Math 3T03 - Topology

Sang Woo Park

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Contents

1 Introduction to topology

1.1 What is topology?

Definition 1.1 (Topology)**.** *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *The union of any collections of elements in τ is also in τ.*
- *The intersection of any finite collection of elements in τ is also in τ.*

Definition 1.2. (X, τ) *is a topological space.*

Definition 1.3. *The elements of τ are called the open sets.*

Example 1.1. Consider $X = \{a, b, c\}$. Is $\tau_1 = \{\varnothing, X, \{a\}\}$ a topology?

Proof. Yes, τ_1 is a topology. Clearly, $\varnothing \in \tau_1$ and $X \in \tau_1$ so it suffices to verify the arbitrary unions and finite intersection axioms.

Let $\{V_\alpha\}_{\alpha \in A}$ be a subcollection of τ_1. Then, we want to show

$$\bigcup_{\alpha \in A} V_\alpha \in \tau_1.$$

If $V_\alpha = \varnothing$ for any $\alpha \in A$, then this does not contribute to the union. Hence, φ can be omittied:

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\substack{\alpha \in A \\ V_\alpha \neq \varnothing}} V_\alpha.$$

If $V_\alpha = X$ for some $\alpha \in A$, then

$$\bigcup_{\alpha \in A} V_\alpha = X \in \tau_1$$

and we are done. So we may assume $V_\alpha \neq X$ for all $\alpha \in A$. Similarly, if $V_\alpha = V_\beta$ for $\alpha \neq \beta$, then we can omit V_β from the union. Since τ_1 only contains $\varnothing$, X and $\{a\}$, we must have

$$\bigcup_{\alpha \in A} V_\alpha = \{a\} \in \tau_1.$$

Similarly, if any element of $\{V_\alpha\}_{\alpha \in A}$ is empty, then

$$\bigcap_{\alpha \in A} V_\alpha = \varnothing \in \tau_1.$$

We can therefore assume $V_\alpha \neq \varnothing$ for all $\alpha \in A$. Again, repetition can be ignored and any V_α that eqauls X can be ignored. Then,

$$\bigcap_{\alpha \in A} V_\alpha = \begin{cases} \{a\} \in \tau_1 \\ X \in \tau_1 \end{cases}$$

$\square$

Example 1.2. Consider $X = \{a, b, c\}$. Is $\tau_2 = \{\varnothing, X, \{a\}, \{b\}\}$ a topology?

Proof. No. Note that $\{a\} \in \tau_2$ and $\{b\} \in \tau_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_2$. $\square$

Example 1.3. Consider $X = \{a, b, c\}$. Then, $\tau_3 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ is a topology.

1.2 Set theory

Definition 1.4. *If X is a set and a is an element, we write $a \in X$.*

Definition 1.5. *If Y is a subset of X, we write $Y \subset X$ or $Y \subseteq X$.*

Example 1.4. Suppose X is a set. Then, *power set* of X is the set $P(X)$ whose elements are all subsets of X. In other words,

$$Y \in P(X) \iff Y \subseteq X$$

Note that $P(X)$ is closed under arbitrary unions and intersections. Hence, $P(X)$ is a topology on X.

Definition 1.6. *$P(X)$ is the discrete topology on X.*

Definition 1.7. *The indiscrete (trivial) topology on X is $\{\varnothing, X\}$.*

Definition 1.8. *Suppose U, V are sets. Then, their union is*

$$U \cup V = \{x | x \in U \text{ or } x \in V\}.$$

Note that if $U_1, U_2, \ldots, U_10$ are sets, their union can be written as follows:

- $U_1 \cup U_2 \cup \cdots \cup U_{10}$
- $\bigcup\limits_{k=1}^{10} U_k$
- $\bigcup\limits_{k=\{1,2,\ldots,10\}} U_k$

Definition 1.9. *Let A be any set. Suppose $\forall \alpha \in A$, I have a set U_α. The union of the U_α over $\alpha \in A$ is*

$$\bigcup_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

Similarly, the intersection is

$$\bigcap_{\alpha \in A} U_\alpha = \{x | \exists \alpha \in A, x \in U_\alpha\}.$$

2 Functions

Definition 2.1. *A function is a rule that assigns to element of a given set A, an element in another set B.*

Often, we use a formula to describe a function.

Example 2.1. $f(x) = \sin(5x)$

Example 2.2. $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Definition 2.2. *A* rule of assignment *is a subset R of the Cartesian product $C \times D$ of two sets with the property that each element of C appears as the first coordinate of at most one ordered pair belonging to R. In other words, a subset R of $C \times D$ is a rule of assignment if it satisfies*

$$\textit{If } (c, d) \in R \textit{ and } (c, d') \in R, d = d'.$$

Definition 2.3. *Given a rule of assignment R, we define the doamin of R to be the subset of C consisting of all first coordinates of elements of R:*

$$\mathrm{domain}(R) \equiv \{c \mid \exists d \in D\, s.t.\, (c, d) \in R\}$$

Definition 2.4. *The image set of R is defined to be the subset of D consisting of all second coordinates.*

$$\mathrm{image}(R) \equiv \{d \mid \exists c \in C\, s.t.\, (c, d) \in R\}$$

Definition 2.5. *A function is a rule of assignment R, together with a set that contains the image set of R. The domain of the rule of assignment is also called the domain of f, and the image of f is defined to be the image set of the rule of assignment.*

Definition 2.6. *The set B is often called the range of f. This is also referred to as the codomain.*

If $A = \mathrm{domain}(f)$, then we write

$$f : A \to B$$

to indicate that f is a function with domain A and codomain B.

Definition 2.7. *Given $a \in A$, we write $f(a) \in B$ for the unique element in B associated to a by the rule of assignment.*

Definition 2.8. *If $S \subseteq A$ is a subset of A, let*

$$f(S)\{f(a) \mid a \in S\} \subseteq B.$$

Definition 2.9. *Given $A_0 \subseteq A$, we can restrict the domain of f to A_0. The restriction is denoted*

$$f|A_0 \equiv \{(a, f(a)) \mid a \in A_0\} \subseteq A \times B.$$

Definition 2.10. *If $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$ is defined to be*

$$\{(a, c) \mid \exists b \in B \text{ s.t. } f(a) = b \text{ and } g(b) = c\} \subseteq A \times C.$$

Definition 2.11. *A function $f : A \to B$ is called injective (or one-to-one) if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$.*

Definition 2.12. *A function $f : A \to B$ is called surjective (or onto) if the image of f equals B, i.e. $f(A) = B.$, i.e. if, for every $b \in B$, there exists $a \in A$ with $f(a) = b$.*

Definition 2.13. *$f : A \to B$ is a bijection if it is one-to-one and onto.*

Definition 2.14. *If $B_0 \subseteq B$ is a subset and $f : A \to B$ is a function, then the preimage of B_0 under f is the subset of A given by*

$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}$$

Example 2.3. If B_0 is disjoint from $f(A)$ then $f^{-1}(B_0) = \varnothing$.

3 Relation

Definition 3.1. *A relation on a set A is a subset R of $A \times A$.*

Given a relation R on A, we will write xRy "x is related to y" to mean $(x, y) \in R$.

Definition 3.2. *An equivalence relation on a set is a relation with the following properties:*

- *Reflexivity. xRx holds $\forall x \in A$.*
- *Symmetric. if xRy then yRx holds $\forall x, y \in A$.*
- *Transitive. if xRy and yRz then xRz holds $\forall x, y, z \in A$.*

Definition 3.3. *Given $a \in A$, let $E(a) = \{x \mid x \sim a\}$ denote the equivalence class of a.*

Remark. $E(a) \subseteq A$ and it is nonempty because $a \in E(a)$.

Proposition 3.1. *If $E(a) \cap E(b) \neq \varnothing$ then $E(a) = E(b)$.*

Proof. IAssume the hypothesis, i.e., suppose $x \in E(a) \cap E(b)$. So $x \sim a$ and $x \sim b$. By symmetry, $a \sim x$ and by transitivity $a \sim b$.

Now suppose that $y \in E(a)$. Then $y \sim a$ but we just saw that $a \sim b$ so $y \sim b$ and $y \in E(b)$. Hence, $E(a) \subseteq E(b)$. Likewise, we can show that $E(b) \subseteq E(a)$. □

Definition 3.4. *A partition of a set is a collection of pairwise disjoint subsets of A whose union is all of A.*

Example 3.1. Consider $A = \{1, 2, 3, 4, 5\}$. Then, $\{1, 2\}$ and $\{3, 4, 5\}$ is a partition of A.

Proposition 3.2. *An equivalence relation in a set determines a partition of A, namely the one with equivalence classes as subsets. Conversely, a partition*[1] *$\{Q_\alpha \mid \alpha \in J\}$ of a set A determines an equivalence relation on A by: $x \sim y$ if and only if $\exists \alpha \in J$ s.t. $x, y \in Q_\alpha$. The equivalence classes of this equivalence relation are precisely the subsets Q_α.*

[1] Note that J is an index set: $Q_\alpha \subseteq A$ for each $\alpha \in J$ and $Q_\alpha \cap Q_\beta = \varnothing$ if $\alpha \neq \beta$. Furthermore, $\cup_{\alpha \in J} = A$.

4 Finite and infinite sets

Definition 4.1. *A nonempty set A is finite if there is a bijection from A to $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{Z}^+$.*

Remark. Consider $n, m \in \mathbb{Z}^+$ with $n \neq m$. Then, there is no bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

Definition 4.2. *Cardinality of a finite set A is defined as follows:*

1. $\text{card}(A) = 0$ *if* $A = \varnothing$.
2. $\text{card}(A) = n$ *if there is a bijection form* A *to* $\{1, 2, \ldots, n\}$.

Note that $\mathbb{Z}^+$ is not finite. How would you prove this? A sneaky approach is that there is a bijection from $\mathbb{Z}^+$ to a proper subset of $\mathbb{Z}^+$.

Consider the shift map:

$$S : \mathbb{Z}^+ \to \mathbb{Z}^+$$

where $S(k) = k + 1$. So S is a bijection from $\mathbb{Z}^+$ to $\{2, 3, \ldots\} \subset \mathbb{Z}^+$, a proper subset of $\mathbb{Z}^+$. On the other hand, any proper sbset B of a finite set A with $\text{card}(A) = n$ has $\text{card}(B) < n$. In particular, there is no bijection from A to B. Therefore, $\mathbb{Z}^+$ is not finite.

Theorem 4.1. *A non-empty set is finite if and only if one of the following holds:*

1. $\exists$ *bijection* $f : A \to \{1, 2, \ldots, n\}$ *for some* $n \in \mathbb{Z}^+$.
2. $\exists$ *injection* $f : A \to \{1, 2, \ldots, N\}$ *for some* $N \in \mathbb{Z}^+$
3. $\exists$ *surjection* $g : \{1, \ldots, N\} \to A$ *for some* $N \in \mathbb{Z}^+$

Remark. Finite unions of finite sets are finite. Finite products of finite sets are also finite.

Recall that if X is a set, then

$$X^w = \pi_{n \in \mathbb{Z}^+} X = \{(x_n) \mid x_n \in X \forall n \in \mathbb{Z}^+\}$$

the set of infinite sequences in X.

If $X = \{a\}$ is a singleton, then X^w is also a singleton. Otherwise, if $\text{card}(X) > 1$, then X^w is an infinite set.

Example 4.1. Consider $X = \{0, 1\}$. Then, X^w is a set of binary sequences.

Definition 4.3. *If there exists a bijection from a set X to $\mathbb{Z}^+$, then X is said to be countably infinite.*

Definition 4.4. *Let $\mathcal{A}$ be a nonempty collection of sets. An indexing family is a set J together with a surjection $f : J \to \mathcal{A}$. We use A_α to denote $f(a) \subseteq \mathcal{A}$. So*

$$\mathcal{A} = \{A_\alpha \mid \alpha \in J\}$$

Most of the time, we will be able to use $\mathcal{Z}^+$ for the index set J.

Definition 4.5 (Cartesian product)**.** *Let $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Finite cartesian product is defined as*

$$A_1 \times \cdots \times A_n = \{(x_1, \dots, x_n) \mid x_i \in A_i\}$$

It is a vector space of n tuples of real numbers.

Definition 4.6 (Infinite Cartesian product)**.** *Infinite Cartesian product $\mathbb{R}^\omega$ is an ω-tuple $x : \mathbb{Z}^+ \to \mathbb{R}$, i.e., a sequence*

$$\begin{aligned} x &= (x_i)_{i=1}^{\infty} = (x_1, x_2, \dots, x_n, \dots) \\ &= (x_i)_{i \in \mathbb{Z}^+} \end{aligned}$$

More generally, if $\mathcal{A} = \{A_i \mid i \in \mathbb{Z}^+\}$. Then,

$$\prod_{i \in \mathbb{Z}^+} A_i = \{(a_i)_{i \in \mathbb{Z}^+} \mid a_i \in A_i\}.$$

Definition 4.7. *A set A is countable if A is finite or it is countably infinite.*

Theorem 4.2 (Criterion for countability)**.** *Suppose A is a non-empty set. Then, the following are equivalent.*

1. *A is countable*
2. *There is an injection $f : A \to \mathbb{Z}^+$*
3. *THere is a surjection $g : \mathbb{Z}^+ \to A$*

Theorem 4.3. *If $C \subseteq \mathbb{Z}^+$ is a subset, then C is countable.*

Proof. Either C is finite or infinite. If C is finite, it follows C is countable.

Assume that C is finite. We will define a bijection $h : \mathbb{Z}^+ \to C$ by induction. Let $h(1)$ be the smallest element in C[2]. Suppose

$$h(1), \dots, h(n-1) \in C$$

and are defined. Let $h(n)$ be the smallest element in $C \setminus \{h(1), \dots h(n-1)\}$. (Note that this set is nonempty.)

We want to show that h is inejctive. Let $n, m \in \mathbb{Z}$ where $n \neq m$. Suppose $n < m$. So $h(n) \in \{h(1), \dots, h(n)\}$ and

$$h(n) \in C \setminus \{h(1), h(2), \dots, h(m-1)\}.$$

So $h(n) \neq h(m)$.

We want to show that h is surjective. Suppose $c \in C$. We will show that $c = h(n)$ for some $n \in \mathbb{Z}^+$. First, notice that $h(\mathbb{Z}^+)$ is not contained in $\{1, 2, \dots, c\}$

[2] $C \neq \varnothing$ and every nonempty subset of $\mathbb{Z}^+$ has a smallest elemnt

because $h(\mathbb{Z}^+)$ is an infinite set. Therefore, there must exists $n \in \mathbb{Z}^+$ such that $h(n) > c$. So let m be the smallest element with $h(m) \geq c$. Then for all $i < m$, $h(i) < c$. Then,

$$c \notin \{h(1), \dots, h(m-1)\}.$$

By definition, $h(m)$ is the smallest element in

$$C \setminus \{h(1), \dots, h(m-1)\}.$$

Then, $h(m) \leq c$. Therefore, $h(m) = c$. □

Corollary 4.1. *Any subset of a countable set is countable.*

Proof. Suppose $A \subset B$ with B countable. Then, there exists an injection $f : B \to \mathbb{Z}^+$. The restriction

$$f|_A : A \to \mathbb{Z}^+$$

is also injective. Therefore, A is countable by the criterion. □

Theorem 4.4. *Any countable union of countable is countable. In other words, if J is countable and A_α is countable for all $\alpha \in J$, then $\cup_{\alpha \in J}$ is countable.*

Theorem 4.5. *A finite product of countable sets is countable.*

Example 4.2. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Note that a countable product of countable sets is not countable.

Example 4.3. Consider $X = \{0, 1\}$. Then, X^ω binary sequence is not countable.

Proof. Suppose $g : \mathbb{Z}^+ \to X^\omega$ is a function. We will show g is not surjective by contructing an element $y \neq g(n)$ for any $n \in \mathbb{Z}^+$.

Write $g(n) = (x_{n1}, x_{n2}, \dots) \in X^\omega$ for each $x_{ni} \in \{0, 1\}$. Define

$$y = (y_1, y_2, \dots)$$

where $y_i = 1 - x_{ii}$. Clearly, $y \in X^\omega$ but $y \neq g(n)$ for any n since $y_n = 1 - x_{nn} \neq x_{nn}$, the n-th entry in g(n). □

Example 4.4. Let X be a set and $\{V_\alpha\}_{\alpha \in I}$ a collection of subsets of X. Show

$$\bigcup_{\alpha \in I} (X - V_\alpha) = X - \bigcap_{\alpha \in I} V_\alpha$$

Proof.

$$\begin{aligned} x \in \bigcup_{\alpha \in I} X - V_\alpha &\iff \exists \alpha \in I, x \in X - V_\alpha \\ &\iff \exists \alpha \in I, x \in X, x \notin V_\alpha \\ &\iff x \in X, \exists \alpha \in I, x \notin V_\alpha \\ &\iff x \in X, x \notin \bigcup_{\alpha \in I} V_\alpha \\ &\iff x \in X - \bigcap_{\alpha in I} V_\alpha \end{aligned}$$

$\square$

5 Topological spaces

Definition 5.1 (Topology)**.** *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

- $\varnothing \in \tau$.
- $X \in \tau$.
- *If* $\{V_\alpha\}_{\alpha \in I} \subseteq \tau$*, then* $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
- *If* $\{V_1, \ldots, V_n\} \subseteq \tau$*, then* $\bigcap_{k=1}^{n} V_k inI$.

Remark. If τ is a topology, then $\{\varnothing, X\} \subseteq \tau$ and $\tau \subseteq \mathcal{P}(X)$, where $\mathcal{P}$ is a power set.

Definition 5.2. *Suppose τ, τ' are topologies on X. We say τ is coarser than τ' if $\tau \subseteq \tau'$. τ' is finer than τ if $\tau \subseteq \tau'$.*

Example 5.1. If X is a set and τ is a topology. Then, τ is coarser than the discrete topology, $\mathcal{P}(X)$, and finer than the trivial topology, $\{\varnothing, X\}$.

Note that if $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$, then $\tau = \tau'$.

Example 5.2. Define τ_f to consist of the subsets U of X wiith the property that $X - U$ is finite or all of X.

Lemma 5.1. *τ_f is a topoology on X. This is defined as the finite complement topology.*

Proof. To show $\varnothing \in \tau_f$, notice that $X - \varnothing = X$. So $\varnothing \in \tau_f$. To show that $X \in \tau_f$, notice that $X - X = \varnothing$, which is finite. So $X \in \tau_f$.

To show that it is closed under arbitrary unions, suppose

$$\{V_\alpha\}_{\alpha \in I} \subseteq \tau_f.$$

It suffices to show that $X - \bigcup_{\alpha \in I} V_\alpha$ is finite. For this, note

$$X - \bigcup_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} (X - V_\alpha).$$

$X - V_\alpha$ is finite so

$$\bigcap_{\alpha \in I} (X - V_\alpha) \subseteq X - V'_\alpha$$

for any $\alpha' \in I$ which implies $\bigcap_{\alpha \in I} X - V_\alpha$ is finite.

For closure under finite intersections, let

$$\{V_1, \ldots, V_n\} \subseteq \tau_f.$$

Then,

$$X - \bigcap_{k=1}^{n} V_k = \bigcup_{k=1}^{n} (X - V_k)$$

is a finite union of finite sets and so it is finite. $\square$

Example 5.3. Consider $X = \mathbb{R}$. Then, $\mathbb{R} - \{0\}$ is open in the finite complement topology. $\mathbb{R} - \{0, 3, 100\}$ is open in τ_f. Note that $(4, \infty)$ is not open in τ_f because its complement is not finite.

Remark. Assume τ is a topology on X.

- (X, τ) is a topological space.
- X is a topological space.
- $U \in \tau$ iff U is open in τ iff U is an open set.

6 Basis for a Topology

Definition 6.1. *Let X be a set. A basis for topology on X is a collection $\mathcal{B}$ of subsets of X satisfying:*

- $\forall x \in X, \exists B \in \mathcal{B}, x \in B$
- $\forall B_1, B_2 \in \mathcal{B}$, *if* $x \in B_1 \cap B_2$ *then* $\exists B_3 \in \mathcal{B}$ *such that* $x \in B_3$ *and* $B_3 \subseteq B_1 \cap B_2$.

Example 6.1. Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_1$ be a collection of interiors of circles in $\mathbb{R}^2$. Then, $\mathcal{B}_1$ is a basis.

Example 6.2. Consider $X = \mathbb{R}^2$. Let $\mathcal{B}_2$ be collection of interiors of axis-parallel rectangles. Then, $\mathcal{B}_2$ is a basis. Note that nontrivial intersection is always a rectangle.

Definition 6.2. *The topology generated by a basis $\mathcal{B}$ is the collection τ satisfying:*

$$U \in \tau \iff \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U.$$

Example 6.3. Consider $X = \mathbb{R}$ with standard topology, τ_{st}. Consider $U = \mathbb{R} - \{0\}$. Then, $U \in \tau_{st}$. Let $x \in U$:

1. (Case 1) $x > 0$. Let $B = (0, x+1)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.
2. (Case 2) $x < 0$. Let $B = (x-1, 0)$. Then, $x \in B \subseteq \mathbb{R} - \{0\}$.

Example 6.4. Let τ_f be finite complement topology on $\mathbb{R}$. If $U \in \tau_f$, then $U \in \tau_{st}$. So $\tau_f \subseteq \tau_{st}$.

Remark. $(3, \infty)$ is open in τ_{st} but not in τ_f.

Lemma 6.1. *If $\mathcal{B}$ is a basis and τ is the topology generated by $\mathcal{B}$, then τ is a topology.*

Proof. $\varnothing \in \tau$ is vacuously true. If $x \in \varnothing$ then $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \varnothing$.

Now, we want to prove that $X \in \tau$. Let $x \in X$. By the axioms for basis, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq X$. Hence, $X \in \tau$.

Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of τ. Let

$$x \in \bigcup_{\alpha \in A} \mathcal{U}_\alpha.$$

Then, $\exists \mathcal{B} \in A$ with $x \in \mathcal{U}_{\mathcal{B}}$. Since $\mathcal{U}_{\mathcal{B}} \in Tau$, $\exists B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \mathcal{U}_{\mathcal{B}}$. TheN, $x \in B$ and $B \subseteq \mathcal{U}_{\mathcal{B}}$. So τ is closed under arbitrary unions.

Let $\{V_1, \ldots, V_n\} \subseteq \tau$. We want to use proof by induction to show that τ is closed under finite intersections. When $n = 1$,

$$\bigcap_{k=1}^{n} V_k = V_1 \in \tau.$$

Assume the claim is true for n. Then,

$$\bigcap_{k=1}^{n+1} V_k = \underbrace{V_{n+1}}_{W} \cap \underbrace{\bigcap_{k=1}^{n} V_k}_{W'}$$

Note $W \in \tau$ and $W' \in \tau$ by the induction hypothesis. If $W \cap W' = \varnothing$ then we are done ($\varnothing \in \tau$). Let $x \in W \cap W'$. Then, $x \in W$ and $x \in W'$, implying that $\exists B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq W$ and $x \in B_2 \subseteq W'$. Hence, $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2 \subseteq W \cap W'$. □

Example 6.5. Consider $X = \mathbb{R}$. Define

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

for $a, b \in \mathbb{R}$. Show

$$\mathcal{B}_{st} = \{(a, b) | a, b \in \mathbb{R}\}$$

is a basis.

Let $x \in \mathbb{R}$. Then,

$$x \in (x - 1, x + 1) \in \mathcal{B}_{st}.$$

For the second axiom, let $(a, b), (c, d) \in \mathcal{B}_{st}$ and assume $x \in (a, b) \cap (c, d)$. Then, $x \in (c, b) \subseteq (a, b) \cap (c, d)$.

Definition 6.3. *The standard topology on $\mathbb{R}$ is the topology τ_{st} generated by $\mathcal{B}_{st}$.*

Lemma 6.2. *Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bases on X and τ and τ' are the topologies they generate. Then, the following are equivalent:*

1. *τ' is finer than τ ($\tau \subseteq \tau'$)*
2. *$\forall x \in X$ and $\forall B \in \mathcal{B}$ with $x \in B$, $\exists B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.*

Proof. First, we want to show that 1 implies 2. Assume $\tau \subseteq \tau'$. Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ generates τ, $B \in \tau$. Then, $B \in \tau'$. Since τ' is generated by $\mathcal{B}'$,

$$\exists B' \in \mathcal{B} \text{ with } x \in B' \subseteq B$$

Other direction is left to readers as an exercise. □

Let $X = \mathbb{R}^2$. Consider $\mathcal{B}_1$, a collection of interior of circles, and $\mathcal{B}_2$, a collection of interior of axis-parallel rectangles. Let τ_j be toplogies generated by $\mathcal{B}_j$ where $j = 1, 2$.

Lemma 6.3. $\tau_1 = \tau_2$

Proof. ($\tau_1 \subseteq \tau_2$). We use the theorem from earlier. Let $x \in \mathbb{R}^2$ and $x \in B_1 \subseteq \mathcal{B}_1$. We have to show that $\exists B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$. Since $x \in B_1$, x sits inside a circle. But we can always construct a rectangle B_2 smaller than B_1 but contains x.

($\tau_2 \subseteq \tau_1$). Let $x \in \mathbb{R}^2$ and $B_2 \in \mathcal{B}_2$ with $x \in B_2$. We can always take $B_1 \in \mathcal{B}_1$ to be a circle inside rectangle B_2 and contains x. So $x \in B_1 \subseteq B_2$. □

Lemma 6.4. *Suppose (X, τ_x) and (Y, τ_Y) are topological spaces. Consider*

$$X \times Y = \{(a, b) | a \in X, b \in Y\}$$

Then,

$$\mathcal{B} = \{U \times V | U \in \tau_X, V \in \tau_Y\}$$

is a basis on $X \times Y$.

Proof. Let $(a, b) \in X \times Y$. Then, $X \times Y \in \mathcal{B}$ and $(a, b) \in X \times Y$. Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and assume

$$(a, b) \in U_1 \times V_1 \cap U_2 \times V_2.$$

Note

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}.$$

So

$$(a, b) \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$$

□

Definition 6.4. *The product topology is the topology generated by this basis.*

Theorem 6.1. *Suppose $\mathcal{B}_x$ and $\mathcal{B}_y$ are bases for τ_x and τ_y, respectively. Then,*

$$\{B_1 \times B_2 | B_1 \in \mathcal{B}_x, B_2 \in \mathcal{B}_y\}$$

is a basis for product topology on $X \times Y$.

Example 6.6. $(X, \tau_x) = (\mathbb{R}, \tau_{st}) = (Y, \tau_Y)$.

Definition 6.5. *The standard topology on $\mathbb{R}^2$ is the product topology of the $(\mathbb{R}, \tau_{st})$.*

Remark. The theoremm tells us that $\mathbb{R}^2$ has a basis

$$\{(a, b) \times (c, d) | a, b, c, d \in \mathbb{R}\}$$

Suppose $\mathcal{B}$ is a basis for a topology τ on X.

- First, if $B \in \mathcal{B}$, then $B \in \tau$. Recall that

  $$U \in \tau \iff \forall x = U, \exists B' = \mathcal{B}, x \in \mathcal{B}' \subseteq U.$$

  So let $x \in B$. Then, $B' = B$ satisfies the above condition ($x \in B \subseteq B$). Hence, $B \in \tau$.

- Let $U \in \tau$. Then, U is a union of elements in $\mathcal{B}$. To prove this, let $U \in \tau$. By definition,

$$\forall x \in U, \exists B_x \in \mathcal{U}$$

such that $x \in B_x \subseteq U$. Then,

$$\{B_x\}_{x \in U} \subseteq \tau.$$

Then,

$$\bigcup_{x \in X_U} B = U.$$

Suppose X, Y are topological spaces and equip $X \times Y$ with the product topology. Define

$$\pi_1 : X \times Y \to X$$
$$\pi_2 : X \times Y \to Y$$

Define

$$\mathcal{S} = \{\pi_1^{-1}(U) | U \subset X\} \cup \{\pi_2^{-1} | V \subset Y\}$$

The collection $\mathcal{S}$ is called a subbasis. The product topology is obtained by considering all unions of finite intersection of elements in $\mathcal{S}$.

Definition 6.6. *If X is a set, a subbasis is a collection $\mathcal{S}$ of subsets whose union is X. This generates a topology by considering all unions of finite intersections of things in $\mathcal{S}$.*

7 The Subspace Topology

Definition 7.1. *Let (X, τ) be a topological space and $Y \subseteq X$ be any subset. The subspace topology on Y is*

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

Lemma 7.1. *τ_Y is a topology.*

Proof.

- $\varnothing \cap Y = \varnothing \in \tau_Y$.
- $X \cap Y = Y \in \tau_Y$.
- (Unions) Let $\{V_\alpha\}_{\alpha \in A} \subseteq \tau_Y$, then $V_\alpha = U_\alpha \cap Y$ for some $U_\alpha \in \tau$. Then,

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in A} U_\alpha \right) \cap Y \in \tau_Y$$

□

Theorem 7.1. *Suppose $\mathcal{B}$ is a basis for X and $Y \subseteq X$. Then,*

$$\{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Example 7.1. Let $X = \mathbb{R}$ with $\mathcal{B} = \{(a, b)|a, b \in \mathbb{R}\}$. Consider

$$Y = [1, \infty).$$

Then, the basis for this topology is given by

$$\mathcal{B}_Y = \{[1, a)|a \in \mathbb{R}, a > 1\} \cup \{(a, b)|a, b \in \mathbb{R}, 1 \leq a < b\}$$

Example 7.2. $(3, 7] \notin \tau_Y$ because there does not exist open $U \in \tau$ ($U \in \mathcal{B}$) such that

$$(3, 7] = U \cap [1, \infty).$$

Proof. To yield contradiction, suppose there exists such U. Then,

$$7 \in (a, b) \cap [1, \infty).$$

So $7 \in (a, b)$ and $7 \in [1, \infty)$. In other words, $a < 7 < b$.

Let $\epsilon = (b - 7)/2$. Then, $a < 7 + \epsilon < b$ so $7 + \epsilon \in (a, b)$ and $7 + \epsilon \in [1, \infty)$. Then,

$$7 + \epsilon \in (a, b) \cap [1, \infty) = (3, 7] \implies 3 < 7 + \epsilon \leq 7,$$

yielding contradiction. □

Example 7.3. Let $X = \mathbb{R}$ and $Y = [0, 1]$. A basis for the subspace on Y is

$$\{(a, b)|0 \leq a < b \leq 1\} \cup \{[a, b)|0 \leq b \leq 1\} \cup \{(a, 1]|0 \leq a \leq 1\}$$

Example 7.4. Consider a rectangle

$$R = [0, 1] \times [2, 6].$$

There are two ways of thinking of topology on this rectangle. Since $R \subseteq \mathbb{R}^2$, we can think of it as a subspace topology. We can also think of it as a product topology. So are they same topologis in this case?

Theorem 7.2. *Suppose X and Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$ are subspaces. Then, the product topology on $A \times B$ equals the topology $A \times B$ inherits as a subset of $X \times Y$.*

Proof. In the subspace topology, basis element is given by

$$(A \times B) \cap (U \times V)$$

where $U \subseteq X$ and $V \subseteq Y$. Likewise, basis element of product topology is given by

$$(A \cap U) \times (B \cap V).$$

But then

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V).$$

□

8 The Order Topology

Let X be a set.

Definition 8.1. *An order relation (or simple order or linear order) is relation R on X so*

1. *(comparability) If $x, y \in X$ and $x \neq y$. Then either xRy or yRx.*
2. *(Nonreflexivity) $\nexists x \in X$ such that xRx.*
3. *(Transitivity)s If xRy and yRz then xRz.*

Example 8.1. Define $X \subseteq \mathbb{R}$ and $R =<$. Then, R is an order relation

Proof.

1. If $x \neq qy$, either $x < y$ or $y < x$.
2. $\nexists x \in X$ such that $x < x$.
3. $x < y, y < z \implies x < z$.

$\square$

If X has an order relation, we often denote it $<$. Suppose $(X, <)$ is a set with an order relation for $a, b \in X$.

Definition 8.2. $(a, b) = \{x \in X | a < x, x < b\}$.

Definition 8.3. $(a, b] = \{x \in X | a < x, x < b, \text{ or } x = b\}$.

Definition 8.4. $[a, b) = \{x \in X | a < x, x < b, \text{ or } x = a\}$.

Definition 8.5. *$a_0 \in X$ is a minimal element if $\forall x \in X$, $a_0 \leq x$.*

Definition 8.6. *$b_0 \in X$ is a maximal element if $\forall x \in X$, $x \leq b_0$.*

Definition 8.7. *Suppose $(X, <_x)$ and $(Y, <_y)$ are simply ordered sets. The dictionary order on $X \times Y$ is the order $<_{x \times y}$ defined as follows:*

$$(a_1, b_1) <_{X \times Y} (a_2, b_2)$$

if $a_1 <_x a_2$ or $a_1 = a_2$ and $b_1 <_y b_2$.

Example 8.2. Define $X = Y = \mathbb{R}$ and $<_x =<_y =<$, usual inequality. Then,

- $(0, 0) <_{X \times Y} (1, -3)$ is true
- $(0, 0) <_{X \times Y} (0, -3)$ is false.

Definition 8.8. *Suppose $(X, <)$ is a simply ordered set. Then, the order topology on X is the one with basis elements: (a, b) for $a, b \in X$, $[a_0, b)$ for $b \in X$ provided a_0 is the minimal element of X, and $(a, b_0]$ for $a \in X$ provided $b_0 \in X$ is the maximal element.*

Example 8.3. Consider $X = [0, 1]$ with $<$, the standard inequality. The order topology is the topology inherited as a subset.

Example 8.4. Consider $(\mathbb{R}^2, <_o)$ where $<_0$ is the dictionary order. The order topology is called the dictionary order topology.

Think about what this set looks like:

$$((0,0),(1,1))$$

Example 8.5. Let X be a set with the discrete topology (all subsets are open). Show that

$$\mathcal{B}_1 = \{\{x\}|x \in X\}$$

is a basis.

Proof. Let $x \in X$ then $x \in \{x\} \subseteq B$. Let $B_1, B_2 \in \mathcal{B}_1$ and $x \in B_1 \cap B_2$. Then,

$$B_1 = \{x\} = B_2$$

so $x \in \{x\} \subseteq B_1 \cap B_2$

Let τ_1 be the topology generated by $\mathcal{B}_1$. We clearly have $\tau_1 \subseteq \tau_d$, where τ_d is the discrete topology. To show that $\tau_d \subseteq \tau_1$, let $U \in \tau_d$ and let $x \in U$. Then, $x \in \{x\} \subseteq U$ and $\{x\} \in \mathcal{B}_1$. So the results follows by a theorem in section 13. $\square$

Example 8.6. Show that

$$\mathcal{B}_2 = \{\{x\} \times (a,b)|x, a, b \in \mathbb{R}\}$$

forms a basis for $\mathbb{R} \times \mathbb{R}$ with the dictionary order topology.

Proof. Assume that it's a basis. We will show that the associated topology τ_2 is the dictionary order topology τ_{DO}.

Note that if $B \in \mathcal{B}_2$, then $\exists x, a, b \in \mathbb{R}$ such that

$$B = \{x\} \times (a,b) = ((x,a),(x,b)) = \{y \in \mathbb{R} \times \mathbb{R}|(x,a) <_{DO} y <_{DO} (x,b)\}.$$

So this is open in τ_{DO}. So $\tau_2 \subseteq \tau_{DO}$.

Let $((a_1, b_1), (a_2, b_2))$ be a basis element in the dictionary order topology. Let $y \in ((a_1, b_1), (a_2, b_2))$. Write $y = (x_1, y_1)$. There are three cases:

1. $a_1 < x_1 < a_2$
2. $a_1 = x_1$
3. $a_2 = x_1$

Here, we will prove case 1. Assume $a_1 < x < a_2$. Then,

$$y \in \{x\} \times (y_1 - 1, y_1 + 1)$$

Then, $\{x\} \times (y_1 - 1, y_1 + 1) \in \mathcal{B}_2$ and also

$$\{x\} \times (y_1 - 1, y_1 + 1) \subseteq ((a_1, b_1), (a_2, b_2))$$

Assuming these the topology on $\mathbb{R}_d \times \mathbb{R}$ has basis

$$\{\{x\} \times (a, b) | x, a, b \in \mathbb{R}\}.$$

This is exactly the basis that generates the dictionary order topology. So two topologies are same. $\square$

Example 8.7. Let $X = \mathbb{R}$. The set

$$\mathcal{B}_\ell = \{[a, b) | a, b \in \mathbb{R}\}$$

is a basis. The topology it generates is called the lower limit topology on $\mathbb{R}$. $\mathbb{R}$ with the lower limit topology is denoted $\mathbb{R}_\ell$.

Remark. Lower limit topology is strictly finer than standard topology, i.e., every open set in standard topology is open in lower limit topology but not vice-versa

Example 8.8. Is $[0, 1)$ open in the standard topology? No $[0, 1)$ is not open in standard topology. If it were, then there would be some $(a, b) \subseteq \mathbb{R}$ such that

$$0 \in (a, b) \subseteq [0, 1).$$

But then

$$0 \in (a, b) \implies a < 0, b > 0$$

So $a/2 \in (a, b)$. But $a/2 < 0$ so $a \notin [0, 1)$.

Example 8.9. Is $(0, 1)$ open in $\mathbb{R}_\ell$? Yes because

$$(0, 1) = \bigcup_{x \in \{y | y > 0\}} [x, 1).$$

9 Interior and Closure

Recall De Morgan's Laws:

- $\bigcup_{\alpha \in I}(X \setminus U_\alpha) = X \setminus \bigcap_{\alpha \in I} U_\alpha$
- $\bigcap_{\alpha \in I}(X \setminus U_\alpha) = X \setminus \bigcup_{\alpha \in I} U_\alpha$

Definition 9.1. *If (X, τ) is a topological space then a subset $A \subseteq X$ is called closed if $X \setminus A$ is open, i.e., if $X \setminus A \in \tau$*

If $\mathcal{C}$ is a collection of all closed sets in X,

1. $X \in \mathcal{C}$
2. $\varnothing \in \mathcal{C}$
3. $\mathcal{C}$ is closed under arbitrary intersection.
4. $\mathcal{C}$ is closed under finite union.

Example 9.1. In $\mathbb{R}$, consider $U_n = [1/n, 1 - 1/n]$ for $n \in \mathbb{Z}^+$. U_n is a closed interval for each $n \in \mathbb{Z}^+$. Hoever,

$$\bigcup_{n \in \mathbb{Z}^+} U_n = (0, 1)$$

is open and not a closed set.

Definition 9.2 (Interior)**.** *Suppose (X, τ) is a topological space and $A \subseteq X$. Define* $\mathrm{Int}(A)$ *to be the union of all open sets contained in A.*

$$\mathrm{Int} A = \bigcup_{\alpha \in J} U_\alpha$$

where $\{U_\alpha, \alpha \in J\}$ consists of all $U_\alpha \in J$ such that $U_\alpha \subseteq A$.

Observe that $\mathrm{Int}A$ is an open set. Notice also that $\mathrm{Int}(A) \subseteq A$. So $\mathrm{Int}A$ is the maximal open set contained in A. In other words, any open set V contained in A is a subset of $\mathrm{Int}A$.

Remark. If A is open, $\mathrm{Int}A = A$.

Definition 9.3. *The closure of A, denoted $\bar{A}$, is defined as the intersection of all closed sets that contain A:*

$$\bar{A} = \bigcap_{\beta \in K} C_\beta,$$

where $\{C_\beta : \beta \in K\}$ consists of all closed sets hat contain A.

Observe that $\bar{A}$ is a closed set because it is an intersection of closed sets. Notice that $A \subseteq \bar{A}$. Then, $\bar{A}$ is the smallest closed set that contains A, i.e. any closed set that contains A must contain $\bar{A}$.

Remark. If A is closed, then $A = \bar{A}$.

Example 9.2. If $U_\alpha, \alpha \in J$ is a collection of sets with $B \subseteq U_\alpha$ for all α, then $B \subseteq \bigcap_{\alpha \in J} U_\alpha$. Further, if $B = U'_\alpha$ for some $\alpha' \in J$, then

$$B = \bigcap_{\alpha \in J} U_\alpha \subset U_{\alpha'}$$

for all α'.

Proposition 9.1.

- $\text{Int}(X - A) = X - \bar{A}$
- $\overline{(X - A)} = X - \text{Int}(A)$.

Example 9.3. Define

$$X = \{a, b, c\}$$
$$\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$$

Then,

$$\mathcal{C} = \{X, \{b, c\}, \{c\}, \varnothing\}$$

Consider $A = \{a\}$. Then, $\text{Int}A = \{a\}$ and $\bar{A} = X$.
Consider $B = \{b\}$. Then, $\text{Int}B = \varnothing$ and $\bar{B} = \{b, c\}$.
Consider $C = \{c\}$. Then, $\text{Int}C = X$ and $\bar{C} = \{c\}$.

Let

$$X = \mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$$

Example 9.4. Consider

$$A = \{re^{i\theta} | 0 \le \theta \le 2\pi, 0 \le r \le 1 \text{or} r1, \theta \in \mathbb{Q}\}$$

Then,

$$\text{Int}A = \{re^{i\theta} | 0 < r < 1\}$$
$$\bar{A} = \{re^{i\theta} | 0 \le r \le 1\}$$

Notice that if $z \in \bar{A}$ and U is an arbitrary open set in $\mathbb{C}$ containing z then $U \cap A \neq \varnothing$.

Theorem 9.1. *If $A \subseteq X$ is any subset then $X \in \bar{A}$ if and only if every open set U containing X intersects A nontrivially.*

Proof. Recall $\bar{A}$ is a closed set, so $X - \bar{A}$ is open. We will show conversely that $x \notin \bar{A}$ if and only if there exists an open set U containing x such that $U \cap A = \varnothing$. But since $X - \bar{A}$ is open, if $x \notin \bar{A}$, then $x \in X - \bar{A}$. Openness of $X - \bar{A}$ now implies that there exists small open set U about x with $U \subseteq X - \bar{A}$. As such, $U \cap \bar{A} = \varnothing$, since $A \subseteq \bar{A}$, it follows that $U \cap A = \varnothing$. $\square$

Definition 9.4. *If $A \subseteq X$ is a subset,*

$$\bar{A} = \{x \in X | U \cap A \neq \varnothing \text{ for every open set } U \text{ about } X\}.$$

Definition 9.5 (Limit point)**.** *A point X is a limit point of the set A if every open set U about x satisfies $(U - \{x\}) \cap A \neq \varnothing$, i.e., we require U intersects A in some point other than x. We write*

$$A' = \{x | x \text{ is a limit point of } A\}.$$

Theorem 9.2. *$\bar{A} = A \cup A'$.*

Proof.

$$(U - \{x\}) \cap A \neq \varnothing \implies U \cap A \neq \varnothing$$

which implies $A \subseteq \bar{A}$. But we already know that $A \subseteq \bar{A}$. Then,

$$A \cup A' \subseteq \bar{A}.$$

Suppose $x \in \bar{A}$. Either $x \in A$, in which case $x \in A \cup A'$ or $x \notin A$. In this case, since $x \in \bar{A}$, every open set U about satisfies $U \cap A \neq \varnothing$. But $x \notin A$ so $x \notin A$. So

$$(U - \{x\}) \cap A \neq \varnothing.$$

So $x \in A'$. □

Let X be a topological space with $Y \subseteq X$ a subset. Regard Y as a topological space using the subspace topology, i.e., $V \subseteq Y$ is open in the subspace topology if and only if there eixsts an open set $U \subseteq X$ with $V = U \cap Y$.

Theorem 9.3. *In the subspace topology on Y, a subset $B \subseteq Y$ is closed if and only if $B = A \cap Y$ for some closed set $A \subseteq X$.*

Proof.

$$\begin{aligned} B \subseteq Y \text{ is closed} &\iff V = Y - B \text{ if open} \\ &\iff \exists U \text{ open in } X \text{ such that } V = U \cap Y \\ &\iff A = X - U \text{ is closed} \end{aligned}$$

□

10 Sequences

Definition 10.1 (Convergnece)**.** *If $\{X_n\}$ is a sequence of points in a topological space X, we say $\{X_n\}$ converges to $x \in X$ if for every neighborhood U about x, there exists integer N such that $n \geq N \implies X_n \in U$.*

Remark. In a general topological space, a sequence may converge to more than one point.

Example 10.1. Consider the trivial topology:

$$\tau_{\text{triv}} = \{\varnothing, X\}.$$

Every sequence converges and it converges to every point in X.

Example 10.2. Given $X = \{a, b, c\}$, consider the following topology:

$$\tau = \{\varnothing, \{a\}, \{a, b\}, X\}.$$

Then, the sequence $X_n = b$ for all n converges to b as well as c but not to a.

Example 10.3. If X has the discrete topology, then show that a sequence $\{X_n\}$ is convergent if and only if $\{X_n\}$ is eventually constant, i.e., $\exists x \in X$ and $N \in \mathbb{Z}^+$ such that

$$n \geq N \implies X_n = X.$$

Definition 10.2. *A topological space A is called Hausdorff if for any two distinct points $x, y \in X$ there exists disjoint open sets U, V with $x \in U$ and $Y \in V$.*

Example 10.4. If $\{X_n\}$ is a sequence of points in Hausdorff space, then if it converges, the point it converges to is unique.

Theorem 10.1. *If (X, τ) is Hausdorff, then every finite subset of X is a closed set.*

Remark. If X is Hausdorff, then every singleton $\{x\}$ is closed.

Example 10.5. Consider

$$\mathcal{C} = \left\{ \left[\frac{1}{n}, 1 - \frac{1}{n} \right] | n \geq 3 \right\}$$

Then,

$$\bigcup_{n=3}^{\infty} C_n = (0, 1)$$

is not closed in $\mathbb{R}$.

Example 10.6. An infinite collection of open sets whose intersection is not necessarily open. Consider

$$U_n = \left(-\frac{1}{n}, \frac{1}{n} \right).$$

Then,

$$\bigcap_{n=1}^{\infty} U_n = \{0\}$$

is closed in standard topology.

Definition 10.3. *A topological space X in which all finite sets are closed is called a T_1 space.*

Theorem 10.2. *If (X, τ) is Hausdorff, then X is a T_1 space. Note that converse is not necessarily true.*

Example 10.7. If X is inifinite and τ is finite complement topology is a T_1 space but not Hausdorff.

Example 10.8. Notice if X is finite, then the every subset $U \subseteq X$ has a finite complement. So in this case, the finite complement topology is the same as the discrete topology. So it is a T_1 space and Hausdorff.

Theorem 10.3. *If X is Hausdorff and $\{x_n\}$ is a convergent sequence converging to $x \in X$, then the point x is unique. In this case, we say x is the limit of the sequence $\{x_n\}$ and write*

$$\lim_{n \to \infty} x_n = x.$$

Theorem 10.4. *If $(X, <)$ is a simply ordered set, then it is Hausdorff in the order topology.*

Theorem 10.5. *If X, Y are two topological spaces annd both are Hausdorff then so is the product $X \times Y$.*

Theorem 10.6. *If X is Hausdorff and $Y \subseteq X$ is a subset, then Y is Hausdorff in subspace topology.*

11 Continuous Functions

What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous?

- $\forall x \in \mathbb{R}$, $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$.
- $\lim_{nto\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)$.
- $\forall x_0 (\forall \epsilon > 0, \exists \delta$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

Definition 11.1. *If $f : X \to Y$ is a function between two topological spaces X, Y, then f is continuous if $f^{-1}(V)$ is open for every open set $V \subseteq Y$.*

Example 11.1. Consider $f : \mathbb{R} \to \mathbb{R}_\ell$, where $\mathbb{R}_\ell$ is the lower limit topology and $f(x) = x$. This map is not continuous. Take $V = [a, b)$. This is open in $\mathbb{R}_\ell$ but $f^{-1}(V) = V$ is not open in the standard topology.

Example 11.2. $g : \mathbb{R}_\ell \to \mathbb{R}$ with $g(x) = x$ is continuous.

Example 11.3. If $f : X \to Y$ is the constant map $f(x) = y_0 \forall x \in X$, then f is continuous for any topology on X, Y.

Theorem 11.1. *Let X, Y be two toplogical spaces and $f : X \to Y$ is a function. If $f : X \to Y$ then the following are equivalent:*

1. *f is continuous*
2. *for all subsets $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$*
3. *for all closed sets $B \subseteq Y$, $f^{-1}(B)$ is closed in X*
4. *for each $x \in X$ if V is open about $f(x)$, then $\exists$ open set U about x such that $f(U) \subseteq V$.*

Example 11.4. Consider $f : X \to Y$ any function. If X has discrete topology, f is automatically continuous.

Proof. Let $U \subseteq Y$ be open. Then, $f^{-1}(U)$ is a subset of X. By the definition of the discrete topology, $f^{-1}(U)$ is therefore open in X. Thus, it is true for all open $U \subseteq Y$, so f is continuous. □

Example 11.5. Consider $Y = [0, 1) \subseteq \mathbb{R}$ with standard topology on $\mathbb{R}$. Find the closure $\bar{Y}$.

We can define closure in three different ways:

1. Intersection of all closed sets containing Y
2. $\bar{Y} = Y \cup \{\text{limit points}\}$
3. $\bar{Y}$ is the smallest closed set containing Y.

Note that $[0,1) \subseteq [0,1]$ and $[0,1]$ is closed in standard topology ($[0,1] = \mathbb{R} - ((-\infty,0) \cup (1,\infty))$). This implies that $\bar{Y} \subseteq [0,1]$.

We also have $[0,1) \subseteq \bar{Y}$. If $\bar{Y} \neq [0,1]$, then $1 \notin \bar{Y}$ and $\bar{Y} = [0,1)$. However, $[0,1)$ is not closed in standard topology and this is a contradiction. Therefore, $Y = [0,1]$.

Example 11.6. Define $X = [0,\infty)$. Consider $X \times X$ with dictionary order topology. Show that $X \times X$ is Hausdorff.

Proof. Let $(x_1,y_1),(x_2,y_2) \in X \times X$ with $(x_1,y_1) \neq (x_2,y_2)$. We may assume $(x_1,y_1) < (x_2,y_2)$. There are 7 cases that we have to consider:

1. $0 < x_1 < x_2$ and $y_1, y_2 > 0$
2. $0 = x_1 < x_2$ and $y_1, y_2 > 0$
3. $0 = x_1 = x_2$ and $0 < y_1 < y_2$
4. $0 = x_1 = x_2$ and $0 = y_1 < y_2$
5. $0 < x_1 < x_2$ and $y_1 = 0$ or $y_2 = 0$
6. $x_1 = x_2 > 0$ and $y_2 > y_1 > 0$
7. $x_1 = x_2 > 0$ and $y_1 = 0$ or $y_2 \neq 0$

For case 1, consider

$$U_1 = \left(\left(x_1, y_1 - \frac{y_1}{2}\right), \left(x_1, y_1 + \frac{y_1}{2}\right)\right)$$
$$U_2 = \left(\left(x_2, y_2 - \frac{y_2}{2}\right), \left(x_2, y_2 + \frac{y_2}{2}\right)\right)$$

Then, there intersection is empty and $(x_1,y_1) \in U_1$ and $(x_2,y_2) \in U_2$. $\square$

Definition 11.2. *Let X and Y be topological spaces. A function $f : X \to Y$ is a homeomorphism if*

1. *f is continuous*
2. *f is bijective (so f^{-1} exists)*
3. *$f^{-1} : Y \to X$ is continuous*

Example 11.7. Suppose X is a topological space. Then, identity function

$$\mathrm{Id} : X \to X$$

with $\mathrm{Id}(x) = x$ is a homeomorphism.

Example 11.8. If $f : X \to Y$ is homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism.

Example 11.9. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$. Then, this is a homeomorphism. Furthermore,

$$f^{-1}(y) = y^{1/3}.$$

12 The Product Topology

Definition 12.1. *Let $\{X_\alpha\}$ be a collection of sets indexed by $\alpha \in J$. Then,*

$$\prod_{\alpha\in J} X_\alpha = \left\{ f : J \to \bigcup_{\alpha \in J} X_\alpha, f(\beta) \in X_\beta \forall \beta \in J \right\}.$$

Example 12.1. Consider $J = \{1, 2\}$. Then,

$$\begin{aligned}\prod_{\alpha\in J} X_\alpha &= \{f : \{1,2\} \to X_1 \cup X_2 | f(1) \in X_1, f(2) \in X_2\} \\ &= \{(f_1, f_2) | f_1 \in X_1, f_2 \in X_2\} \\ &= X_1 \times X_2\end{aligned}$$

More generally, if $J = \{1, 2, \dots, n\}$, then

$$\prod_{\alpha\in J} X_\alpha = X_1 \times X_2 \times \cdots \times X_n$$

If $J = \mathbb{Z}_+ = \{1, 2, 3, \dots\}$, then we can write

$$\prod_{\alpha\in J} = X_1 \times X_2 \times X_3 \times \cdots$$

If $J = \mathbb{Z}_+$ and $X_\alpha = X \forall \alpha \in J$, then

$$\prod_{\alpha\in J} X_\alpha = X \times X \times \cdots = X^\omega.$$

Example 12.2. Consider $J = \mathbb{R}$ and let $X_\alpha = \mathbb{R} \forall \alpha \in \mathbb{R}$. Then,

$$\begin{aligned}\prod_{\alpha\in J} X_\alpha &= \prod_{\alpha\in\mathbb{R}} \mathbb{R} \\ &= \left\{ f : \mathbb{R} \to \bigcup_{\alpha in \mathbb{R}} \mathbb{R} \,\middle|\, f(x) \in \mathbb{R} \forall x \in \mathbb{R} \right\} \\ &= \{f : \mathbb{R} \to \mathbb{R} | f \text{ is a function}\}.\end{aligned}$$

Definition 12.2 (Box topology)**.** *Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The box topology on $\prod X_\alpha$ is the one with basis*

$$\left\{ \prod_{\alpha\in J} U_\alpha \,\middle|\, U_\alpha \subseteq X_\alpha \text{ open} \right\}$$

Lemma 12.1. *If $\mathcal{B}_\alpha$ is a basis for X_α, then*

$$\left\{ \prod_{\alpha\in J} B_\alpha \,\middle|\, B_\alpha \in \mathcal{B}_\alpha \right\}$$

is a basis for the box topology.

Definition 12.3. *For $\beta in J$, define*

$$\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$$

Example 12.3. Consider $J = \{1, 2\}$. Then,

$$\prod_{\alpha\in J} X_\alpha = X_1 \times X_2.$$

Furthermore,

$$\pi_1 : X_1 \times X_2 \to X_1$$
$$\pi_2 : X_2 \times X_2 \to X_2$$

Definition 12.4 (Product topology)**.** *The product topology on $\prod_{\alpha\in J} X_\alpha$ is the one with subbasis given by*

$$\left\{\pi_\beta^{-1}(U_\beta) \,\middle|\, U_\beta \subseteq X_\beta \text{ open }, \beta \in J\right\}$$

If $\prod X_\alpha$ is equipped with the product topology, we call $\prod X_\alpha$ the product space.

Lemma 12.2. *If $\mathcal{B}_\alpha$ is a basis for X_α, then*

$$\left\{\pi_\beta^{-1}(B_\beta) \,\middle|\, B_\beta \in \mathcal{B}_\beta, \beta \in J\right\}$$

is a subbasis for the product topology.

Remark. If J is finite, then the box topology on $\prod X_\alpha$ equals the product topology.

Example 12.4. Consider $J = \mathbb{Z}_+$ with $X_\alpha = \mathbb{R}$. Then,

$$\prod_{\alpha\in J} X_\alpha = \mathbb{R}^\omega.$$

For example, a box topology can be

$$\prod_{\alpha\in J} B_\alpha = B_1 \times B_2 \times B_3 \times \cdots .$$

Then,

$$(-1, 1) \times (-2, 2) \times \cdots \times (-n, n) \times \cdots$$

is a basis element in box topology.

On the other hand,

$$(-1, 1) \times (-2, 2) \times \cdots \times (-n, n) \times \cdots$$

is not open in product topology.

$$(-1, 1) \times (-2, 2) \times \cdots \times (-n, n) \times \mathbb{R} \times \cdots \times \mathbb{R} \cdots$$

is open in product topology.

Lemma 12.3. *The collection*

$$\left\{ \prod_{\alpha \in J} U_\alpha \,\Big|\, U_\alpha \subseteq X_\alpha \textit{ is open }, U_\alpha = X_\alpha \textit{ for all but a finite number of } \alpha \in J \right\}$$

forms a subbasis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

Example 12.5.

$$(-1,1) \times (-1,1) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

is open in the product topology on $\mathbb{R}^\omega$ but

$$(-1,1) \times (-1,1) \times \cdots \times (-1,1) \times \cdots$$

is not open in product topology.

If we define $\pi_1 : \mathbb{R}^\omega \to \mathbb{R}$ where $\pi_1(x_1, x_2, \ldots,) \to x_1$. Then,

$$\pi_1^{-1}((-1,1)) = (-1,1) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

is open in product topology. Likewise, we have

$$\pi_2^{-1}((-1,1)) = \mathbb{R} \times (-1,1) \times \mathbb{R} \times \cdots$$

open in product topology. Then,

$$\pi_1^{-1}((-1,1)) \times \pi_2^{-1}((-1,1)) = (-1,1) \times (-1,1) \times \mathbb{R} \times \mathbb{R} \times \cdots .$$

is open in product topology. Note that we can only take finite intersections.

Example 12.6. Consider $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ with the product topology. Show

$$(-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdots$$

is not open.

Solution. Recall that U is open if and only if $\forall x \in U$ there exists $B \in \mathcal{B}$ a basis element such that

$$x \in B \subseteq U.$$

Observe that

$$(0, 0, \ldots, 0) \in U = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdot.$$

Assume B is a basis element containing x. Then, we can write

$$B = V_1 \times V_2 \times \cdots \times V_m \times \mathbb{R} \times \mathbb{R} \times \cdots$$

where $V_l \subseteq \mathbb{R}$ open and $1 \le l \le m$. If $B \subseteq U$, then the following condition must be satisfied:

$$\mathbb{R} \subseteq \left(-\frac{1}{m+1}, \frac{1}{m+1}\right).$$

This is false. So $B \not\subseteq U$. Therefore, U is not open. $\square$

Example 12.7. Is the following function continuous? $f : \mathbb{R} \to \mathbb{R}^2$ where

$$f(t) = (\cos(t), t^2 + e^t)$$

Theorem 12.1. *Let $\{X_\alpha\}$ be topological space and give $\prod X_\alpha$ the product topology. Let*

$$f : A \to \prod X_\alpha$$

be any function. Then, f is continuous if and only if $\pi_\alpha \circ f : A \to X_\alpha$ is continuous $\forall \alpha \in J$.

Example 12.8. Let $J = \mathbb{Z}_+$. Then,

$$\prod X_\alpha = X_1 \times X_2 \times \cdots$$

Consider $f : A \to X_1 \times X_2 \times \cdots$ with

$$\begin{aligned} f(a) &= (f_1(a), f_2(a), \dots) \\ f_1 &= \pi_1 \circ f \\ f_2 &= \pi_2 \circ f \\ &\vdots \end{aligned}$$

Note that $f_\alpha = \pi_\alpha \circ f$ is the α-th component of f.

Example 12.9. Recall that the product topology has subbasis

$$\mathcal{S} = \{\prod_\alpha^{-1}(U) \,|\, U \subseteq X_\alpha \text{ open }, \alpha \in J\}$$

So $\pi_\beta : \prod X_\alpha \to X_\beta$is continuous $\forall \beta \in J$.

Now, we will prove the theorem.

Proof. ($\Rightarrow$) If f is continuous, then $\pi_\alpha \circ f$ is continuous because composition of continuous functions is continuous.

($\Leftarrow$) Assume $\pi_\alpha \circ f$ is continuous $\forall \alpha \in J$. Let $\pi_\alpha^{-1}(U) \subseteq \prod X_\alpha$ be a subbasis element. Then,

$$f^{-1}\left(\pi_\alpha^{-1}(U)\right) = (\pi_\alpha \circ f)^{-1}(U).$$

Then, this is open because the composition map $(\pi_\alpha \circ f)$ is assumed to be continuous and $U \subseteq X_\alpha$ is open. $\square$

Recall that a basis for product topology is

$$\prod_{\alpha \in J} U_\alpha$$

where $U_\alpha \subseteq X$ where $U_\alpha = X_\alpha$ for all but a finite number of $\alpha \in J$. Note

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = U_\beta$ if $\alpha = \beta$ and $U_\alpha = X_\alpha$ if $\alpha \neq \beta$.

Example 12.10. Consider $\prod X_\alpha = \mathbb{R} \times \mathbb{R}$. Then,

$$\pi_1^{-1}((-1,1)) = (-1,1) \times \mathbb{R}.$$

On the other hand, the basis from $\mathcal{S}$ is

$$\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}) = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = U_{\alpha_j}$ if $\alpha = \alpha_j$ and $U_\alpha = X_\alpha$ otherwise. Hence,

$$x \in \pi_\beta^{-1}(U_\beta)$$

if and only if

$$\pi_\beta(x) \in U_\beta$$

if and only if the β-composition of x is in U_β (no restriction on the other components).

Example 12.11. Consider $f : \mathbb{R} \to \mathbb{R}^\omega$ where

$$f(t) = (t, t, t, \dots).$$

By the previous theorem, this is continuous in the product topology on $\mathbb{R}^\omega$. However, this is not continuous with the box topology on $\mathbb{R}^\omega$.

Proof. Consider

$$U = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdots$$

We will show that $f^{-1}(U)$ is not open in $\mathbb{R}$.

We claim that $f^{-1}(U) = \{0\}$. If $t \in f^{-1}(U)$, then $f(t) \in U$. Note that

$$(t, t, t, \dots) \in U$$

So

$$\begin{gathered} t \in (-1,1) \\ t \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ \vdots \\ t \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ \vdots \end{gathered}$$

So $|t| < 1/n \,\forall n$. This implies that $t = 0$. Conversely, $f(0) \in U$. □

13 Metric Topology

Definition 13.1. *Let X be a set. A metric on X is a function $d : X \times X \to \mathbb{R}$ satisfying*

1. $d(x, y) \geq 0 \forall x, y \in X$ *and* $d(x, y) = 0 \iff x = y$
2. $\forall x, y \in X, d(x, y) = d(y, x)$
3. *(triangle inequality)* $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$

Definition 13.2. (X, d) *is a metric space.*

We can think $d(x, y)$ as the "distance" between x and y.

Definition 13.3. *Let $x \in X$ and $\epsilon > 0$. We can define*

$$B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\},$$

the ball of radius epsilon centered at x, relative to the metric d.

Example 13.1. Define $d : X \times X \to \mathbb{R}$ as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

We want to verify three axioms.

1. $1 \leq 0$ and $d(x, y) = 0$ iff $x = y$
2. $x = y \iff y = x$ and $x \neq y \iff y \neq x$
3. Note that $d(x, z), d(x, y), d(y, z)$ are all either equal to 0 or 1. The triangle inequality can only fail if

   $$d(x, z) = 1 \text{ and } d(x, y) + d(y, z) = 0$$

   Note that

   $$d(x, y) + d(y, z) = 0 \implies d(x, y) = 0, d(y, z) = 0 \implies x = y, y = z$$

   This implies that $x = z$ by transitivity and $d(x, z) = 0$. This yields contradiction because we assumed $d(x, z) = 1$. So the triangle inequality does not fail.

Example 13.2. Let $X = \mathbb{R}$ and define d as above. Sketch the ball with radius 1 centered at 0. Sketch $B_d(0, 2)$.

Note that

$$B_d(0, 1) = \{y \in \mathbb{R} | d(0, y) < 1\} = \{0\}$$

and

$$B_d(0, 2) = \{y \in \mathbb{R} | d(0, y) < 2\} = \mathbb{R}.$$

Example 13.3. Let $X = \mathbb{R}$ and define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where

$$d(x, y) = |x - y|.$$

We want to check that d is a metric.

1. $d(x, y) = |x - y| \geq 0$. Furthermore,
$$d(x, y) = 0 \iff x - y = 0 \iff x = y.$$

2. $$d(x, y) = |x - y| = | - y + x| = | - (y - x)| = |y - x| = d(y, x)$$

3. $$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$$

Example 13.4. Define d as in previous example. Then,

$$B_l(3, \epsilon) = \{y \in \mathbb{R} | d(3, y) < \epsilon\} = \{y \in \mathbb{R} || 3 - y| < \epsilon\}$$

Note that $|3 - y| < \epsilon$ is equivalent to $3 - y < \epsilon$ and $-(3 - y) < \epsilon$. Rearranging, we have

$$B_l(3, \epsilon) = \{y \in \mathbb{R} | - \epsilon + 3 < y < \epsilon + 3\} = (3 - \epsilon, 3 + \epsilon)$$

Lemma 13.1. *Let (X, d) be a metric space. The collection*

$$B = \{B_l(X, \epsilon) | x \in X, \epsilon \in (0, \infty)\}$$

forms a basis. The induced topology is called the metric topology if (X, τ) is a topological space and τ is induced from a metric as above ($\exists$ a metric d so τ is in the metric associated to d then we call (X, τ) metrizable)

Example 13.5. Consider X with

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

The metric topology is the discrete topology. $B_d(x, 1) = \{x\}$ are contained in the basis.

Example 13.6. Consider $X = \mathbb{R}$ with $d(x, y) = |x - y|$. Then, the metric topology is the standard topology and the basis is $B_d(x, \epsilon) = (x - \epsilon, x + \epsilon)$.

Example 13.7. Consider $X = \mathbb{R}^n$ where $n \geq 1$. Then, for $x, y \in X$, we can write

$$x = (x_1, x_2, \ldots, x_n)$$
$$y = (y_1, y_2, \ldots, y_n)$$

Let

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Lemma 13.2. *This is a metric called the Euclidean metric. For $n = 2$, $B_d(x, \epsilon)$ is a circle centered at x with radius ϵ.*

Example 13.8. Consider $X = \mathbb{R}^n$ and define

$$\rho(x, y) = (|x_1 - y_1|, \ldots, |x_n - y_n|)$$

When $n = 2$ $B_\rho(X, \epsilon)$ forms a square whose sidelength is 2ϵ. For example,

$$B_\rho(0, \epsilon) = \{y|\max(|y_1|, |y_2|) < \epsilon\}$$

Example 13.9. Show every metric space is Hausdorff.

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Set $\epsilon = \frac{1}{2}d(x_1, x_2)$. Since $x_1 \neq x_2$, so $\epsilon > 0$. Hence,

$$x_1 \in B_d(x_1, \epsilon) \text{ and } x_2 \in B_d(x_2, \epsilon)$$

Also,

$$B_d(x_1, \epsilon) \cap B_d(x_2, \epsilon) = \varnothing$$

because if $z \in B_d(x_1, \epsilon) \cap B_d(x_2, \epsilon)$, then

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, z) + d(z, x_2) \\ &< \epsilon + \epsilon \\ &= d(x_1, x_2) \end{aligned}$$

which is a contradiction. □

14 Comparing metrics

Consider $X = \mathbb{R}^n$ and $x, y \in X$ such that

$$\begin{aligned} x &= (x_1, \ldots, x_n) \\ y &= (y_1, \ldots, y_n) \end{aligned}$$

Recall that Euclidean metric is defined as

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n + y_n)^2}$$

and the square metric is defined by

$$\begin{aligned} \rho(x, y) &= \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} \\ &= \max_i\{|x_i - y_i|\} \end{aligned}$$

Consider $B_d(0, \epsilon)$. By definition,

$$\begin{aligned} B_d(0, \epsilon) &= \{y|d(0, y) < \epsilon\} \\ &= \{(y_1, \ldots, y_n)|\sqrt{y_1^2 + \cdots + y_n^2} < \epsilon\} \\ &= \{(y_1, \ldots, y_n)|y_1^2 + \cdots + y_n^2 < \epsilon^2\} \end{aligned}$$

is the interior of a radius-ϵ sphere. On the other hand,

$$\begin{aligned} B_\rho(0,\epsilon) &= \{(y_1,\ldots,y_n) | \max_i\{|x_i - y_i|\}\} \\ &= (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times\cdots\times(-\epsilon,\epsilon) \end{aligned}$$

Hence, when $n = 2$, $B_d(0,\epsilon)$ is a circle of radius ϵ and $B_\rho(0,\epsilon)$ is a square whose side lengths are 2ϵ. We know how to compare two toplogies generated by these metrics but how can we compare the metrics?

Lemma 14.1. *Suppose d and d' are metrics on X and τ and τ' are their respective topologies. The following are equivalent:*

1. $\tau \subseteq \tau'$
2. $\forall x \in X, \forall \epsilon > 0, \exists \delta > 0$ *such that* $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.

Proof. Assume $\tau \subseteq \tau'$. Let $x \in X$ and $\epsilon > 0$. By Lemma 13.3, there exists a basis element B' for τ' so

$$x \in B' \subseteq B_d(x,\epsilon).$$

We have $B' = B_{d'}(y,r)$ for some $y \in X$ and $r > 0$. Let $\delta = r - d'(x,y)$. Since $x \in B_{d'}(y,r)$, we have $\delta > 0$.

We claim that $B_{d'}(x,\delta) \subseteq B_{d'}(y,r)$. Let $z \in B_{d'}(x,\delta)$. Then,

$$\begin{aligned} d'(z,y) &\leq d'(,x) + d'(x,y) \\ &< \delta + d'(x,y) \\ &= r \end{aligned}$$

This implies that $z \in B_{d'}(y,r)$. Hence,

$$B_{d'}(x,\delta) \subseteq B' \subseteq B_d(x,\epsilon)$$

So this proves the second statement.

Now, assume that the second statement is true. Let $B_d(x,\epsilon)$ be a τ-basis element. Let $y \in B_d(x,\epsilon)$ then the assumed statement implies that we can find $\delta > 0$ such that

$$y \in B_{d'}(y,\delta) \subseteq B_d(x,\epsilon)$$

Then, $\tau \subseteq \tau'$ follows by Lemma 13.3. ☐

Theorem 14.1. *The topologies on $\mathbb{R}^n$ coming from the Euclidean metric and the square metric are equivalent. These both equal the product topology.*

Proof. Let τ_ρ be the topology coming from the square metric ρ and τ_d that coming from the euclidean metric d. Note

$$\begin{aligned} \rho(x,y) &= \max_i\{|x_i - y_i|\} \\ &= |x_{i_*} - y_{y_*}| \text{ for some } i_* \in \{1,2,\ldots,n\} \\ &= \sqrt{(x_{i_*} - y_{i_*})^2} \\ &\leq \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2} \\ &= d(x,y) \end{aligned}$$

So

$$\begin{aligned} B_d(x,\epsilon) &= \{y | d(x,y) < \epsilon\} \\ &\subseteq \{y | \rho(x,y) < \epsilon\} \\ &= B_\rho(x,\epsilon). \end{aligned}$$

By the previous lemma, $\tau_\rho \subseteq \tau_d$.

For the reverse inequality,

$$\begin{aligned} d(x,y) &= \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2} \\ &= \sqrt{(x_{i_*}-y_{i_*})^2 + \cdots + (x_{i_*}-y_{i_*})^2} \\ &= \sqrt{n(x_{i_*}-y_{i_*})^2} \\ &= \sqrt{n}\rho(x,y). \end{aligned}$$

This implies that $B_\rho(x, \epsilon/\sqrt{n}) \subseteq B_d(x,\epsilon)$. Therefore, $\tau_d \subseteq \tau_\rho$

Finally, we want to prove that ρ-topology equals the product topology. Let $(a_1,b_1) \times \cdots \times (a_n,b_n)$ be a product topology basis element and

$$(x_1,\ldots,x_n) \in (a_1,b_1) \times \cdots \times (a_n,b_n).$$

Set

$$\epsilon = \min_{i \in \{1,\ldots,n\}} \{|x_i - a_i|, |x_i - b_i|\}$$

Then,

$$\begin{aligned} (x_1,\ldots,x_n) &\in (x_1-\epsilon, x_1+\epsilon) \times \cdots \times (x_n-\epsilon, x_n+\epsilon) \\ &\subseteq (a_1,b_1) \times \cdots \times (a_n,b_n), \end{aligned}$$

where

$$(x_1-\epsilon, x_1+\epsilon) \times \cdots \times (x_n-\epsilon, x_n+\epsilon) = B_\rho((x_1,\ldots,x_n),\epsilon)$$

Conversely, let

$$B_\rho((x_1,\ldots,x_n),\epsilon) = (x_1-\epsilon, x_1+\epsilon) \times \cdots \times (x_n-\epsilon, x_n+\epsilon)$$

and $y \in B_\rho(x,\epsilon)$. Then,

$$y \in B_\rho(x,\epsilon) \subseteq B_\rho(x,\epsilon)$$

$\square$

Example 14.1. For $1 \le p < \infty$, then

$$d_p((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = ((x_1-y_1)^p + \cdots + (x_n-y_n))^{1/p}$$

on $\mathbb{R}^n$. This is a metric on $\mathbb{R}^n$.

We want to draw $B_{d_p}(0,1)$ for different values of p. When $p=1$, we have a rhombus. When $p=2$, we have a circle. When $p>2$, we get a "fat" circle. Note that

$$B_{d_p}(0,1) = \{(x,y) | x^p + y^p = 1\} \implies y = \pm(1-x^p)^{1/p}$$

As $p \to \infty$, we get a square. Hence,

$$\lim_{p \to \infty} d_p(x,y) = \rho(x,y)$$

Example 14.2. Consider

$$X = C^0([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

For $1 \leq p < \infty$, define

$$d_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p dx \right)^{1/p}.$$

This is a metric on X. When $p = 2$, d_2 is very important in physics.

Theorem 14.2. *Suppose (X, d_X) and (Y, d_Y) are metric spaces (with the metric topology). Let $f : X \to Y$ be a function. Then, f is continuous if and only if*

$$\forall x \in X \ \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } d_X(x,y) < \delta \implies d_y(f(x), f(y)) < \epsilon$$

Proof. Assume f is continuous. Let $x \in X$ and $\epsilon > 0$. Since f is continuous and $B_{d_y}(f(x), \epsilon)$ is open, it follows that $f^{-1}(B_{d_y} f(x), \epsilon)$ is open in X. Note that

$$x \in f^{-1}(B_{d_y} f(x), \epsilon) \implies \exists \delta > 0$$

such that

$$x \in B_{d_x}(x, \delta) \subseteq f^{-1}(B_{d_y} f(x), \epsilon).$$

Assume $y \in X$ satisfies $d_x(x,y) < \delta$. Then, $y \in B_{d_x}(x, \delta)$, so

$$y \in f^{-1}(B_{d_y} f(x), \epsilon).$$

Hence, $f(y) \in B_{d_y}(f(x), \epsilon)$. So

$$d_y(f(x), f(y)) < \epsilon.$$

Conversely, assume the $\delta - \epsilon$ condition. Let $U \subseteq Y$ be open. If $U = \varnothing$ then $f^{-1}(\varnothing) = \varnothing$ is open. More generally, if $f^{-1}(U) = \varnothing$, then we're done. We may assume $\exists x \in f^{-1}(U)$. Note $f(x) \in U$ since U is open. Then, there exists $\epsilon > 0$ such that

$$f(x) \in B_{d_y}(f(x), \epsilon) \subseteq U.$$

The $\delta - \epsilon$ condition implies that there exists $\delta > 0$ such that

$$d_x(x,y) < \delta \implies d_y(f(x), f(y)) < \epsilon.$$

So $B_x(x, \delta) \subseteq f^{-1}(U)$. That is, for every $x \in f^{-1}(U)$, there exists $\delta > 0$ so that

$$x \in B_{d_x}(x, \delta) \subseteq f^{-1}(U).$$

So $f^{-1}(U)$ is open. □

Example 14.3. Show the function $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$m(a, b) = a + b$$

is continuous.

Proof. Let $d_{\mathbb{R}\times\mathbb{R}}$ be a square metric. Consider $(a, b), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$. Then,

$$d_{\mathbb{R}\times\mathbb{R}}((a, b), (a_2, b_2)) < \delta \implies |a - a_2| + |b - b_2| < \delta.$$

Note that

$$\begin{aligned} d_{\mathbb{R}}(m(a, b), m(a_2, b_2)) &= |m(a, b) - m(a_2, b_2)| \\ &= |a + b - a_2 - b_2| \\ &= |a - a_2 + b - b_2| \\ &\leq |a - a_2| + |b - b_2| < \delta < \epsilon \end{aligned}$$

□

Similarly, we can show that subtraction, multiplication, and divisions (as long as we don't divide by 0) are continuous functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Corollary 14.1. *Suppose $f, g : X \to \mathbb{R}$ are continuous where X is a topological space. Then,*

$$\begin{aligned} f + g : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto f(x) + g(x) \\ f - g : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto f(x) - g(x) \\ f \cdot g : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto f(x)g(x) \end{aligned}$$

are continuous with the product topologyon $X \times X$.

Proof. Note that $f + g$ is the composition of

$$\begin{aligned} X \times X &\to \mathbb{R} \times \mathbb{R} \\ (x, y) &\mapsto (f(x), g(y)) \\ \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (s, t) &\mapsto s + t \end{aligned}$$

These are continuous so the composition is continuous. □

Definition 14.1. *Let X be a set and (Y, d) a metric space. Suppose*

$$f_n : X \to Y$$

is a sequence of functions. Then, $(f_n)_{n\in\mathbb{Z}_+}$ converge uniformly to $f : X \to Y$ if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}_+ \; \forall n \geq N \; \forall x \in X \;\; d(f_n(x), f(x)) < \epsilon$$

Theorem 14.3. *Sppose X is a topological space and (Y, d) is a metric space. Suppose (f_n) is a sequence of functions converging uniformly to $f : X \to Y$. If each f_n is continuous then f is continuous.*

Proof. Let $U \subseteq Y$ be open. We want to show that $f^{-1}(U)$ is open. It is sufficient to show that

$$\forall x \in f^{-1}(U)\ \exists V \text{ open in } X \text{ such that } x \in V \subseteq f^{-1}(U)$$

$\square$