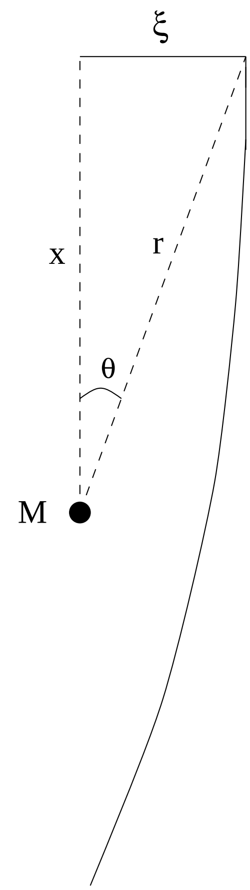
Gravitational Lenses

Consider a photon moving past a point of mass, $\mathcal{M}$, with an starting "impact parameter," ξ. From classical Newtonian gravity, the photon will undergo an acceleration perpendicular to the direction of its motion. Under the Born approximation, the amount of this deflection can be calculated simply by integration along the path.



$$\frac{dv_\perp}{dt} = \frac{G\mathcal{M}}{r^2}\sin\theta \qquad (10.01)$$

If we substitute $dx = c\,dt$, then

$$v_\perp = \frac{G\mathcal{M}}{c}\int_{-\infty}^{\infty}\frac{1}{x^2+\xi^2}\cdot\frac{\xi}{(x^2+\xi^2)^{1/2}}\,dx$$

$$= \frac{G\mathcal{M}\xi}{c}\int_{-\infty}^{\infty}\left(x^2+\xi^2\right)^{-3/2}\,dx \quad (10.02)$$

The integral is analytic, and works out to $2/\xi^2$. So

$$v_\perp = \frac{2G\mathcal{M}}{\xi c} \qquad (10.03)$$

and the Newtonian deflection angle (in the limit of a small deflection) is

$$\alpha = \frac{v}{c} = \frac{2G\mathcal{M}}{\xi c^2} \qquad (10.04)$$

In the general relativistic case, however, gravity affects both the spatial and time component of photon's path, so that the actual bending is twice this value. Thus, we define the angle of deflection, otherwise known as the Einstein angle, as

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{10.05}$$

To understand the geometry of a gravitational lens, let's first define a few terms. Let

$$\begin{array}{lcl} D_d & = & \text{distance from the observer to the lens} \\ D_s & = & \text{distance from the observer to the light source} \\ D_{ds} & = & \text{distance from the lens to the source} \\ \vec{\beta} & = & \text{true angle between the lens and the source} \\ \vec{\theta} & = & \text{observed angle between the lens and the source.} \\ \vec{\xi} & = & \text{distance from the lens to a passing light ray} \\ \vec{\alpha} & = & \text{the Einstein angle of deflection} \end{array}$$

Note that β, θ, α, and ξ are all vectors, and calculations must deal with negative, as well as positive angles. For a simple point-source lens, the vectorial components of the angles make no difference, but in systems where the lens is complex (*i.e.*, several galaxies/clusters along the line-of-sight) the deflection angles must be added vectorially.

In practice, α, β, and θ are all very small, so small angle approximations can be used. Also, D_d, D_s, and D_{ds} are all much bigger than ξ. Thus, the deflection can be considered instantaneous (*i.e.*, we can use a "thin lens" approximation).

The geometry of a simple gravitational lens system is layed out in the figure. From $\triangle OSI$ and the law of sines,

$$\frac{\sin(180 - \alpha)}{D_s} = \frac{\sin(\theta - \beta)}{D_{ds}} \tag{10.06}$$

Since all the angles are small, $\sin(\theta - \beta) \approx \theta - \beta$, and $\sin(180 - \alpha) = \sin\alpha \approx \alpha$. So

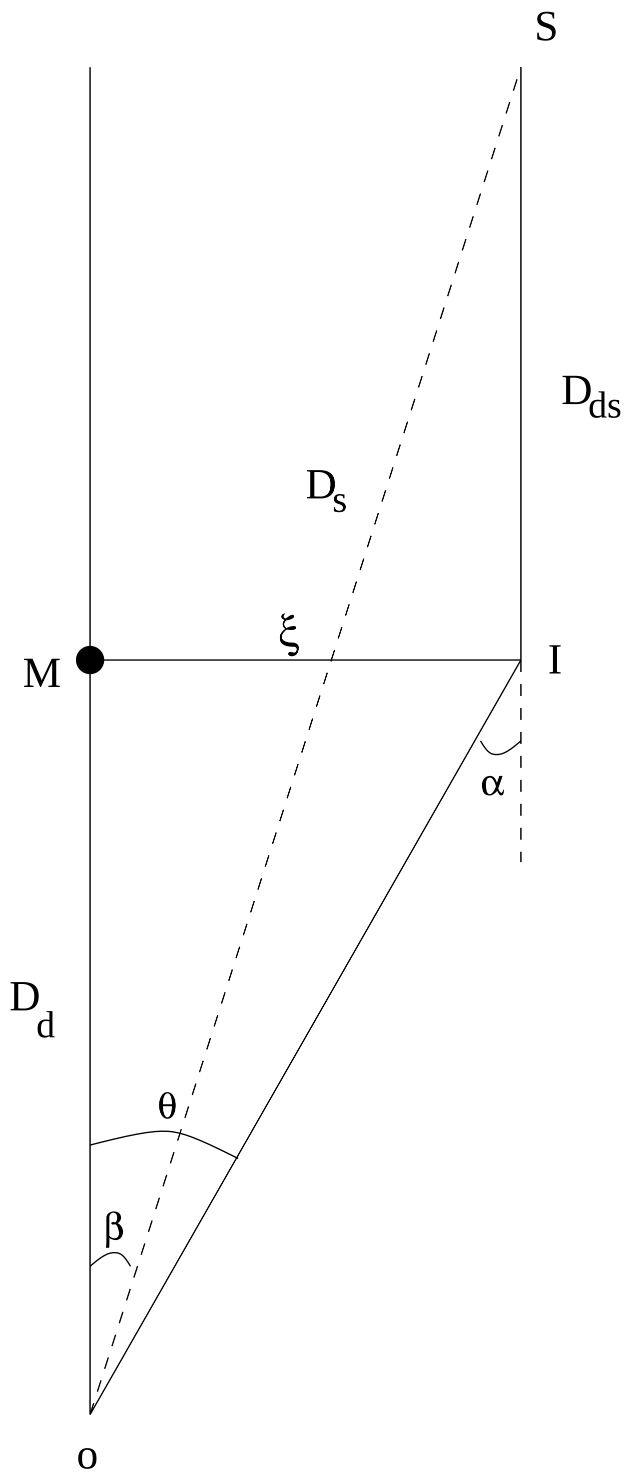
$$\vec{\beta} = \vec{\theta} - \frac{D_{ds}}{D_s}\vec{\alpha} \tag{10.07}$$

(Again, for single point lenses, the fact that the angles are vectors are irrelevant. For simplicity, I will therefore drop the vector signs for the rest of these derivations.)

Since we have already seen that

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{10.05}$$

and, from the simple geometry of small angles, $\theta = \xi / D_d$,



$$\beta = \theta - \left(\frac{4G\mathcal{M}}{c^2}\frac{D_{ds}}{D_d D_s}\right) \cdot \frac{1}{\theta} \tag{10.08}$$

In other words, we have a relation between β and θ. But note: for a given value of β, there is more than one value of θ that will satisfy the equation. This is a general theorum of lenses. For non-transparent lenses (such as the Schwarzschild lens being considered) there will always be an even number of images; for transparent lenses, there is always an odd number of images.

To simplify, let's define the characteristic bending angle, α_0, as a quantity that depends only on the mass of the lens and the distances involved

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2}\frac{D_{ds}}{D_d D_s}\right)^{1/2} \tag{10.09}$$

so that

$$\beta = \theta - \frac{\alpha_0^2}{\theta} \tag{10.10}$$

If β and θ are co-planar (as they will be for a point source lens), then (10.10) is equivalent to the quadratic equation

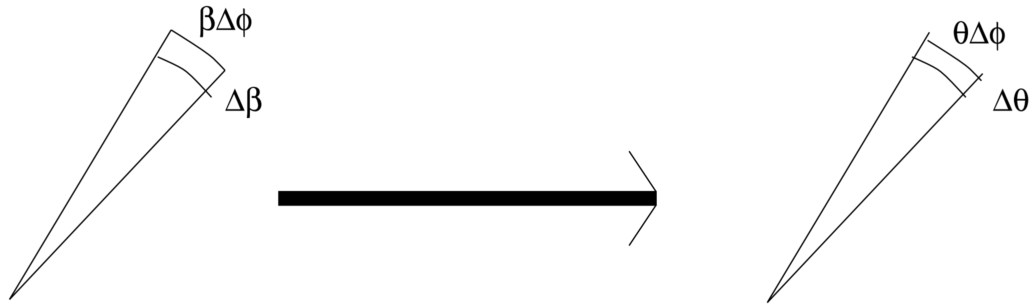
$$\theta^2 - \beta\theta - \alpha_0^2 = 0 \tag{10.11}$$

whose solution is

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{10.12}$$

This defines the location of the gravitional lens images as a function of the true position of the source with respect to the lens, β, and α_0.

Magnification for a Schwarzschild Lens



The relation

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{10.12}$$

implies that at least one image will be magnified. To see this, let's lay out a polar coordinate system with the lens at the center, and consider the light passing through a differential area, dA. At the lens, this area element is $dA = \beta\Delta\phi\,\Delta\beta$. However, due to the gravitational lens, the angles are distorted, so that the area the observer see is $dA' = \theta\Delta\phi\,\Delta\theta$. Thus, the lens has "focussed" the light from area dA to area dA'. The magnification will therefore be the ratio of the two areas

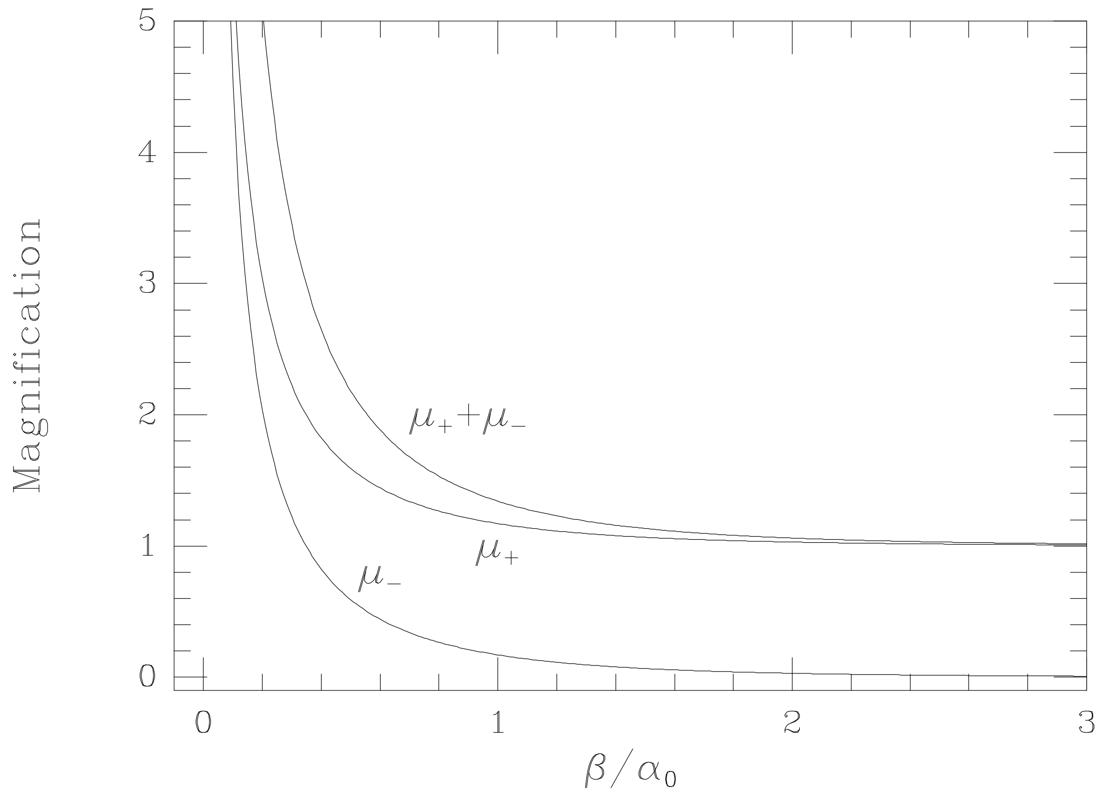
$$\mu = \frac{\theta\Delta\phi\,\Delta\theta}{\beta\Delta\phi\Delta\beta} = \frac{\theta}{\beta}\frac{\Delta\theta}{\Delta\beta} = \frac{\theta}{\beta}\frac{d\theta}{d\beta} \tag{10.13}$$

The last term is just the derivative of (10.12), or

$$\frac{d\theta}{d\beta} = \frac{1}{2}\left(1 \pm \frac{\beta}{\sqrt{4\alpha_0^2 + \beta^2}}\right) \tag{10.14}$$

which means that the magnification is

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2 + \beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2 + \beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{10.15}$$



Note the limits: as $\beta \longrightarrow \infty$, $\mu_+ \longrightarrow 1$ and $\mu_- \longrightarrow 0$; in other words, there is no lensing. On the other hand, when $\beta \longrightarrow 0$, both μ_+ and μ_- go to infinity. As you can see from the curves, when $\beta < \alpha_0$, gravitational lens magnification is important.

There are a few other relations that should be kept in mind. Since the magnification of a lens is given by

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{10.15}$$

the brightness ratio of two images is

$$\nu = \frac{\mu_+}{\mu_-} = \left\{\frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}+\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}-\beta}\right\}^2 \tag{10.16}$$

which simplies through (10.12) to

$$\nu = \frac{\mu_+}{\mu_-} = \left(\frac{\theta_1}{\theta_2}\right)^2 \tag{10.17}$$

Some other convenient relations (which can be proved with some simple algebra) are

$$\theta_1 - \theta_2 = \left(4\alpha_0^2 + \beta^2\right)^{1/2} \tag{10.18}$$

$$\beta = \theta_1 - \theta_2 \tag{10.19}$$

and

$$\theta_1\theta_2 = -\alpha_0^2 \tag{10.20}$$

(Again, remember, that the angles are vector quantities, so angles such as θ can be positive or negative.)

General Scale Lengths for Lenses

It's useful to keep in mind some of the numbers that are typical of gravitational lens problems. Remember that the key value is α_0,

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2} \tag{10.09}$$

For the MACHO project, which tries to detect stars in the Large Magellanic Cloud ($D_s = 50$ kpc) that are lensed by stellar-type objects in the Milky Way,

$$\alpha_0 = 4 \times 10^{-4} \left(\frac{\mathcal{M}}{\mathcal{M}_\odot} \right)^{1/2} \left(\frac{D_{ds}}{D_d} \right)^{1/2} \text{ arcsec} \tag{10.21}$$

In this case, the image separations $\theta \sim \alpha_0$ are sub-milliarcsec, so you will never be able to resolve the two separately lensed images. But, with both images landing on top of each other, the source can be (greatly) magnified.

For quasar lensing by a typical large elliptical galaxy ($10^{12}\mathcal{M}_\odot$) in an Einstein-de Sitter universe,

$$\alpha_0 = 1.6 \left(\frac{\mathcal{M}}{10^{12}\mathcal{M}_\odot} \right)^{1/2} \left(\frac{z_s - z_d}{z_s z_d} \right)^{1/2} h^{1/2} \text{ arcsec} \tag{10.22}$$

where h is the Hubble Constant in units of 100 km/s/Mpc. Here, the typical scale-length is a little over an arcsec, so two separate images can be resolved.