# Math 3T03 - Topology

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## Contents

# 1 Introduction to topology

## 1.1 What is topology?

**Definition 1.1** (Topology)**.** *Let $X$ be a set. A topology on $X$ is a collection $\tau$ of subsets of $X$ satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *The union of any collections of elements in $\tau$ is also in $\tau$.*
- *THe intersection of any finite collection oi elements in $\tau$ is also in $\tau$.*

**Definition 1.2.** $(X, \tau)$ *is a topological space.*

**Definition 1.3.** *The elements of $\tau$ are called the open sets.*

**Example 1.1.** Consider $X = \{a, b, c\}$. Is $\tau_1 = \{\varnothing, X, \{a\}\}$ a topology?

*Proof.* Yes, $\tau_1$ is a topology. Clearly, $\varnothing \in \tau_1$ and $X \in \tau_1$ so it suffices to verify the arbitrary unions and finite intersection axioms.

Let $\{V_\alpha\}_{\alpha \in A}$ be a subcollection of $\tau_1$. Then, we want to show

$$\bigcup_{\alpha \in A} V_\alpha \in \tau_1.$$

If $V_\alpha = \varnothing$ for any $\alpha \in A$, then this does not contribute to the union. Hence, $\varphi$ can be omittied:

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\substack{\alpha \in A \\ V_\alpha \neq \varnothing}} V_\alpha.$$

If $V_\alpha = X$ for some $\alpha \in A$, then

$$\bigcup_{\alpha \in A} V_\alpha = X \in \tau_1$$

and we are done. So we may assume $V_\alpha \neq X$ for all $\alpha \in A$. Similarly, if $V_\alpha = V_\beta$ for $\alpha \neq \beta$, then we can omit $V_\beta$ from the union. Since $\tau_1$ only contains $\varnothing$, $X$ and $\{a\}$, we must have

$$\bigcup_{\alpha \in A} V_\alpha = \{a\} \in \tau_1.$$

Similarly, if any element of $\{V_\alpha\}_{\alpha \in A}$ is empty, then

$$\bigcap_{\alpha \in A} V_\alpha = \varnothing \in \tau_1.$$

We can therefore assume $V_\alpha \neq \varnothing$ for all $\alpha \in A$. Again, repetition can be ignored and any $V_\alpha$ that eqauls $X$ can be ignored. Then,

$$\bigcap_{\alpha \in A} V_\alpha = \begin{cases} \{a\} \in \tau_1 \\ X \in \tau_1 \end{cases}$$

$\square$

**Example 1.2.** Consider $X = \{a, b, c\}$. Is $\tau_2 = \{\varnothing, X, \{a\}, \{b\}\}$ a topology?

*Proof.* No. Note that $\{a\} \in \tau_2$ and $\{b\} \in \tau_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_2$. $\square$

**Example 1.3.** Consider $X = \{a, b, c\}$. Then, $\tau_3 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ is a topology.

## 1.2 Set theory

**Definition 1.4.** *If $X$ is a set and $a$ is an element, we write $a \in X$.*

**Definition 1.5.** *If $Y$ is a subset of $X$, we write $Y \subset X$ or $Y \subseteq X$.*

**Example 1.4.** Suppose $X$ is a set. Then, *power set* of $X$ is the set $P(X)$ whose elements are all subsets of $X$. In other words,

$$Y \in P(X) \iff Y \subseteq X$$

Note that $P(X)$ is closed under arbitrary unions and intersections. Hence, $P(X)$ is a topology on $X$.

**Definition 1.6.** *$P(X)$ is the discrete topology on $X$.*

**Definition 1.7.** *The indiscrete (trivial) topology on $X$ is $\{\varnothing, X\}$.*

**Definition 1.8.** *Suppose $U, V$ are sets. Then, their union is*

$$U \cup V = \{x | x \in U \text{ or } x \in V\}.$$

Note that if $U_1, U_2, \ldots, U_10$ are sets, their union can be written as follows:

- $U_1 \cup U_2 \cup \cdots \cup U_{10}$
- $\bigcup\limits_{k=1}^{10} U_k$
- $\bigcup\limits_{k=\{1,2,\ldots,10\}} U_k$

**Definition 1.9.** *Let $A$ be any set. Suppose $\forall \alpha \in A$, I have a set $U_\alpha$. The union of the $U_\alpha$ over $\alpha \in A$ is*

$$\bigcup_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

*Similarly, the intersection is*

$$\bigcap_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

## 2 Functions

**Definition 2.1.** *A function is a rule that assigns to element of a given set $A$, an element in another set $B$.*

Often, we use a formula to describe a function.

**Example 2.1.** $f(x) = \sin(5x)$

**Example 2.2.** $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

**Definition 2.2.** *A* rule of assignment *is a subset $R$ of the Cartesian product $C \times D$ of two sets with the property that each element of $C$ appears as the first coordinate of at most one ordered pair belonging to $R$. In other words, a subset $R$ of $C \times D$ is a rule of assignment if it satisfies*

$$\textit{If } (c,d) \in R \textit{ and } (c,d') \in R, d = d'.$$

**Definition 2.3.** *Given a rule of assignment $R$, we define the doamin of $R$ to be the subset of $C$ consisting of all first coordinates of elements of $R$:*

$$\text{domain}(R) \equiv \{c \mid \exists d \in D \, s.t. \, (c,d) \in R\}$$

**Definition 2.4.** *The image set of $R$ is defined to be the subset of $D$ consisting of all second coordinates.*

$$\text{image}(R) \equiv \{d \mid \exists c \in C \, s.t. \, (c,d) \in R\}$$

**Definition 2.5.** *A function is a rule of assignment $R$, together with a set that contains the image set of $R$. The domain of the rule of assignment is also called the domain of $f$, and the image of $f$ is defined to be the image set of the rule of assignment.*

**Definition 2.6.** *The set $B$ is often called the range of $f$. This is also referred to as the codomain.*

If $A = \text{domain}(f)$, then we write

$$f : A \to B$$

to indicate that $f$ is a function with domain $A$ and codomain $B$.

**Definition 2.7.** *Given $a \in A$, we write $f(a) \in B$ for the unique element in $B$ associated to $a$ by the rule of assignment.*

**Definition 2.8.** *If $S \subseteq A$ is a subset of $A$, let*

$$f(S)\{f(a) \mid a \in S\} \subseteq B.$$

**Definition 2.9.** *Given $A_0 \subseteq A$, we can restrict the domain of $f$ to $A_0$. The restriction is denoted*

$$f|A_0 \equiv \{(a, f(a)) \mid a \in A_0\} \subseteq A \times B.$$

**Definition 2.10.** *If* $f : A \to B$ *and* $g : B \to C$ *then* $g \circ f : A \to C$ *is defined to be*

$$\{(a, c) \mid \exists b \in B \text{ s.t. } f(a) = b \text{ and } g(b) = c\} \subseteq A \times C.$$

**Definition 2.11.** *A function* $f : A \to B$ *is called injective (or one-to-one) if* $f(a) = f(a')$ *implies* $a = a'$ *for all* $a, a' \in A$.

**Definition 2.12.** *A function* $f : A \to B$ *is called surjective (or onto) if the image of* $f$ *equals* $B$*, i.e.* $f(A) = B$*., i.e. if, for every* $b \in B$*, there exists* $a \in A$ *with* $f(a) = b$.

**Definition 2.13.** $f : A \to B$ *is a bijection if it is one-to-one and onto.*

**Definition 2.14.** *If* $B_0 \subseteq B$ *is a subset and* $f : A \to B$ *is a function, then the preimage of* $B_0$ *under* $f$ *is the subset of* $A$ *given by*

$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}$$

**Example 2.3.** If $B_0$ is disjoint from $f(A)$ then $f^{-1}(B_0) = \varnothing$.

# 3 Relation

**Definition 3.1.** *A relation on a set $A$ is a subset $R$ of $A \times A$.*

Given a relation $R$ on $A$, we will write $xRy$ "$x$ is related to $y$" to mean $(x, y) \in R$.

**Definition 3.2.** *An equivalence relation on a set is a relation with the following properties:*

- *Reflexivity. $xRx$ holds $\forall x \in A$.*
- *Symmetric. if $xRy$ then $yRx$ holds $\forall x, y \in A$.*
- *Transitive. if $xRy$ and $yRz$ then $xRz$ holds $\forall x, y, z \in A$.*

**Definition 3.3.** *Given $a \in A$, let $E(a) = \{x \mid x \sim a\}$ denote the equivalence class of $a$.*

*Remark.* $E(a) \subseteq A$ and it is nonempty because $a \in E(a)$.

**Proposition 3.1.** *If $E(a) \cap E(b) \neq \varnothing$ then $E(a) = E(b)$.*

*Proof.* IAssume the hypothesis, i.e., suppose $x \in E(a) \cap E(b)$. So $x \sim a$ and $x \sim b$. By symmetry, $a \sim x$ and by transitivity $a \sim b$.

Now suppose that $y \in E(a)$. Then $y \sim a$ but we just saw that $a \sim b$ so $y \sim b$ and $y \in E(b)$. Hence, $E(a) \subseteq E(b)$. Likewise, we can show that $E(b) \subseteq E(a)$. □

**Definition 3.4.** *A partition of a set is a collection of pairwise disjoint subsets of $A$ whose union is all of $A$.*

**Example 3.1.** Consider $A = \{1, 2, 3, 4, 5\}$. Then, $\{1, 2\}$ and $\{3, 4, 5\}$ is a partition of $A$.

**Proposition 3.2.** *An equivalence relation in a set determines a partition of $A$, namely the one with equivalence classes as subsets. Conversely, a partition*[1] *$\{Q_\alpha \mid \alpha \in J\}$ of a set $A$ determines an equivalence relation on $A$ by: $x \sim y$ if and only if $\exists \alpha \in J$ s.t. $x, y \in Q_\alpha$. The equivalence classes of this equivalence relation are precisely the subsets $Q_\alpha$.*

[1] Note that $J$ is an index set: $Q_\alpha \subseteq A$ for each $\alpha \in J$ and $Q_\alpha \cap Q_\beta = \varnothing$ if $\alpha \neq \beta$. Furthermore, $\cup_{\alpha \in J} = A$.

# 4 Finite and infinite sets

**Definition 4.1.** *A nonempty set $A$ is finite if there is a bijection from $A$ to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{Z}^+$.*

*Remark.* Consider $n, m \in \mathbb{Z}^+$ with $n \neq m$. Then, there is no bijection from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, m\}$.

**Definition 4.2.** *Cardinality of a finite set $A$ is defined as follows:*

1. $\mathrm{card}(A) = 0$ *if* $A = \varnothing$.
2. $\mathrm{card}(A) = n$ *if there is a bijection form $A$ to* $\{1, 2, \dots, n\}$.

Note that $\mathbb{Z}^+$ is not finite. How would you prove this? A sneaky approach is that there is a bijection from $\mathbb{Z}^+$ to a proper subset of $\mathbb{Z}^+$.

Consider the shift map:

$$S : \mathbb{Z}^+ \to \mathbb{Z}^+$$

where $S(k) = k + 1$. So $S$ is a bijection from $\mathbb{Z}^+$ to $\{2, 3, \dots\} \subset \mathbb{Z}^+$, a proper subset of $\mathbb{Z}^+$. On the other hand, any proper sbset $B$ of a finite set $A$ with $\mathrm{card}(A) = n$ has $\mathrm{card}(B) < n$. In particular, there is no bijection from $A$ to $B$. Therefore, $\mathbb{Z}^+$ is not finite.

**Theorem 4.1.** *A non-empty set is finite if and only if one of the following holds:*

1. *$\exists$ bijection $f : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{Z}^+$.*
2. *$\exists$ injection $f : A \to \{1, 2, \dots, N\}$ for some $N \in \mathbb{Z}^+$*
3. *$\exists$ surjection $g : \{1, \dots, N\} \to A$ for some $N \in \mathbb{Z}^+$*

*Remark.* Finite unions of finite sets are finite. Finite products of finite sets are also finite.

Recall that if $X$ is a set, then

$$X^w = \pi_{n \in \mathbb{Z}^+} X = \{(x_n) \mid x_n \in X \forall n \in \mathbb{Z}^+\}$$

the set of infinite sequences in $X$.

If $X = \{a\}$ is a singleton, then $X^w$ is also a singleton. Otherwise, if $\mathrm{card}(X) > 1$, then $X^w$ is an infinite set.

**Example 4.1.** Consider $X = \{0, 1\}$. Then, $X^w$ is a set of binary sequences.

**Definition 4.3.** *If there exists a bijection from a set $X$ to $\mathbb{Z}^+$, then $X$ is said to be countably infinite.*

**Definition 4.4.** *Let $\mathcal{A}$ be a nonempty collection of sets. An indexing family is a set $J$ together with a surjection $f : J \to \mathcal{A}$. We use $A_\alpha$ to denote $f(a) \subseteq \mathcal{A}$. So*

$$\mathcal{A} = \{A_\alpha \mid \alpha \in J\}$$

Most of the time, we will be able to use $\mathcal{Z}^+$ for the index set $J$.

**Definition 4.5** (Cartesian product)**.** *Let $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Finite cartesian product is defined as*

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) \mid x_i \in A_i\}$$

*It is a vector space of $n$ tuples of real numbers.*

**Definition 4.6** (Infinite Cartesian product)**.** *Infinite Cartesian product $\mathbb{R}^\omega$ is an $\omega$-tuple $x : \mathbb{Z}^+ \to \mathbb{R}$, i.e., a sequence*

$$\begin{aligned} x &= (x_i)_{i=1}^{\infty} = (x_1, x_2, \ldots, x_n, \ldots) \\ &= (x_i)_{i \in \mathbb{Z}^+} \end{aligned}$$

*More generally, if $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Then,*

$$\prod_{i \in \mathbb{Z}^+} A_i = \{(a_i)_{i \in \mathbb{Z}^+} \mid a_i \in A_i\}.$$

**Definition 4.7.** *A set $A$ is countable if $A$ is finite or it is countably infinite.*

**Theorem 4.2** (Criterion for countability)**.** *Suppose $A$ is a non-empty set. Then, the following are equivalent.*

1. *$A$ is countable*
2. *There is an injection $f : A \to \mathbb{Z}^+$*
3. *THere is a surjection $g : \mathbb{Z}^+ \to A$*

**Theorem 4.3.** *If $C \subseteq \mathbb{Z}^+$ is a subset, then $C$ is countable.*

*Proof.* Either $C$ is finite or infinite. If $C$ is finite, it follows $C$ is countable.

Assume that $C$ is finite. We will define a bijection $h : \mathbb{Z}^+ \to C$ by induction. Let $h(1)$ be the smallest element in $C$[2]. Suppose

$$h(1), \ldots, h(n-1) \in C$$

and are defined. Let $h(n)$ be the smallest element in $C \setminus \{h(1), \ldots h(n-1)\}$. (Note that this set is nonempty.)

We want to show that $h$ is inejctive. Let $n, m \in \mathbb{Z}$ where $n \neq m$. Suppose $n < m$. So $h(n) \in \{h(1), \ldots, h(n)\}$ and

$$h(n) \in C \setminus \{h(1), h(2), \ldots, h(m-1)\}.$$

So $h(n) \neq h(m)$.

We want to show that $h$ is surjective. Suppose $c \in C$. We will show that $c = h(n)$ for some $n \in \mathbb{Z}^+$. First, notice that $h(\mathbb{Z}^+)$ is not contained in $\{1, 2, \ldots, c\}$

[2] $C \neq \varnothing$ and every nonempty subset of $\mathbb{Z}^+$ has a smallest elemnt

because $h(\mathbb{Z}^+)$ is an infinite set. Therefore, there must exists $n \in \mathbb{Z}^+$ such that $h(n) > c$. So let $m$ be the smallest element with $h(m) \geq c$. Then for all $i < m$, $h(i) < c$. Then,

$$c \notin \{h(1), \dots, h(m-1)\}.$$

By definition, $h(m)$ is the smallest element in

$$C \setminus \{h(1), \dots, h(m-1)\}.$$

Then, $h(m) \leq c$. Therefore, $h(m) = c$. □

**Corollary 4.1.** *Any subset of a countable set is countable.*

*Proof.* Suppose $A \subset B$ with $B$ countable. Then, there exists an injection $f : B \to \mathbb{Z}^+$. The restriction

$$f|_A : A \to \mathbb{Z}^+$$

is also injective. Therefore, $A$ is countable by the criterion. □

**Theorem 4.4.** *Any countable union of countable is countable. In other words, if $J$ is countable and $A_\alpha$ is countable for all $\alpha \in J$, then $\cup_{\alpha \in J}$ is countable.*

**Theorem 4.5.** *A finite product of countable sets is countable.*

**Example 4.2.** $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Note that a countable product of countable sets is not countable.

**Example 4.3.** Consider $X = \{0, 1\}$. Then, $X^\omega$ binary sequence is not countable.

*Proof.* Suppose $g : \mathbb{Z}^+ \to X^\omega$ is a function. We will show $g$ is not surjective by contructing an element $y \neq g(n)$ for any $n \in \mathbb{Z}^+$.

Write $g(n) = (x_{n1}, x_{n2}, \dots) \in X^\omega$ for each $x_{ni} \in \{0, 1\}$. Define

$$y = (y_1, y_2, \dots)$$

where $y_i = 1 - x_{ii}$. Clearly, $y \in X^\omega$ but $y \neq g(n)$ for any $n$ since $y_n = 1 - x_{nn} \neq x_{nn}$, the $n$-th entry in g(n). □

**Example 4.4.** Let $X$ be a set and $\{V_\alpha\}_{\alpha \in I}$ a collection of subsets of $X$. Show

$$\bigcup_{\alpha \in I} (X - V_\alpha) = X - \bigcap_{\alpha \in I} V_\alpha$$

*Proof.*

$$\begin{aligned}
x \in \bigcup_{\alpha \in I} X - V_\alpha &\iff \exists \alpha \in I, x \in X - V_\alpha \\
&\iff \exists \alpha \in I, x \in X, x \notin V_\alpha \\
&\iff x \in X, \exists \alpha \in I, x \notin V_\alpha \\
&\iff x \in X, x \notin \bigcup_{\alpha \in I} V_\alpha \\
&\iff x \in X - \bigcap_{\alpha in I} V_\alpha
\end{aligned}$$

$\square$

# 5 Topology

**Definition 5.1** (Topology). *Let $X$ be a set. A topology on $X$ is a collection $\tau$ of subsets of $X$ satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *If* $\{V_\alpha\}_{\alpha \in I} \subseteq \tau$, *then* $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
- *If* $\{V_1, \ldots, V_n\} \subseteq \tau$, *then* $\bigcap_{k=1}^{n} V_k in I$.

*Remark.* If $\tau$ is a topology, then $\{\varnothing, X\} \subseteq \tau$ and $\tau \subseteq \mathcal{P}(X)$, where $\mathcal{P}$ is a power set.

**Definition 5.2.** *Suppose $\tau, \tau'$ are topologies on $X$. We say $\tau$ is coarser than $\tau'$ if $\tau \subseteq \tau'$. $\tau'$ is finer than $\tau$ if $\tau \subseteq \tau'$.*

**Example 5.1.** If $X$ is a set and $\tau$ is a topology. Then, $\tau$ is coarser than the discrete topology, $\mathcal{P}(X)$, and finer than the trivial topology, $\{\varnothing, X\}$.

Note that if $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$, then $\tau = \tau'$.

**Example 5.2.** Define $\tau_f$ to consist of the subsets $U$ of $X$ wiith the property that $X - U$ is finite or all of $X$.

**Lemma 5.1.** *$\tau_f$ is a topoology on $X$. This is defined as the finite complement topology.*

*Proof.* To show $\varnothing \in \tau_f$, notice that $X - \varnothing = X$. So $\varnothing \in \tau_f$. To show that $X \in \tau_f$, notice that $X - X = \varnothing$, which is finite. So $X \in \tau_f$.

To show that it is closed under arbitrary unions, suppose

$$\{V_\alpha\}_{\alpha \in I} \subseteq \tau_f.$$

It suffices to show that $X - \bigcup_{\alpha \in I} V_\alpha$ is finite. For this, note

$$X - \bigcup_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} (X - V_\alpha).$$

$X - V_\alpha$ is finite so

$$\bigcap_{\alpha \in I} (X - V_\alpha) \subseteq X - V'_\alpha$$

for any $\alpha' \in I$ which implies $\bigcap_{\alpha \in I} X - V_\alpha$ is finite.

For closure under finite intersections, let

$$\{V_1, \ldots, V_n\} \subseteq \tau_f.$$

Then,

$$X - \bigcap_{k=1}^{n} V_k = \bigcup_{k=1}^{n} (X - V_k)$$

is a finite union of finite sets and so it is finite. $\square$

**Example 5.3.** Consider $X = \mathbb{R}$. Then, $\mathbb{R} - \{0\}$ is open in the finite complement topology. $\mathbb{R} - \{0, 3, 100\}$ is open in $\tau_f$. Note that $(4, \infty)$ is not open in $\tau_f$ because its complement is not finite.

*Remark.* Assume $\tau$ is a topology on $X$.

- $(X, \tau)$ is a topological space.
- $X$ is a topological space.
- $U \in \tau$ iff $U$ is open in $\tau$ iff $U$ is an open set.

# 6 Basis

**Definition 6.1.** *Let $X$ be a set. A basis for topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$ satisfying:*

- $\forall x \in X, \exists B \in \mathcal{B}, x \in B$
- $\forall B_1, B_2 \in \mathcal{B}$, *if* $x \in B_1 \cap B_2$ *then* $\exists B_3 \in \mathcal{B}$ *such that* $x \in B_3$ *and* $B_3 \subseteq B_1 \cap B_2$.

**Example 6.1.** Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_1$ be a collection of interiors of circles in $\mathbb{R}^2$. Then, $\mathcal{B}_1$ is a basis.

**Example 6.2.** Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_2$ be collection of interiors of axis-parallel rectangles. Then, $\mathcal{B}_2$ is a basis. Note that nontrivial intersection is always a rectangle.

**Definition 6.2.** *The topology generated by a basis $\mathcal{B}$ is the collection $\tau$ satisfying:*

$$U \in \tau \iff \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U.$$

**Example 6.3.** Consider $X = \mathbb{R}$ with standard topology, $\tau_{st}$. Consider $U = \mathbb{R} - \{0\}$. Then, $U \in \tau_{st}$. Let $x \in U$:

1. (Case 1) $x > 0$. Let $B = (0, x+1)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.
2. (Case 2) $x < 0$. Let $B = (x-1, 0)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.

**Example 6.4.** Let $\tau_f$ be finite complement topology on $\mathbb{R}$. If $U \in \tau_f$, then $U \in \tau_{st}$. So $\tau_f \subseteq \tau_{st}$.

*Remark.* $(3, \infty)$ is open in $\tau_{st}$ but not in $\tau_f$.

**Lemma 6.1.** *If $\mathcal{B}$ is a basis and $\tau$ is the topology generated by $\mathcal{B}$, then $\tau$ is a topology.*

*Proof.* $\varnothing \in \tau$ is vacuously true. If $x \in \varnothing$ then $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \varnothing$.

Now, we want to prove that $X \in \tau$. Let $x \in X$. By the axioms for basis, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq X$. Hence, $X \in \tau$.

Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of $\tau$. Let

$$x \in \bigcup_{\alpha \in A} \mathcal{U}_\alpha.$$

Then, $\exists \mathcal{B} \in A$ with $x \in \mathcal{U}_{\mathcal{B}}$. Since $\mathcal{U}_{\mathcal{B}} \in Tau$, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \mathcal{U}_{\mathcal{B}}$. TheN, $x \in B$ and $B \subseteq \mathcal{U}_{\mathcal{B}}$. So $\tau$ is closed under arbitrary unions.

Let $\{V_1, \ldots, V_n\} \subseteq \tau$. We want to use proof by induction to show that $\tau$ is closed under finite intersections. When $n = 1$,

$$\bigcap_{k=1}^{n} V_k = V_1 \in \tau.$$

Assume the claim is true for $n$. Then,

$$\bigcap_{k=1}^{n+1} V_k = \underbrace{V_{n+1}}_{W} \cap \underbrace{\bigcap_{k=1}^{n} V_k}_{W'}$$

Note $W \in \tau$ and $W' \in \tau$ by the induction hypothesis. If $W \cap W' = \varnothing$ then we are done ($\varnothing \in \tau$). Let $x \in W \cap W'$. Then, $x \in W$ and $x \in W'$, implying that $\exists B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq W$ and $x \in B_2 \subseteq W'$. Hence, $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2 \subseteq W \cap W'$. □

**Example 6.5.** Consider $X = \mathbb{R}$. Define

$$(a,b) = \{x \in \mathbb{R} | a < x < b\}$$

for $a, b \in \mathbb{R}$. Show

$$\mathcal{B}_{st} = \{(a,b) | a, b \in \mathbb{R}\}$$

is a basis.

Let $x \in \mathbb{R}$. Then,

$$x \in (x-1, x+1) \in \mathcal{B}_{st}.$$

For the second axiom, let $(a,b), (c,d) \in \mathcal{B}_{st}$ and assume $x \in (a,b) \cap (c,d)$. Then, $x \in (c,b) \subseteq (a,b) \cap (c,d)$.

**Definition 6.3.** *The standard topology on $\mathbb{R}$ is the topology $\tau_{st}$ generated by $\mathcal{B}_{st}$.*

**Lemma 6.2.** *Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bases on $X$ and $\tau$ and $\tau'$ are the topologies they generate. Then, the following are equivalent:*

1. *$\tau'$ is finer than $\tau$ ($\tau \subseteq \tau'$)*

2. *$\forall x \in X$ and $\forall B \in \mathcal{B}$ with $x \in B$, $\exists B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.*

*Proof.* First, we want to show that 1 implies 2. Assume $\tau \subseteq \tau'$. Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ generates $\tau$, $B \in \tau$. Then, $B \in \tau'$. Since $\tau'$ is generated by $\mathcal{B}'$,

$$\exists B' \in \mathcal{B} \text{ with } x \in B' \subseteq B$$

Other direction is left to readers as an exercise. □

Let $X = \mathbb{R}^2$. Consider $\mathcal{B}_1$, a collection of interior of circles, and $\mathcal{B}_2$, a collection of interior of axis-parallel rectangles. Let $\tau_j$ be toplogies generated by $\mathcal{B}_j$ where $j = 1, 2$.

**Lemma 6.3.** $\tau_1 = \tau_2$

*Proof.* ($\tau_1 \subseteq \tau_2$). We use the theorem from earlier. Let $x \in \mathbb{R}^2$ and $x \in B_1 \subseteq \mathcal{B}_1$. We have to show that $\exists B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$. Since $x \in B_1$, $x$ sits inside a circle. But we can always construct a rectangle $B_2$ smaller than $B_1$ but contains $x$.

($\tau_2 \subseteq \tau_1$). Let $x \in \mathbb{R}^2$ and $B_2 \in \mathcal{B}_2$ with $x \in B_2$. We can always take $B_1 \in \mathcal{B}_1$ to be a circle inside rectangle $B_2$ and contains $x$. So $x \in B_1 \subseteq B_2$. □

**Lemma 6.4.** *Suppose $(X, \tau_x)$ and $(Y, \tau_Y)$ are topological spaces. Consider*

$$X \times Y = \{(a, b) | a \in X, b \in Y\}$$

*Then,*

$$\mathcal{B} = \{U \times V | U \in \tau_X, V \in \tau_Y\}$$

*is a basis on $X \times Y$.*

*Proof.* Let $(a, b) \in X \times Y$. Then, $X \times Y \in \mathcal{B}$ and $(a, b) \in X \times Y$. Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and assume

$$(a, b) \in U_1 \times V_1 \cap U_2 \times V_2.$$

Note

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}.$$

So

$$(a, b) \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$$

□

**Definition 6.4.** *The product topology is the topology generated by this basis.*

**Theorem 6.1.** *Suppose $\mathcal{B}_x$ and $\mathcal{B}_y$ are bases for $\tau_x$ and $\tau_y$, respectively. Then,*

$$\{B_1 \times B_2 | B_1 \in \mathcal{B}_x, B_2 \in \mathcal{B}_y\}$$

*is a basis for product topology on $X \times Y$.*

**Example 6.6.** $(X, \tau_x) = (\mathbb{R}, \tau_{st}) = (Y, \tau_Y)$.

**Definition 6.5.** *The standard topology on $\mathbb{R}^2$ is the product topology of the $(\mathbb{R}, \tau_{st})$.*

*Remark.* The theoremm tells us that $\mathbb{R}^2$ has a basis

$$\{(a, b) \times (c, d) | a, b, c, d \in \mathbb{R}\}$$

# 7 Subspace topology

**Definition 7.1.** *Let $(X, \tau)$ be a topological space and $Y \subseteq X$ be any subset. The subspace topology on $Y$ is*

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

**Lemma 7.1.** *$\tau_Y$ is a topology.*

*Proof.*

- $\varnothing \cap Y = \varnothing \in \tau_Y$.
- $X \cap Y = Y \in \tau_Y$.
- (Unions) Let $\{V_\alpha\}_{\alpha \in A} \subseteq \tau_Y$ , then $V_\alpha = U_\alpha \cap Y$ for some $U_\alpha \in \tau$. Then,

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) = \left( \bigcup_{\alpha \in A} U_\alpha \right) \cap Y \in \tau_Y$$

□

**Theorem 7.1.** *Suppose $\mathcal{B}$ is a basis for $X$ and $Y \subseteq X$. Then,*

$$\{B \cap Y | B \in \mathcal{B}\}$$

*is a basis for the subspace topology on $Y$.*

**Example 7.1.** Let $X = \mathbb{R}$ with $\mathcal{B} = \{(a, b) | a, b \in \mathbb{R}\}$. Consider

$$Y = [1, \infty).$$

Then, the basis for this topology is given by

$$\mathcal{B}_Y = \{[1, a) | a \in \mathbb{R}, a > 1\} \cup \{(a, b) | a, b \in \mathbb{R}, 1 < a < b\}$$

**Example 7.2.** $(3, 7] \notin \tau_Y$ because there does not exist open $U \in \tau$ ($U \in \mathcal{B}$) such that

$$(3, 7] = U \cap [1, \infty).$$

*Proof.* To yield contradiction, suppose there exists such $U$. Then,

$$7 \in (a, b) \cap [1, \infty).$$

So $7 \in (a, b)$ and $7 \in [1, \infty)$. In other words, $a < 7 < b$.

Let $\epsilon = (b - 7)/2$. Then, $a < 7 + \epsilon < b$ so $7 + \epsilon \in (a, b)$ and $7 + \epsilon \in [1, \infty)$. Then,

$$7 + \epsilon \in (a, b) \cap [1, \infty) = (3, 7] \implies 3 < 7 + \epsilon \leq 7,$$

yielding contradiction. □

**Example 7.3.** Let $X = \mathbb{R}$ and $Y = [0, 1]$. A basis for the subspace on $Y$ is

$$\{(a, b) | 0 < a < b < 1\} \cup \{[a, b) | 0 < b < 1\} \cup \{(a, 1] | 0 < a < 1\}$$