Math 4A03 - Real analysis

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Course Outline

- Office hours: Tuesday 11:30 - 1:30

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1 Metric space theory

1.1 Introduction

Example 1.1.1. $(\mathbb{R}, |\cdot|)$ is a metric space, where $|\cdot|$ denotes absolute value.

Example 1.1.2. $(\mathbb{R}^n, \text{Euclidean norm})$ is a metric space.

Example 1.1.3. $C[a,b]$ denotes space of continuous functions on $[a,b]$ and is an infinite dimensional metric space.

1.2 Metric spaces

In elementary analysis, we went over the notion of real numbers, sequences, connvergence, continuity, integration and differentiation. An important tool was the absolute value as it allowed us to measure distance between two points.

Definition 1.1 (Absolute value)**.** *The absolute value between x and y is written as $|x-y|$ and measures distance on number line.*

Maurice Frechet noted that much of analysis may be extended to to abstract sets M, provided there is a reasonable definition of distance between points. In other words, much of elementary analysis depends not on specific properties of $\mathbb{R}$, but only on its metric properties and can be abstracted.

Recall that a sequence $(X_n)_{n\in\mathbb{N}}$ converges to x in $\mathbb{R}$ if $\forall\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$. In order to extend this notion, we need to define distance. What is a reasonable way to define distance?

Definition 1.2 (Distance)**.** *A function $d : M \times M \to \mathbb{R}$ is a* metric *on M if the following holds:*

(i) $0 \geq d(x,y) < \infty \quad \forall x, y \in M$.

(ii) $d(x,y) = 0$ *iff* $x = y$. *If this condition does not hold, d is called a pseudo-metric.*

(iii) $d(x,y) = d(y,x) \forall x, y \in M$.

(iv) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in M$. *This generalizes the triangle inequality.*

Remark. In $\mathbb{R}^2$, the smallest distance between two points is the line.

Definition 1.3. *We say (M,d) define a* Metric space*.*

Example 1.2.1. $(M = \mathbb{R},\ d(x,y) = |x-y|)$ is a valid metric space.

Example 1.2.2. $d_e(x,y) = |e^{-x} - e^{-y}|$ is a metric on $\mathbb{R}$.

Example 1.2.3 (Discrete metric)**.**

$$d_{\text{disc}}(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

is a valid metric on any set M. In particular, note that any set M has at least one metric.

Of course, changing the metric d changes the geometry of disks, our notion of convergence, continuity, etc. For example, $(\mathbb{R}, d(x, y) = |x - y|)$ and $(\mathbb{R}, d_{\text{disc}})$ are very different as Metric spaces.

1.3 Normed vector spaces

There is a natural metric to choose in this case to keep the inherent structure of the space.

Definition 1.4 (Norm)**.** *Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ with the following properties:*

(i) $0 \leq \|v\| < \infty \quad \forall v \in V$.

(ii) $\|v\| = 0 \iff v = 0$.

(iii) $\|\alpha v\| = |\alpha|\|v\| \quad \forall \alpha \in \mathbb{R}, v \in V$.

(iv) $\|v + w\| \leq \|v\| + \|w\|$.

Proposition 1.1. *If V is a vector space with norm $\|\cdot\|$, then (V, d) is a metric space with $d(v, w) = \|v - w\|$. We call this metric the* usual *metric.*

Note that

$$\begin{aligned} d(v, w) &= \|v - w\|\|v - u + u - w\| \\ &\leq \|v - u\| + \|u - w\| \\ &= d(v, u) + d(u, w) \end{aligned}$$

Clearly, $\|x\| = |x|$ define a norm on $\mathbb{R}$ so the usual metric is an example of this proposition. However, d_{e} and d_{disc}, defined above, are valid metric which don't come from norms.

Example 1.3.1. $d(x, y) = (x - y)^2$ is not a metric as it fails to satisfy the triangle inequality. For example, consider $x = 0$ and $y = 1$. Then,

$$d(0, 1) > d(0, 1/2) + d(1/2, 1).$$

Example 1.3.2. $d(x, y) = x^2 - y^2$ is not a metric because $d(x, y) = 0$ iff $x = \pm y$.

Example 1.3.3. $d(x, y) = \sqrt{|x - y|}$ is a metric. In particular, the triangle inequality holds:

$$\sqrt{|x - y|} \leq \sqrt{|x - z + z - y|} \leq \sqrt{|x - z| + |z - y|}$$

Example 1.3.4. $d(x, y) = \frac{|x-y|}{1+|x-y|}$ is a metric. Let$F(t) = t/(1 + t)$. Then, we can show that

$$F(a + b) \leq F(a) + F(b).$$

Example 1.3.5. Consider a Euclidean space $\mathbb{R}^n$. Then, the usual metric is given by

$$d(x, y) = \sqrt{\sum_{j=1}^{N} |x_j - y_j|^2}.$$

Hence, we have the Euclidean norm:

$$\|x\|_2 = \sqrt{\sum_{j=1}^{N} |x_j|^2}$$

We can write this using the inner product:

$$\|x\|_2 = \langle x, x \rangle$$

To verify that this is indeed a norm, we need *Cauchy-Schwartz inequality*:

$$|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$$

To prove the inequality, let $t \in \mathbb{R}$. Then, for all $x, y \in \mathbb{R}^N$, we have

$$\begin{aligned} 0 \leq \|x + ty\|_2^2 &= \langle x + ty, x + ty \rangle \\ &= \|x\|^2 + t^2\|y\|^2 + 2t\langle x, y \rangle \quad \forall t \in \mathbb{R} \end{aligned}$$

This is a quadratic in t so the discriminant must be non-positive. In other words,

$$At^2 + Bt + C \geq 0 \iff B^2 - 4AC \leq 0.$$

Hence,

$$4(\langle x, y \rangle)^2 \leq 4\|x\|^2\|y\|^2 \implies |\langle x, y \rangle| \leq \|x\|\|y\|$$

Now, the trinagle inequality follows easily for $\|x\| = \sqrt{\langle x, x \rangle}$:

$$\begin{aligned} \|x + y\|^2 &= \|x\| + \|y\|^2 + 2\langle x, y \rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

Note that other norms are possible on $\mathbb{R}^n$. For example,

$$d_1(x, y) = \sum_{j=1}^{N} |x_j - y_j|$$

is a metric where the associated norm is given by $\|x\|_1 = \sum_{j=1}^N |x_j|$.

$$d_p(x, y) = \left(\sum_{i=1}^N |x_j - y_j|^p \right)^{1/p}$$

is a metric where the associated norm is given by $\|x\|_p = \left(\sum_{j=1}^N |x_j|^p \right)^{1/p}$. Finally,

$$d_\infty(x, y) = \|x - y\|_\infty$$

is a norm where $\|x\|_\infty = \sup_{1 \le j \le N} |x_j|$.

1.4 Neighborhood

Now that we have a distance designed on a space (M, d), we can study its geometry.

Definition 1.5. *The open ball $B_r(x)$ of radius $r > 0$ centered at x is*

$$B_r(x) = \{y \in M \mid d(x, y) < r\}.$$

Definition 1.6. *A set $A \subseteq M$ is bounded if there exists a ball $B_R(x)$ with $A \subseteq B_R(x)$.*

Definition 1.7. *A set $\mathcal{U}$ is a neighborhood of x if there exists $r > 0$ with $B_r(x) \le \mathcal{U}$.*

Example 1.4.1. In $\mathbb{R}$ equipped with the usual metric, $B_r(x) = (x - r, x + r)$, an open interval. In $\mathbb{R}^2$ and $\mathbb{R}^3$, $B_r(x)$ is the round disk and a spherical ball centered at x.

Changing metric changes the shape of the ball.

Example 1.4.2. First, consider $(\mathbb{R}^2, d_\infty)$. Then,

$$B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^2 \mid d_\infty(\vec{x}, \vec{y}) < r\}$$

Note that

$$\begin{aligned} d_\infty(\vec{x}, \vec{y}) < r &\iff \sup(|x_1 - y_1|, |x_2 - y_2|) < r \\ &\iff |x_1 - y_1|, |x_2 - y_2| < r \end{aligned}$$

Therefore, $B_r(\vec{x})$ is a square.

Example 1.4.3. Consider a set M equipped with discrete topology. If $r \le 1$,

$$d_{\text{disc}}(x, y) < r \le 1 \implies y = x.$$

Hence, $B_r(x) = \{x\}$. If $r > 1$,

$$d_{\text{disc}} < r \implies y \in M.$$

Hence, $B_r(x) = M$. Therefore, all sets are open and all sets are closed.

Definition 1.8. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in (M,d). We say $X_m \to X$ for some $x \in M$ if $\forall\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $d(X_n, X) < \epsilon$ for all $n \in N$. Equivalently, $X_n \to X$ if $\forall\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_n \in B_\epsilon(x)$ for all $n \geq N$. Or, for every neighborhood $\mathcal{U}$ of x there exists $N \in \mathbb{N}$ with $X_n \in \mathcal{U}$ for all $n \geq N$.*

In general, the choice of the metric may affect convergence.

Example 1.4.4. Let $M = \mathbb{R}$ equipped with discrete topology. Suppose $X_n \to X$, choose $\epsilon = 1/2$. Then, there exists $N \in \mathbb{N}$ such that $d(X_n, X) < \epsilon$ for all $n \geq N$. In other words, $X_n = x$ for all $n \geq N$. So $\{X_n\}$ is convergent in d_{disc} iff $\{X_n\}$ is eventually constant. For example, $X_n = 1/n$ does not converge.

Example 1.4.5. Let $M = \mathbb{R}^2$ and $d = d_\infty$. Take a ball $B^\infty_\epsilon(x)$. Note that

$$B^2_{\sqrt{2}\epsilon}(\epsilon) \subset B^\infty_\epsilon(x) \subset B^2_{\sqrt{2}\epsilon}(\epsilon)$$

So convergence in sup norm is equivalent to convergence in Euclidean metric.

If $X_n \to X$ in $(\mathbb{R}^2, d_2)$ then $X_n \to X$ in $(\mathbb{R}^2, d_\infty)$. Conversely, if $X_n \to X$ in $(\mathbb{R}^2, d_\infty)$ then $X_n \to X$ in $(\mathbb{R}^2, d_2)$. Hence, $(\mathbb{R}^2, d_\infty)$ and $(\mathbb{R}^2, d_2)$ are equivalent in the sense that they have the same convergent and divergent sequences.

Theorem 1.1. *Let $\|\cdot\|$ be any norm in $\mathbb{R}^m$. Then, $(\mathbb{R}^N, \|x-y\|)$ is equivalent to the Euclidean metric on $\mathbb{R}^N$. More precisely, one can show that there exists $c_1, c_2 > 0$ with*

$$c_1\|x\|_2 \leq \|x\| \leq c_2\|x\|_2 \quad \forall x \in \mathbb{R}^N.$$

1.5 Sequence spaces

Infinite dimensional spaces are more interesting. Let

$$W = \{x = (x_1, x_2, x_3, \dots) | x_m \in \mathbb{R}\}.$$

W is a vector space on $\mathbb{R}$. We can define family of norms on subspaces of W.

Definition 1.9. $\ell^p = \{x \in W | \sum_{k=1}^\infty |x_k|^p \infty\}$ *for* $1 \leq p < \infty$ *with norm*

$$\|x\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$$

Definition 1.10. $\ell^\infty = \{x \in W | \sup_k |x_k| < \infty\}$ *with norm*

$$\|x\|_p = \sup_k \|x_k\|.$$

Let's review a series of non-negative terms. $S_n = \sum_{k=1}^n a_k$ where $a_k \geq 0$ forms a sequence $(S_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, since each $a_k \geq 0$,

$$S_{n+1} = S_n + a_{n+1} \geq S_n$$

Sequence is monotone increasing.

If $S_n \to S$, we say S_n converges. If $S_n \to \infty$, we say S_n diverges. If S_n is bounded, it converges to

$$S = \sup_n S_n = \lim S_n$$

Theorem 1.2. *ℓ_∞ is a normed vector spaces.*

Proof.
- $\|x\|_\infty \geq 0$; $\|x|_\infty = 0$ iff $\sup_k |x_k| = 0$
- $\|\alpha x\|_\infty = |\alpha|\|x\|_\infty$.
- Let $x, y \in \ell_\infty$. Then,

$$\begin{aligned}\|x_j + y_j\| &\leq |x_j| + |y_j| \\ &\leq \|x\|_\infty + \|y\|_\infty\end{aligned}$$

Hence, $\|x\|_\infty + \|y\|_\infty$ is an upper bound for $\|x_j + y_j\|$. Therefore,

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$$

□

Theorem 1.3. *ℓ_p is a normed vector spaces.*

Proof. For $1 \leq p < \infty$, call

$$\left(\sum_{k=0}^{N} |x_k|^p \right)^{1/p} = \|x\|_{N,p},$$

the p-norm of the N-tuple. So the triangle inequality is given by

$$\begin{aligned}\|x + y\|_{N,p} &\leq \|x\|_{N,p} + \|y\|_{N,p} \\ &\leq \|x\|_p + \|y\|_p\end{aligned}$$

for all $N \in \mathbb{N}$.

Now, the sequence $(\|x+y\|_{N,p})_{N\in\mathbb{N}}$ is monotonically increasing and bounded above so its limit exists.

$$\|x + y\|_p = \lim_{N\to\infty} \|x + y\|_{N,p} \leq \|x\|_p + \|y\|_p$$

□

Example 1.5.1. Consider a sequence (X_n) such that further subsequence converges to X. Show that the entire sequence X_n converges to X.

Instead, we can prove its contrapositive. Suppose X_n doesn't converge to X, i.e., there exists $\epsilon > 0$ and $k \in \mathbb{N}$ and exists $m_k > k$ such that $d(X_{m_k}, X) > \epsilon$. Then, any subsequence of $\{X_{m_k}\}$ converge to X.

1.6 Spaces of continuous functions

Another important normed vector space is

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

with sup norm:

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

Since f is continuous on $[a,b]$, the supremum is really a maximum. This defines a norm.

Example 1.6.1. Another useful norm on $C([a,b])$ is the L^1-norm:

$$\|f\|_1 = \int_a^b |f(x)|dx.$$

Verify that this is a norm.

These define very different metric spaces. Convergence in $\|\cdot\|_\infty$ implies uniform convergence. Note that convergence in $\|\cdot\|_\infty$ implies convergence in L^1 but not other way around.

1.7 Open and closed

Definition 1.11. *Let (M,d) be a metric space, $\mathcal{U} \subseteq M$. We say $x \in \mathcal{U}$ is an interior point of $\mathcal{U}$ if there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{U}$.*

Definition 1.12. *We say that the set $\mathcal{U}$ is open if every $x \in \mathcal{U}$ is an interior point.*

Example 1.7.1. In any metric space, the vall $B_r(x)$ is always open.

Proof. Let $y \in B_r(x)$ and let $s = d(y,x) < r$. Let $\epsilon = r - s$. We claim that $B_\epsilon(y) \subseteq B_r(x)$.

Let $z \in B_\epsilon(y)$. Then, $d(y,z) < \epsilon$ and

$$\begin{aligned} d(x,z) &\le d(x,y) + d(y,z) \\ &< s + \epsilon \\ &= r \end{aligned}$$

This implies that $z \in B_r(x)$ and therefore,

$$B_\epsilon(y) \subseteq B_r(x).$$

□

Example 1.7.2. Which are the open sets on (M, d_{disc})? Since $B_{1/2}(x) = \{x\}$, for every set $S \subseteq M$, every point in $x \in S$ is an interior point so all subset of M are open.

Theorem 1.4.

- *Let* $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ *be an arbitrary collection of open sets. Then,* $\mathcal{U} = \bigcup_{\alpha\in I}\mathcal{U}_\alpha$ *is open.*
- *If* $\mathcal{U}_1, \ldots, \mathcal{U}_n$ *are each open, then so is their intersection.*

Proof. First, let $x \in \mathcal{U}$. Then, there exists $\alpha_0 \in I$ such that $x \in \mathcal{U}_{\alpha_0}$ where U_{α_0} is open. So there exists $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq \mathcal{U}_{\alpha_0} \subseteq \mathcal{U}$$

and x is an interior point of $\mathcal{U}$. This is true for all $x \in \mathcal{U}$ and so $\mathcal{U}$ is open.

Second, let $x \in \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n$. Then, $x \in \mathcal{U}_j$ for all $j \in 1, \ldots, n$. Then, there exists $\epsilon_j > 0$ with $B_{\epsilon_j}(x) \subseteq \mathcal{U}_j$. Let $\epsilon = \min_j \epsilon_j > 0$. Then,

$$B_\epsilon(x) \subseteq B_{\epsilon_j}(x) \subseteq \mathcal{U}_j \,\forall j.$$

□

Example 1.7.3.

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

is not open in the standard topology in $\mathbb{R}$.

Theorem 1.5. *In* $\mathbb{R}$ *with the usual norm, if* $\mathcal{U}$ *is open, then* $\mathcal{U}$ *is the disjoint union of countably many open intervals.*

Proof. Let $x \in \mathcal{U}$ and define

$$\begin{aligned} a_x &= \inf\{a : (a, x] \subset \mathcal{U}\} \\ b_x &= \inf\{b : [x, b) \subset \mathcal{U}\} \end{aligned}$$

Since $\mathcal{U}$ is open, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq \mathcal{U}$ such that $a_x < x < b_x$. Also,

$$x \in I_x = (a_x, b_x) \subseteq \mathcal{U}$$

and gives the longest interval containing x lying inside $\mathcal{U}$ (think about this; it follows fom the definition of the interval).

We claim the following: for all $y \in \mathcal{U}$, either $I_y = I_x$ or $I_y \cap I_x = \varnothing$. If $I_y \cap I_x \neq \varnothing$, then $I_x \cup I_y$ is open, contains x, contradicting the definition of $I_x = (a_x, b_x)$ and $I_x = I_y$.

Therefore, intervals $\{I_x\}$ are disjoints. To see that there are only countably many, choose a rational number $q_x \in I_x$. The distinct intervals are disjoint and so the q_x are distinct but $\mathbb{Q}$ is countable. There is no such classification in $\mathbb{R}$. □

Definition 1.13. *A set* $F \subseteq M$ *is closed if its complement* $M \setminus F$ *is open.*

Note that a set can be both open and closed or can be neither open nor closed.

Example 1.7.4. $(0, 1]$ is neither open nor closed in the standard topology.

Theorem 1.6. *If $\{F_\alpha \mid \alpha \in I\}$ is an arbitrary collection of closed sets, $\bigcap_{\alpha\in I} F_\alpha$ is closed.*

Theorem 1.7. *If $F_1, F_2, \dots, F_m$ are closed, their union is closed.*

Definition 1.14. *x is a limit point of a set $A \subseteq M$ if $\forall \epsilon > 0$, there exists $y \in A$ with $y \neq x$ and $y \in B_\epsilon(x)$.*

Lemma 1.1. *Equivalently, x is a limit point of A iff there exists a sequence $\{x_n\}$ in A with $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \to x$.*

Proof. Let $\epsilon = 1/n$, then there exists $x_n \in A$, $x_n \neq x$ such that $x_n \in B_{1/n}(x)$. In other words, $d(x_n, x) < 1/n$ for all n. This is equivalent to saying that $x_n \to x$.

Now, assume that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N$ and $x_n \neq x$. Therefore, for all $\epsilon > 0$ there exists $y \in A$ with $y \neq x$ and $y \in B_\epsilon(x)$. □

In fact, the sequence x_n may be chosen to be distinct.

Example 1.7.5. 0 is a limit point of $A = \{1/n\}_{n\in\mathbb{N}}$ but $1/n$ is not a limit point of A.

Example 1.7.6. Consider $A = (-1, 1)\cup\{2\}$. Limit points are given by $\{-1, 1\}$.

Lemma 1.2. *F is closed if and only if $B_\epsilon(x) \cap F \neq \varnothing$ for all $\epsilon > 0$ implies $x \in F$.*

Proof. F is closed iff $M \setminus F$ is open. Let $x \in M \setminus F$. Then, there exists $\epsilon > 0$ such that $B_\epsilon(x) \in M \setminus F$. In other words, $B_\epsilon(x) \cap F = \varnothing$. So if $B_\epsilon(x) \cap F \neq \varnothing$ for all $\epsilon > 0$. Then, $x \in F$. □

Theorem 1.8. *A is closed iff A contains all its limit points.*

Proof. Suppose A is closed and $x \in M \setminus A$, which is open. Then, there exists $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq M \setminus A \iff B_\epsilon(x) \cap A = \varnothing$$

Then, x is not a limit point of A. So all limit points of A are inside A.

Suppose A contains all its limit points and consider $x \in A$. Then, x is not a limit point. Then, there exists $\epsilon > 0$ such that $B_\epsilon \cap A = \varnothing$ and

$$B_\epsilon(x) \subseteq M \setminus A.$$

Therefore, x is an interior point of $M \setminus A$. This is true for all $x \in M \setminus A$ and so $M \setminus A$ is open, implying that A is closed. □

Corollary 1.1. *A is closed iff whenever $x_n \to x$ with $(x_n)_{n\in\mathbb{N}} \subseteq A$, then $x \in A$.*

Corollary 1.2. *A is closed iff*

$$B_\epsilon(x) \cap A \neq \varnothing\, \forall \epsilon > 0 \implies x \in A.$$

Definition 1.15. *For every set $A \subseteq M$,*

(i) the interior of A, A° or $\operatorname{int}(A)$ is the largest open set contained in A:

$$\operatorname{int}(A) = \bigcup \{\mathcal{U}_{open} \subseteq A\}.$$

(ii) The closure of A, $\bar{A}$ is the smallest closed set which contains A:

$$\bar{A} = \bigcap \{F \ closed |\, A \subseteq F\}$$

Proposition 1.2. *A° is a set of all interior points of A.*

Proof. Note that A° is a subset of all interior points of A. If $x \in A^\circ$, then as A° is open, there exists $\epsilon > 0$ such that

$$B_\epsilon(x) \subset A^\circ \subseteq A.$$

Now, we want to show that other direction. Since if x is an interior point then there exists $\epsilon > 0$ such that $B_\epsilon \subset A$ and this is an open set. Since A° is the largest open set, $A \subseteq A^\circ$. □

Proposition 1.3. *$x \in \bar{A}$ iff $B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$.*

Proof. Suppose $B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$. Hence, $B_\epsilon(x) \cap \bar{A} \neq \varnothing$ since $A \subseteq \bar{A}$. But $\bar{A}$ is closed so $x \in \bar{A}$.

If $x \in \bar{A}$ and let $\epsilon > 0$ be given. Assume that $B_\epsilon(x) \cap A = \varnothing$. Then, $A \subseteq (B_\epsilon(x))^c$. Because definition of $\bar{A}$, we have

$$\bar{A} \subseteq (B_\epsilon(x))^c.$$

This is a contradiction because $x \notin (B_\epsilon(x))^c$. □

Theorem 1.9. *Let A' be the set of all limit points of A. Then,*

$$\bar{A} = A \cup A'.$$

Proof. Recall that $x \in \bar{A}$ iff $B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$. Let $x \in \bar{A}$. If $x \in A$, there is nothing to prove. If $x \notin A$, we want to show that $x \in A'$.

We have $B_\epsilon(x) \cap A \neq$ for all $\epsilon > 0$. In other words, there exists $y \neq x$ such that $y \in A$ and $y \in B_\epsilon(x)$. This means that x is a limit point by definition. □

Example 1.7.7. Consider $A = (-1, 1) \cup \{2\}$. Then, $\bar{A} = [-1, 1] \cup \{2\}$.

Corollary 1.3. *$x \in \bar{A}$ iff there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x$.*

Example 1.7.8. Consider $\ell_2, \|\cdot\|_2$. Let $A = \{x \in \ell_2 : |x_{32}| \leq 1\}$.

We can easily show that the set A is not open. For example,

$$(0, 0, 0, \dots, 0, 1, 0, 0, \dots).$$

is not an interior point. For all $\epsilon > 0$, let

$$y = (0, 0, 0, \dots, 0, 1 + \epsilon/2, 0, \dots) \in \ell^2.$$

Then,

$$\|x - y\|_{\ell_2} = \frac{\epsilon}{2} < \epsilon,$$

i.e., $y \in B_\epsilon(x)$ but $y \notin A$. Moreover, we have

$$A^\circ = \{x \in \ell^2 : |x_{32}| < 1\}$$

and the proof for this is left as an exercise.

The set A is closed. Let $x^{(n)}$ be any sequence in A which converges $x^{(n)} \to x$. We must show that $x \in A$. Note that

$$|x_{32} - x_{32}^{(n)}| \leq \|x - x_{32}^{(n)}\|_{\ell_2}$$

because

$$\sqrt{\sum |x_j - x_j^{(n)}|^2} = \|x - x^{(n)}\|_{\ell_2}.$$

So

$$|x_{32} - x_{32}^{(n)}| \to 0$$

as $n \to \infty$. But $x^{(n)} \in A$ so $|x_{32}^{(n)}| \leq 1$ for all n. Hence, $|x_{32}| \leq 1$ also $x \in A$. So x is closed.

Example 1.7.9. Consider

$$\begin{aligned} B &= \{x \in \ell_2 : |x_j| \leq 1 \forall j\} \\ &= \bigcap_{j=1}^{\infty} \{x \in \ell_2 : |x_j| \leq 1\} \end{aligned}$$

This is an intersection of closed sets so B is closed.

Example 1.7.10. Prove that

$$B^\circ = \{x \in \ell^2 : |x_j| < 1 \, \forall j\}.$$

Example 1.7.11. Consider a set of truncated sequence

$$C = \{x \in \ell^2 : \exists k \in \mathbb{N} \text{ with } x_j = 0 \, \forall j \geq k\}$$

Is this set open? For $x \in C$ let

$$y = (y_1, y_2, \dots)$$

with $y_j = x_j + \frac{\epsilon^2}{2^j}$. So $y_j = \epsilon^2/2^j$ for $j \geq k$ and $y \notin C$. But

$$\|y - x\|_2 = \sqrt{\sum_{j=1}^{\infty} \left(\frac{\epsilon^4}{4^j}\right)} \leq \frac{\epsilon}{2} < \epsilon$$

so $y \in B_\epsilon(x)$ for all $\epsilon > 0$ with $y \notin C$. So x is not an interior point. And so $\text{int}(C) = \varnothing$.

Is this set closed? Let $y \in \ell_2$. Let

$$y^{(n)} = (y_1, y_2, \dots, y_n, \dots) \in C \,\forall n.$$

By practice problem 40 from page 48,

$$y^{(n)} \to y$$

in ℓ_2-norm. Hence, $y \in \bar{C}$ and $\bar{C} = \ell_2$ and so $\bar{C} \neq C$. So C is not closed. However, we say that the set C is dense in ℓ_2.

Example 1.7.12. We are given $(\ell_\infty, \|\cdot\|_\infty)$. Let

$$D = \{x \in \ell_\infty : \exists k \in \mathbb{N}, x_j = 0 \,\forall j \geq k\}.$$

By practice problem 40 from page 48, D is not dense in ℓ_∞, and so its closure is a subset of ℓ_∞ but different from ℓ_∞.

We claim that

$$\bar{D} = c_0 = \{x \in \ell_\infty : x_j \to 0 \text{ as } j \to \infty\}.$$

Suppose $x^{(n)} \in D$ and $x^{(n)} \to x$ in ℓ_∞-norm. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_j^{(n)} - x_j \leq \|x^{(n)} - x\|_\infty < \epsilon,$$

for all $n \geq N$. Since $x^{(n)} \in D$, there exists k such that $x_j^{(n)} = 0$ for all $j \geq k$, i.e.,

$$|x_j| = |x_j - x_j^{(N)}| < \epsilon.$$

This impliesi that $x_j \to 0$ in ℓ_∞-norm and $\bar{D} \subseteq c_0$.

Conversely, if $x \in c_0$, let

$$x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots) \in D.$$

Then,

$$\begin{aligned}\|x - x^{(n)}\|_\infty &= \sup_{j \geq 1} |x_j - x_j^{(n)}| \\ &= \sup_{j \geq n+1} |x_j|.\end{aligned}$$

Since $x \in c_0$, for all $\epsilon > 0$, there exists N such that for all $j \geq N$, $|x_j| < \epsilon$. Thus, for all $n \geq N$,

$$\|x - x^{(n)}\|_\infty < \epsilon,$$

i.e., $x^{(n)} \to x$ in ℓ_∞-norm.

Example 1.7.13. Consider a space of continuous functions $N = C([0,1])$ equipped with a sup-norm. Then,

$$\begin{aligned} B_2(f) &= \{g \in C([0,1]) \,:\, \|g - f\|_\infty < \epsilon\} \\ &= \{g \in C([0,1]) \,:\, f(x) - \epsilon < g(x) < f(x) + \epsilon \,\forall x \in [0,1].\} \end{aligned}$$

Example 1.7.14. Consider a space of continuous functions $N = C([0,1])$ equipped with a sup-norm. Then,

$$\begin{aligned} B_2(f) &= \{g \in C([0,1]) \,:\, \|g - f\|_\infty < \epsilon\} \\ &= \{g \in C([0,1]) \,:\, f(x) - \epsilon < g(x) < f(x) + \epsilon \,\forall x \in [0,1].\} \end{aligned}$$

Then, consider

$$A = \{f \in N \,:\, 0 < f(x) < 1, \forall x \in [0,1]\}.$$

This set is open. Let $f \in A$. Since $[0,1]$ is closed and bounded, f attains its maximum and minimum. Let

$$\begin{aligned} m_1 &= \min_{[0,1]} f(x) = f(a) > 0 \\ M_2 &= \max_{[0,1]} f(x) = f(b) < 1 \end{aligned}$$

This implies that $0 < m_1 \leq f(x) \leq M_2 < 1$ for all $x \in [0,1]$.

Let $\epsilon < \min\{m_1, 1 - M_2\}$.Then, if $g \in B_\epsilon(f)$, we have

$$0 < m_1 - \epsilon \leq f(x) - \epsilon < g(x) < f(x) + \epsilon \leq M_2 + \epsilon < 1.$$

Thus, $g \in A$. So $B_\epsilon(f) \subseteq A$ and f is an interior point. Then, for all $f \in A$, f is an interior point and A is open.

On the other hand,

$$\bar{A} = \{f \in C([0,1]) \,:\, 0 \leq f(x) \leq 1\}.$$

Note that if $f_n \in A$, $f_n \to f$ then $0 \leq f(x) \leq 1$. To verify, if $0 \leq f(x) \leq 1$, let $f_n(x) \in A$ such that

$$f_n(x) = \begin{cases} 1 - \frac{1}{n} & \text{if } f(x) > 1 - \frac{1}{n} \\ f(x) & \text{if } \frac{1}{n} \leq f(x) \leq 1 - \frac{1}{n} \\ \frac{1}{n} & \text{if } f(x) \leq \frac{1}{n} \end{cases}$$

Then, $f_n \in A$ for all n. In particular,

$$\|f - f_n\| = \frac{1}{n} \to 0,$$

i.e.,

$$f_n \to f \in (C([0,1]), \|\cdot\|_\infty).$$

So $f \in A'$, i.e., $\bar{A} \subseteq A'$. So $\bar{A}$ is indeed the clousre of A.

Example 1.7.15. Consider

$$B = \{f \in C[0,1] \,:\, 0 \le f(x) \le x\}$$

the set. This set is closed. What about its interior?

Definition 1.16. *Let $A \subseteq M$ in a metric space (M,d). We say that x is a boundary point of A,*

$$x \in \partial A = \text{bound}(A)$$

if every neighborhood of x intersects with both A and A^c. In other words, for all $\epsilon > 0$, there exists $z \in A$ and $y \in A^c$ with $y, z \in B_\epsilon(x)$.

Remark. $\partial A = \bar{A} \cap \bar{A}^c = \bar{A} \setminus A^\circ$.

Example 1.7.16. For the above example, we have

$$\partial A = \{f \in C[0,1] \,:\, \exists x \in [0,1] \text{ such that } f(x) = 0 \text{ or } f(x) = 1\}$$

Let

$$f_\epsilon(x) = \begin{cases} 1 - \frac{\epsilon}{4} & f(x) > 1 - \frac{\epsilon}{4} \\ f(x) & \frac{\epsilon}{4} \le f(x) \le 1 - \frac{\epsilon}{4} \\ \frac{\epsilon}{4} & f(x) < \frac{\epsilon}{4} \end{cases}$$

Then, $\|f - f_\epsilon\|_\infty = \epsilon/4 < \epsilon$, i.e., $f_\epsilon \in A$ with $f_\epsilon \in B_\epsilon(f)$. Furthermore,

$$B_\epsilon(f) \cap A^c \neq \varnothing, B_\epsilon(f) \cap A \neq \varnothing$$

1.8 Continuity

Definition 1.17. *Suppose (M,d) and (N,ρ) are both metric spaces and $f : M \to N$ is a function. We say f is continuous at $x \in M$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in M$ iwth $d(x,y) < \delta$, we have $\rho(f(x), f(y)) < \epsilon$. If f is continuous at every point $x \in M$, we say f is continuous in M.*

Theorem 1.10. *The following are equivalent for any $f : (M,d) \to (N,\rho)$:*

1. *f is continuous on M*
2. *if $x_n \to x$ in (M,d) then $f(x_n) \to f(x)$ is (N,ρ)*
3. *whenever $E \subseteq N$ is closed, $f^{-1}(E)$ is closed in M*
4. *whenever $V \subseteq N$ is open, $f^{-1}(V)$ is open.*

Proof. ($1 \implies 2$) For all $\epsilon > 0$, let $\delta > 0$ be such that $\rho(f(x), f(y)) < \epsilon$ for all $x, y \in M$ with $d(x,y) < \delta$. Then, let $N_0 \in \mathbb{N}$ with $d(x, x_n) < \delta$ for all $n \ge N$. So for all $n \ge N_0$, $\rho(f(x), f(x_n)) < \epsilon$.

($2 \implies 3$) Suppose $E \subseteq N$ is closed and $(x_n)_{n\in\mathbb{N}} \subset f^{-1}(E)$ and $x_n \to x \in M$ (to show that $x \in f^{-1}(E)$). By (2), $f(x_n) \to f(x)$. Since E is closed, $f(x) \in E$, i.e., $x \in f^{-1}(E)$.

(3 $\implies$ 4) Let

$$\begin{aligned} f^{-1}(A^c) &= \{x \in M \mid f(x) \in A^c\} \\ &= \{x \in M \mid f(x) \in A\}^c \\ &= \big(f^{-1}(A)\big)^c \end{aligned}$$

If $V \subseteq N$ is open, V^c is closed so $f^{-1}(V^c)$ is closed, implying that

$$f^{-1}(V) = \big(f^{-1}(V)\big)^c$$

is open.

(4 $\implies$ 1) Let $\epsilon > 0$ and $B_\epsilon(f(x)) \subseteq N$. Then, $f^{-1}(V)$ is open in N, containing x, implying that there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(V)$. Then, for all $y \in B_\delta(x)$,

$$f(y) \in B_\epsilon(f(x)),$$

i.e., $d(y,x) < \delta$, implying that $\rho(f(x), f(y)) < \epsilon$. □

Example 1.8.1. Let M be any set with d be discrete metric. Let N be any set and ρ be any metric. Consider a function $f : M \to N$. Then, for $V \subseteq N$

$$f^{-1}(V) = \{x \in M \mid f(x) \in V\}$$

is open in M. So any function f is continuous.

Remark. If $\mathcal{U} \subseteq M$ is open,

$$f(\mathcal{U}) = \{y \in N \,:\, y = f(x), x \in \mathcal{U}\}$$

may not be open.

Example 1.8.2. Consider $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 5$.

Remark. If V is closed, $f(V)$ may not be closed.

Example 1.8.3. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = e^{-x}$ for $V = [0, \infty)$. Then, $f(V) = (0, 1]$.

Remark. Suppose $f(x) = x$ on $A = [-1, 1]$, i.e.,

$$f : [-1, 1] \to \mathbb{R}.$$

Then, f is continuous on A but

$$\begin{aligned} f^{-1} &= \{x \in A = [-1,1] \,:\, f(x) \in (0,2)\} \\ &= (0, 1]. \end{aligned}$$

Note that $(0, 1]$ is open relative to A (subset topology).

Definition 1.18. *Let $A \subset M$ be a metric space. We say that $\mathcal{U} \subseteq A$ is open relative to A if there exists an open set $V \subset M$ with*

$$\mathcal{U} = A \cap V.$$

Similarly, $W \subseteq A$ is closed relative to A if there exists a closed set $Z \subset M$ with $W = Z \cap A$. The theore on continuity holds in subsets of M, using the concept of relative topology.

Theorem 1.11. *Let $f : A \subseteq M \to N$. f is continuous on A if and only if for all open set $S \subseteq N$, $f^{-1}(S)$ is open relative to A, if and only if for all closed set $T \subseteq N$, $f^{-1}(T)$ is closed relative to A.*

Example 1.8.4. Consider a function $f : \ell_2 \to \mathbb{R}$ where $f(x) = x_{32}$.

we claim that f is continuous. Again, use $|x_{32}| \leq \|x\|_2$. So if $x, y \in \ell_2$,

$$\rho(f(x), f(y)) = |f(x) - f(y)| = |x_{32} - y_{32}| \leq \|x - y\|_{32} = d(x, y).$$

So for all $\epsilon > 0$, let $\delta = \epsilon$. Then, $d(x, y) < \delta$ implies that $\rho(f(x), f(y)) < \epsilon$.

Note that f is Lipschitz continuous.

Consider

$$\begin{aligned} A &= \{x \in \ell_2 \,:\, |x|_{32} \leq 1\} \\ &= \{x \in \ell_2 \,:\, -1 \leq f(x) \leq 1\} \\ &= f^{-1}([-1, 1]) \end{aligned}$$

Since $[-1, 1]$ is closed and f is continuousm $f^{-1}([-1, 1]) = A$ is closed.

Example 1.8.5. Suppose f is continuous with $f(0) > 0$. Then does there exist a such that $f(x) > 0$ for all $x \in (-a, a)$?

Let $\epsilon = f(a)/2$ and consider $\theta = (f(0) - \epsilon, f(0) + \epsilon)$. Since f is continuous, $f^{-1}(\theta)$ is open and contains 0. Therefore, there exists $B_a(0) \subset f^{-1}(\theta)$ such that $f(x) \in \theta$, meaning that

$$f(x) \geq \frac{f(0)}{2} > 0.$$

1.9 Homeomorphism and Isometry

If $f, g : (M, d) \to \mathbb{R}$ are continuous, so are cf, $f + g$, fg. So $C(M, \mathbb{R})$, set of continuous functions, is an algebra, a vector space but also closed under multiplication. We'd like to define a norm on $C(M, \mathbb{R})$, such as the sup-norm. But is $\sup |f(x)| < \infty$?

We destinguish two special kinds of facts for such function:

$$f : (M, d) \to (N, \rho).$$

Definition 1.19. *f is a homeomorphism if f is continuous, one-to-one, continuous and f^{-1} is continuous.*

If f is a homeomorphism, the metric space (N,ρ) is essentially the same as (M,d) for some open sets, consequently, homeomorphism is equivalent to topological equivalent.

Definition 1.20. *f is an isometry if $\rho(f(x),f(y)) = d(x,y)$, i.e., distances are preserved. The geometry of M and $f(M)$ are the same.*

1.10 Completeness

Definition 1.21. *We say $(X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (M,d) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ with*

$$d(x_n, x_m) < \epsilon\, \forall m, n \geq N.$$

Remark. In $\mathbb{R}$ a sequence is convergent if and only if it is Cauchy.

This property was the completeness of $\mathbb{R}$. In any metric space, convergence implies Cauchy: since $x_m \to x$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

for all $n, m > N$.

Definition 1.22 (Completeness)**.** *We call (M,d) complete if every Cauchy sequence is convergent.*

Example 1.10.1. $\mathbb{R}^k$ is complete. If $x_m = (x_m^1, x_m^2, \ldots, x_m^k)$ is a Cauchy sequence in $(\mathbb{R}, \text{usual Euclidean})$. But $|x_n^i - x_m^i| \leq \|x_n - x_m\| < \epsilon$ for all $n, m \geq N$. Hence, each coordinate sequences $(x_n^i)_{n\in\mathbb{N}}$, $i = 1, \ldots, k$ is Cauchy. So $x_n^i \to x^i$ for some $x^i \in \mathbb{R}$. So candidate for the limit $(x_n)_{n\in\mathbb{N}}$ is $(x^1, \ldots, x^k) \in \mathbb{R}^k$. It is easy to show that

$$\|x_n - x\| \to 0.$$

Example 1.10.2. $(M, \text{discrete})$ is complete. Cauchy sequences are eventually convergent.

Example 1.10.3. Consider $M = C([-1,1])$ with

$$d(f,g) = \int_{-1}^{1} |f(x) - g(x)| dx.$$

For example, take

$$f_n(x) = \begin{cases} 1 & x \geq 1/n \\ \text{linear} & -1/n \leq x \leq 1/n \\ -1 & x \leq -1/n \end{cases}$$

Then,

$$d(f_n, f_m) = \left| \frac{1}{m} - \frac{1}{n} \right|$$

is Cauchy. The limit is discontinuous. So this is not convergent because convergence requires the limit to be in the set. So this space is incomplete.

Example 1.10.4. ℓ_1, ℓ_2, c_0 and ℓ_∞ are complete. Let's do

$$\ell_2 = \{x : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$$

Take a sequence $X_n = (X_n(1), X_n(2), \dots)$, which is Cauchy in ℓ_2. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|X_n - X_m\|_2 < \epsilon/2$ for all $m, n \geq N$. First, note that

$$\begin{aligned} |X_n(k) - X_m(k)| &= \sqrt{|X_n(k) - X_m(k)|^2} \\ &\leq \|X_n - X_m\|_2 \\ &< \frac{\epsilon}{2}. \end{aligned}$$

So for fixed $k \in \mathbb{N}$, $(X_n(k))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. So there exists

$$X(k) = \lim_{n\to\infty} X_n(k) \, \forall k \in \mathbb{N}$$

Our candidate ofr the limit is

$$X = (X(1), X(2), \dots).$$

To show, we have to show that $X_n \to X$ in ℓ_2.

Verify that $X \in \ell_2$, for fixed $k \in \mathbb{N}$:

$$\sum_{k=1}^{K} (x(k))^2 = \lim_{n\to\infty} \sum_{k=1}^{K} (X_n(k))^2 \leq \|X_n\|^2$$

Since (X_n) is Cauchy, it is bounded (from 3A03). Then,

$$\sum_{k=1}^{K} (x(k))^2 \leq M$$

for any $K \in \mathbb{N}$. Take $K \to \infty$ then

$$\|x\|_{\ell^2}^2 \leq M < \infty \implies x \in \ell_2$$

Next, verify that $X_n \to X$ in ℓ_2-norm. For all K fixed,

$$\begin{aligned} \sum_{k=1}^{K} |X(k) - X_n(k)|^2 &= \lim_{n\to\infty} \sum_{k=1}^{K} |X_m(k) - X_n(k)|^2 \\ &\leq \lim_{m\to\infty} \|X_m - X_n\|_{\ell_2}^2 < \frac{\epsilon^2}{4} \end{aligned}$$

for all $n \geq N$ and all $k \in \mathbb{N}$. Let $K \to \infty$,

$$\|X - X_m\|_{\ell_2}^2 < \frac{\epsilon^2}{4}$$

for all $n \geq N$. So the convergence has been proved.

Theorem 1.12. *If (M, d) is a complete metric space and $A \subseteq M$ is closed, then (A, d) is itself a complete metric space.*

Corollary 1.4. *c_0 is a complete metric space.*

1.11 Compactness

Theorem 1.13 (Bolzano-Weierstrass Theorem)**.** *In $\mathbb{R}$ with usual metric, any bounded sequence contains a convergent subsequence.*

Example 1.11.1. Consider $M = \ell_\infty$. Let

$$x_n = (0, 0, \dots, 1, 0, \dots)$$

Then, $d(x_n, x_m) = 2$ for all $n \neq m$. So there rae no Cauchy subsequences.

Example 1.11.2. Consider $\mathbb{R}^n$ with Euclidean metric. If $X_k \in \mathbb{R}^n$, then $X_k = (X_k^1, \dots, X_k^n)$ is bounded by M and

$$|X_k^j| \leq \|X_k\| \leq M$$

for all $k \in \mathbb{N}$ and for all $j = 1, 2, \dots, n$. So $(X_k^j)_{k \in \mathbb{N}}$ is bounded in $\mathbb{R}$ for all j. Then, thre exists a subsequence $k_{1,m}$ with

$$X^1_{k_{1,m}} \to x^1.$$

and there exists a subsequence $k_{2,m}$ of $k_{1,m}$ such that

$$x^2_{k_{2,m}} \to x^2$$

and

$$x^1_{k_{2,m}} \to x^1.$$

We continue. Then, there exists a subsequence $k_{n,m}$ with the desired property:

$$X^j_{k_{n,m}} \to X^j$$

for all j. So

$$X_{k_{n,m}} \to x$$

in Euclidean.

We will restrict to special subset of (M, d) to get this.

Definition 1.23. *$K \subseteq M$ is called sequentially compact if every sequence in K contains a subsequence which converges to a point in K. If this holds for $K = M$, we call M a sequentially compact metric space.*

So we've just shown that a closed, bounded subset of $\mathbb{R}^n$ is sequentially compact. If K is closed and bounded in $\mathbb{R}$, let $(x_k)_{k \in \mathbb{N}}$ be any sequence in K, then $(x_k)_{k \in \mathbb{N}}$ bounded implies that there exists a convergent subsequence $x_{k_m} \to x$ where $x \in M$. If K is closed, then $x \in K$.

Theorem 1.14 (Heine-Borel theorem)**.** *In $\mathbb{R}^n$, a set is sequentially compact iff it is closed and bounded.*

Proposition 1.4. *In any metric space, if K is sequentially compact, then K is closed and bounded.*

Proof. First, if $(x_n)_{n\in\mathbb{N}}$ is convergent, $x_n \to x$ with $x_n \in K$ for all n, K is sequentially compact implies that there is a subsequence that converges to $x_* \in K$. Then, the whole sequence converges to $x = x_*$. This holds for any convergent sequences so K is closed.

Suppose K is unbounded. Let $x_1 \in K$. Then, there exists $x_2 \in K$ such that $d(x_1, x_2) \geq 1$. Also, there exists $x_2 \in K$ such htat $d(x_1, x_3) \geq 1 + d(x_1, x_2)$. Then,

$$d(x_2, x_3) \geq d(x_1, d_3) - d(x_1, x_2) \geq 1.$$

Then, there exists $x_4 \in K$ such that $d(x_1, x_4) \geq d(x_1, x_3) + 1$. Then,

$$\begin{aligned} d(x_2, x_4) &\geq d(x_1, x_4) - d(x_1, x_2) \\ &\geq d(x_1, x_3) + 1 - d(x_1, x_2) \\ &\geq 1 + d(x_1, x_2) + 1 - d(x_1, x_2) \\ &= 2 \end{aligned}$$

implying that $d(x_3, x_4) \geq 1$. We can continue this and get a sequence such that $d(x_n, x_m) \geq 1$ for all $n \neq m$, i.e., with no Cauchy subsequence and hence no convergent subsequence. This is a contradiction and K must be bounded. □

Definition 1.24. *An open cover of a set $A \subseteq M$ is a collection $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of open sets U_α with*

$$A \subseteq \bigcup_{\alpha\in I} U_\alpha$$

Example 1.11.3. If A is open, $\{\mathcal{U} = A\}$ is an open cover.

Example 1.11.4. Taking $I = A$, $\{\mathcal{U}_\alpha = B_\epsilon(\alpha)\}_{\alpha\in A}$ is an open cover for any $\epsilon > 0$.

Definition 1.25. *We say $K \subseteq M$ is compact if every open cover of K contains a finite subcover, i.e., if $K \subseteq \bigcup \mathcal{U}_\alpha$, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_N$ with*

$$K \subseteq \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_N}$$

Note that the subcover is a finite colllection from the original collection.

Example 1.11.5. Consider $M = \mathbb{R}$ equipped with the usual metric. Consider $A = (0, 1]$. Let

$$\mathcal{U}_\alpha = \left(\alpha - \frac{1}{10}, \alpha + \frac{1}{10}\right),$$

where $\alpha \in (0, 1]$, is an open cover. Then, $\mathcal{U}_{1/10}, \mathcal{U}_{2/10}, \ldots, \mathcal{U}_1$ is a finite subcover.

However, A is not compact. Consider $V_\alpha = (\alpha/2, 2)$. Then, this doesn't have a finite subcoer. Let

$$(V_{\alpha_1}, \ldots, V_{\alpha_N})$$

be any finite subcover. Then,

$$\inf (V_{\alpha_1} \cup \dots \cup V_{\alpha_N}) = \frac{\alpha_1}{2} > 0.$$

Then, $(0, \alpha_1/2]$ is not conatined within the collection so $(0, 1]$ is not compact.

Theorem 1.15. *In any (M, d), K is compact iff K is sequentially compact.*

Proof. First, assume that K is (covering) compact. (Lemma 1) if $F \subseteq K$ is closed, K is closed then F is compact. Let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be any open cover of F. Then, $\{\mathcal{U}_\alpha\}_{\alpha \in I} \cup \{F^c\}$ is an open cover of K. Then, K is compact implies that

$$\mathcal{U}_{\alpha_1} \cup \dots \cup \mathcal{U}_{\alpha_n} \cup F^c$$

covers K and

$$\mathcal{U}_{\alpha_1} \cup \dots \cup \mathcal{U}_{\alpha_n}$$

covers K.

Now assume (X_n) is a sequence in K with no convergent subsequences. Let $F = \{X_n : n \in \mathbb{N}\}$. Then, F has infinitely many elements and no limit points. So F is closed and F is compact.

Since F has no limit point, for all $y_n \in F$, there exists $\epsilon_n > 0$ such tht $B_{\epsilon_n}(y_n)$ contains F other than y_n itself. $\{B_{\epsilon_m}(y_m)\}$ is an open cover that has finite subcover. This is a contradiction. □

Definition 1.26. *We say a set A is totally bounded if $\forall \epsilon > 0$, there exists finite colection $X_1, \dots, X_n \in A$ with*

$$A \subseteq B_\epsilon(x_1) \cup \dots \cup B_\epsilon(x_N)$$

This is like compactness but for the special class of open cover by balls of fixed radius.

Remark. We proved that if A is totally bounded then any sequence must have a convergent subsequence. If A is closed, then A is sequentially compact.

Proposition 1.5. *Assume K is sequentially compact. Then, K is totally bounded.*

Proof. Suppose no. Then, there exists $\epsilon > 0$ for which a finite collection of balls can not cover K. Take $y_1 \in K$ then $B_\epsilon(y_1)$ does not cover K. In other words, there exists $y_2 \in K \setminus B_\epsilon(y_1)$. So $d(y_2, y_1) \geq \epsilon$. Now, $\{B_\epsilon(y_1), B_\epsilon(y_2)\}$ does not cover K. So we can define a ball around y_3 again.

Continued, we get a sequence $(y_n)_{n \in \mathbb{N}} \subset K$, where

$$d(y_n, y_m) \geq \epsilon$$

with no convergent subsequences. □

We are ready to show that K sequentially compact implies K covering compact. Assume K is sequentially compact and $\{\mathcal{U}_\alpha\}$ is an open cover.

Lemma 1.3. *There exists $r > 0$ such that for all $y \in K$, there exists $\alpha \in I$ with $B_r(y) \subseteq \mathcal{U}_\alpha$.*

Proof. If not, then for all $r = 1/n$, there exists $y_n \in K$ such that $B_{y_n}(y_n)$ is not contained in any $\mathcal{U}_\alpha$. K sequentially compact implies that there exists subsequence $y_{m_j} \to y_0 \in K$ but $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ covers K so there exists α_0 with $y_0 \in \mathcal{U}_{\alpha_0}$. Choose $\epsilon > 0$ with $B_\epsilon(y_0) \subseteq \mathcal{U}_{\alpha_0}$ but $y_{m_j} \to y_0$. So there exists $\tau \in \mathbb{N}$ such that $y_{m_j} \in B_{\epsilon/2}(y_0)$ for all $j \geq \tau$. As $d(y_m, y) < \epsilon/2$ and as for j large enough, $1/n < \epsilon/2$. Therefore,

$$B_{1/m_j}(y_{m_j}) \subseteq B_\epsilon(y_0) \subseteq \mathcal{U}_{\alpha_0}$$

This is a contradiction. $\square$

To complete the proof of the theorem, let $r > 0$ be as in the above lemma and apply the proposition: there exists $y_1, \ldots, y_N \in K$ such that

$$K \subseteq B_r(y_1) \cup \cdots \cup B_r(y_N)$$

By the above lemma, $\forall j = 1, \ldots, N$, there exists α_j such that

$$B_r(y) \subseteq \mathcal{U}_{\alpha_j} \implies K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}$$

So we have shown that

1. If K is compact then K is sequentially compact
2. If K is sequentially compact then K is totally bounded
3. If K is sequentially compact and totally bounded then K is compact

Example 1.11.6. Show that

$$\{x \in \ell_2 \,|\, |x(k)| \leq 1/k\}$$

is compact.

Theorem 1.16. *If $f : M \to N$ is continuous and $K \subseteq M$ is compact, then $f(K)$ is compact in N.*

Proof. Let $\{\mathcal{U}_\alpha\}$ be an open cover of $f(K)$,

$$f(K) \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha,$$

i.e., for all $x \in K$, there exists $\alpha_x \in i$ such that $f(x) \in \mathcal{U}_{\alpha_x}$ iff

$$x \in f^{-1}(\mathcal{U}_{\alpha_x})$$

for some $\alpha_x \in I$. Then,

$$K \subseteq \bigcup_{\alpha \in I} f^{-1}(\mathcal{U}_\alpha)$$

is an open cover of K. So K is compact implies that there exists a finite subcover of K. So

$$K \subseteq f^{-1}(\mathcal{U}_{\alpha_1}) \cup \cdots \cup f^{-1}(\mathcal{U}_{\alpha_N})$$

So for all $x \in K$, $x \in f^{-1}(\mathcal{U}_{\alpha_j})$ for some j iff $f(x) \in \mathcal{U}_{\alpha_j}$ iff

$$f(K) \subseteq \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_N},$$

a finite subcover. □

Corollary 1.5. *If f is continuous and K is compact, then $f(K)$ is bounded. In particular, if $f : K \subseteq M \to \mathbb{R}$ is continuous, there exists $R > 0$ such that $|f(x)| \leq R$ for all $x \in K$.*

Theorem 1.17. *Suppose $f : M \to \mathbb{R}$ is continuous, $K \subseteq M$ comapct. Then, f attains its maximum and minimum value on K, i.e., there exists x_* and x^* in K such that $f(x_*) = \inf f(x)$ and $f(x^*) = \sup f(x)$*

Proof. Let $S = \sup f(x) < \infty$ by corollary. By definition of supremum, there exists a sequence $x_n \in K$, $f(x_n) \to S$. Since K ois compact, there exists a subseeunce and $x^* \in K$ such taht $x_{n_k} \to x^*$. Since f is continuous, $f(x_{n_k}) \to f(x^*) = S$. □

Recall the definition of uniform continuity.

Definition 1.27. *f is uniformly continuous on $S \subseteq M$ if $\forall \epsilon > 0$, there exists $\delta > 0$s uch that for all $x, y \in S$, $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$*

Theorem 1.18. *Let $K \subseteq M$ be compact. Then, if f is continuous on K, f is uniformly continuos on K*

Example 1.11.7. Consider

$$K = \{x \in \ell_2 \,|\, |x(k)| \leq 1/k\}$$

This set is compact. Show that this set is sequentially compact and show that is closed and totally bounded.

Note that for $x \in K$,

$$\|x\|_{\ell_2}^2 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

For all $\epsilon > 0$ there exists N such that

$$\sum_{N+1}^{\infty} \frac{1}{k^2} < \epsilon.$$

Let $g_N : \mathbb{R}^N \to \ell_2$. This is continuous. Note that

$$A_N = \{x \in \mathbb{R}^N : |x_n| \leq \frac{1}{n}, n = 1, \ldots, N\}$$

is compact. Then, $g_N(A_n) = k_N$ is compact. For $x \in K$, denote

$$x^{(N)} = (x(1), x(2), \dots, x(N), 0, 0, \dots)$$

so that $x^{(N)} \in k_N \subseteq \ell_2$ is compact. Then, there exists $x_1, \dots, x_m \in K_N$ such that

$$k_N \subseteq \bigcup_{j=1}^{m} B_{\epsilon/2}(x_j).$$

Then, $d(x, x^{(N)}) < \epsilon/2$. So

$$K \subseteq \bigcup_{j=1}^{m} B_{\epsilon}(x_j)$$

and K is totally bounded.

On the other hand, we can use a diagonal argument.

Theorem 1.19. *Let $K \subseteq M$ be compact. Then, if f is continuous on K then f is uniformly continuous on K*

Proof. Recall the definition of uniform continuity: for all $\epsilon > 0$, there exists $\delta > 0$ so that $\forall x, y,\ inS$ such that $d(x,y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$. For any $\epsilon > 0$ and $\forall x \in K$, there exists $\delta(x) > 0$ so that if $y \in K$, $d(x,y) < \delta(x)$ then $\rho(f(x), f(y)) < \epsilon/2$. Consider $\mathcal{U}_x = B_{\frac{\delta(x)}{2}}(x)$, where $x \in K$. Then $\{\mathcal{U}_x\}_{x \in K}$ is an open cover of K. Because K is compact, there exists a finite subcover

$$K \subseteq \mathcal{U}_{x_1} \cup \cdots \cup \mathcal{U}_{x_N}.$$

Let $\delta = \frac{1}{2}(\delta(x_1), \dots, \delta(x_N)) > 0$ as there are finitely many terms. Let $x, y \in K$ such that $d(x,y) \in \delta$. Since $x \in K$, we have

$$x \in \mathcal{U}_x = B_{\frac{\delta(x)}{2}}(x_j)$$

for some $j \in \{1, 2, \dots, N\}$. Since $d(x,y) < \delta < \frac{1}{2}\delta(x_j)$, then $y \in B_{\delta(x_j)}(x_j)$. Since

$$\begin{aligned} d(y, x_j) &\leq d(y, x) + d(x, x_j) \\ &\leq \delta + \frac{\delta(x_j)}{2} \\ &\leq \frac{\delta(x_j)}{2} + \frac{\delta(x_j)}{2} \\ &= \delta(x_j). \end{aligned}$$

Therefore,

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_j)) + \rho(f(x_j), f(y)) \leq \epsilon.$$

$\square$

2 Space of continuous functions

2.1 Sets of functions

The simplest set of function is $C(X, y)$, the class of continuous functions $f : (X, d) \to (Y, \rho)$. First, we recall some important notion of convergence.

Definition 2.1. *Let $f_n : X \to Y$ be any sequence of functions and $f : X \to Y$. We say that $f_n \to f$ pointwise on X if $f_n(x) \to f(x)$ for each fixed $x \in X$, i.e., for all $x \in X$, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ with $\rho(f(x), f_n(x)) < \epsilon$ for all $n \geq N$.*

Example 2.1.1. Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} -1 & x \leq -\frac{1}{n} \\ nx & -\frac{1}{n} < x < \frac{1}{n} \\ 1 & x \geq \frac{1}{n}. \end{cases}$$

Then,

$$f_n(x) \to f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases}$$

Note that f_n are continuous but the limit is not.

We see that pointwise limit doesn't give a good notion of convergence. The proper convergence in uniform convergence, derived from supremum norm.

Definition 2.2. *$f_n \to f$ uniformly on X if $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \epsilon$ for all $n \geq N$ and $\forall x \in X$.*

Theorem 2.1. *If $(f_n)_{n \in \mathbb{N}} \subseteq C(M, N)$ and $f_n \to f$ uniformly on M then $f \in C(M, N)$.*

Proof. Let $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that $\rho(f_m(x), f(x)) < \epsilon/3$ for all $x \in M$ (by uniform convergence). Since f_n is continuous, for all $x \in M$, there exists $\delta(x) > 0$ such that if $y \in M$ then $d(x, y) < \delta$ implies $\rho(f_m(x), f_m(y)) < \epsilon/3$. By the triangle inequality,

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f_m(x)) + \rho(f_m(x), f_m(y)) + \rho(f_m(y), f(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

□

So pointwise does not preserve continuity but uniform convergence does. So the notion of uniform convergence is a good metric for (M, N).

Now, let's get back to $(C([-1,1]), \|\cdot\|_\infty)$. We claim that this is complete. To show this, take a Cauchy sequence and show that in converges in general in our space.

Consider $\|f_m - f_n\|_\infty < \epsilon$ for all $n, m > N \in \mathbb{N}$. Then,

$$\sup_{x\in[-1,1]} \|f_m - f_n\|_\infty < \epsilon.$$

So for all $x \in [-1,1]$, $|f_m - f_n| < \epsilon$. Given a fixed $x_0 \in [-1,1]$, $|f_m(x_0) - f_n(x_0)| < \epsilon$ is Cauchy, i.e., $\{f_n(x_0)\}_{n\in\mathbb{N}}$ is Cauchy. Since $\mathbb{R}$ is complete,

$$f_n(x_0) \to f(x_0)$$

for some $f(x_0) \in \mathbb{R}$. Hence, our sequence $f_n(x_0) \to f(x_0)$ converges pointise.

We now need to show that $f(x)$ is continuous and that $f_n \to f$ in $\|\cdot\|_\infty$. Since $\forall x \in [-1,1]$, we have

$$|f_n(x) - f_m(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)|$$

By our definition then, $|f_n(x) - f(x)| < \epsilon$ for all x and for all $n \geq N$, since $\{f_n\}$ is Cauchy. So

$$\sup_{x\in[-1,1]} |f_n(x) - f(x)| < \epsilon,$$

for all $n \geq N$. So $f_n \to f$ in sup-norm.

Since $f_n \to f$ in sup-norm, it's uniform. By our previous theorem, f is continuous.

Remark. We can generalize to see that $(C([a,b]), \|\cdot\|_\infty)$ is complete.

The question is whether we can generalize function so that we can have $C(M,N)$¿

We have $f_n \to f$ uniformly if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sup \rho(f_n, f) < \epsilon$$

for all $n \geq N$. So this could define a distance on $C(M,N)$ but we have to be careful.

Example 2.1.2. Consider $f, g : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$ and $g(x) = 0$. Then,

$$\rho(f(x), g(x)) = |x| = \infty.$$

So we don't have finite distance.

Hence, we need to just have our functions to be (1) bounded or (2) our space has to be compact.

First, consider (M,d) to be a compact space. Then, $f :\to (N,\rho)$ is bounded if f is continuous, For example, for $N = \mathbb{R}$, we may define $\|f\|_\infty = \sup |f(x)| < \infty$ when M is compact.

Second, we can restrict to $C_b(M,N)$.

Theorem 2.2.

- *The space of bounded continuous functions is complete with* $\|\cdot\|_\infty$ *provided that* N *is complete.*
- $C(M,N)$ *is complete under the* sup-*norm provided that* M *is compact and* N *is complete.*

Example 2.1.3. Consider

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

Note that $f_n(x)$ is maximized when $x = 1/n$ and we have $f_n(x) = 1/2$. So

$$f_n(x) = \frac{n}{n^2}\frac{x}{(x^2+1/n^2)}$$

and converges pointwise to 0. So $f_n \to 0$.

If f_n converges to a pointwise limit, does it also converge uniformly? Note that uniform convergence implies pointwise convergence (prove this) but not necessarily otherwise.

If we take $(0,1]$, then $f_n \to 0$ So consider $[\delta, 1]$ for $\delta > 0$. Note that

$$f_N(\delta) = \frac{N\delta}{1+N^2\delta^2} = \frac{1}{N}\frac{\delta}{\delta+1/N^2}$$

So choose N such that $f_n(\delta) < \epsilon$ for all $n \geq N$. Since $f_N(\delta)$ is max, we have uniform convergence on $[\delta, 1]$.

Example 2.1.4. Consider

$$f_n(x) = \frac{nx}{1+nx}.$$

We can rewrite it as

$$f_n(x) = \frac{n}{n}\left(\frac{x}{x+1/n}\right)$$

and so $f_n \to 0$ for $x = 0$ but $f_n \to 1$ for any other x. Since our limit is not continuous, we don't have uniform convergence on $[0,1]$. However, we do have uniform convergence on $[\delta, 1]$ for all $\delta > 0$.

We need to show that $\|f_n - 1\|_\infty < \epsilon$ for all $n \geq N$. Since f_n is increasing, difference is biggest at δ. So pick N so that $|f_N(\delta) - 1| < \epsilon$ and we have uniform convergence for all $x \in [\delta, 1]$.

We have two important consequences of uniform convergence.

Theorem 2.3. *Suppose* f_n *are continuous on* $[a,b] \subseteq \mathbb{R}$ *and* $f_n \to f$ *uniformly. Then,*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof.

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \leq |b-a|\sup|f_n - f|$$
$$< |b-a|\left(\frac{\epsilon}{|b-a|}\right)$$

Therefore,

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| < \epsilon.$$

□

Example 2.1.5. Pointwise limit is not enough for the above. Consider a function that looks like a triangle which takes a value of 2_n at $x = 2^{-n}$ and there are straight lines from this point to $x = 0$ and $x = 2^{-(n-1)}$. Then,

$$\int_0^1 f_n(x)dx = \frac{1}{2}2^n 2^{n-1} = 1$$

for all n. But the pointwise limit is $f(x) = 0$ and $\int 0 \neq 1$. So we don't have uniform convergence and have a problem.

2.2 Integration and differentiation

We need to review Riemann integral:

- If $f : [a,b] \to \mathbb{R}$ is continuous, f is Rieman integrable also if f is piecewise continuous and bounded
- If f, g are Riemann integrable, $f \leq g$, then for all $x \in [a,b]$ then $\int_a^b f dx \leq \int_a^b g dx$. In particular, if they are bounded, ie., $|f| \leq M$ then we have

$$|\int f(x)dx| \leq \int |f(x)|dx \leq M(b-a)$$

- If f is continuous on $[a,b]$ then for $c \in [a,b]$,

$$F(x) = \int_c^x f(t)dt$$

  is $C^1([a,b])$ and $F'(x) = f(x)$ with $F(c) = 0$. This is also referred to a the fundamental theorem of calculus.
- Integral of a sum is the sum of integrals
- We can split up integrals

Example 2.2.1. Here are some useful limits that can be proven using L'Hopital:

- $n^{1/n} \to 1$
- $t^{1/n} \to 1$ for all $t > 0$
- $(1 + t/n)^n \to e^t$ for all t
- $n^p/(1 + t)^n \to 0$ for all $p \in \mathbb{R}$

Example 2.2.2. Show that $\int_0^{1-1/n} f_n \to \int_0^1 f$ if $f_n \to f$ uniformly on $[0, 1]$.

Proof. Observe that

$$\left| \int_0^{1-\frac{1}{n}} f_n(x)dx - \int_0^1 f(x)dx \right| = \left| \int_0^{1-\frac{1}{n}} (f_n(x) - f(x)) - \int_{1-\frac{1}{n}}^1 f(x)dx \right|$$

Note that

$$\int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f_n(x)\psi_[0, 1 - 1/n](x)dx$$

where

$$\psi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then,

$$\begin{aligned} \left| \int_0^{1-\frac{1}{n}} (f_n(x) - f(x)) - \int_{1-\frac{1}{n}}^1 f(x)dx \right| &\leq \int_0^{1-\frac{1}{n}} |f_n(x) - f(x)|dx + \int_{1-\frac{1}{n}}^1 |f(x)|dx \\ &< \frac{\epsilon}{2} + \|f\|_\infty \left(1 - \frac{1}{n}\right) \\ &< \epsilon \end{aligned}$$

for large enough n. □

Note that uniform convergence isn't enough to interchange limits of integration if the domin is unbounded. For example, consider a function which is 0 everywhere but has a straight line from $(n, 0)$ to $(2n, 1/n)$ to $(3n, 0)$. Then, $f_n \to 0$ uniformly. Indeed for all $\epsilon > 0$ we can choose N so that $\|f_N\|_\infty = 1/N < \epsilon$ by construction. However,

$$\int_0^\infty f_n(x)dx = 1$$

So

$$\lim_{n\to\infty} \int f_n dx \neq \int \lim_{n\to\infty} f_n dx.$$

So our theorem from before depend on our interval being compact. What about differentiation? Is it enough to have uniform convergence?

Example 2.2.3. Consider

$$f_n(x) = \frac{1}{\sqrt{n}} \sin(nx) \to 0.$$

Then,

$$f_n'(x) = \sqrt{n} \cos(nx)$$

does not even converge pointwise. So we need more.

Theorem 2.4. *Suppose $f_n \in C^1([a,b])$ such that $\exists g : [a,b] \to \mathbb{R}$ with $f_n' \to g$ uniformly on $[a,b]$ and there exists $x_0 \in [a,b]$ and $y_0 \in \mathbb{R}$ such that $f_n(x_0) \to y_0$ such that $f_n(x_0) \to y_0$. Then, there exists $f \in C^1[a,b]$ such that $f_n \to f$ uniformly and $f' = g$.*

Proof. By fundamental theorem of calculus,

$$f_n(x) = f_n(x_d) + \int_{x_0}^{x} f_n'(t)dt.$$

For any fixed $x \in [a,b)$, we may define

$$\lim_{n\to\infty} f_n(x) = y_0 + \int_{1/n}^{x} g(t)dt = f(x)$$

By F.T.C., $f'(x) = g(x)$ is continuous since f_n' is continuous and $f_n' \to g$ uniformly. So $f \in C^1([a,b])$.

Note that $y_0 = f(x_0)$. It remains to show that $f_n \to f$ uniformly. Note that

$$\begin{aligned} |f(x) - f_n(x)| &= \left| y_0 - f_n(x_0) + \int_{x_0}^{x} (g(t) - f_n'(t))dt \right| \\ &\le |y_0 - f_n(x_0)| + \int_{x_0}^{x} |g(t) - f_n'(t)|dt \end{aligned}$$

So for all $\epsilon > 0$, there exists N such that $\forall n \ge N$, both

$$|f_n(x_0) - y_0| < \frac{\epsilon}{2}$$

and

$$h(t) - f_n'(t) < \frac{\epsilon}{2(b-a)}$$

for all $t \in [a,b]$. Therefore, for all $n \ge N$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} + \int_a^b \frac{\epsilon}{2(b-a)} dt = \epsilon.$$

In other words, $\|f_n - f\|_\infty \to 0$. $\square$

2.3 Weierstrass approximation theorem

For images in vector spaces N, we may define uniform convergence of series.

Definition 2.3 (Uniform convergence of series)**.** *$f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on M if the partial sums, $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly on M.*

Theorem 2.5 (Weierstrass M-test)**.** *Suppose $(f_n)_{n\in\mathbb{N}}$ in $(C(M,N), \sup)$ are bounded functions, $f_n : M \to N$ and N is complete normed vector space. Let $M_n = \|f_n\|_\infty$. If $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_n f_n$ converges uniformly on M.*

Proof. Define the partial sum $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Then,

$$\|S_n - S_m\|_\infty = \| \sum_{k=m+1}^{n} f_k \| \leq \sum_{k=m+1}^{n} \|f_k\|_\infty.$$

And since $\sum_{k=m+1}^{n} \|f_k\|_\infty < \infty$, we have for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ with

$$\|S_n - S_m\|_\infty \leq \sum_{k=m+1}^{n} \|f_k\|_\infty < \epsilon$$

for all $n, m \geq N$. So $(S_n)_{n\in\mathbb{N}}$ is cauchy in sup-norm. Since N is complete, $S_n = \sum_{k=1}^{n} f_k$ is convergent in sup-norm. $\square$

Corollary 2.1. *If each f_n is continuous M compact or if each $f_n \in C_b(M,N)$, N complete, then if $\sum \|f_n\|_\infty < \infty$ then $f = \sum f_n$ is continuous.*

Corollary 2.2. *If each f_n is Riemann integrable on $[a,b]$ then f is Riemann integrable and*

$$\int_a^b f dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx.$$

Corollary 2.3. *If each $f_n \in C^1[a,b]$ and $\sum_1^\infty f_n'$ converges uniformly on $[a,b]$ and $\sum_1^\infty f_n(x)$ is convergent, then $f'(x) = \sum_1^\infty f'(x)$ and $f \in C^1[a,b]$.*

Application of Weierstrass M-test is a function which is everywhere continuous and nowhere differentiable. Start with $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = |x|$ for $-1 \leq x \leq 1$ and $g(x) = g(x-2)$ otherwise. Then, g is continuous on $\mathbb{R}$ and 2-periodic. Also $|g(x) - g(y)| \leq |x - y|$. Define

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k g(4^k x)$$

Since $\|g_k(x)\|_\infty \leq (3/4)^k$ over $\mathbb{R}$ and $\sum_{k=0}^{\infty} (3/4)^k$ converges, so by Weierstrass M-test, the series converges uniformly on $\mathbb{R}$. Therefore, f is continuous on $\mathbb{R}$.

Take any $x \in \mathbb{R}$. Let $n \in \mathbb{N}$ and choose $\delta_n = \pm\frac{1}{2}4^{-m}$ with sign chosen so that there are no integers between $4^n x$ and $4^n(x+\delta_n) = 4^n \pm 1/2$. With this choice of δ_n, we will look at

$$\lim_{n\to\infty} \frac{f(x+\delta_n) - f(x)}{\delta_n}.$$

If it convergnes, it will tend to $f'(x)$.

Observe that

$$\left|g(4^k(x+\delta_n)) - g(4^k x)\right| = \begin{cases} 0 & \text{if } k \geq n \\ \frac{1}{2} & \text{if } k = n \\ \leq |4^k \delta_n| & \text{if } k < n \text{ by } g \text{ lipsitchiz} \end{cases}$$

Next,

$$\begin{aligned}\frac{f(x+\delta_n) - f(x)}{\delta_n} &= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k \left(\frac{g(4^k(x+\delta_n)) - g(4^k x)}{\delta_n}\right) \\ &= \underbrace{\pm\left(\frac{3}{4}\right)^m \frac{\frac{1}{2}}{\pm\frac{1}{2}4^{-n}}}_{3^n} + \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^k \left(\frac{g(4^k(x+\delta_n)) - g(4^k x)}{\delta_n}\right)\end{aligned}$$

where

$$\begin{aligned}&\left|\sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^k \left(\frac{g(4^k(x+\delta_n)) - g(4^k x)}{\delta_n}\right)\right| \\ &\leq \sum_{k=0}^{n-1} \frac{3^k}{4^k} \frac{|4^k \delta_n|}{|\delta_n|} \\ &= \sum_{k=0}^{n-1} 3^k \\ &\leq \frac{3^n}{2}.\end{aligned}$$

So

$$\frac{f(x+\delta_n) - f(x)}{\delta_n} \geq 3^n - \frac{3^n}{2} = \frac{3^n}{2} \to \infty.$$

So $f'(x)$ does not exist, i.e., f is not differentiable at any $x \in \mathbb{R}$.

Theorem 2.6 (The Weierstrass approximation theorem)**.** *For any continuous function $f : [a,b] \to \mathbb{R}$, there exists a sequence $p_n(x)$ of polynomials with $p_n \to f$ unifomrly on $[a,b]$. In other words, the class of all polynomials is a dense set in $C([a,b])$ with* sup*-norm. We say that $(C[a,b], \|\cdot\|_\infty)$ is separable since there is a countable dense subset.*

Corollary 2.4. *$C^\infty([a,b])$ is dense in $C[a,b]$.*

In the text, a proof is given involving a special family of polynomials, the Berstein polynomials:

$$B_n(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

where $n \in \mathbb{N}$. The proof we will use is based on the idea of an approximatioin to the identity. We introduce a special sequence of polynomials:

$$Q_n(t) = C_n(1-t^2)^n, \quad t \in [-1,1]$$

where $C_n = 1/\left(\int_{-1}^{1}(1-t^2)^n dt\right)$ so that

$$\int_{-1}^{1} Q_n(t)dt = 1.$$

To estimate the magnitutde of C_n, we compute

$$\begin{aligned}
\int_{-1}^{1}(1-t^2)^n dt &= 2\int_0^1 (1-t^2)^n dt \\
&\geq 2\int_0^{1/\sqrt{n}} (1-t^2)^n dt \\
&\geq 2\left(1-\frac{1}{n}\right)^n \frac{1}{\sqrt{n}}
\end{aligned}$$

Or use binomials and use the fact that: $(1-t^2)^n \geq 1 - nt^2$. Then, $C_n < c\sqrt{n}$ for some $c > e/2$. So

$$0 \leq Q_n(t) \leq c\sqrt{n}(1-t^2)^n.$$

If $1 \geq |t| \geq \delta > 0$, then

$$0 \leq Q_n(t) \leq c\sqrt{n}(1-\delta^2)^n$$

So $Q_n(t) \to 0$ uniformly in $\{t \in [-1,1] : |t| \geq\}$ Since

$$\int_{-1}^{2} Q_n(t) = 1,$$

this means that $Q_n(0) \to \infty$. We call any family of

Inutitively, $Q_n \to \delta_0$, the Dirac delta. It concentrates all the area under the graph in a neighbordhoof of $t = 0$ as $n \to \infty$.

We now reduce the problem a little. First, we may assume that $[a, b] = [0, 1]$. If not, consider $g(x) = f(x-a)/(b-a)$, which is continuous on $[0, 1]$. Next, assume $f(0) = 0 = f(1)$. If not, let

$$g(x) = f(x) - f(0) - x(f(1) - f(0)).$$

If $\|p_n - g\|_\infty < \epsilon$, then

$$\|p_n + f(0) + x(f(1) - f(0)) - f(x)\|_\infty < \epsilon$$

and $p_n + f(0) + x(f(1) - f(0))$ is also a polynomial. The advantage is that we can extend $f : \mathbb{R} \to \mathbb{R}$ by letting $f(x) \equiv 0$ if $x < 0$ and $x > 1$ and f is still continuous on $\mathbb{R}$.

Recall the definition of uniform continuity: for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, y with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Lemma 2.1. *f is uniformly continuous on $\mathbb{R}$.*

Proof. We proved any continuous function on a closed bounded interval is uniformly continuous. The constant $f(x) \equiv 0$ is uniformly continuous on $[0, 1]^c$ so f is uniformly continuous on $\mathbb{R}$ since if $|x - y| < \delta \geq 1$, then $x, y \in [-1, 2]$ or obth $x, y \in \mathbb{R} \setminus [0, 1]$ and f is uniformly continuous on $[-1, 2]$. $\square$

Define on $[0, 1]$,

$$P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t)dt.$$

By a change of variable, $s = t + x$ and $f(s) = 0$ if $s \notin [0, 1]$, i.e., $-x \leq t \leq 1 - x$, so we have

$$\begin{aligned} P_n(x) &= \int_{-x}^{1-x} f(x + t)Q_n(t)dt \\ &= \int_{0}^{1} f(s)Q_n(s - x)dx \end{aligned}$$

Note $P_n(x)$ is a polynomial.

$$\begin{aligned} Q_n(s - x) &= c_n(1 - (s - x)^2)^n \\ &= q_{2n}(s)x^{2n} + q_{2n-1}(s)x^{2n-1} + \cdots + q_0(s) \end{aligned}$$

is a polynomial in x with coefficients. Therefore,

$$P_n(x) = \left[\int_0^1 f(s)q_{2n}(s)ds\right] x^{2n} + \cdots + \left[\int_0^1 f(s)q_0(s)ds\right]$$

We must show that $p_n(x) \to f(x)$ uniformly on $[0, 1]$. We will need

- For all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall x \in [0, 1]$, with $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon/2$
- Also, f is bounded so there exists $M > 0$ such that $|f(x)| \leq M \forall x \in [0, 1]$.
- $Q_n \to 0$ uniformly on $A_\delta = [-1, 1] \setminus [-\delta, \delta]$, i.e., there exists $N \in \mathbb{N}$ with $|Q_n(t)| < \epsilon/(8M)$ for all $n \geq N$ and for all $t \in A_\delta$.

Then,

$$\begin{aligned}|f(x) - P_n(x)| &= \left| f(x) \int_{-1}^{1} Q_n(t)dt - \int_{-1}^{1} f(x+t)Q_n(t)dt \right| \\ &= \left| \int_{-1}^{1} (f(x) - f(x+t))Q_n(t)dt \right|\end{aligned}$$

Now, we divide the integral into 2 pieces:

$$\left| \int_{-\delta}^{\delta} (f(x) - f(x+t))Q_n(t)dt \right|$$

and we use uniform continuity.

Since $x + t = y$, $|x - y| = |x - (x+t)| = |t| < \delta$. So the integral is less than

$$\frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t)dt \leq \frac{\epsilon}{2} \int_{-1}^{1} Q_n(t)dt = \frac{\epsilon}{2}$$

Also,

$$\left| \int_{A_\delta} (f(x) - f(x+t))Q_n(t)dt \right| \leq 2M \frac{\epsilon}{8M} \int_{A_\delta} 1dt \leq \frac{\epsilon}{2}.$$

Therefore, $|f(x) - P_n(x)| < \epsilon$ for all $x \in [0,1]$ for all $n \geq N$. So $P_n(x)$ uniformly converges on $[0,1]$.

Example 2.3.1. Consider f, g with same moment then $f = g$, i.e.,

$$\int_a^b x^n f(x) = \int_a^b x^n g(x)$$

for all n.

Example 2.3.2. If $f \in C([a,b])$ and $\int_a^b x^n f(x)dx = 0$ for all $n \in \mathbb{N}$ then $f(x) \equiv 0$.

Proof. There exists p_n such that $p_n \to f$ (Weirstrass). Hypothesis: $\int_a^b p_n f dx = 0$ for every polynomial. Since $p_n \to f$, we have $\int_a^b f^2 dx = 0$ and since $f^2 \geq 0$, we have $f \equiv 0$. □

On the proof of Weirstrass, here's what we have actually proven:

1. If we have a family $Q_n(x)$ that is integrable function with the property that

   - $Q_n(s) \geq 0$ for all $s \in \mathbb{R}$
   - $\int Q_n(s)ds = 1$
   - $\forall \delta > 0$, let $A_\delta = \mathbb{R} \setminus (-\delta, \delta)$, then

$$\lim_{n \to \infty} \int_{A_\delta} A_n(s)ds = 0$$

Note that we chose Q_n to be polynomial but it doensn't have to be.

2. Suppose f is uniformly bounded on $S \subseteq \mathbb{R}$ and uniformly continuous on S, varnishing outside of S. Then,

$$\int_{\mathbb{R}} f(s)Q_n(s-x)dx \to f$$

uniformly as $n \to \infty$.

Example 2.3.3. Consider

$$G_t(y) = \frac{1}{\sqrt{4\pi t}} e^{-y^2/(4t)}$$

where $t > 0$. This is analogous to Q_n with $n = 1/t$. As $t \to 0^+$, $G_t \to \infty$. Then,

- $G_t \geq 0$ because this is exponential
- $\int G_t ds = 1$ because G is normal distribution function
- Away from the origin for our integral goes to 0

So

$$u(x,t) = \int G_t(x-s)f(x)ds \to f(x)$$

for all f uniformly continuos, uniformly bounded on $\mathbb{R}$. But,

$$\frac{d}{dt}G_t(x) = \frac{d^2}{dx^2}G_t(x)$$

holds for all $t > 0$ and for all $x \in \mathbb{R}$. So $u(x,y)$ solves the heat equation with initla conditions given by $f(x)$.

Example 2.3.4. Consider

$$P_y(s) = \frac{1}{\pi} \cdot \frac{1}{s^2+y^2}$$

Now, $n = 1/y$. As $y \to 0^+$, $n \to \infty$. Checking conditions is left to the readers as an exercise. Then,

$$u(x,y) = \int_{\mathbb{R}} P_y(s-x)f(s)ds \to f(x)$$

for f uniformly bounded, uniformly continuous. Hence,

$$\frac{d^2}{dx^2}P_y(x) + \frac{d^2}{dy}P_y(x) = 0$$

for $y > 0$ so $u(x,y)$ solves the Laplacian and is a Harmonic extension of f to the upper half plane.

Suppose $f(x, y)$ is continuous on $[a, b] \times [c, d] = R$ and $f(x, \cdot)$ is differentiable on $[c, d]$ and f_y is continuous on R. Then,

$$F_y = \int_a^b f(x, y)dx$$

is C^1 and obtained by

$$F_y' = \int_a^b f_y(x, y)dx.$$

We will study

$$\frac{F(y+h) - F(y)}{h} = \int_a^b \frac{f(x, y+h) - f(x, y)}{h} dx$$

We need uniform convergence of $\frac{f(x,y+h)-f(x,y)}{h}$ for $f_y(x, y)$. By MUT, consider $f_y(x, \phi)$ for $y < \phi < y + h$. Then, $f_y(x, \phi)$ is uniformly continuous in ϕ, i.e., for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f_y(x, z) - f_y(x, y)| < \epsilon$$

if $|x - y| < \delta$. So

$$|f_y(x, \phi) - f_y(x, y)| < \epsilon$$

uniformly. So we have uniform convergence of $f_y(x, \phi)$ to $f_y(x, y)$.

3 Fourier Series

3.1 Fourier Series

Fourier series is about approximation of function, but by trigonometric polynomial. The natrual norm is not supremum but the L^2-norm. Take $C([-\pi, \pi])$ and define the norm

$$\|f\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx\right)^{1/2}$$

This norm is special in that it is associated to an inner product:

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

with $\|f\|_2 = \sqrt{\langle f, f\rangle}$. Our special class of f will be the trigonometric polynomials.

Call

$$\tau_n = \left\{T(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\right\}$$

and

$$\tau = \bigcup_n \tau_n$$

Then, each τ_n is a linear subspace of $C([-\pi, \pi])$. A calculation reveals that the building blocks:

$$\phi_k(x) = \begin{cases} \cos kx & k \geq 1 \\ 1 & \end{cases}, \quad \psi_k(x) = \sin kx$$

form an orthogonal family in L^2-inner product. Then,

$$\langle \phi_k, \psi_j\rangle = 0 = \langle \phi_k, \phi_j\rangle = \langle \psi_k, \psi_j\rangle$$

provided that $k \neq j$. Furthermore,

$$\langle \phi_k, \phi_k\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos kx)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(1 + \cos 2kx)dx = 1$$

However, note that $\langle 1, 1\rangle = 2$.

Given a Riemann integrable f, we get an element of τ_n by orthogonal projection. Let

$$a_0 = \langle f, \phi_0\rangle$$
$$a_k = \langle f, \phi_k\rangle$$
$$b_k = \langle f, \psi_k\rangle$$

and

$$S_n(f) = \frac{\alpha_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

the n-th Fourier sum. $S_n(f) \in \tau_n$ where $n \in \mathbb{N}$. As τ_n are linear subspaces and this is obtained by orthogonal projection, we expect this to be the best approximation of f in τ_n.

Note that $S_n(f)(x) = S_n(f)(x + 2\pi)$ for all $n \in \mathbb{N}$. So $S_n(f)$ is continuous and 2π-periodic. We write $g \in C^{2\pi}$ for $g : \mathbb{R} \to \mathbb{R}$ that is continuous and 2π-periodic.

We want to compare f with $S_n(f)$ so we will restrict to $f \in C^{2\pi}$. You can also think of $f \in C([-\pi, \pi])$ as an element of $C^{2\pi}$ by periodic extension of f to $\mathbb{R}$. Note that this may crease discontinuity in f and of f' at multiples of π. So we can think of the Fourier series as a $2 - \pi$ periodic continuous extensions of f.

If $T \in \tau_n$, then $f - S_n \perp T$, i.e.,

$$\langle f - S_n(f), T\rangle = 0$$

for all $T \in \tau_n$. In deed, write this as

$$T(x) = \alpha_k\phi_k + \beta_k\psi_k$$

then

$$\langle f - S_n(f), \alpha_k\phi_k + \beta_k\psi_k\rangle = \alpha_k\langle f, \phi_k\rangle + \beta_k\langle f, \psi_k\rangle - \alpha_k\langle S_n(f), \phi_k\rangle - \beta_k\langle S_n(f), \psi_k\rangle,$$

where

$$\begin{aligned} a_k &= \langle f, \phi_k\rangle = \langle S_n(f), \phi_k\rangle \\ b_k &= \langle f, \psi_k\rangle = \langle S_n(f), \psi_k\rangle \end{aligned}$$

and so the entire thing goes to zero.

As the inner product is linear, this works for all $T \in \tau_n$.

Proposition 3.1. *For all $f \in C^{2\pi}$, and for all $n \in \mathbb{N}$,*

$$\|f - S_n(f)\| \leq \|f - T\|_2 \, \forall T \in \tau_n$$

and equality holds iff $T = S_n(f)$.

Proof. Let $T \in \tau_n$. Then,

$$f - T = (f - S_n(f)) + (S_n(f) - T)$$

Then, by Pythagorean,

$$\|f - T\|_2^2 = \|f - S_n(f)\|_2^2 + \|S_n(f) - T\|_2^2 \quad \geq \|f - S_n(f)\|_2^2$$

and equality holds iff $T = S_n(f)$. ∎

3.2 Parseval's identity

Next, we will calculate the error in the approximation by $S_n(f)$:

$$\begin{aligned}
\|f - S_n(f)\|_2^2 &= \langle f - S_n(f), f - S_n(f)\rangle \\
&= \langle f - S_n(f), f\rangle - \langle f - S_n(f), S_n(f)\rangle \\
&= \|f\|_2^2 - \left[\frac{a_0}{2}\langle \phi_0, f\rangle + \sum_{k=1}^{n}(a_k\langle \cos kx, f\rangle + b_k\langle \sin kx, f\rangle)\right] \\
&= \|f\|_2^2 - \left[\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right],
\end{aligned}$$

where

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) = \|S_n(f)\|_2^2$$

We can make two conclusions from this.

Theorem 3.1 (Bessel's inequality)**.** *For all $n \in \mathbb{N}$,*

$$\|S_n(f)\|_2^2 = \frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \|f\|_2^2$$

In particular, if $f \in L^2$, the Fourier coefficients are square summable, i.e., in ℓ_2.

Theorem 3.2 (Riemann lemma)**.** *If $f \in L^2$, then*

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0.$$

In other words,

$$\lim_{k\to\infty}\int_{-\pi}^{\pi} f(x)\cos kx dx = 0$$

Theorem 3.3 (L^2-convergence criterion)**.** *$S_n(f) \to f$ in L^2-norm if and only if*

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \|f\|_2^2.$$

This is Parseval's identity.

We will prove this later for $f \in C^{2\pi}$:

Theorem 3.4. *If f is Riemann integrable on $(-\pi, \pi)$ then $S_n(f) \to f$ in the L^2-norm.*

Think of the association

$$f \in C^{2\pi} \mapsto (a_0, \{a_k\}, \{b_k\}),$$

where $a_0 = \langle f, \phi_0 \rangle$, $a_k = \langle f, \phi_k \rangle$, $a_0 = \langle f, \psi_0 \rangle$. Then, we can think of this map as a linear function $f_c \in \mathcal{F}$:

$$C^{2\pi} \to \mathbb{R} \times \ell_2 \times \ell_2$$

with $\|(a_0, \{a_k\}, \{b_k\})\|^2 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$.

By Bessel's inequality,

$$\|\mathcal{F}(f)\|^2 = \|(a_0, \{a_k\}, \{b_k\})\|^2 \leq |\cdot|\|f\|_2^2,$$

implying that

$$\|\mathcal{F}(f-g)\|^2 \leq |\cdot|\|f-g\|_2^2.$$

So $\mathcal{F}$ is Lipschitz continuous map. By Parseval, $\mathcal{F}$ is an isometry as it preserves the norm:

$$\|\mathcal{F}(f)\| = \|f\|_{L^2}$$

Another consequence of Parseval's identity is that $\mathcal{F}$ is one-to-one. If f and g have the same Fourier coefficients, then

$$\mathcal{F}(f-g) = \mathcal{F}(f) - \mathcal{F}(g) = (0, \{0\}, \{0\}).$$

But $\|f-g\|_2^2 = \|F(f-g)\|^2 = 0$, implying that $f = g$.

Let $f \in C^{2\pi}$. Does $S_n(f) \to f$ pointwise or uniformly? Given the Fourier series, can we recover f?

Theorem 3.5. *Suppose the Fourier coefficients of f are in ℓ_1:*

$$\sum_{k=1}^{\infty} |a_k| + |b_k| < \infty.$$

Then, $S_n(f) \to f$ uniformly.

Theorem 3.6. *Let $g_k(x) = a_k \cos kx + b_k \sin kx$. Then,*

$$|g_k(x)| \leq |a_k| + |b_k|$$

for all $x \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

By Weierstrass M-test,

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b + k \sin kx)$$

converges uniformly and $g \in C^{2\pi}$. Since g has all the same Fourier coefficient as f, $f = g$ and $S_n(f) \to f$ uniformly.

Indeed,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos kx dx = \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} S_n(f) \cos kx dx.$$

Example 3.2.1. Show that if $f \in C^{2\pi} \cap C^1(\mathbb{R})$, then $S_n(f) \to f$ uniformly. We can use Cauchy-Schwarz to prove this.

We observe that there are many sequences in ℓ_2 which are not in ℓ_1, e.g., $a_k = 1/k$. So this suggests that not every $f \in C^{\pi}$ has uniform convergent Fourier Series. This is the major difficulty with Fourier series – pointwise and uniform convergence.

If it usually convenient to express Fourier series using complex notation:

$$e^{ikx} = \cos kx \pm i \sin kx.$$

Hence, the k-th order term for f,

$$a_k \cos kx + b_k \sin kx = \underbrace{\left(\frac{a_k}{2} + \frac{b_k}{2i}\right)}_{\hat{f}(k)} e^{ikx} + \underbrace{\left(\frac{a_k}{2} - \frac{b_k}{2i}\right)}_{\hat{f}(-k)} e^{-ikx}$$

and $\hat{f}(0) = a_0/2$. So we can write

$$f(x) \sim \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}$$

with

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

Since

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \text{ and } b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx$$

and so

$$\frac{a_k}{2} + \frac{b_k}{2i} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos kx - i \sin kx) dx.$$

Introducing the complex inner product,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \bar{g}(x) dx.$$

So we see $\langle f, e^{IKX} \rangle = \hat{f}(k)$. And Parseval's identity becomes

$$\|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2$$

so the fourier series is a linear isometry: $C^{2\pi}(\mathbb{C}) \to \ell^2(\mathbb{Z})$ – bi-infinite sequence square summable.

Riemann's lemma needs

$$\lim_{|k| \to \infty} \int_{-\pi}^{\pi} f(x) e^{ikx} dx = 0.$$

To explore pointwise and uniform convergence, we rewrite the partial sum.

$$
\begin{aligned}
S_n(f) &= \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \\
&= \sum_{|k|\leq n} \hat{f}(k)e^{ikx} \\
&= \sum_{|k|\leq n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dte^{ikx} \\
&= \int_{-\pi}^{\pi} \underbrace{\frac{1}{2\pi}\sum_{|k|\leq n} e^{ik(x-t)}}_{D_n(x-t)} f(t)dt \\
&= \int_{-\pi}^{\pi} D_n(x-t)f(t)dt \\
&= D_n * f(x)
\end{aligned}
$$

This is a convolution!

Note that

$$
\begin{aligned}
D_n(t) &= \frac{1}{2\pi}\sum_{|k|\leq n} e^{ikt} \\
&= \frac{1}{2\pi}e^{-int}\sum_{j=0}^{2n} e^{ijt} \\
&= \frac{1}{2\pi}e^{-int}\frac{e^{it(2n+1)}-1}{e^{it}-1} \\
&= \frac{1}{2\pi}e^{-int}\frac{e^{it(2n+1)}-1}{e^{it/2}(2i\sin t(t/2))} \\
&= \frac{1}{2\pi}\frac{e^{i(n+1/2)t}-e^{-i(n+1/2)t}}{2i\sin(t/2)},
\end{aligned}
$$

i.e.,

$$
D_n(t) = \frac{1}{2\pi}\frac{\sin(n+1/2)t}{\sin(t/2)}
$$

is the Dirichlet kernel.

Here are some nice properties:

1. $D_n(t) = D_n(-t)$

2. $\int_{-\pi}^{\pi} D_n(t)dt = 1$ because

$$\begin{aligned}\int_{-\pi}^{\pi} D_n(t)dt &= \frac{1}{2\pi}\sum_{|k|\leq n}\int_{-\pi}^{\pi} e^{ikt}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1dt + \sum_{k\neq 0}\int_{-\pi}^{\pi} e^{ikt} \\ &= 1\end{aligned}$$

3. $|D_n(t)| \leq \frac{n+1/2}{\pi} = D_n(0)$. Indeed,

$$\begin{aligned}|D_n(t)| &= \left|\frac{1}{2\pi}\left[1 + \sum_{k=1}^{n}(e^{ikt} + e^{-ikt})\right]\right| \\ &= \frac{1}{2\pi}(1+2n)\end{aligned}$$

4. $|D_n(t)| \geq \frac{\sin(n+1/2)t}{\pi|t|}$ since $|\sin(t/2)| \leq |t/2|$.
5. Lebegue number is defined as $\lambda_n = \int_{-\pi}^{\pi} |D_n(t)|dt \geq \frac{4}{\pi^2}\log n$.
6. We do have concentration at $t = 0$, but $|D_n(t)|$ is not very small outside a neighborhood of $t = 0$. This is very different from the weierstrass kernel.

Practice problems

Example 3.2.2. f is continuous on $[0,1]$. Find the limit, and prove your answer:

$$\lim_{n\to\infty}\int_0^1 (n+1)x^n f(x)dx$$

Proof. First, f is continuous on $[0,1]$ so there exists $\delta > 0$ such that $|f(x) - f(1)| < \epsilon/2$ when $x \in (1-\delta, 1]$. Second, f is bounded on $[0,1]$ so there exists $M = \|f\|_\infty < \infty$. By triangle inequality, ... Then,

$$\begin{aligned}
\left|\int_0^1 (n+1)x^n f(x)dx - f(1)\right| &= \left|\int_0^1 (n+1)x^n f(x)dx - \int_0^1 (n+1)x^n f(1)dx\right| \\
&= \left|\int_0^1 (n+1)x^n (f(x)-f(1))dx\right| \\
&\le \int_0^1 (n+1)x^n |f(x)-f(1)|dx \\
&= \int_0^{1-\delta} (n+1)x^n |f(x)-f(1)|dx + \int_{1-\delta}^1 (n+1)x^n |f(x)-f(1)|dx \\
&\le \int_0^{1-\delta} (n+1)x^n (2M)dx + \int_{1-\delta}^1 (n+1)x^n \left(\frac{\epsilon}{2}\right) dx \\
&\le 2M\left(\frac{\epsilon}{4M}\right) + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

Example 3.2.3. If $\sum_{n=1}^\infty |a_n| < \infty$, show that $\sum_{n=1}^\infty a_n e^{-nx}$ is uniformly convergent on $[0,\infty)$.

Proof. Let $f_n(x) = a_n e^{-nx}$. Since $e^{-nx} \le 1$ for all $x \in [0,\infty)$, we have

$$\begin{aligned}
|f_n(x)| &= |a_n e^{-nx}| \\
&= |a_n| \cdot |e^{-nx}| \\
&\le |a_n|
\end{aligned}$$

So $\|f_n\|_\infty = |a_n|$. Then,

$$\sum_{n=1}^\infty |f_n(x)| \le \sum_{n=1}^\infty |a_n| < \infty$$

By M-test, $\sum a_n e^{-nx}$ converges uniformly on $[0,\infty)$. □

Example 3.2.4. If we assume only that a_n is bounded, show that $\sum a_n e^{-nx}$ is uniformly convergent on $[\delta,\infty)$ for every $\delta > 0$.

Proof. Since (a_n) is only assumed to be bounded, $\sum a_n$ need not converge (for example, consider $a_n = 1/n$). So we can't use $|a_n|e^{-nx} \leq |a_n|e^{-n\cdot 0} = |a_n|$ as in part (a).

Fix $\delta > 0$ and set $f_n(x) = a_n e^{-nx}$. Let $N = \sup|a_n|$. Then,

$$|f_n(x)| \leq Me^{-n\delta}$$

for all $x \in [\delta, \infty)$. So $\|f_n\|_\infty \leq Me^{-n\delta}$ Choose $N \in \mathbb{N}$ large enough so that for all $n \geq N$, we hvae $e^{-n\delta} \leq 1/n^2$. For such N, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |f_n(x)| &= \sum_{n=1}^{N-1} |f(x)| + \sum_{n=N}^{\infty} |f(x)| \\ &< \sum_{n=1}^{N-1} Me^{-n\delta} + \sum_{n=N}^{\infty} \frac{M}{n^2} < \infty \end{aligned}$$

□

Example 3.2.5. Let $M = (0, \infty)$ with $d(x, y) = |1/x - 1/y|$. Is (M, d) complete? Justify your answer.

Proof. Consider the sequence $(x_n) = (n)_{n=1}^{\infty}$ and let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that for all $n, m \geq N$, $1/n, 1/m < \epsilon/2$. By the triangle inequality,

$$d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \epsilon$$

So (x_n) is Cauchy in (M, d). But $x_n \to 0$ as $n \to \infty$ in (M, d) and $0 \notin M$. □

Example 3.2.6. If $f : (0, 1) \to \mathbb{R}$ is uniformly continuous, show that $\lim_{x\to 0^+} f(x)$ exists. Conclude that f is bounded on $(0, 1)$.

Proof. Let $(x_n) \subset (0, 1)$ be a decreasing sequence such that $x_n \to 0$ in $\mathbb{R}$. Thus, (x_n) is Cauchy.

Since uniformly continuous functions map Cauchy sequences to Cauchy sequences, $(f(x_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $c_1 \in \mathbb{R}$ such that $\lim_{n\to\infty} f(x_n) = c_1$. So $\lim_{x\to 0^+} f(x)$ exists. In the same way, $\lim_{x\to 1^{-1}} = c_2$ exists.

Defin $\bar{f} : [0, 1] \to \mathbb{R}$ by

$$\bar{f}(x) = \begin{cases} c_1 & \text{if } x = 0 \\ f(x) & \text{if } x \in (0, 1) \\ c_2 & \text{if } x = 1 \end{cases}$$

By definition, $\bar{f}$ is continuous on the compact set $[0, 1]$ so $\bar{f}$ is bounded on $[0, 1]$ and thus bounded on $(0, 1)$. But since $\bar{f}(x) = f(x)$ on $(0, 1)$, we have that f is bounded on $(0, 1)$. □

Example 3.2.7. Let (X, d) and (Y, ρ) be metric spaces, and let $f, f_n : X \to Y$ with $f_n \to f$ uniformly on X. If each f_n is continuous at $x \in X$, and if $x_n \to x$ in X, prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Proof. Since f_n is continuous, f is continuous. Let $\epsilon > 0$ be given. By continuity of f at $x \in X$, there exists $\delta > 0$ so that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon/2$. Using the fact that $x_n \to x$, we can choose $N_1 \in \mathbb{N}$ large enough so that for all $n \geq N_1$, $d(x, x_n) < \delta$ implies $\rho(f(x), f(x_n)) < \epsilon/2$. Since $f_n \to f$ on X, we can find $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $\rho(f_n(x_n), f(x)) < \epsilon/2$ independent of $x \in X$. In particular, $\rho(f_n(x_n), f(x_n)) < \epsilon/2$ for any fixed $x_k \in (x_n)_{n=1}^{\infty}$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$ we have by the triangle inequality,

$$\rho(f_n(x_n), f(x)) \leq \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f(x)) < \epsilon$$

□

Example 3.2.8. Show that there cannot be a sequence of polynomials P_n for which $P_n \to \sin x$ uniformly on $\mathbb{R}$.

Proof. Suppose in order to derive a contradiction that there exists a sequence of polynomials P_n such that $P_n \to \sin x$ on $\mathbb{R}$. Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{R}$ such that for all $n \geq N$ and for all $x \in \mathbb{R}$, we have

$$-\epsilon < P_n(x) - \sin(x) < \epsilon$$

Rewriting, we have

$$-\epsilon - 1 < P_n(x) < \epsilon + 1$$

So $|P_n(x)| < \epsilon + 1$. However, polynomials are unbounded and this is a contradiction. □