Math 3T03 - Topology

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Contents

1 Introduction to topology

1.1 What is topology?

Definition 1.1 (Topology)**.** *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *The union of any collections of elements in τ is also in τ.*
- *The intersection of any finite collection of elements in τ is also in τ.*

Definition 1.2. (X, τ) *is a topological space.*

Definition 1.3. *The elements of τ are called the open sets.*

Example 1.1. Consider $X = \{a, b, c\}$. Is $\tau_1 = \{\varnothing, X, \{a\}\}$ a topology?

Proof. Yes, τ_1 is a topology. Clearly, $\varnothing \in \tau_1$ and $X \in \tau_1$ so it suffices to verify the arbitrary unions and finite intersection axioms.

Let $\{V_\alpha\}_{\alpha \in A}$ be a subcollection of τ_1. Then, we want to show

$$\bigcup_{\alpha \in A} V_\alpha \in \tau_1.$$

If $V_\alpha = \varnothing$ for any $\alpha \in A$, then this does not contribute to the union. Hence, φ can be omittied:

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\substack{\alpha \in A \\ V_\alpha \neq \varnothing}} V_\alpha.$$

If $V_\alpha = X$ for some $\alpha \in A$, then

$$\bigcup_{\alpha \in A} V_\alpha = X \in \tau_1$$

and we are done. So we may assume $V_\alpha \neq X$ for all $\alpha \in A$. Similarly, if $V_\alpha = V_\beta$ for $\alpha \neq \beta$, then we can omit V_β from the union. Since τ_1 only contains $\varnothing$, X and $\{a\}$, we must have

$$\bigcup_{\alpha \in A} V_\alpha = \{a\} \in \tau_1.$$

Similarly, if any element of $\{V_\alpha\}_{\alpha \in A}$ is empty, then

$$\bigcap_{\alpha \in A} V_\alpha = \varnothing \in \tau_1.$$

We can therefore assume $V_\alpha \neq \varnothing$ for all $\alpha \in A$. Again, repetition can be ignored and any V_α that eqauls X can be ignored. Then,

$$\bigcap_{\alpha \in A} V_\alpha = \begin{cases} \{a\} \in \tau_1 \\ X \in \tau_1 \end{cases}$$

□

Example 1.2. Consider $X = \{a, b, c\}$. Is $\tau_2 = \{\varnothing, X, \{a\}, \{b\}\}$ a topology?

Proof. No. Note that $\{a\} \in \tau_2$ and $\{b\} \in \tau_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_2$. $\square$

Example 1.3. Consider $X = \{a, b, c\}$. Then, $\tau_3 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ is a topology.

1.2 Set theory

Definition 1.4. *If X is a set and a is an element, we write $a \in X$.*

Definition 1.5. *If Y is a subset of X, we write $Y \subset X$ or $Y \subseteq X$.*

Example 1.4. Suppose X is a set. Then, *power set* of X is the set $P(X)$ whose elements are all subsets of X. In other words,

$$Y \in P(X) \iff Y \subseteq X$$

Note that $P(X)$ is closed under arbitrary unions and intersections. Hence, $P(X)$ is a topology on X.

Definition 1.6. *$P(X)$ is the discrete topology on X.*

Definition 1.7. *The indiscrete (trivial) topology on X is $\{\varnothing, X\}$.*

Definition 1.8. *Suppose U, V are sets. Then, their union is*

$$U \cup V = \{x | x \in U \text{ or } x \in V\}.$$

Note that if $U_1, U_2, \ldots, U_10$ are sets, their union can be written as follows:

- $U_1 \cup U_2 \cup \cdots \cup U_{10}$
- $\bigcup_{k=1}^{10} U_k$
- $\bigcup_{k=\{1,2,\ldots,10\}} U_k$

Definition 1.9. *Let A be any set. Suppose $\forall \alpha \in A$, I have a set U_α. The union of the U_α over $\alpha \in A$ is*

$$\bigcup_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

Similarly, the intersection is

$$\bigcap_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

2 Functions

Definition 2.1. *A function is a rule that assigns to element of a given set A, an element in another set B.*

Often, we use a formula to describe a function.

Example 2.1. $f(x) = \sin(5x)$

Example 2.2. $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Definition 2.2. *A* rule of assignment *is a subset R of the Cartesian product $C \times D$ of two sets with the property that each element of C appears as the first coordinate of at most one ordered pair belonging to R. In other words, a subset R of $C \times D$ is a rule of assignment if it satisfies*

$$\textit{If } (c, d) \in R \textit{ and } (c, d') \in R, d = d'.$$

Definition 2.3. *Given a rule of assignment R, we define the doamin of R to be the subset of C consisting of all first coordinates of elements of R:*

$$\text{domain}(R) \equiv \{c \mid \exists d \in D\, s.t.\, (c, d) \in R\}$$

Definition 2.4. *The image set of R is defined to be the subset of D consisting of all second coordinates.*

$$\text{image}(R) \equiv \{d \mid \exists c \in C\, s.t.\, (c, d) \in R\}$$

Definition 2.5. *A function is a rule of assignment R, together with a set that contains the image set of R. The domain of the rule of assignment is also called the domain of f, and the image of f is defined to be the image set of the rule of assignment.*

Definition 2.6. *The set B is often called the range of f. This is also referred to as the codomain.*

If $A = \text{domain}(f)$, then we write

$$f : A \to B$$

to indicate that f is a function with domain A and codomain B.

Definition 2.7. *Given $a \in A$, we write $f(a) \in B$ for the unique element in B associated to a by the rule of assignment.*

Definition 2.8. *If $S \subseteq A$ is a subset of A, let*

$$f(S)\{f(a) \mid a \in S\} \subseteq B.$$

Definition 2.9. *Given $A_0 \subseteq A$, we can restrict the domain of f to A_0. The restriction is denoted*

$$f|A_0 \equiv \{(a, f(a)) \mid a \in A_0\} \subseteq A \times B.$$

Definition 2.10. *If $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$ is defined to be*

$$\{(a, c) \mid \exists b \in B \text{ s.t. } f(a) = b \text{ and } g(b) = c\} \subseteq A \times C.$$

Definition 2.11. *A function $f : A \to B$ is called injective (or one-to-one) if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$.*

Definition 2.12. *A function $f : A \to B$ is called surjective (or onto) if the image of f equals B, i.e. $f(A) = B$., i.e. if, for every $b \in B$, there exists $a \in A$ with $f(a) = b$.*

Definition 2.13. *$f : A \to B$ is a bijection if it is one-to-one and onto.*

Definition 2.14. *If $B_0 \subseteq B$ is a subset and $f : A \to B$ is a function, then the preimage of B_0 under f is the subset of A given by*

$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}$$

Example 2.3. If B_0 is disjoint from $f(A)$ then $f^{-1}(B_0) = \varnothing$.

3 Relation

Definition 3.1. *A relation on a set A is a subset R of $A \times A$.*

Given a relation R on A, we will write xRy "x is related to y" to mean $(x, y) \in R$.

Definition 3.2. *An equivalence relation on a set is a relation with the following properties:*

- *Reflexivity. xRx holds $\forall x \in A$.*
- *Symmetric. if xRy then yRx holds $\forall x, y \in A$.*
- *Transitive. if xRy and yRz then xRz holds $\forall x, y, z \in A$.*

Definition 3.3. *Given $a \in A$, let $E(a) = \{x \mid x \sim a\}$ denote the equivalence class of a.*

Remark. $E(a) \subseteq A$ and it is nonempty because $a \in E(a)$.

Proposition 3.1. *If $E(a) \cap E(b) \neq \varnothing$ then $E(a) = E(b)$.*

Proof. IAssume the hypothesis, i.e., suppose $x \in E(a) \cap E(b)$. So $x \sim a$ and $x \sim b$. By symmetry, $a \sim x$ and by transitivity $a \sim b$.

Now suppose that $y \in E(a)$. Then $y \sim a$ but we just saw that $a \sim b$ so $y \sim b$ and $y \in E(b)$. Hence, $E(a) \subseteq E(b)$. Likewise, we can show that $E(b) \subseteq E(a)$. $\square$

Definition 3.4. *A partition of a set is a collection of pairwise disjoint subsets of A whose union is all of A.*

Example 3.1. Consider $A = \{1, 2, 3, 4, 5\}$. Then, $\{1, 2\}$ and $\{3, 4, 5\}$ is a partition of A.

Proposition 3.2. *An equivalence relation in a set determines a partition of A, namely the one with equivalence classes as subsets. Conversely, a partition[1] $\{Q_\alpha \mid \alpha \in J\}$ of a set A determines an equivalence relation on A by: $x \sim y$ if and only if $\exists \alpha \in J$ s.t. $x, y \in Q_\alpha$. The equivalence classes of this equivalence relation are precisely the subsets Q_α.*

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[1] Note that J is an index set: $Q_\alpha \subseteq A$ for each $\alpha \in J$ and $Q_\alpha \cap Q_\beta = \varnothing$ if $\alpha \neq \beta$. Furthermore, $\cup_{\alpha \in J} = A$.

4 Finite and infinite sets

Definition 4.1. *A nonempty set A is finite if there is a bijection from A to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{Z}^+$.*

Remark. Consider $n, m \in \mathbb{Z}^+$ with $n \neq m$. Then, there is no bijection from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, m\}$.

Definition 4.2. *Cardinality of a finite set A is defined as follows:*

1. $\mathrm{card}(A) = 0$ *if* $A = \varnothing$.
2. $\mathrm{card}(A) = n$ *if there is a bijection form A to $\{1, 2, \dots, n\}$.*

Note that $\mathbb{Z}^+$ is not finite. How would you prove this? A sneaky approach is that there is a bijection from $\mathbb{Z}^+$ to a proper subset of $\mathbb{Z}^+$.

Consider the shift map:

$$S : \mathbb{Z}^+ \to \mathbb{Z}^+$$

where $S(k) = k + 1$. So S is a bijection from $\mathbb{Z}^+$ to $\{2, 3, \dots\} \subset \mathbb{Z}^+$, a proper subset of $\mathbb{Z}^+$. On the other hand, any proper sbset B of a finite set A with $\mathrm{card}(A) = n$ has $\mathrm{card}(B) < n$. In particular, there is no bijection from A to B. Therefore, $\mathbb{Z}^+$ is not finite.

Theorem 4.1. *A non-empty set is finite if and only if one of the following holds:*

1. $\exists$ *bijection* $f : A \to \{1, 2, \dots, n\}$ *for some* $n \in \mathbb{Z}^+$.
2. $\exists$ *injection* $f : A \to \{1, 2, \dots, N\}$ *for some* $N \in \mathbb{Z}^+$
3. $\exists$ *surjection* $g : \{1, \dots, N\} \to A$ *for some* $N \in \mathbb{Z}^+$

Remark. Finite unions of finite sets are finite. Finite products of finite sets are also finite.

Recall that if X is a set, then

$$X^w = \pi_{n \in \mathbb{Z}^+} X = \{(x_n) \mid x_n \in X \forall n \in \mathbb{Z}^+\}$$

the set of infinite sequences in X.

If $X = \{a\}$ is a singleton, then X^w is also a singleton. Otherwise, if $\mathrm{card}(X) > 1$, then X^w is an infinite set.

Example 4.1. Consider $X = \{0, 1\}$. Then, X^w is a set of binary sequences.

Definition 4.3. *If there exists a bijection from a set X to $\mathbb{Z}^+$, then X is said to be countably infinite.*

Definition 4.4. *Let $\mathcal{A}$ be a nonempty collection of sets. An indexing family is a set J together with a surjection $f : J \to \mathcal{A}$. We use A_α to denote $f(a) \subseteq \mathcal{A}$. So*

$$\mathcal{A} = \{A_\alpha \mid \alpha \in J\}$$

Most of the time, we will be able to use $\mathcal{Z}^+$ for the index set J.

Definition 4.5 (Cartesian product)**.** *Let $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Finite cartesian product is defined as*

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) \mid x_i \in A_i\}$$

It is a vector space of n tuples of real numbers.

Definition 4.6 (Infinite Cartesian product)**.** *Infinite Cartesian product $\mathbb{R}^\omega$ is an ω-tuple $x : \mathbb{Z}^+ \to \mathbb{R}$, i.e., a sequence*

$$\begin{aligned} x &= (x_i)_{i=1}^{\infty} = (x_1, x_2, \ldots, x_n, \ldots) \\ &= (x_i)_{i \in \mathbb{Z}^+} \end{aligned}$$

More generally, if $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Then,

$$\prod_{i \in \mathbb{Z}^+} A_i = \{(a_i)_{i \in \mathbb{Z}^+} \mid a_i \in A_i\}.$$

Definition 4.7. *A set A is countable if A is finite or it is countably infinite.*

Theorem 4.2 (Criterion for countability)**.** *Suppose A is a non-empty set. Then, the following are equivalent.*

1. *A is countable*
2. *There is an injection $f : A \to \mathbb{Z}^+$*
3. *THere is a surjection $g : \mathbb{Z}^+ \to A$*

Theorem 4.3. *If $C \subseteq \mathbb{Z}^+$ is a subset, then C is countable.*

Proof. Either C is finite or infinite. If C is finite, it follows C is countable.

Assume that C is finite. We will define a bijection $h : \mathbb{Z}^+ \to C$ by induction. Let $h(1)$ be the smallest element in C[2]. Suppose

$$h(1), \ldots, h(n-1) \in C$$

and are defined. Let $h(n)$ be the smallest element in $C \setminus \{h(1), \ldots h(n-1)\}$. (Note that this set is nonempty.)

We want to show that h is inejctive. Let $n, m \in \mathbb{Z}$ where $n \neq m$. Suppose $n < m$. So $h(n) \in \{h(1), \ldots, h(n)\}$ and

$$h(n) \in C \setminus \{h(1), h(2), \ldots, h(m-1)\}.$$

So $h(n) \neq h(m)$.

We want to show that h is surjective. Suppose $c \in C$. We will show that $c = h(n)$ for some $n \in \mathbb{Z}^+$. First, notice that $h(\mathbb{Z}^+)$ is not contained in $\{1, 2, \ldots, c\}$

[2] $C \neq \varnothing$ and every nonempty subset of $\mathbb{Z}^+$ has a smallest elemnt

because $h(\mathbb{Z}^+)$ is an infinite set. Therefore, there must exists $n \in \mathbb{Z}^+$ such that $h(n) > c$. So let m be the smallest element with $h(m) \geq c$. Then for all $i < m$, $h(i) < c$. Then,

$$c \notin \{h(1), \dots, h(m-1)\}.$$

By definition, $h(m)$ is the smallest element in

$$C \setminus \{h(1), \dots, h(m-1)\}.$$

Then, $h(m) \leq c$. Therefore, $h(m) = c$. □

Corollary 4.1. *Any subset of a countable set is countable.*

Proof. Suppose $A \subset B$ with B countable. Then, there exists an injection $f : B \to \mathbb{Z}^+$. The restriction

$$f|_A : A \to \mathbb{Z}^+$$

is also injective. Therefore, A is countable by the criterion. □

Theorem 4.4. *Any countable union of countable is countable. In other words, if J is countable and A_α is countable for all $\alpha \in J$, then $\cup_{\alpha \in J}$ is countable.*

Theorem 4.5. *A finite product of countable sets is countable.*

Example 4.2. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Note that a countable product of countable sets is not countable.

Example 4.3. Consider $X = \{0, 1\}$. Then, X^ω binary sequence is not countable.

Proof. Suppose $g : \mathbb{Z}^+ \to X^\omega$ is a function. We will show g is not surjective by contructing an element $y \neq g(n)$ for any $n \in \mathbb{Z}^+$.

Write $g(n) = (x_{n1}, x_{n2}, \dots) \in X^\omega$ for each $x_{ni} \in \{0, 1\}$. Define

$$y = (y_1, y_2, \dots)$$

where $y_i = 1 - x_{ii}$. Clearly, $y \in X^\omega$ but $y \neq g(n)$ for any n since $y_n = 1 - x_{nn} \neq x_{nn}$, the n-th entry in g(n). □

Example 4.4. Let X be a set and $\{V_\alpha\}_{\alpha \in I}$ a collection of subsets of X. Show

$$\bigcup_{\alpha \in I} (X - V_\alpha) = X - \bigcap_{\alpha \in I} V_\alpha$$

Proof.

$$\begin{aligned} x \in \bigcup_{\alpha \in I} X - V_\alpha &\iff \exists \alpha \in I, x \in X - V_\alpha \\ &\iff \exists \alpha \in I, x \in X, x \notin V_\alpha \\ &\iff x \in X, \exists \alpha \in I, x \notin V_\alpha \\ &\iff x \in X, x \notin \bigcup_{\alpha \in I} V_\alpha \\ &\iff x \in X - \bigcap_{\alpha in I} V_\alpha \end{aligned}$$

$\square$

5 Topological spaces

Definition 5.1 (Topology)**.** *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *If* $\{V_\alpha\}_{\alpha \in I} \subseteq \tau$, *then* $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
- *If* $\{V_1, \ldots, V_n\} \subseteq \tau$, *then* $\bigcap_{k=1}^{n} V_k inI$.

Remark. If τ is a topology, then $\{\varnothing, X\} \subseteq \tau$ and $\tau \subseteq \mathcal{P}(X)$, where $\mathcal{P}$ is a power set.

Definition 5.2. *Suppose τ, τ' are topologies on X. We say τ is coarser than τ' if $\tau \subseteq \tau'$. τ' is finer than τ if $\tau \subseteq \tau'$.*

Example 5.1. If X is a set and τ is a topology. Then, τ is coarser than the discrete topology, $\mathcal{P}(X)$, and finer than the trivial topology, $\{\varnothing, X\}$.

Note that if $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$, then $\tau = \tau'$.

Example 5.2. Define τ_f to consist of the subsets U of X wiith the property that $X - U$ is finite or all of X.

Lemma 5.1. *τ_f is a topoology on X. This is defined as the finite complement topology.*

Proof. To show $\varnothing \in \tau_f$, notice that $X - \varnothing = X$. So $\varnothing \in \tau_f$. To show that $X \in \tau_f$, notice that $X - X = \varnothing$, which is finite. So $X \in \tau_f$.

To show that it is closed under arbitrary unions, suppose

$$\{V_\alpha\}_{\alpha \in I} \subseteq \tau_f.$$

It suffices to show that $X - \bigcup_{\alpha \in I} V_\alpha$ is finite. For this, note

$$X - \bigcup_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} (X - V_\alpha).$$

$X - V_\alpha$ is finite so

$$\bigcap_{\alpha \in I} (X - V_\alpha) \subseteq X - V'_\alpha$$

for any $\alpha' \in I$ which implies $\bigcap_{\alpha \in I} X - V_\alpha$ is finite.

For closure under finite intersections, let

$$\{V_1, \ldots, V_n\} \subseteq \tau_f.$$

Then,

$$X - \bigcap_{k=1}^{n} V_k = \bigcup_{k=1}^{n} (X - V_k)$$

is a finite union of finite sets and so it is finite. □

Example 5.3. Consider $X = \mathbb{R}$. Then, $\mathbb{R} - \{0\}$ is open in the finite complement topology. $\mathbb{R} - \{0, 3, 100\}$ is open in τ_f. Note that $(4, \infty)$ is not open in τ_f because its complement is not finite.

Remark. Assume τ is a topology on X.

- (X, τ) is a topological space.
- X is a topological space.
- $U \in \tau$ iff U is open in τ iff U is an open set.

6 Basis for a Topology

Definition 6.1. *Let X be a set. A basis for topology on X is a collection $\mathcal{B}$ of subsets of X satisfying:*

- $\forall x \in X, \exists B \in \mathcal{B}, x \in B$
- $\forall B_1, B_2 \in \mathcal{B}$*, if* $x \in B_1 \cap B_2$ *then* $\exists B_3 \in \mathcal{B}$ *such that* $x \in B_3$ *and* $B_3 \subseteq B_1 \cap B_2$*.*

Example 6.1. Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_1$ be a collection of interiors of circles in $\mathbb{R}^2$. Then, $\mathcal{B}_1$ is a basis.

Example 6.2. Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_2$ be collection of interiors of axis-parallel rectangles. Then, $\mathcal{B}_2$ is a basis. Note that nontrivial intersection is always a rectangle.

Definition 6.2. *The topology generated by a basis $\mathcal{B}$ is the collection τ satisfying:*

$$U \in \tau \iff \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U.$$

Example 6.3. Consider $X = \mathbb{R}$ with standard topology, τ_{st}. Consider $U = \mathbb{R} - \{0\}$. Then, $U \in \tau_{st}$. Let $x \in U$:

1. (Case 1) $x > 0$. Let $B = (0, x+1)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.
2. (Case 2) $x < 0$. Let $B = (x-1, 0)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.

Example 6.4. Let τ_f be finite complement topology on $\mathbb{R}$. If $U \in \tau_f$, then $U \in \tau_{st}$. So $\tau_f \subseteq \tau_{st}$.

Remark. $(3, \infty)$ is open in τ_{st} but not in τ_f.

Lemma 6.1. *If $\mathcal{B}$ is a basis and τ is the topology generated by $\mathcal{B}$, then τ is a topology.*

Proof. $\varnothing \in \tau$ is vacuously true. If $x \in \varnothing$ then $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \varnothing$.

Now, we want to prove that $X \in \tau$. Let $x \in X$. By the axioms for basis, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq X$. Hence, $X \in \tau$.

Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of τ. Let

$$x \in \bigcup_{\alpha \in A} \mathcal{U}_\alpha.$$

Then, $\exists \mathcal{B} \in A$ with $x \in \mathcal{U}_{\mathcal{B}}$. Since $\mathcal{U}_{\mathcal{B}} \in Tau$, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \mathcal{U}_{\mathcal{B}}$. TheN, $x \in B$ and $B \subseteq \mathcal{U}_{\mathcal{B}}$. So τ is closed under arbitrary unions.

Let $\{V_1, \ldots, V_n\} \subseteq \tau$. We want to use proof by induction to show that τ is closed under finite intersections. When $n = 1$,

$$\bigcap_{k=1}^{n} V_k = V_1 \in \tau.$$

Assume the claim is true for n. Then,

$$\bigcap_{k=1}^{n+1} V_k = \underbrace{V_{n+1}}_{W} \cap \underbrace{\bigcap_{k=1}^{n} V_k}_{W'}$$

Note $W \in \tau$ and $W' \in \tau$ by the induction hypothesis. If $W \cap W' = \varnothing$ then we are done ($\varnothing \in \tau$). Let $x \in W \cap W'$. Then, $x \in W$ and $x \in W'$, implying that $\exists B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq W$ and $x \in B_2 \subseteq W'$. Hence, $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2 \subseteq W \cap W'$. □

Example 6.5. Consider $X = \mathbb{R}$. Define

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

for $a, b \in \mathbb{R}$. Show

$$\mathcal{B}_{st} = \{(a, b) | a, b \in \mathbb{R}\}$$

is a basis.

Let $x \in \mathbb{R}$. Then,

$$x \in (x - 1, x + 1) \in \mathcal{B}_{st}.$$

For the second axiom, let $(a, b), (c, d) \in \mathcal{B}_{st}$ and assume $x \in (a, b) \cap (c, d)$. Then, $x \in (c, b) \subseteq (a, b) \cap (c, d)$.

Definition 6.3. *The standard topology on $\mathbb{R}$ is the topology τ_{st} generated by $\mathcal{B}_{st}$.*

Lemma 6.2. *Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bases on X and τ and τ' are the topologies they generate. Then, the following are equivalent:*

1. *τ' is finer than τ ($\tau \subseteq \tau'$)*
2. *$\forall x \in X$ and $\forall B \in \mathcal{B}$ with $x \in B$, $\exists B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.*

Proof. First, we want to show that 1 implies 2. Assume $\tau \subseteq \tau'$. Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ generates τ, $B \in \tau$. Then, $B \in \tau'$. Since τ' is generated by $\mathcal{B}'$,

$$\exists B' \in \mathcal{B} \text{ with } x \in B' \subseteq B$$

Other direction is left to readers as an exercise. □

Let $X = \mathbb{R}^2$. Consider $\mathcal{B}_1$, a collection of interior of circles, and $\mathcal{B}_2$, a collection of interior of axis-parallel rectangles. Let τ_j be toplogies generated by $\mathcal{B}_j$ where $j = 1, 2$.

Lemma 6.3. $\tau_1 = \tau_2$

Proof. ($\tau_1 \subseteq \tau_2$). We use the theorem from earlier. Let $x \in \mathbb{R}^2$ and $x \in B_1 \subseteq \mathcal{B}_1$. We have to show that $\exists B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$. Since $x \in B_1$, x sits inside a circle. But we can always construct a rectangle B_2 smaller than B_1 but contains x.

($\tau_2 \subseteq \tau_1$). Let $x \in \mathbb{R}^2$ and $B_2 \in \mathcal{B}_2$ with $x \in B_2$. We can always take $B_1 \in \mathcal{B}_1$ to be a circle inside rectangle B_2 and contains x. So $x \in B_1 \subseteq B_2$. $\square$

Lemma 6.4. *Suppose (X, τ_x) and (Y, τ_Y) are topological spaces. Consider*

$$X \times Y = \{(a, b) | a \in X, b \in Y\}$$

Then,

$$\mathcal{B} = \{U \times V | U \in \tau_X, V \in \tau_Y\}$$

is a basis on $X \times Y$.

Proof. Let $(a, b) \in X \times Y$. Then, $X \times Y \in \mathcal{B}$ and $(a, b) \in X \times Y$. Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and assume

$$(a, b) \in U_1 \times V_1 \cap U_2 \times V_2.$$

Note

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}.$$

So

$$(a, b) \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$$

$\square$

Definition 6.4. *The product topology is the topology generated by this basis.*

Theorem 6.1. *Suppose $\mathcal{B}_x$ and $\mathcal{B}_y$ are bases for τ_x and τ_y, respectively. Then,*

$$\{B_1 \times B_2 | B_1 \in \mathcal{B}_x, B_2 \in \mathcal{B}_y\}$$

is a basis for product topology on $X \times Y$.

Example 6.6. $(X, \tau_x) = (\mathbb{R}, \tau_{st}) = (Y, \tau_Y)$.

Definition 6.5. *The standard topology on $\mathbb{R}^2$ is the product topology of the $(\mathbb{R}, \tau_{st})$.*

Remark. The theoremm tells us that $\mathbb{R}^2$ has a basis

$$\{(a, b) \times (c, d) | a, b, c, d \in \mathbb{R}\}$$

Suppose $\mathcal{B}$ is a basis for a topology τ on X.

- First, if $B \in \mathcal{B}$, then $B \in \tau$. Recall that

  $$U \in \tau \iff \forall x = U, \exists B' = \mathcal{B}, x \in \mathcal{B}' \subseteq U.$$

  So let $x \in B$. Then, $B' = B$ satisfies the above condition ($x \in B \subseteq B$). Hence, $B \in \tau$.

- Let $U \in \tau$. Then, U is a union of elements in $\mathcal{B}$. To prove this, let $U \in \tau$. By definition,
$$\forall x \in U, \exists B_x \in \mathcal{U}$$
such that $x \in B_x \subseteq U$. Then,
$$\{B_x\}_{x \in U} \subseteq \tau.$$
Then,
$$\bigcup_{x \in X_U} B = U.$$

Suppose X, Y are topological spaces and equip $X \times Y$ with the product topology. Define
$$\pi_1 : X \times Y \to X$$
$$\pi_2 : X \times Y \to Y$$
Define
$$\mathcal{S} = \{\pi_1^{-1}(U) | U \subset X\} \cup \{\pi_2^{-1} | V \subset Y\}$$
The collection $\mathcal{S}$ is called a subbasis. The product topology is obtained by considering all unions of finite intersection of elements in $\mathcal{S}$.

Definition 6.6. *If X is a set, a subbasis is a collection $\mathcal{S}$ of subsets whose union is X. This generates a topology by considering all unions of finite intersections of things in $\mathcal{S}$.*

7 The Subspace Topology

Definition 7.1. *Let (X, τ) be a topological space and $Y \subseteq X$ be any subset. The subspace topology on Y is*

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

Lemma 7.1. *τ_Y is a topology.*

Proof.

- $\varnothing \cap Y = \varnothing \in \tau_Y$.
- $X \cap Y = Y \in \tau_Y$.
- (Unions) Let $\{V_\alpha\}_{\alpha \in A} \subseteq \tau_Y$, then $V_\alpha = U_\alpha \cap Y$ for some $U_\alpha \in \tau$. Then,

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in A} U_\alpha \right) \cap Y \in \tau_Y$$

□

Theorem 7.1. *Suppose $\mathcal{B}$ is a basis for X and $Y \subseteq X$. Then,*

$$\{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Example 7.1. Let $X = \mathbb{R}$ with $\mathcal{B} = \{(a, b) | a, b \in \mathbb{R}\}$. Consider

$$Y = [1, \infty).$$

Then, the basis for this topology is given by

$$\mathcal{B}_Y = \{[1, a) | a \in \mathbb{R}, a > 1\} \cup \{(a, b) | a, b \in \mathbb{R}, 1 \leq a < b\}$$

Example 7.2. $(3, 7] \notin \tau_Y$ because there does not exist open $U \in \tau$ ($U \in \mathcal{B}$) such that

$$(3, 7] = U \cap [1, \infty).$$

Proof. To yield contradiction, suppose there exists such U. Then,

$$7 \in (a, b) \cap [1, \infty).$$

So $7 \in (a, b)$ and $7 \in [1, \infty)$. In other words, $a < 7 < b$.

Let $\epsilon = (b - 7)/2$. Then, $a < 7 + \epsilon < b$ so $7 + \epsilon \in (a, b)$ and $7 + \epsilon \in [1, \infty)$. Then,

$$7 + \epsilon \in (a, b) \cap [1, \infty) = (3, 7] \implies 3 < 7 + \epsilon \leq 7,$$

yielding contradiction. □

Example 7.3. Let $X = \mathbb{R}$ and $Y = [0, 1]$. A basis for the subspace on Y is

$$\{(a, b)|0 \leq a < b \leq 1\} \cup \{[a, b)|0 \leq b \leq 1\} \cup \{(a, 1]|0 \leq a \leq 1\}$$

Example 7.4. Consider a rectangle

$$R = [0, 1] \times [2, 6].$$

There are two ways of thinking of topology on this rectangle. Since $R \subseteq \mathbb{R}^2$, we can think of it as a subspace topology. We can also think of it as a product topology. So are they same topologis in this case?

Theorem 7.2. *Suppose X and Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$ are subspaces. Then, the product topology on $A \times B$ equals the topology $A \times B$ inherits as a subset of $X \times Y$.*

Proof. In the subspace topology, basis element is given by

$$(A \times B) \cap (U \times V)$$

where $U \subseteq X$ and $V \subseteq Y$. Likewise, basis element of product topology is given by

$$(A \cap U) \times (B \cap V).$$

But then

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V).$$

□

8 The Order Topology

Let X be a set.

Definition 8.1. *An order relation (or simple order or linear order) is relation R on X so*

1. *(comparability) If $x, y \in X$ and $x \neq y$. Then either xRy or yRx.*
2. *(Nonreflexivity) $\nexists x \in X$ such that xRx.*
3. *(Transitivity)s If xRy and yRz then xRz.*

Example 8.1. Define $X \subseteq \mathbb{R}$ and $R =<$. Then, R is an order relation

Proof.

1. If $x \neq qy$, either $x < y$ or $y < x$.
2. $\nexists x \in X$ such that $x < x$.
3. $x < y, y < z \implies x < z$.

$\square$

If X has an order relation, we often denote it $<$. Suppose $(X, <)$ is a set with an order relation for $a, b \in X$.

Definition 8.2. $(a, b) = \{x \in X | a < x, x < b\}$.

Definition 8.3. $(a, b] = \{x \in X | a < x, x < b,$ *or* $x = b\}$.

Definition 8.4. $[a, b) = \{x \in X | a < x, x < b,$ *or* $x = a\}$.

Definition 8.5. $a_0 \in X$ *is a minimal element if* $\forall x \in X$, $a_0 \leq x$.

Definition 8.6. $b_0 \in X$ *is a maximal element if* $\forall x \in X$, $x \leq b_0$.

Definition 8.7. *Suppose $(X, <_x)$ and $(Y, <_y)$ are simply ordered sets. The dictionary order on $X \times Y$ is the order $<_{x \times y}$ defined as follows:*

$$(a_1, b_1) <_{X \times Y} (a_2, b_2)$$

if $a_1 <_x a_2$ or $a_1 = a_2$ and $b_1 <_y b_2$.

Example 8.2. Define $X = Y = \mathbb{R}$ and $<_x = <_y = <$, usual inequality. Then,

- $(0, 0) <_{X \times Y} (1, -3)$ is true
- $(0, 0) <_{X \times Y} (0, -3)$ is false.

Definition 8.8. *Suppose $(X, <)$ is a simply ordered set. Then, the order topology on X is the one with basis elements: (a, b) for $a, b \in X$, $[a_0, b)$ for $b \in X$ provided a_0 is the minimal element of X, and $(a, b_0]$ for $a \in X$ provided $b_0 \in X$ is the maximal element.*

Example 8.3. Consider $X = [0, 1]$ with $<$, the standard inequality. The order topology is the topology inherited as a subset.

Example 8.4. Consider $(\mathbb{R}^2, <_o)$ where $<_0$ is the dictionary order. The order topology is called the dictionary order topology.

Think about what this set looks like:

$$((0, 0), (1, 1))$$

Example 8.5. Let X be a set with the discrete topology (all subsets are open). Show that

$$\mathcal{B}_1 = \{\{x\}|x \in X\}$$

is a basis.

Proof. Let $x \in X$ then $x \in \{x\} \subseteq B$. Let $B_1, B_2 \in \mathcal{B}_1$ and $x \in B_1 \cap B_2$. Then,

$$B_1 = \{x\} = B_2$$

so $x \in \{x\} \subseteq B_1 \cap B_2$

Let τ_1 be the topology generated by $\mathcal{B}_1$. We clearly have $\tau_1 \subseteq \tau_d$, where τ_d is the discrete topology. To show that $\tau_d \subseteq \tau_1$, let $U \in \tau_d$ and let $x \in U$. Then, $x \in \{x\} \subseteq U$ and $\{x\} \in \mathcal{B}_1$. So the results follows by a theorem in section 13. $\square$

Example 8.6. Show that

$$\mathcal{B}_2 = \{\{x\} \times (a, b)|x, a, b \in \mathbb{R}\}$$

forms a basis for $\mathbb{R} \times \mathbb{R}$ with the dictionary order topology.

Proof. Assume that it's a basis. We will show that the associated topology τ_2 is the dictionary order topology τ_{DO}.

Note that if $B \in \mathcal{B}_2$, then $\exists x, a, b \in \mathbb{R}$ such that

$$B = \{x\} \times (a, b) = ((x, a), (x, b)) = \{y \in \mathbb{R} \times \mathbb{R}|(x, a) <_{DO} y <_{DO} (x, b)\}.$$

So this is open in τ_{DO}. So $\tau_2 \subseteq \tau_{DO}$.

Let $((a_1, b_1), (a_2, b_2))$ be a basis element in the dictionary order topology. Let $y \in ((a_1, b_1), (a_2, b_2))$. Write $y = (x_1, y_1)$. There are three cases:

1. $a_1 < x_1 < a_2$
2. $a_1 = x_1$
3. $a_2 = x_1$

Here, we will prove case 1. Assume $a_1 < x < a_2$. Then,

$$y \in \{x\} \times (y_1 - 1, y_1 + 1)$$

Then, $\{x\} \times (y_1 - 1, y_1 + 1) \in \mathcal{B}_2$ and also

$$\{x\} \times (y_1 - 1, y_1 + 1) \subseteq ((a_1, b_1), (a_2, b_2))$$

Assuming these the topology on $\mathbb{R}_d \times \mathbb{R}$ has basis

$$\{\{x\} \times (a, b) | x, a, b \in \mathbb{R}\}.$$

This is exactly the basis that generates the dictionary order topology. So two topologies are same. $\square$

Example 8.7. Let $X = \mathbb{R}$. The set

$$\mathcal{B}_\ell = \{[a, b) | a, b \in \mathbb{R}\}$$

is a basis. The topology it generates is called the lower limit topology on $\mathbb{R}$. $\mathbb{R}$ with the lower limit topology is denoted $\mathbb{R}_\ell$.

Remark. Lower limit topology is strictly finer than standard topology, i.e., every open set in standard topology is open in lower limit topology but not vice-versa

Example 8.8. Is $[0, 1)$ open in the standard topology? No $[0, 1)$ is not open in standard topology. If it were, then there would be some $(a, b) \subseteq \mathbb{R}$ such that

$$0 \in (a, b) \subseteq [0, 1).$$

But then

$$0 \in (a, b) \implies a < 0, b > 0$$

So $a/2 \in (a, b)$. But $a/2 < 0$ so $a \notin [0, 1)$.

Example 8.9. Is $(0, 1)$ open in $\mathbb{R}_\ell$? Yes because

$$(0, 1) = \bigcup_{x \in \{y | y > 0\}} [x, 1).$$

9 Interior and Closure

Recall De Morgan's Laws:

- $\bigcup_{\alpha\in I}(X \setminus U_\alpha) = X \setminus \bigcap_{\alpha\in I} U_\alpha$
- $\bigcap_{\alpha\in I}(X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in I} U_\alpha$

Definition 9.1. *If (X, τ) is a topological space then a subset $A \subseteq X$ is called closed if $X \setminus A$ is open, i.e., if $X \setminus A \in \tau$*

If $\mathcal{C}$ is a collection of all closed sets in X,

1. $X \in \mathcal{C}$
2. $\varnothing \in \mathcal{C}$
3. $\mathcal{C}$ is closed under arbitrary intersection.
4. $\mathcal{C}$ is closed under finite union.

Example 9.1. In $\mathbb{R}$, consider $U_n = [1/n, 1 - 1/n]$ for $n \in \mathbb{Z}^+$. U_n is a closed interval for each $n \in \mathbb{Z}^+$. Hoever,

$$\bigcup_{n\in\mathbb{Z}^+} U_n = (0, 1)$$

is open and not a closed set.

Definition 9.2 (Interior)**.** *Suppose (X, τ) is a topological space and $A \subseteq X$. Define* $\mathrm{Int}(A)$ *to be the union of all open sets contained in A.*

$$\mathrm{Int}A = \bigcup_{\alpha\in J} U_\alpha$$

where $\{U_\alpha, \alpha \in J\}$ consists of all $U_\alpha \in J$ such that $U_\alpha \subseteq A$.

Observe that $\mathrm{Int}A$ is an open set. Notice also that $\mathrm{Int}(A) \subseteq A$. So $\mathrm{Int}A$ is the maximal open set contained in A. In other words, any open set V contained in A is a subset of $\mathrm{Int}A$.

Remark. If A is open, $\mathrm{Int}A = A$.

Definition 9.3. *The closure of A, denoted $\bar{A}$, is defined as the intersection of all closed sets that contain A:*

$$\bar{A} = \bigcap_{\beta\in K} C_\beta,$$

where $\{C_\beta : \beta \in K\}$ consists of all closed sets hat contain A.

Observe that $\bar{A}$ is a closed set because it is an intersection of closed sets. Notice that $A \subseteq \bar{A}$. Then, $\bar{A}$ is the smallest closed set that contains A, i.e. any closed set that contains A must contain $\bar{A}$.

Remark. If A is closed, then $A = \bar{A}$.

Example 9.2. If $U_\alpha, \alpha \in J$ is a collection of sets with $B \subseteq U_\alpha$ for all α, then $B \subseteq \bigcap_{\alpha \in J} U_\alpha$. Further, if $B = U'_\alpha$ for some $\alpha' \in J$, then

$$B = \bigcap_{\alpha \in J} U_\alpha \subset U_{\alpha'}$$

for all α'.

Proposition 9.1.

- $\mathrm{Int}(X - A) = X - \bar{A}$
- $\overline{(X - A)} = X - \mathrm{Int}(A)$.

Example 9.3. Define

$$X = \{a, b, c\}$$
$$\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$$

Then,

$$\mathcal{C} = \{X, \{b, c\}, \{c\}, \varnothing\}$$

Consider $A = \{a\}$. Then, $\mathrm{Int}A = \{a\}$ and $\bar{A} = X$.
Consider $B = \{b\}$. Then, $\mathrm{Int}B = \varnothing$ and $\bar{B} = \{b, c\}$.
Consider $C = \{c\}$. Then, $\mathrm{Int}C = X$ and $\bar{C} = \{c\}$.

Let

$$X = \mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$$

Example 9.4. Consider

$$A = \{re^{i\theta} | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1 \mathrm{or} r 1, \theta \in \mathbb{Q}\}$$

Then,

$$\mathrm{Int}A = \{re^{i\theta} | 0 < r < 1\}$$
$$\bar{A} = \{re^{i\theta} | 0 \leq r \leq 1\}$$

Notice that if $z \in \bar{A}$ and U is an arbitrary open set in $\mathbb{C}$ containing z then $U \cap A \neq \varnothing$.

Theorem 9.1. *If $A \subseteq X$ is any subset then $X \in \bar{A}$ if and only if every open set U containing X intersects A nontrivially.*

Proof. Recall $\bar{A}$ is a closed set, so $X - \bar{A}$ is open. We will show conversely that $x \notin \bar{A}$ if and only if there exists an open set U containing x such that $U \cap A = \varnothing$. But since $X - \bar{A}$ is open, if $x \notin \bar{A}$, then $x \in X - \bar{A}$. Openness of $X - \bar{A}$ now implies that there exists small open set U about x with $U \subseteq X - \bar{A}$. As such, $U \cap \bar{A} = \varnothing$, since $A \subseteq \bar{A}$, it follows that $U \cap A = \varnothing$. $\square$

Definition 9.4. *If $A \subseteq X$ is a subset,*

$$\bar{A} = \{x \in X | U \cap A \neq \varnothing \textit{ for every open set } U \textit{ about } X\}.$$

Definition 9.5 (Limit point)**.** *A point X is a limit point of the set A if every open set U about x satisfies $(U - \{x\}) \cap A \neq \varnothing$, i.e., we require U intersects A in some point other than x. We write*

$$A' = \{x | x \textit{ is a limit point of } A\}.$$

Theorem 9.2. *$\bar{A} = A \cup A'$.*

Proof.

$$(U - \{x\}) \cap A \neq \varnothing \implies U \cap A \neq \varnothing$$

which implies $A \subseteq \bar{A}$. But we already know that $A \subseteq \bar{A}$. Then,

$$A \cup A' \subseteq \bar{A}.$$

Suppose $x \in \bar{A}$. Either $x \in A$, in which case $x \in A \cup A'$ or $x \notin A$. In this case, since $x \in \bar{A}$, every open set U about satisfies $U \cap A \neq \varnothing$. But $x \notin A$ so $x \notin A$. So

$$(U - \{x\}) \cap A \neq \varnothing.$$

So $x \in A'$. □

Let X be a topological space with $Y \subseteq X$ a subset. Regard Y as a topological space using the subspace topology, i.e., $V \subseteq Y$ is open in the subspace topology if and only if there eixsts an open set $U \subseteq X$ with $V = U \cap Y$.

Theorem 9.3. *In the subspace topology on Y, a subset $B \subseteq Y$ is closed if and only if $B = A \cap Y$ for some closed set $A \subseteq X$.*

Proof.

$$\begin{aligned} B \subseteq Y \text{ is closed} &\iff V = Y - B \text{ if open} \\ &\iff \exists U \text{ open in } X \text{ such that } V = U \cap Y \\ &\iff A = X - U \text{ is closed} \end{aligned}$$

□

10 Sequences

Definition 10.1 (Convergnece). *If $\{X_n\}$ is a sequence of points in a topological space X, we say $\{X_n\}$ converges to $x \in X$ if for every neighborhood U about x, there exists integer N such that $n \geq N \implies X_n \in U$.*

Remark. In a general topological space, a sequence may converge to more than one point.

Example 10.1. Consider the trivial topology:

$$\tau_{\text{triv}} = \{\varnothing, X\}.$$

Every sequence converges and it converges to every point in X.

Example 10.2. Given $X = \{a, b, c\}$, consider the following topology:

$$\tau = \{\varnothing, \{a\}, \{a, b\}, X\}.$$

Then, the sequence $X_n = b$ for all n converges to b as well as c but not to a.

Example 10.3. If X has the discrete topology, then show that a sequence $\{X_n\}$ is convergent if and only if $\{X_n\}$ is eventually constant, i.e., $\exists x \in X$ and $N \in \mathbb{Z}^+$ such that

$$n \geq N \implies X_n = X.$$

Definition 10.2. *A topological space A is called Hausdorff if for any two distinct points $x, y \in X$ there exists disjoint open sets U, V with $x \in U$ and $Y \in V$.*

Example 10.4. If $\{X_n\}$ is a sequence of points in Hausdorff space, then if it converges, the point it converges to is unique.

Theorem 10.1. *If (X, τ) is Hausdorff, then every finite subset of X is a closed set.*

Remark. If X is Hausdorff, then every singleton $\{x\}$ is closed.

Example 10.5. Consider

$$\mathcal{C} = \left\{ \left[\frac{1}{n}, 1 - \frac{1}{n} \right] | n \geq 3 \right\}$$

Then,

$$\bigcup_{n=3}^{\infty} C_n = (0, 1)$$

is not closed in $\mathbb{R}$.

Example 10.6. An infinite collection of open sets whose intersection is not necessarily open. Consider

$$U_n = \left(-\frac{1}{n}, \frac{1}{n} \right).$$

Then,

$$\bigcap_{n=1}^{\infty} U_n = \{0\}$$

is closed in standard topology.

Definition 10.3. *A topological space X in which all finite sets are closed is called a T_1 space.*

Theorem 10.2. *If (X, τ) is Hausdorff, then X is a T_1 space. Note that converse is not necessarily true.*

Example 10.7. If X is inifinite and τ is finite complement topology is a T_1 space but not Hausdorff.

Example 10.8. Notice if X is finite, then the every subset $U \subseteq X$ has a finite complement. So in this case, the finite complement topology is the same as the discrete topology. So it is a T_1 space and Hausdorff.

Theorem 10.3. *If X is Hausdorff and $\{x_n\}$ is a convergent sequence converging to $x \in X$, then the point x is unique. In this case, we say x is the limit of the sequence $\{x_n\}$ and write*

$$\lim_{n\to\infty} x_n = x.$$

Theorem 10.4. *If $(X, <)$ is a simply ordered set, then it is Hausdorff in the order topology.*

Theorem 10.5. *If X, Y are two topological spaces annd both are Hausdorff then so is the product $X \times Y$.*

Theorem 10.6. *If X is Hausdorff and $Y \subseteq X$ is a subset, then Y is Hausdorff in subspace topology.*

11 Continuous Functions

What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous?

- $\forall x \in \mathbb{R}$, $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$.
- $\lim_{nto\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)$.
- $\forall x_0 (\forall \epsilon > 0, \exists \delta$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

Definition 11.1. *If $f : X \to Y$ is a function between two topological spaces X, Y, then f is continuous if $f^{-1}(V)$ is open for every open set $V \subseteq Y$.*

Example 11.1. Consider $f : \mathbb{R} \to \mathbb{R}_\ell$, where $\mathbb{R}_\ell$ is the lower limit topology and $f(x) = x$. This map is not continuous. Take $V = [a, b)$. This is open in $\mathbb{R}_\ell$ but $f^{-1}(V) = V$ is not open in the standard topology.

Example 11.2. $g : \mathbb{R}_\ell \to \mathbb{R}$ with $g(x) = x$ is continuous.

Example 11.3. If $f : X \to Y$ is the constant map $f(x) = y_0 \forall x \in X$, then f is continuous for any topology on X, Y.

Theorem 11.1. *Let X, Y be two toplogical spaces and $f : X \to Y$ is a function. If $f : X \to Y$ then the following are equivalent:*

1. *f is continuous*
2. *for all subsets $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$*
3. *for all closed sets $B \subseteq Y$, $f^{-1}(B)$ is closed in X*
4. *for each $x \in X$ if V is open about $f(x)$, then $\exists$ open set U about x such that $f(U) \subseteq V$.*

Example 11.4. Consider $f : X \to Y$ any function. If X has discrete topology, f is automatically continuous.

Proof. Let $U \subseteq Y$ be open. Then, $f^{-1}(U)$ is a subset of X. By the definition of the discrete topology, $f^{-1}(U)$ is therefore open in X. Thus, it is true for all open $U \subseteq Y$, so f is continuous. $\square$

Example 11.5. Consider $Y = [0, 1) \subseteq \mathbb{R}$ with standard topology on $\mathbb{R}$. Find the closure $\bar{Y}$.

We can define closure in three different ways:

1. Intersection of all closed sets containing Y
2. $\bar{Y} = Y \cup \{\text{limit points}\}$
3. $\bar{Y}$ is the smallest closed set containing Y.

Note that $[0,1) \subseteq [0,1]$ and $[0,1]$ is closed in standard topology ($[0,1] = \mathbb{R} - ((-\infty,0) \cup (1,\infty))$). This implies that $\bar{Y} \subseteq [0,1]$.

We also have $[0,1) \subseteq \bar{Y}$. If $\bar{Y} \neq [0,1]$, then $1 \notin \bar{Y}$ and $\bar{Y} = [0,1)$. However, $[0,1)$ is not closed in standard topology and this is a contradiction. Therefore, $Y = [0,1]$.

Example 11.6. Define $X = [0,\infty)$. Consider $X \times X$ with dictionary order topology. Show that $X \times X$ is Hausdorff.

Proof. Let $(x_1,y_1),(x_2,y_2) \in X \times X$ with $(x_1,y_1) \neq (x_2,y_2)$. We may assume $(x_1,y_1) < (x_2,y_2)$. There are 7 cases that we have to consider:

1. $0 < x_1 < x_2$ and $y_1, y_2 > 0$
2. $0 = x_1 < x_2$ and $y_1, y_2 > 0$
3. $0 = x_1 = x_2$ and $0 < y_1 < y_2$
4. $0 = x_1 = x_2$ and $0 = y_1 < y_2$
5. $0 < x_1 < x_2$ and $y_1 = 0$ or $y_2 = 0$
6. $x_1 = x_2 > 0$ and $y_2 > y_1 > 0$
7. $x_1 = x_2 > 0$ and $y_1 = 0$ or $y_2 \neq 0$

For case 1, consider

$$U_1 = \left(\left(x_1, y_1 - \frac{y_1}{2}\right), \left(x_1, y_1 + \frac{y_1}{2}\right)\right)$$
$$U_2 = \left(\left(x_2, y_2 - \frac{y_2}{2}\right), \left(x_2, y_2 + \frac{y_2}{2}\right)\right)$$

Then, there intersection is empty and $(x_1,y_1) \in U_1$ and $(x_2,y_2) \in U_2$. $\square$

Definition 11.2. *Let X and Y be topological spaces. A function $f : X \to Y$ is a homeomorphism if*

1. *f is continuous*
2. *f is bijective (so f^{-1} exists)*
3. *$f^{-1} : Y \to X$ is continuous*

Example 11.7. Suppose X is a topological space. Then, identity function

$$\mathrm{Id} : X \to X$$

with $\mathrm{Id}(x) = x$ is a homeomorphism.

Example 11.8. If $f : X \to Y$ is homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism.

Example 11.9. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$. Then, this is a homeomorphism. Furthermore,

$$f^{-1}(y) = y^{1/3}.$$

12 The Product Topology

Definition 12.1. *Let $\{X_\alpha\}$ be a collection of sets indexed by $\alpha \in J$. Then,*

$$\prod_{\alpha\in J} X_\alpha = \left\{ f : J \to \bigcup_{\alpha \in J} X_\alpha, f(\beta) \in X_\beta \forall \beta \in J \right\}.$$

Example 12.1. Consider $J = \{1, 2\}$. Then,

$$\begin{aligned}\prod_{\alpha\in J} X_\alpha &= \{f : \{1,2\} \to X_1 \cup X_2 | f(1) \in X_1, f(2) \in X_2\} \\ &= \{(f_1, f_2) | f_1 \in X_1, f_2 \in X_2\} \\ &= X_1 \times X_2\end{aligned}$$

More generally, if $J = \{1, 2, \dots, n\}$, then

$$\prod_{\alpha\in J} X_\alpha = X_1 \times X_2 \times \cdots \times X_n$$

If $J = \mathbb{Z}_+ = \{1, 2, 3, \dots\}$, then we can write

$$\prod_{\alpha\in J} = X_1 \times X_2 \times X_3 \times \cdots$$

If $J = \mathbb{Z}_+$ and $X_\alpha = X \forall \alpha \in J$, then

$$\prod_{\alpha\in J} X_\alpha = X \times X \times \cdots = X^\omega.$$

Example 12.2. Consider $J = \mathbb{R}$ and let $X_\alpha = \mathbb{R} \forall \alpha \in \mathbb{R}$. Then,

$$\begin{aligned}\prod_{\alpha\in J} X_\alpha &= \prod_{\alpha\in\mathbb{R}} \mathbb{R} \\ &= \left\{ f : \mathbb{R} \to \bigcup_{\alpha in \mathbb{R}} \mathbb{R} \,\middle|\, f(x) \in \mathbb{R} \forall x \in \mathbb{R} \right\} \\ &= \{f : \mathbb{R} \to \mathbb{R} | f \text{ is a function}\}.\end{aligned}$$

Definition 12.2 (Box topology)**.** *Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The box topology on $\prod X_\alpha$ is the one with basis*

$$\left\{ \prod_{\alpha\in J} U_\alpha \,\middle|\, U_\alpha \subseteq X_\alpha \textit{ open} \right\}$$

Lemma 12.1. *If $\mathcal{B}_\alpha$ is a basis for X_α, then*

$$\left\{ \prod_{\alpha\in J} B_\alpha \,\middle|\, B_\alpha \in \mathcal{B}_\alpha \right\}$$

is a basis for the box topology.

Definition 12.3. *For $\beta in J$, define*

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

Example 12.3. Consider $J = \{1, 2\}$. Then,

$$\prod_{\alpha \in J} X_\alpha = X_1 \times X_2.$$

Furthermore,

$$\pi_1 : X_1 \times X_2 \to X_1$$
$$\pi_2 : X_2 \times X_2 \to X_2$$

Definition 12.4 (Product topology)**.** *The product topology on $\prod_{\alpha \in J} X_\alpha$ is the one with subbasis given by*

$$\left\{ \pi_\beta^{-1}(U_\beta) \,\middle|\, U_\beta \subseteq X_\beta \text{ open }, \beta \in J \right\}$$

If $\prod X_\alpha$ is equipped with the product topology, we call $\prod X_\alpha$ the product space.

Lemma 12.2. *If $\mathcal{B}_\alpha$ is a basis for X_α, then*

$$\left\{ \pi_\beta^{-1}(B_\beta) \,\middle|\, B_\beta \in \mathcal{B}_\beta, \beta \in J \right\}$$

is a subbasis for the product topology.

Remark. If J is finite, then the box topology on $\prod X_\alpha$ equals the product topology.

Example 12.4. Consider $J = \mathbb{Z}_+$ with $X_\alpha = \mathbb{R}$. Then,

$$\prod_{\alpha \in J} X_\alpha = \mathbb{R}^\omega.$$

For example, a box topology can be

$$\prod_{\alpha \in J} B_\alpha = B_1 \times B_2 \times B_3 \times \cdots .$$

Then,

$$(-1, 1) \times (-2, 2) \times \cdots \times (-n, n) \times \cdots$$

is a basis element in box topology.

On the other hand,

$$(-1, 1) \times (-2, 2) \times \cdots \times (-n, n) \times \cdots$$

is not open in product topology.

$$(-1, 1) \times (-2, 2) \times \cdots \times (-n, n) \times \mathbb{R} \times \cdots \times \mathbb{R} \cdots$$

is open in product topology.

Lemma 12.3. *The collection*

$$\left\{ \prod_{\alpha \in J} U_\alpha \,\middle|\, U_\alpha \subseteq X_\alpha \text{ is open }, U_\alpha = X_\alpha \text{ for all but a finite number of } \alpha \in J \right\}$$

forms a subbasis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

Example 12.5.

$$(-1,1) \times (-1,1) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

is open in the product topology on $\mathbb{R}^\omega$ but

$$(-1,1) \times (-1,1) \times \cdots \times (-1,1) \times \cdots$$

is not open in product topology.

If we define $\pi_1 : \mathbb{R}^\omega \to \mathbb{R}$ where $\pi_1(x_1, x_2, \ldots,) \to x_1$. Then,

$$\pi_1^{-1}((-1,1)) = (-1,1) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

is open in product topology. Likewise, we have

$$\pi_2^{-1}((-1,1)) = \mathbb{R} \times (-1,1) \times \mathbb{R} \times \cdots$$

open in product topology. Then,

$$\pi_1^{-1}((-1,1)) \times \pi_2^{-1}((-1,1)) = (-1,1) \times (-1,1) \times \mathbb{R} \times \mathbb{R} \times \cdots .$$

is open in product topology. Note that we can only take finite intersections.

Example 12.6. Consider $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ with the product topology. Show

$$(-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdots$$

is not open.

Solution. Recall that U is open if and only if $\forall x \in U$ there exists $B \in \mathcal{B}$ a basis element such that

$$x \in B \subseteq U.$$

Observe that

$$(0,0,\ldots,0) \in U = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdot.$$

Assume B is a basis element containing x. Then, we can write

$$B = V_1 \times V_2 \times \cdots \times V_m \times \mathbb{R} \times \mathbb{R} \times \cdots$$

where $V_l \subseteq \mathbb{R}$ open and $1 \leq l \leq m$. If $B \subseteq U$, then the following condition must be satisfied:

$$\mathbb{R} \subseteq \left(-\frac{1}{m+1}, \frac{1}{m+1}\right).$$

This is false. So $B \not\subseteq U$. Therefore, U is not open. $\square$

Example 12.7. Is the following function continuous? $f : \mathbb{R} \to \mathbb{R}^2$ where

$$f(t) = (\cos(t), t^2 + e^t)$$

Theorem 12.1. *Let $\{X_\alpha\}$ be topological space and give $\prod X_\alpha$ the product topology. Let*

$$f : A \to \prod X_\alpha$$

be any function. Then, f is continuous if and only if $\pi_\alpha \circ f : A \to X_\alpha$ is continuous $\forall \alpha \in J$.

Example 12.8. Let $J = \mathbb{Z}_+$. Then,

$$\prod X_\alpha = X_1 \times X_2 \times \cdots$$

Consider $f : A \to X_1 \times X_2 \times \cdots$ with

$$\begin{aligned} f(a) &= (f_1(a), f_2(a), \dots) \\ f_1 &= \pi_1 \circ f \\ f_2 &= \pi_2 \circ f \\ &\vdots \end{aligned}$$

Note that $f_\alpha = \pi_\alpha \circ f$ is the α-th component of f.

Example 12.9. Recall that the product topology has subbasis

$$\mathcal{S} = \{\prod_\alpha^{-1}(U) \,|\, U \subseteq X_\alpha \text{ open }, \alpha \in J\}$$

So $\pi_\beta : \prod X_\alpha \to X_\beta$is continuous $\forall \beta \in J$.

Now, we will prove the theorem.

Proof. ($\Rightarrow$) If f is continuous, then $\pi_\alpha \circ f$ is continuous because composition of continuous functions is continuous.

($\Leftarrow$) Assume $\pi_\alpha \circ f$ is continuous $\forall \alpha \in J$. Let $\pi_\alpha^{-1}(U) \subseteq \prod X_\alpha$ be a subbasis element. Then,

$$f^{-1}\left(\pi_\alpha^{-1}(U)\right) = (\pi_\alpha \circ f)^{-1}(U).$$

Then, this is open because the composition map $(\pi_\alpha \circ f)$ is assumed to be continuous and $U \subseteq X_\alpha$ is open. $\square$

Recall that a basis for product topology is

$$\prod_{\alpha \in J} U_\alpha$$

where $U_\alpha \subseteq X$ where $U_\alpha = X_\alpha$ for all but a finite number of $\alpha \in J$. Note

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = U_\beta$ if $\alpha = \beta$ and $U_\alpha = X_\alpha$ if $\alpha \neq \beta$.

Example 12.10. Consider $\prod X_\alpha = \mathbb{R} \times \mathbb{R}$. Then,

$$\pi_1^{-1}((-1,1)) = (-1,1) \times \mathbb{R}.$$

On the other hand, the basis from $\mathcal{S}$ is

$$\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}) = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = U_{\alpha_j}$ if $\alpha = \alpha_j$ and $U_\alpha = X_\alpha$ otherwise. Hence,

$$x \in \pi_\beta^{-1}(U_\beta)$$

if and only if

$$\pi_\beta(x) \in U_\beta$$

if and only if the β-composition of x is in U_β (no restriction on the other components).

Example 12.11. Consider $f : \mathbb{R} \to \mathbb{R}^\omega$ where

$$f(t) = (t, t, t, \dots).$$

By the previous theorem, this is continuous in the product topology on $\mathbb{R}^\omega$. However, this is not continuous with the box topology on $\mathbb{R}^\omega$.

Proof. Consider

$$U = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdots$$

We will show that $f^{-1}(U)$ is not open in $\mathbb{R}$.

We claim that $f^{-1}(U) = \{0\}$. If $t \in f^{-1}(U)$, then $f(t) \in U$. Note that

$$(t, t, t, \dots) \in U$$

So

$$\begin{aligned} &t \in (-1,1) \\ &t \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ &\vdots \\ &t \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ &\vdots \end{aligned}$$

So $|t| < 1/n \, \forall n$. This implies that $t = 0$. Conversely, $f(0) \in U$. ∎

13 Metric Topology

Definition 13.1. *Let X be a set. A metric on X is a function $d : X \times X \to \mathbb{R}$ satisfying*

1. $d(x, y) \geq 0 \forall x, y \in X$ *and* $d(x, y) = 0 \iff x = y$
2. $\forall x, y \in X, d(x, y) = d(y, x)$
3. *(triangle inequality)* $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$

Definition 13.2. (X, d) *is a metric space.*

We can think $d(x, y)$ as the "distance" between x and y.

Definition 13.3. *Let $x \in X$ and $\epsilon > 0$. We can define*

$$B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\},$$

the ball of radius epsilon centered at x, relative to the metric d.

Example 13.1. Define $d : X \times X \to \mathbb{R}$ as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

We want to verify three axioms.

1. $1 \leq 0$ and $d(x, y) = 0$ iff $x = y$
2. $x = y \iff y = x$ and $x \neq y \iff y \neq x$
3. Note that $d(x, z), d(x, y), d(y, z)$ are all either equal to 0 or 1. The triangle inequality can only fail if

   $$d(x, z) = 1 \text{ and } d(x, y) + d(y, z) = 0$$

   Note that

   $$d(x, y) + d(y, z) = 0 \implies d(x, y) = 0, d(y, z) = 0 \implies x = y, y = z$$

   This implies that $x = z$ by transitivity and $d(x, z) = 0$. This yields contradiction because we assumed $d(x, z) = 1$. So the triangle inequality does not fail.

Example 13.2. Let $X = \mathbb{R}$ and define d as above. Sketch the ball with radius 1 centered at 0. Sketch $B_d(0, 2)$.

Note that

$$B_d(0, 1) = \{y \in \mathbb{R} | d(0, y) < 1\} = \{0\}$$

and

$$B_d(0, 2) = \{y \in \mathbb{R} | d(0, y) < 2\} = \mathbb{R}.$$

Example 13.3. Let $X = \mathbb{R}$ and define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where

$$d(x, y) = |x - y|.$$

We want to check that d is a metric.

1. $d(x, y) = |x - y| \geq 0$. Furthermore,

$$d(x, y) = 0 \iff x - y = 0 \iff x = y.$$

2. $$d(x, y) = |x - y| = |-y + x| = |-(y - x)| = |y - x| = d(y, x)$$

3. $$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$$

Example 13.4. Define d as in previous example. Then,

$$B_l(3, \epsilon) = \{y \in \mathbb{R} | d(3, y) < \epsilon\} = \{y \in \mathbb{R} || 3 - y| < \epsilon\}$$

Note that $|3 - y| < \epsilon$ is equivalent to $3 - y < \epsilon$ and $-(3 - y) < \epsilon$. Rearranging, we have

$$B_l(3, \epsilon) = \{y \in \mathbb{R} | -\epsilon + 3 < y < \epsilon + 3\} = (3 - \epsilon, 3 + \epsilon)$$

Lemma 13.1. *Let (X, d) be a metric space. The collection*

$$B = \{B_l(X, \epsilon) | x \in X, \epsilon \in (0, \infty)\}$$

forms a basis. The induced topology is called the metric topology if (X, τ) is a topological space and τ is induced from a metric as above ($\exists$ a metric d so τ is in the metric associated to d then we call (X, τ) metrizable)

Example 13.5. Consider X with

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

The metric topology is the discrete topology. $B_d(x, 1) = \{x\}$ are contained in the basis.

Example 13.6. Consider $X = \mathbb{R}$ with $d(x, y) = |x - y|$. Then, the metric topology is the standard topology and the basis is $B_d(x, \epsilon) = (x - \epsilon, x + \epsilon)$.

Example 13.7. Consider $X = \mathbb{R}^n$ where $n \geq 1$. Then, for $x, y \in X$, we can write

$$x = (x_1, x_2, \ldots, x_n)$$
$$y = (y_1, y_2, \ldots, y_n)$$

Let

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Lemma 13.2. *This is a metric called the Euclidean metric. For $n = 2$, $B_d(x, \epsilon)$ is a circle centered at x with radius ϵ.*

Example 13.8. Consider $X = \mathbb{R}^n$ and define

$$\rho(x, y) = (|x_1 - y_1|, \dots, |x_n - y_n|)$$

When $n = 2$ $B_\rho(X, \epsilon)$ forms a square whose sidelength is 2ϵ. For example,

$$B_\rho(0, \epsilon) = \{y | \max(|y_1|, |y_2|) < \epsilon\}$$

Example 13.9. Show every metric space is Hausdorff.

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Set $\epsilon = \frac{1}{2}d(x_1, x_2)$. Since $x_1 \neq x_2$, so $\epsilon > 0$. Hence,

$$x_1 \in B_d(x_1, \epsilon) \text{ and } x_2 \in B_d(x_2, \epsilon)$$

Also,

$$B_d(x_1, \epsilon) \cap B_d(x_2, \epsilon) = \varnothing$$

because if $z \in B_d(x_1, \epsilon) \cap B_d(x_2, \epsilon)$, then

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, z) + d(z, x_2) \\ &< \epsilon + \epsilon \\ &= d(x_1, x_2) \end{aligned}$$

which is a contradiction. $\square$

14 Comparing metrics

Consider $X = \mathbb{R}^n$ and $x, y \in X$ such that

$$\begin{aligned} x &= (x_1, \dots, x_n) \\ y &= (y_1, \dots, y_n) \end{aligned}$$

Recall that Euclidean metric is defined as

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n + y_n)^2}$$

and the square metric is defined by

$$\begin{aligned} \rho(x, y) &= \max\{|x_1 - y_1|, \dots, |x_n - y_n|\} \\ &= \max_i\{|x_i - y_i|\} \end{aligned}$$

Consider $B_d(0, \epsilon)$. By definition,

$$\begin{aligned} B_d(0, \epsilon) &= \{y | d(0, y) < \epsilon\} \\ &= \{(y_1, \dots, y_n) | \sqrt{y_1^2 + \cdots + y_n^2} < \epsilon\} \\ &= \{(y_1, \dots, y_n) | y_1^2 + \cdots + y_n^2 < \epsilon^2\} \end{aligned}$$

is the interior of a radius-ϵ sphere. On the other hand,

$$\begin{aligned} B_\rho(0,\epsilon) &= \{(y_1,\ldots,y_n)|\max_i\{|x_i-y_i|\}\} \\ &= (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times\cdots\times(-\epsilon,\epsilon) \end{aligned}$$

Hence, when $n=2$, $B_d(0,\epsilon)$ is a circle of radius ϵ and $B_\rho(0,\epsilon)$ is a square whose side lengths are 2ϵ. We know how to compare two toplogies generated by these metrics but how can we compare the metrics?

Lemma 14.1. *Suppose d and d' are metrics on X and τ and τ' are their respective topologies. The following are equivalent:*

1. $\tau \subseteq \tau'$
2. $\forall x\in X, \forall\epsilon>0, \exists\delta>0$ *such that* $B_{d'}(x,\delta)\subseteq B_d(x,\epsilon)$.

Proof. Assume $\tau\subseteq\tau'$. Let $x\in X$ and $\epsilon>0$. By Lemma 13.3, there exists a basis element B' for τ' so

$$x\in B'\subseteq B_d(x,\epsilon).$$

We have $B'=B_{d'}(y,r)$ for some $y\in X$ and $r>0$. Let $\delta=r-d'(x,y)$. Since $x\in B_{d'}(y,r)$, we have $\delta>0$.

We claim that $B_{d'}(x,\delta)\subseteq B_{d'}(y,r)$. Let $z\in B_{d'}(x,\delta)$. Then,

$$\begin{aligned} d'(z,y) &\leq d'(,x)+d'(x,y) \\ &< \delta+d'(x,y) \\ &= r \end{aligned}$$

This implies that $z\in B_{d'}(y,r)$. Hence,

$$B_{d'}(x,\delta)\subseteq B'\subseteq B_d(x,\epsilon)$$

So this proves the second statement.

Now, assume that the second statement is true. Let $B_d(x,\epsilon)$ be a τ-basis element. Let $y\in B_d(x,\epsilon)$ then the assumed statement implies that we can find $\delta>0$ such that

$$y\in B_{d'}(y,\delta)\subseteq B_d(x,\epsilon)$$

Then, $\tau\subseteq\tau'$ follows by Lemma 13.3. ☐

Theorem 14.1. *The topologies on $\mathbb{R}^n$ coming from the Euclidean metric and the square metric are equivalent. These both equal the product topology.*

Proof. Let τ_ρ be the topology coming from the square metric ρ and τ_d that coming from the euclidean metric d. Note

$$\begin{aligned} \rho(x,y) &= \max_i\{|x_i-y_i|\} \\ &= |x_{i_*}-y_{y_*}| \text{ for some } i_*\in\{1,2,\ldots,n\} \\ &= \sqrt{(x_{i_*}-y_{i_*})^2} \\ &\leq \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2} \\ &= d(x,y) \end{aligned}$$

So

$$\begin{aligned} B_d(x,\epsilon) &= \{y|d(x,y)<\epsilon\} \\ &\subseteq \{y|\rho(x,y)<\epsilon\} \\ &= B_\rho(x,\epsilon). \end{aligned}$$

By the previous lemma, $\tau_\rho \subseteq \tau_d$.

For the reverse inequality,

$$\begin{aligned} d(x,y) &= \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2} \\ &= \sqrt{(x_{i_*}-y_{i_*})^2+\cdots+(x_{i_*}-y_{i_*})^2} \\ &= \sqrt{n(x_{i_*}-y_{i_*})^2} \\ &= \sqrt{n}\rho(x,y). \end{aligned}$$

This implies that $B_\rho(x,\epsilon/\sqrt{n}) \subseteq B_d(x,\epsilon)$. Therefore, $\tau_d \subseteq \tau_\rho$

Finally, we want to prove that ρ-topology equals the product topology. Let $(a_1,b_1)\times\cdots\times(a_n,b_n)$ be a product topology basis element and

$$(x_1,\ldots,x_n) \in (a_1,b_1)\times\cdots\times(a_n,b_n).$$

Set

$$\epsilon = \min_{i\in\{1,\ldots,n\}}\{|x_i-a_i|,|x_i-b_i|\}$$

Then,

$$\begin{aligned} (x_1,\ldots,x_n) &\in (x_1-\epsilon,x_1+\epsilon)\times\cdots\times(x_n-\epsilon,x_n+\epsilon) \\ &\subseteq (a_1,b_1)\times\cdots\times(a_n,b_n), \end{aligned}$$

where

$$(x_1-\epsilon,x_1+\epsilon)\times\cdots\times(x_n-\epsilon,x_n+\epsilon) = B_\rho((x_1,\ldots,x_n),\epsilon)$$

Conversely, let

$$B_\rho((x_1,\ldots,x_n),\epsilon) = (x_1-\epsilon,x_1+\epsilon)\times\cdots\times(x_n-\epsilon,x_n+\epsilon)$$

and $y \in B_\rho(x,\epsilon)$. Then,

$$y \in B_\rho(x,\epsilon) \subseteq B_\rho(x,\epsilon)$$

□

Example 14.1. For $1 \le p < \infty$, then

$$d_p((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = ((x_1-y_1)^p+\cdots+(x_n-y_n))^{1/p}$$

on $\mathbb{R}^n$. This is a metric on $\mathbb{R}^n$.

We want to draw $B_{d_p}(0,1)$ for different values of p. When $p=1$, we have a rhombus. When $p=2$, we have a circle. When $p>2$, we get a "fat" circle. Note that

$$B_{d_p}(0,1) = \{(x,y)|x^p+y^p=1\} \implies y = \pm(1-x^p)^{1/p}$$

As $p \to \infty$, we get a square. Hence,

$$\lim_{p\to\infty} d_p(x,y) = \rho(x,y)$$

Example 14.2. Consider

$$X = C^0([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

For $1 \le p < \infty$, define

$$d_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p dx\right)^{1/p}.$$

This is a metric on X. When $p = 2$, d_2 is very important in physics.

Theorem 14.2. *Suppose (X, d_X) and (Y, d_Y) are metric spaces (with the metric topology). Let $f : X \to Y$ be a function. Then, f is continuous if and only if*

$$\forall x \in X \ \forall \epsilon > 0 \ \exists \delta > 0 \textit{ such that } d_X(x,y) < \delta \implies d_y(f(x), f(y)) < \epsilon$$

Proof. Assume f is continuous. Let $x \in X$ and $\epsilon > 0$. Since f is continuous and $B_{d_y}(f(x), \epsilon)$ is open, it follows that $f^{-1}(B_{d_y} f(x), \epsilon)$ is open in X. Note that

$$x \in f^{-1}(B_{d_y} f(x), \epsilon) \implies \exists \delta > 0$$

such that

$$x \in B_{d_x}(x, \delta) \subseteq f^{-1}(B_{d_y} f(x), \epsilon).$$

Assume $y \in X$ satisfies $d_x(x,y) < \delta$. Then, $y \in B_{d_x}(x, \delta)$, so

$$y \in f^{-1}(B_{d_y} f(x), \epsilon).$$

Hence, $f(y) \in B_{d_y}(f(x), \epsilon)$. So

$$d_y(f(x), f(y)) < \epsilon.$$

Conversely, assume the $\delta - \epsilon$ condition. Let $U \subseteq Y$ be open. If $U = \varnothing$ then $f^{-1}(\varnothing) = \varnothing$ is open. More generally, if $f^{-1}(U) = \varnothing$, then we're done. We may assume $\exists x \in f^{-1}(U)$. Note $f(x) \in U$ since U is open. Then, there exists $\epsilon > 0$ such that

$$f(x) \in B_{d_y}(f(x), \epsilon) \subseteq U.$$

The $\delta - \epsilon$ condition implies that there exists $\delta > 0$ such that

$$d_x(x,y) < \delta \implies d_y(f(x), f(y)) < \epsilon.$$

So $B_x(x, \delta) \subseteq f^{-1}(U)$. That is, for every $x \in f^{-1}(U)$, there exists $\delta > 0$ so that

$$x \in B_{d_x}(x, \delta) \subseteq f^{-1}(U).$$

So $f^{-1}(U)$ is open. $\square$

Example 14.3. Show the function $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$m(a, b) = a + b$$

is continuous.

Proof. Let $d_{\mathbb{R}\times\mathbb{R}}$ be a square metric. Consider $(a, b), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$. Then,

$$d_{\mathbb{R}\times\mathbb{R}}((a, b), (a_2, b_2)) < \delta \implies |a - a_2| + |b - b_2| < \delta.$$

Note that

$$\begin{aligned} d_{\mathbb{R}}(m(a, b), m(a_2, b_2)) &= |m(a, b) - m(a_2, b_2)| \\ &= |a + b - a_2 - b_2| \\ &= |a - a_2 + b - b_2| \\ &\leq |a - a_2| + |b - b_2| < \delta < \epsilon \end{aligned}$$

$\square$

Similarly, we can show that subtraction, multiplication, and divisions (as long as we don't divide by 0) are continuous functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Corollary 14.1. *Suppose $f, g : X \to \mathbb{R}$ are continuous where X is a topological space. Then,*

$$\begin{aligned} f + g : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto f(x) + g(x) \\ f - g : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto f(x) - g(x) \\ f \cdot g : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto f(x)g(x) \end{aligned}$$

are continuous with the product topologyon $X \times X$.

Proof. Note that $f + g$ is the composition of

$$\begin{aligned} X \times X &\to \mathbb{R} \times \mathbb{R} \\ (x, y) &\mapsto (f(x), g(y)) \\ \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (s, t) &\mapsto s + t \end{aligned}$$

These are continuous so the composition is continuous. $\square$

Definition 14.1. *Let X be a set and (Y, d) a metric space. Suppose*

$$f_n : X \to Y$$

is a sequence of functions. Then, $(f_n)_{n\in\mathbb{Z}_+}$ converge uniformly to $f : X \to Y$ if

$$\forall \epsilon > 0 \ \exists N \in \mathbb{Z}_+ \ \forall n \geq N \ \forall x \in X \ d(f_n(x), f(x)) < \epsilon$$

Theorem 14.3. *Sppose X is a topological space and (Y, d) is a metric space. Suppose (f_n) is a sequence of functions converging uniformly to $f : X \to Y$. If each f_n is continuous then f is continuous.*

Proof. Let $V \subseteq Y$ be open. We want to show that $f^{-1}(V) \subseteq X$ is open. It is sufficient to show that

$$\forall x_0 \in f^{-1}(V)\ \exists U \text{ open in } X \text{ such that } x_0 \in U \subseteq f^{-1}(V)$$

Consider $V = B_d(f(x_0), \epsilon) \subseteq Y$. Then, take

$$U = f_n^{-1}(B_d(f_n(x_0), \epsilon/3)) \subseteq X$$

is open. We want to show that

1. $x_0 \in f_n^{-1}(B_d(f_n(x_0), \epsilon/3))$
2. $f_n^{-1}(B_d(f_n(x_0), \epsilon/3)) \subseteq f^{-1}(B_d(f(x_0), \epsilon))$

The first part is true if and only if $f_n(x_0) \in B_d(f_n(x_0), \epsilon/3)$. This is clearly true because $d(f_n(x_0), f_n(x_0)) = 0 < \epsilon/3$.

Now, we want to prove the second statement. Let $x \in f_n^{-1}(B_d(f_n(x_0), \epsilon/3))$.

$$\begin{aligned} x \in f^{-1}(B_d(f(x_0), \epsilon)) &\iff f(x) \in B_d(f(x_0), \epsilon) \\ &\iff d(f(x), f(x_0)) < \epsilon. \end{aligned}$$

Hence, it suffices to show that this is true.

Observe that

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_n(x)) + d(f_n(x), f(x_0)) \\ &\leq d(f(x), f_n(x)) + d(f_n(x_0), f(x_0)) + d(f_n(x), f_n(x_0)) \end{aligned}$$

Since $f_n(x) \in B_d(f_n(x_0), \epsilon/3)$, it folows that $d(f_n(x), f_n(x_0)) < \epsilon/3$. Since f_n converges to f uniformly, there exists N such that

$$d(f_n(y), f(y)) < \epsilon/3$$

for all $y \in X$ and for all $n \geq N$. For such n, we have

$$\begin{aligned} d(f_n(x_0), f(x_0)) &< \epsilon/3 \\ d(f_n(x), f(x)) &< \epsilon/3 \end{aligned}$$

Then, it follows that

$$d(f(x), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

15 Connectedness

Definition 15.1. *Suppose X is a topological space. A separation of X is a pair $U, V \subseteq X$ such that*

- $U \neq \varnothing$ *is open*
- $V \neq \varnothing$ *is open*
- $U \cap V = \varnothing$
- $U \cup V = X$

Definition 15.2. *X is connected if it does not have a separation.*

Lemma 15.1. *X is connected if and only if the following holds: If $U \subseteq X$ is open and closed, then $U = \varnothing$ or $U = X$.*

Proof. ($\Rightarrow$) Assume X is connected and $U \subseteq X$ is open and closed. Then, $V = U^c = X - U$ is also open and closed. So U, V are both open and $X = U \cup V$ and $U \cap V = \varnothing$. But X is connected so either $U = \varnothing$ or $V = \varnothing$.

($\Leftarrow$) Assume that X satisfies the condtion of the lemma. Suppose U and V are open, non-empty sets such that $U \cap V = \varnothing$ and $U \cup V = X$. Then, $V = U^c$ so U is open and closed. So $U = \varnothing$ (contradition) or $U = X$ implying that $V = \varnothing$ (again, contradiction). $\square$

Example 15.1. Consider $X = \{1, 2\}$ with discrete topology. This is not connected:

$$U = \{1\}, V = \{2\}$$

satisfies the conditions.

Example 15.2. Consider $X = \{1, 2\}$ with trivial topology. Then, X is connected. Assume $U \subseteq X$ is both open and closed. Since we are in a trivial topology, $U = \varnothing$ or $U = X$.

Example 15.3. $\mathbb{R}$ under standard topology and $(a, b), [a, b], (a, b], [a, b)$ under subspace topology are connected.

Example 15.4. $\mathbb{Q}$ with subspace topology is not conneceted. In particular, $U = (-\infty, \pi) \cap \mathbb{Q}$, $V = (\pi, \infty) \cap \mathbb{Q}$ is a separation.

Theorem 15.1. *Suppose X is connected and $f : X \to Y$ is continuous. Then, $\mathrm{Im}(f) = \{f(x) | x \in X\}$ is connected in the subspace topology coming from Y.*

Proof. Set $z = \mathrm{Im}(f) \subseteq Y$ with the subspace topology. Define

$$\begin{aligned} g : X &\to Z \\ x &\mapsto f(x) \end{aligned}$$

we claim that g is continuous. Let $U \subseteq Z$ be open. Then, $U = Z \cap U'$ where $U' \subseteq Y$ is open. Then, $g^{-1}(U) = f^{-1}(U')$ is open in X. We therefore assume f is surjective.

Now, we want to prove the theorem. Let $U \subseteq Y$ be closed, open and nonempty. It suffices to show that $U = Y$. If f is continuous, $f^{-1}(U)$ is open and closed. If f is surjective and $U \neq \varnothing$. Then, $f^{-1}(U) \neq \varnothing$. By the lmma and connectivity of X, $f^{-1}(U) = X$. This implies that $U = f(X) = Y$. □

Theorem 15.2. *Suppose $A \subseteq X$ is a subspace and A is connected. If $A \subseteq B \subseteq \bar{A}$, then B is connected.*

Proof. Assume B is not connceted so there is a separation U, V of B. Then,

$$\begin{aligned} U &= B \cap U' \\ V &= B \cap V' \end{aligned}$$

for $U', V' \subseteq X$ open. Note that $U' \cap A, V' \cap A$ are open in A, disjoint, and cover A. It follows $U' \cap A = \varnothing$ or $V' \cap A = \varnothing$. Assume $V' \cap A = \varnothing$. Then, $\bar{A} \subseteq \bar{U}'$. In fact, since $A \subseteq B$, we actually have $A \subseteq U = U' \cap B$ so $\bar{A} \subseteq \bar{U}$. Hence, $B \subseteq \bar{U}$. Then, $V \neq \varnothing$ so there exists $b \in V \subseteq B$. So $\exists Q \subseteq B$ open where $b \in Q \subseteq V$. But $B \subseteq \bar{U}$ so $b \in \bar{U}$. So every neighborhood of b intersects U. But $U \cap V = \varnothing$. So this is a contradiction. □

Remark. All intervals in $\mathbb{R}$ are connected.

Example 15.5. Consider

$$A = \{(x, \sin(1/x)) \mid x \in (0, 1/\pi)\} \subseteq \mathbb{R}^2.$$

Then, A is the image of

$$\begin{aligned} f : (0, 1/\pi) &\to \mathbb{R}^2 \\ x &\mapsto (x, \sin(1/x)). \end{aligned}$$

This is continuos. So A is connceted and hence $\bar{A} \subseteq \mathbb{R}^2$ is connected. (A is the topologist's sine curve).

Theorem 15.3. *Suppose X is a topological space and $\{A_\alpha\}$ is a collection of connected subspaces ($A_\alpha \subseteq X$ and A_α is connected). If $\bigcap_\alpha A_\alpha \neq \varnothing$ then $\bigcup_\alpha A_\alpha$ is conneceted.*

Proof. Assume U, V is a separation of $\bigcup_\alpha A_\alpha$. Let $p \in \bigcap_\alpha A_\alpha$. WLOG, we may assume $p \in U$. Fix α. Then, once of $U \cap A_\alpha$ or $V \cap A_\alpha$ is empty because A_α is connected. Since $p \in U \cap A_\alpha$, we must have $V \cap A_\alpha = \varnothing$. So $U \cap A_\alpha = A_\alpha$ so $A_\alpha \subseteq U$. Then,

$$\bigcup_\alpha A_\alpha \subseteq U.$$

Since U, V is a separation, $\bigcup A_\alpha = U \cup V$ so $V = \varnothing$. This is a contradiction. □

Theorem 15.4. *Suppose X, Y are conncted. Then, $X \times Y$ is connected.*

Proof. Fix $(a, b) \in X \times Y$. For $x \in X$, consider

$$T_X = \{x\} \times Y \cup X \times \{b\}.$$

Then, $X \times \{b\}$ is homeomorphic to X so it is connceted. Likewise, $\{x\} \times Y$ is homeomorphic to Y so it is connceted. Since $(x, b) \in X \times \{x\} \cap \{x\} \times Y$, so T_x is conncted by the previous theorem. Note that

$$\bigcup_{x \in X} T_x = X \times Y, \bigcap_{x \in X} T_x \neq \varnothing.$$

By the pervious theorem, it follows $X \times Y$ is connected. $\square$

Corollary 15.1. $\mathbb{R}^n$ *is conncected* $\forall n \geq 1$.

In summary, here are some useful facts related to connectedness:

- $\mathbb{R}$ (and its subintervals) is connceted
- If $A \subseteq X$ is connected, then $\bar{A}$ is connected.
- If $A_\alpha \subseteq X$ are connceted and $\bigcap_\alpha A_\alpha \neq \varnothing$ then $\bigcup_\alpha A_\alpha$ is connected
- If X, Y are connected, then $X \times Y$ is connected

Corollary 15.2. *If* $X_1, \ldots, X_n$ *are connected, then* $X_1 \times \cdots \times X_n$ *is connceted.*

Proof. Use the previous bullet and induction. $\square$

Consider

$$\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots = \prod_{\mathbb{Z}_+} \mathbb{R}.$$

Theorem 15.5. $\mathbb{R}^\omega$ *is not connected in the box topology.*

Proof. Let

$$U = \{\mathbf{x} = (x_1, x_2, \ldots) \mid \mathbf{x} \text{ is bounded}\}.$$

and

$$V = \{\mathbf{x} \mid \mathbf{x} \text{ is unbounded}\}.$$

Then, we have

$$\begin{aligned} \mathbb{R}^\omega &= U \cup V \\ U \cap V &= \varnothing \\ U &\neq \varnothing \\ V &\neq \varnothing \end{aligned}$$

Let $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$. Then,

$$\mathbf{x} \in W = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots \times (x_n - 1, x_n + 1) \times \cdots$$

If $\mathbf{x}$ is bounded, then everything in W is bounded. Then,

$$\mathbf{x} \in U \implies \mathbf{x} \in W \subseteq U$$

So U is open.

If $\mathbf{x}$ is unbounded, then everything in W is unbounded so

$$\mathbf{x} \in V \implies \mathbf{x} \in W \subseteq V$$

and hence V is open. So U, V is a separation. $\square$

Theorem 15.6. *$\mathbb{R}^\omega$ is connected in the product topology.*

Proof. Define

$$\tilde{\mathbb{R}}^n = \{(x_1, x_2, \dots, x_n, 0, 0, \dots)\} \subseteq \mathbb{R}^\omega$$

Then, $\tilde{\mathbb{R}}^n$ is homeomorphic to $\mathbb{R}^n$ so $\mathbb{R}^n$ is conncected.

$$(0, 0, 0, \dots) \in \tilde{\mathbb{R}}^n \ \forall n$$

so $\bigcap_{n \in \mathbb{Z}_+} \tilde{\mathbb{R}}^n \neq \varnothing$. Then, $\mathbb{R}^\infty = \bigcup_n \tilde{\mathbb{R}}^n$ is connceted. Hence, $\overline{\mathbb{R}^\infty} \subseteq \mathbb{R}^\omega$ is connected.

Now, we claim that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Let $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega$ and U a basis element for $\mathbb{R}^\omega$ with $\mathbf{x} \in U$. We want to show that $U \cap \mathbb{R}^\omega \neq \varnothing$. we have

$$U = U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$$

Notice $\mathbf{y} = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ is in $\tilde{\mathbb{R}^n}$ and hence $\mathbf{y} \in \mathbb{R}^\omega$. Then, $x_1 \in U_1, \dots, x_n \in U_n, 0 \in \mathbb{R}$ so $\mathbf{y} \in U$. Therefore, $U \cap \mathbb{R}^\omega \neq \varnothing$. $\square$

Definition 15.3. *Let X be a topological space and $x_0, x_1 \in X$. A path in X from x_0 to x_1 is a continuous function*

$$\gamma : [a, b] \to X$$

such that $\gamma(a) = x_0$ and $\gamma(b) = x_1$.

Example 15.6. $\gamma(t) = tx_1 + (1-t)x_0$ where $t \in [0, 1]$.

Definition 15.4. *X is path-connected if $\forall x_0, x_1 \in X$, there exists path in X from x_0 to x_1.*

Theorem 15.7. *If X is path-connceted then X is connected.*

The converse of the theorem is false.

Example 15.7. Consider

$$A = \{(x, \sin(1/x)) \mid x \in (0, 1/\pi)\}.$$

Last time, we showed that $\bar{A}$ is connected. However, $\bar{A}$ is not path connected.

Consider $x_0 = (0,0) \in \bar{A}$ and $x_1 = (1/\pi, 0) \in \bar{A}$. Suppose $\gamma : [0,1] \to \bar{A}$ is a path from x_0 to x_1. Then, we can write $\gamma(t) = (x(t), y(t))$. So

$$\begin{aligned} x &: [0,1] \to [0, 1/\pi] \\ y &: [0,1] \to [-1, 1] \end{aligned}$$

are continuous. The set $\{t \in [0,1] \mid x(t) = 0\}$ is closed. Let $b = \sup\{t \in [0,1] \mid x(t) = 0\}$. Then, $x(b) = 0$ and $x(t) > 0$ for $t \in (b, 1]$ with $y(t) = \sin(1/x(t))$. By reparameterizing, we may assume that $b = 0$. For each n, find $0 < u < x(1/n)$ so $\sin(1/u) = (-1)^n$. By the intermediate value theorem, there exists $0 < t_n < 1/n$ so $x(t_n) = u$. Note that

$$\begin{aligned} y(t_n) &= \sin\left(\frac{1}{x(t_n)}\right) \\ &= \sin\left(\frac{1}{u}\right) \\ &= (-1)^n \end{aligned}$$

Then,

$$\lim_{n\to\infty} y(t_n) = \lim_{t\to 0} y(t) = y(0).$$

But $\lim_{n\to\infty}(-1)^n$ does not exist.

16 Homeomorphism

Definition 16.1. *A homeomorphism is a function $f : X \to Y$ that satisfies the following conditions:*

- *continuous*
- *invertible (bijective)*
- *f^{-1} is also continuous*

Example 16.1. Consider

$$\begin{aligned} f : [0, 2\pi) &\to S^1 \\ \theta &\mapsto (\cos\theta, \sin\theta) \end{aligned}$$

Here, inverse breaks circle so f is continuous but f^{-1} is not continuous.

Definition 16.2. *X, Y are homeomorphic if there exists homeomorphism from X to Y. If this is true, we write $X \cong Y$ and say that X is homeomorphic to Y.*

Theorem 16.1. *Assume $X \cong Y$. Then X is connected iff Y is connected.*

Theorem 16.2 (Intermediate value theorem)**.** *Suppose $f : X \to \mathbb{R}$ is continuous. Assume X is connected. Let $x, y \in X$ and $r \in \mathbb{R}$ such that r is between $f(x)$ and $f(y)$. Then, there exists $z \in X$ such that $f(z) = r$.*

Proof. Consider $(-\infty, r)$ and (r, ∞). Those are open in $\mathbb{R}$, nonempty, and disjoint. Let $U = f^{-1}((-\infty, r))$ and $V = f^{-1}((-\infty, r))$. These are also open, nonempty, and disjoint. But X is connected so U, V can't be a separation. So $U \cup V \neq X$ and hence $\exists z \in X - (U \cup V)$. Then, $f(z) \notin (-\infty, r)$ and $f(z) \notin (r, \infty)$. Therefore, $f(z) = r$. $\square$

Theorem 16.3. $\mathbb{R}$ *is connected.*

Proof. Suppose $A \subseteq \mathbb{R}$ is nonempty and bounded above. Then, there exists a least upper bound M for A and any other upper bound M' satisfies $M \leq M'$. We write ^{A}f or this LUB.

Suppose U, V is a separation of $\mathbb{R}$. So there exists $a \in U$ and $b \in U$. Then, WLOG, we may assume $a < b$. Sets

$$U_0 = U \cap [a, b]$$
$$V_0 = V \cap [a, b]$$

form a separation of $[a, b]$. Notice $U_0 \neq \varnothing$ and $x \in U_0$ satisfies $x \leq b$. So $C = \sup(U_0)$ exists. Then, $c \in U$ or $C \in V$. In fact, $c \geq a$ and $c \leq b$ so $c \in U_0$ or $c \in V_0$. Assume $c \in V_0$. Since $V_0$is open in $[a, b]$ so there exists $< c$ such that $(d, c] \subseteq V_0$. Note if $x < g$ then $x \notin U_0$ because c is an upper bound of U_0. This implies that $[c, b] \subseteq V_0$. Then,

$$(d, c] \cup [c, b] \subseteq V_0 \implies (d, b] \subseteq V_0.$$

But then d is an upper bound for U_0 so $c \leq d$. This is a contradiction.

Now, suppose $c \in U_0$. Then, Y_0 is open in $[a, b]$. Then, there exists $e > c$ such that $[c, e) \subseteq U_0$. Let $f = (e + c)/2$. Then, $c < f < e$ and $f \in [c, e)$. Then, $f \in U_0$. Then, $c \geq f$ because it is an upper bound. This is a contradiction. $\square$

Example 16.2. $(-\pi/2, \pi/2)$ is connected.

Proof. $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ is a homeomorphism. $\square$

Example 16.3. $(0, 1) \cong (a, b)$ if $a < b$.

Proof. $f : (0, 1) \to (a, b)$ given by $f(t) = (1 - t)a + b$ is a homeomorphism. $\square$

Corollary 16.1. $(0, 1)$ *is connected.*

Corollary 16.2. (a, b) *is connected for any* $a < b$.

Recall that if A is connected then $\forall A \subseteq B \subseteq \bar{A}$, B is connected. We apply this with $A = (a, b)$. Then, $[a, b)$ or $(a, b]$ or $[a, b]$ are connected.

Corollary 16.3. *Suppose* $I \subseteq \mathbb{R}$ *is a subspace. Then,* I *is connected if and only if* $I = \mathbb{R}$ *or* I *is an interval or* I *is any of the follwowing:*

$$(a, \infty), (-\infty, b), [a, \infty), (-\infty, b].$$

Example 16.4. $\mathbb{R}^n - \{0\}$ is path-connected for $n \geq 2$.

Proof. Let $x, y \in \mathbb{R}^n - \{0\}$.

- (Case 1) The line l passing through x and y does not contain 0. Then, $\gamma : [0,1] \to \mathbb{R}^n - \{0\}$ given by $\gamma(t) = (1-t)x + ty$ has image in l and hence in $\mathbb{R}^n - \{0\}$. So this is a path from x to y.

- (Case 2) $0 \in l$. Since $n \geq 2$, $\mathbb{R}^n \neq \mathbb{R}$ and $\mathbb{R}^n \neq l$. So there exists $z \in (\mathbb{R}^n - \{0\}) - l$. Define $\gamma : [0,1] \to \mathbb{R}^n - \{0\}$ by concatenating a path from x to z with one from z to y.

□

Example 16.5. Write down a formula for the path that goes from x to z and from z to y.

Definition 16.3. *For $n \geq 1$, define the n-sphere:*

$$S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$$

Lemma 16.1. *S^n is path connected.*

Proof. Consider

$$\begin{aligned} f : \mathbb{R}^n - \{0\} &\to S^n \\ x &\mapsto \frac{x}{|x|} \end{aligned}$$

where

$$|x| = \sqrt{x_0^2 + \cdots + x_n^2}$$

f is continuous and $\mathrm{Im}(f) = S^n$. Since $\mathbb{R}^{n+1} - \{0\}$ is path-connected, so S^n is path-connected. □

17 Compactness

Definition 17.1. *Let X be a topological space. A cover of X is a collection A of subsets of X whose union equals X. The cover A is open if all elements in A are open in X.*

Definition 17.2. *If A is a cover of X, a subcover is $A' \subseteq A$ that is a cover of X. A' is an open subcover if it is an open cover.*

Example 17.1. $A' = \{B_d(0, 2n) | n \in \mathbb{Z}_+\}$ is an open subcover of X.

Definition 17.3. *X is a compact if every open cover of X has a subcover that is finite (consists of a finite number of elements of the cover).*

Example 17.2. $\mathbb{R}^n$ is not compact.

Proof. Consider $A = \{B_d(0,n)|n \in \mathbb{Z}_+\}$. This is an open cover. Suppose $A \subseteq A$ is a subcover that is finite. Then,

$$A' = \{B_d(0,n_1), B_d(0,n_2), B_d(0,n_3), \dots, B_d(0,n_k)\}$$

Let $N = \max\{n_1, n_2, \dots, n_k\}$. Then,

$$(N, 0, 0, \dots, 0) \in \mathbb{R}^n$$

is not in $B_d(0,n_1), B_d(0,n_2), \dots, B_d(0,n_k)$. So A' is not a cover. $\square$

Example 17.3. Prove that $(0,1]$ is not compact by considering

$$A = \left\{ \left(\frac{1}{n}, 1 \right] |n \in \mathbb{Z}_+ \right\}.$$

Example 17.4. $X = \{1\}$ is compact.

Proof. Let $\mathcal{A}$ be any open cover. Then, $\exists A \in \mathcal{A}$, so $1 \in A$. Then, $\mathcal{A}' = \{A\}$ is a finite subcover. $\square$

Example 17.5. $X = \{0\} \cup \{1/n|n \in \mathbb{Z}_+\}$ is compact.

Proof. Let $\mathcal{A}$ be an open cover. Then, $\exists A \in \mathcal{A}$ such that $0 \in A$. A is open in the subspace topology so $A = X \cap U$ where $U \subseteq \mathbb{R}$ is open. Then, $\exists N \in \mathbb{Z}_+$ so $1/n \in U \forall n > N$. Hence, $1/n \in A \forall n > N$. $\forall 1 \leq n \leq N$, $\exists A_n \in \mathcal{A}$ such that $1/n \in A_n$. Then,

$$A' = \{A, A_1, \dots, A_N\}$$

is a finite subcover. $\square$

Example 17.6. All closed intervals $[a,b] \subseteq \mathbb{R}$ where $a, b \in \mathbb{R}$ are compact.

Theorem 17.1. *Suppose X is compact and $Y \subseteq X$ is closed. Then, Y is compact.*

Proof. Let $\mathcal{A}$ be an open cover of Y. Then, $B = \mathcal{A} \cup \{X - Y\}$ is an open cover of X. Since X is compact, there exists finite subcover $B' \subseteq B$. If $\{X - Y\} \notin B'$ then B' is a finite subcover of $\mathcal{A}$. If $\{X - Y\} \in B'$ then simply omit this to get

$$B'' = B' - \{X - Y\}$$

a finite subcover of A. $\square$

Theorem 17.2. *Suppose X is Hausdorff and $Y \subseteq X$ is compact. Then, Y is closed in X.*

Proof. It suffices to show that $X - Y$ is open. Let $x \in X - Y$. We want to find open $U \subseteq X - Y$ so $x \in U \subseteq X - Y$. X is Hausdorff by assumption so $\forall y \in Y$, there exists $U_y, V_y \subseteq X$ open so $U_y \cap V_y = \varnothing$ and $x \in U_y$ and $y \in V_y$. Then,

$$A = \{V_y \cap Y | y \in Y\}$$

is an open cover of Y. So since Y is compact, there exists finite usbcover

$$A' = \{V_{y_1} \cap Y, \dots, V_{y_N} \cap Y\}.$$

Then,

$$U_{Y_1} \cap U_{Y_2} \cap \cdots \cap U_{Y_N}$$

is open, contains X and so we are done. □

Theorem 17.3. *Suppose X is compact and $f : X \to Y$ is continuous. Then,*

$$\text{Im}(f) = f(Y) \subseteq Y$$

is compact.

Proof. Let $\mathcal{A}$ be an open cover of $\text{Im}(f)$. Then, write

$$\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$$

Since A_α is open in $\text{Im}(f)$, we can write

$$A_\alpha = B_\alpha \cap \text{Im}(f)$$

where B_α is open in Y. Then, $f^{-1}(B_\alpha) \subseteq X$ is open in X so

$$\{f^{-1}(B_\alpha) | \alpha \in J\}$$

is an open cover of X. Then,

$$\{A_{\alpha_1}, \dots, A_{\alpha_N}\}$$

is a finite subcover of $\text{Im}(f)$. □

Example 17.7. Show that compact ness is a topological property. If $X \cong Y$ then X is compact if and only if Y is compact.

Proof. Assume $X \cong Y$, so there exists homeomorphism $f : X \to Y$. Assume X in compact. Then, $\text{Im}(f)$ is compact. But then, $\text{Im}(f) = Y$. So Y is comapct. Conversely, use the same argument with $f^{-1} : Y \to X$ which is a homeomorphism. □

Example 17.8. Suppose X is compact, Y is Hausdorff, and $f : X \to Y$ is bijective and continuous. Then, f is a homeomorphism. (Hint: it suffices to show that $f(C)$ is closed for all closed $C \subseteq X$).

Proof. To show that the hint is valid, show that f^{-1} is continuous. Then, $(f^{-1})^{-1}(C) = f(C)$. Suppose that the hint is true. Suppose C is closed. Then, C is compact and $f(C)$ is comapct. Furthermore, $f(C)$ is closed. □

Definition 17.4. *Suppose $\mathcal{A}$ is a collection of open sets in X and $Y \subset X$ is a subspace. Then, A is a cover of Y if*

$$Y \subseteq \bigcup_{A \in \mathcal{A}} A$$

Lemma 17.1 (Tube lemma)**.** *Given*

$$\{x\} \times Y \subseteq \bigcup_{j=1}^{N} A_{x,j}$$

and Y compact, there exists $W_x \subseteq X$ open so

$$\{x\} \times Y \subseteq W_x \times Y \subseteq \bigcup_{j=1}^{N} A_{x,j}$$

Proof. Since each $A_{x,j}$ is open, $\forall y \in Y$, there exists $U_y \times V_y \subseteq X \times Y$ such that

$$(x, y) \in U_y \times V_y \subseteq \bigcup A_{x,j}$$

Then, $\{U_y \times V_y | y \in Y\}$ is an open cover of $\{x\} \times Y$, implying that there exists a finite subcover, which we can write

$$\{U_{y_1} \times V_{y_1}, \ldots, U_{y_M} \times V_{y_M}\}$$

Then, lte

$$W_x = U_{y_1} \cap \cdots \cap U_{y_M}.$$

This satisfies the condition of the lemma. $\square$

Theorem 17.4. *Suppose X and Y are compact. Then, $X \times Y$ is compact.*

Proof. Let $\mathcal{A}$ be an open cover of $X \times Y$. Fix $x \in X$. Then, $\{x\} \times Y \cong Y$ so $\{x\} \times Y$ is compact. A is a cover of $\{x\} \times Y$. So there exists a finite subcover

$$\{A_{x,1}, A_{x,2}, \ldots, A_{x,N}\} \subseteq \mathcal{A}$$

Note that $\forall x \in X$, the tube lemma gives an open $W_x \subseteq X$ with $x \in W_x$. Then,

$$\{W_x | x \in X\}$$

is an open cover of X. Then, X compact implies that there exists finite subcover

$$\{W_{x_1}, \ldots, W_{x_2}\}.$$

Then,

$$\begin{aligned} \{&A_{x_1,1}, A_{x_1,2}, \ldots, A_{x_1,N} \\ &A_{x_2,1}, A_{x_2,2}, \ldots, A_{x_2,N} \\ &\vdots \\ &A_{x_L,1}, A_{x_L,2}, \ldots, A_{x_L,N}\} \end{aligned}$$

is a finite subcover of $x \times Y$. $\square$

Corollary 17.1. *Suppose $X_1, \ldots, X_n$ are compact. Then, $X_1 \times \cdots \times X_n$ is compact.*

Proof. We use the previous theorem and induction on n. When $n = 1$, the statement is obviously true. If X_1 is compact, then X_1 is compact. Now, assume the result is holds for $n - 1$. Let $X_1, \ldots, X_n$ be compact. Then,

$$X_1 \times \cdots X_{n-1}$$

is compact. Then,

$$X_1 \times X_{n-1} \times X_n = (X_1 \times \cdots \times X_{n-1}) \times X_n$$

is compact by the theorem. $\square$

Example 17.9. Suppose $\{X_\alpha\}_{\alpha \in J}$ is a collection of compact sets. Is $\prod_{\alpha \in J} X_\alpha$ compact with the product topology? Depends on the axiom, this may not be true. If we assume the axiom of choice, then, by Tychonoff, this is true. Axiom of choice states that you can make an infinite number of arbitrary choices.