countable sets is countable.*

Example 4.2. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Note that a countable product of countable sets is not countable.

Example 4.3. Consider $X = \{0, 1\}$. Then, X^ω binary sequence is not countable.

Proof. Suppose $g : \mathbb{Z}^+ \to X^\omega$ is a function. We will show g is not surjective by contructing an element $y \neq g(n)$ for any $n \in \mathbb{Z}^+$.

Write $g(n) = (x_{n1}, x_{n2}, \dots) \in X^\omega$ for each $x_{ni} \in \{0, 1\}$. Define

$$y = (y_1, y_2, \dots)$$

where $y_i = 1 - x_{ii}$. Clearly, $y \in X^\omega$ but $y \neq g(n)$ for any n since $y_n = 1 - x_{nn} \neq x_{nn}$, the n-th entry in g(n). □

Example 4.4. Let X be a set and $\{V_\alpha\}_{\alpha \in I}$ a collection of subsets of X. Show

$$\bigcup_{\alpha \in I} (X - V_\alpha) = X - \bigcap_{\alpha \in I} V_\alpha$$

Proof.

$$\begin{aligned} x \in \bigcup_{\alpha \in I} X - V_\alpha &\iff \exists \alpha \in I, x \in X - V_\alpha \\ &\iff \exists \alpha \in I, x \in X, x \notin V_\alpha \\ &\iff x \in X, \exists \alpha \in I, x \notin V_\alpha \\ &\iff x \in X, x \notin \bigcup_{\alpha \in I} V_\alpha \\ &\iff x \in X - \bigcap_{\alpha in I} V_\alpha \end{aligned}$$

$\square$

5 Topology

Definition 5.1 (Topology)**.** *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *If* $\{V_\alpha\}_{\alpha \in I} \subseteq \tau$, *then* $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
- *If* $\{V_1, \ldots, V_n\} \subseteq \tau$, *then* $\bigcap_{k=1}^{n} V_k inI$.

Remark. If τ is a topology, then $\{\varnothing, X\} \subseteq \tau$ and $\tau \subseteq \mathcal{P}(X)$, where $\mathcal{P}$ is a power set.

Definition 5.2. *Suppose τ, τ' are topologies on X. We say τ is coarser than τ' if $\tau \subseteq \tau'$. τ' is finer than τ if $\tau \subseteq \tau'$.*

Example 5.1. If X is a set and τ is a topology. Then, τ is coarser than the discrete topology, $\mathcal{P}(X)$, and finer than the trivial topology, $\{\varnothing, X\}$.

Note that if $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$, then $\tau = \tau'$.

Example 5.2. Define τ_f to consist of the subsets U of X wiith the property that $X - U$ is finite or all of X.

Lemma 5.1. *τ_f is a topoology on X. This is defined as the finite complement topology.*

Proof. To show $\varnothing \in \tau_f$, notice that $X - \varnothing = X$. So $\varnothing \in \tau_f$. To show that $X \in \tau_f$, notice that $X - X = \varnothing$, which is finite. So $X \in \tau_f$.

To show that it is closed under arbitrary unions, suppose

$$\{V_\alpha\}_{\alpha \in I} \subseteq \tau_f.$$

It suffices to show that $X - \bigcup_{\alpha \in I} V_\alpha$ is finite. For this, note

$$X - \bigcup_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} (X - V_\alpha).$$

$X - V_\alpha$ is finite so

$$\bigcap_{\alpha \in I} (X - V_\alpha) \subseteq X - V'_\alpha$$

for any $\alpha' \in I$ which implies $\bigcap_{\alpha \in I} X - V_\alpha$ is finite.

For closure under finite intersections, let

$$\{V_1, \ldots, V_n\} \subseteq \tau_f.$$

Then,

$$X - \bigcap_{k=1}^{n} V_k = \bigcup_{k=1}^{n} (X - V_k)$$

is a finite union of finite sets and so it is finite. $\square$

Example 5.3. Consider $X = \mathbb{R}$. Then, $\mathbb{R} - \{0\}$ is open in the finite complement topology. $\mathbb{R} - \{0, 3, 100\}$ is open in τ_f. Note that $(4, \infty)$ is not open in τ_f because its complement is not finite.

Remark. Assume τ is a topology on X.

- (X, τ) is a topological space.
- X is a topological space.
- $U \in \tau$ iff U is open in τ iff U is an open set.

6 Basis

Definition 6.1. *Let X be a set. A basis for topology on X is a collection $\mathcal{B}$ of subsets of X satisfying:*

- $\forall x \in X, \exists B \in \mathcal{B}, x \in B$
- $\forall B_1, B_2 \in \mathcal{B}$, *if* $x \in B_1 \cap B_2$ *then* $\exists B_3 \in \mathcal{B}$ *such that* $x \in B_3$ *and* $B_3 \subseteq B_1 \cap B_2$.

Example 6.1. Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_1$ be a collection of interiors of circles in $\mathbb{R}^2$. Then, $\mathcal{B}_1$ is a basis.

Example 6.2. Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_2$ be collection of interiors of axis-parallel rectangles. Then, $\mathcal{B}_2$ is a basis. Note that nontrivial intersection is always a rectangle.

Definition 6.2. *The topology generated by a basis $\mathcal{B}$ is the collection τ satisfying:*

$$U \in \tau \iff \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U.$$

Example 6.3. Consider $X = \mathbb{R}$ with standard topology, τ_{st}. Consider $U = \mathbb{R} - \{0\}$. Then, $U \in \tau_{st}$. Let $x \in U$:

1. (Case 1) $x > 0$. Let $B = (0, x+1)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.
2. (Case 2) $x < 0$. Let $B = (x-1, 0)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.

Example 6.4. Let τ_f be finite complement topology on $\mathbb{R}$. If $U \in \tau_f$, then $U \in \tau_{st}$. So $\tau_f \subseteq \tau_{st}$.

Remark. $(3, \infty)$ is open in τ_{st} but not in τ_f.

Lemma 6.1. *If $\mathcal{B}$ is a basis and τ is the topology generated by $\mathcal{B}$, then τ is a topology.*

Proof. $\varnothing \in \tau$ is vacuously true. If $x \in \varnothing$ then $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \varnothing$.

Now, we want to prove that $X \in \tau$. Let $x \in X$. By the axioms for basis, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq X$. Hence, $X \in \tau$.

Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of τ. Let

$$x \in \bigcup_{\alpha \in A} \mathcal{U}_\alpha.$$

Then, $\exists \mathcal{B} \in A$ with $x \in \mathcal{U}_\mathcal{B}$. Since $\mathcal{U}_\mathcal{B} \in Tau$, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \mathcal{U}_\mathcal{B}$. TheN, $x \in B$ and $B \subseteq \mathcal{U}_\mathcal{B}$. So τ is closed under arbitrary unions.

Let $\{V_1, \ldots, V_n\} \subseteq \tau$. We want to use proof by induction to show that τ is closed under finite intersections. When $n = 1$,

$$\bigcap_{k=1}^{n} V_k = V_1 \in \tau.$$

Assume the claim is true for n. Then,

$$\bigcap_{k=1}^{n+1} V_k = \underbrace{V_{n+1}}_{W} \cap \underbrace{\bigcap_{k=1}^{n} V_k}_{W'}$$

Note $W \in \tau$ and $W' \in \tau$ by the induction hypothesis. If $W \cap W' = \varnothing$ then we are done ($\varnothing \in \tau$). Let $x \in W \cap W'$. Then, $x \in W$ and $x \in W'$, implying that $\exists B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq W$ and $x \in B_2 \subseteq W'$. Hence, $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2 \subseteq W \cap W'$. □

Example 6.5. Consider $X = \mathbb{R}$. Define

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

for $a, b \in \mathbb{R}$. Show

$$\mathcal{B}_{st} = \{(a, b) | a, b \in \mathbb{R}\}$$

is a basis.

Let $x \in \mathbb{R}$. Then,

$$x \in (x - 1, x + 1) \in \mathcal{B}_{st}.$$

For the second axiom, let $(a, b), (c, d) \in \mathcal{B}_{st}$ and assume $x \in (a, b) \cap (c, d)$. Then, $x \in (c, b) \subseteq (a, b) \cap (c, d)$.

Definition 6.3. *The standard topology on* $\mathbb{R}$ *is the topology* τ_{st} *generated by* $\mathcal{B}_{st}$.

Lemma 6.2. *Suppose* $\mathcal{B}$ *and* $\mathcal{B}'$ *are bases on* X *and* τ *and* τ' *are the topologies they generate. Then, the following are equivalent:*

1. τ' *is finer than* τ ($\tau \subseteq \tau'$)
2. $\forall x \in X$ *and* $\forall B \in \mathcal{B}$ *with* $x \in B$, $\exists B' \in \mathcal{B}'$ *with* $x \in B' \subseteq B$.

Proof. First, we want to show that 1 implies 2. Assume $\tau \subseteq \tau'$. Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ generates τ, $B \in \tau$. Then, $B \in \tau'$. Since τ' is generated by $\mathcal{B}'$,

$$\exists B' \in \mathcal{B} \text{ with } x \in B' \subseteq B$$

Other direction is left to readers as an exercise. □

Let $X = \mathbb{R}^2$. Consider $\mathcal{B}_1$, a collection of interior of circles, and $\mathcal{B}_2$, a collection of interior of axis-parallel rectangles. Let τ_j be toplogies generated by $\mathcal{B}_j$ where $j = 1, 2$.

Lemma 6.3. $\tau_1 = \tau_2$

Proof. ($\tau_1 \subseteq \tau_2$). We use the theorem from earlier. Let $x \in \mathbb{R}^2$ and $x \in B_1 \subseteq \mathcal{B}_1$. We have to show that $\exists B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$. Since $x \in B_1$, x sits inside a circle. But we can always construct a rectangle B_2 smaller than B_1 but contains x.

($\tau_2 \subseteq \tau_1$). Let $x \in \mathbb{R}^2$ and $B_2 \in \mathcal{B}_2$ with $x \in B_2$. We can always take $B_1 \in \mathcal{B}_1$ to be a circle inside rectangle B_2 and contains x. So $x \in B_1 \subseteq B_2$. $\square$

Lemma 6.4. *Suppose (X, τ_x) and (Y, τ_Y) are topological spaces. Consider*

$$X \times Y = \{(a, b) | a \in X, b \in Y\}$$

Then,

$$\mathcal{B} = \{U \times V | U \in \tau_X, V \in \tau_Y\}$$

is a basis on $X \times Y$.

Proof. Let $(a, b) \in X \times Y$. Then, $X \times Y \in \mathcal{B}$ and $(a, b) \in X \times Y$. Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and assume

$$(a, b) \in U_1 \times V_1 \cap U_2 \times V_2.$$

Note

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}.$$

So

$$(a, b) \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$$

$\square$

Definition 6.4. *The product topology is the topology generated by this basis.*

Theorem 6.1. *Suppose $\mathcal{B}_x$ and $\mathcal{B}_y$ are bases for τ_x and τ_y, respectively. Then,*

$$\{B_1 \times B_2 | B_1 \in \mathcal{B}_x, B_2 \in \mathcal{B}_y\}$$

is a basis for product topology on $X \times Y$.

Example 6.6. $(X, \tau_x) = (\mathbb{R}, \tau_{st}) = (Y, \tau_Y)$.

Definition 6.5. *The standard topology on $\mathbb{R}^2$ is the product topology of the $(\mathbb{R}, \tau_{st})$.*

Remark. The theoremm tells us that $\mathbb{R}^2$ has a basis

$$\{(a, b) \times (c, d) | a, b, c, d \in \mathbb{R}\}$$

6.1 Connecting basis to topology

Suppose $\mathcal{B}$ is a basis for a topology τ on X.

- First, if $B \in \mathcal{B}$, then $B \in \tau$. Recall that

  $$U \in \tau \iff \forall x = U, \exists B' = \mathcal{B}, x \in \mathcal{B}' \subseteq U.$$

  So let $x \in B$. Then, $B' = B$ satisfies the above condition ($x \in B \subseteq B$). Hence, $B \in \tau$.

- Let $U \in \tau$. Then, U is a union of elements in $\mathcal{B}$. To prove this, let $U \in \tau$. By definition,

  $$\forall x \in U, \exists B_x \in \mathcal{U}$$

  such that $x \in B_x \subseteq U$. Then,

  $$\{B_x\}_{x \in U} \subseteq \tau.$$

  Then,

  $$\bigcup_{x \in X_U} B = U.$$

7 Subspace topology

Definition 7.1. *Let (X, τ) be a topological space and $Y \subseteq X$ be any subset. The subspace topology on Y is*

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

Lemma 7.1. *τ_Y is a topology.*

Proof.

- $\varnothing \cap Y = \varnothing \in \tau_Y$.
- $X \cap Y = Y \in \tau_Y$.
- (Unions) Let $\{V_\alpha\}_{\alpha \in A} \subseteq \tau_Y$, then $V_\alpha = U_\alpha \cap Y$ for some $U_\alpha \in \tau$. Then,

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in A} U_\alpha \right) \cap Y \in \tau_Y$$

$\square$

Theorem 7.1. *Suppose $\mathcal{B}$ is a basis for X and $Y \subseteq X$. Then,*

$$\{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Example 7.1. Let $X = \mathbb{R}$ with $\mathcal{B} = \{(a, b) | a, b \in \mathbb{R}\}$. Consider

$$Y = [1, \infty).$$

Then, the basis for this topology is given by

$$\mathcal{B}_Y = \{[1, a) | a \in \mathbb{R}, a > 1\} \cup \{(a, b) | a, b \in \mathbb{R}, 1 \leq a < b\}$$

Example 7.2. $(3, 7] \notin \tau_Y$ because there does not exist open $U \in \tau$ $(U \in \mathcal{B})$ such that

$$(3, 7] = U \cap [1, \infty).$$

Proof. To yield contradiction, suppose there exists such U. Then,

$$7 \in (a, b) \cap [1, \infty).$$

So $7 \in (a, b)$ and $7 \in [1, \infty)$. In other words, $a < 7 < b$.

Let $\epsilon = (b - 7)/2$. Then, $a < 7 + \epsilon < b$ so $7 + \epsilon \in (a, b)$ and $7 + \epsilon \in [1, \infty)$. Then,

$$7 + \epsilon \in (a, b) \cap [1, \infty) = (3, 7] \implies 3 < 7 + \epsilon \leq 7,$$

yielding contradiction. $\square$

Example 7.3. Let $X = \mathbb{R}$ and $Y = [0, 1]$. A basis for the subspace on Y is

$$\{(a, b)|0 \leq a < b \leq 1\} \cup \{[a, b)|0 \leq b \leq 1\} \cup \{(a, 1]|0 \leq a \leq 1\}$$

Example 7.4. Consider a rectangle

$$R = [0, 1] \times [2, 6].$$

There are two ways of thinking of topology on this rectangle. Since $R \subseteq \mathbb{R}^2$, we can think of it as a subspace topology. We can also think of it as a product topology. So are they same topologis in this case?

Theorem 7.2. *Suppose X and Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$ are subspaces. Then, the product topology on $A \times B$ equals the topology $A \times B$ inherits as a subset of $X \times Y$.*

Proof. In the subspace topology, basis element is given by

$$(A \times B) \cap (U \times V)$$

where $U \subseteq X$ and $V \subseteq Y$. Likewise, basis element of product topology is given by

$$(A \cap U) \times (B \cap V).$$

But then

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V).$$

□

8 Order topology

Let X be a set.

Definition 8.1. *An order relation (or simple order or linear order) is relation R on X so*

1. *(comparability) If $x, y \in X$ and $x \neq y$. Then either xRy or yRx.*
2. *(Nonreflexivity) $\nexists x \in X$ such that xRx.*
3. *(Transitivity)s If xRy and yRz then xRz.*

Example 8.1. Define $X \subseteq \mathbb{R}$ and $R =<$. Then, R is an order relation

Proof.

1. If $x \neq qy$, either $x < y$ or $y < x$.
2. $\nexists x \in X$ such that $x < x$.
3. $x < y, y < z \implies x < z$.

□

If X has an order relation, we often denote it $<$. Suppose $(X, <)$ is a set with an order relation for $a, b \in X$.

Definition 8.2. $(a, b) = \{x \in X | a < x, x < b\}$.

Definition 8.3. $(a, b] = \{x \in X | a < x, x < b, \text{ or } x = b\}$.

Definition 8.4. $[a, b) = \{x \in X | a < x, x < b, \text{ or } x = a\}$.

Definition 8.5. *$a_0 \in X$ is a minimal element if $\forall x \in X$, $a_0 \leq x$.*

Definition 8.6. *$b_0 \in X$ is a maximal element if $\forall x \in X$, $x \leq b_0$.*

Definition 8.7. *Suppose $(X, <_x)$ and $(Y, <_y)$ are simply ordered sets. The dictionary order on $X \times Y$ is the order $<_{x \times y}$ defined as follows:*

$$(a_1, b_1) <_{X \times Y} (a_2, b_2)$$

if $a_1 <_x a_2$ or $a_1 = a_2$ and $b_1 <_y b_2$.

Example 8.2. Define $X = Y = \mathbb{R}$ and $<_x =<_y =<$, usual inequality. Then,

- $(0, 0) <_{X \times Y} (1, -3)$ is true
- $(0, 0) <_{X \times Y} (0, -3)$ is false.

Definition 8.8. *Suppose $(X, <)$ is a simply ordered set. Then, the order topology on X is the one with basis elements: (a, b) for $a, b \in X$, $[a_0, b)$ for $b \in X$ provided a_0 is the minimal element of X, and $(a, b_0]$ for $a \in X$ provided $b_0 \in X$ is the maximal element.*

Example 8.3. Consider $X = [0, 1]$ with $<$, the standard inequality. The order topology is the topology inherited as a subset.

Example 8.4. Consider $(\mathbb{R}^2, <_o)$ where $<_0$ is the dictionary order. The order topology is called the dictionary order topology.

Think about what this set looks like:

$$((0, 0), (1, 1))$$