Gravitational Lens Time Delay

If the source behind a gravitational lens is variable, then, of course, the images we see will also be variable. However, the images will not necessarily vary at the same time: in general, there will be a *time delay* between the two events. This comes from two components.

First, there is a purely geometrical time delay. Consider the value of $D_{ds} + D_d - D_s$ for each image of the lens. From $\triangle OSI$ and the law of cosines

$$D_s^2 = D_d^2 + D_{ds}^2 - 2D_d D_{ds} \cos(180 - \alpha) \tag{11.01}$$

If we substitute

$$\cos(180 - \alpha) = -\cos\alpha = -\left\{1 - 2\sin^2(\alpha/2)\right\} \tag{11.02}$$

then

$$D_s^2 = D_d^2 + D_{ds}^2 + 2D_d D_{ds} - 4D_d D_{ds} \sin^2(\alpha/2) \tag{11.03}$$

or

$$\begin{aligned} 4D_d D_{ds} \sin^2(\alpha/2) &= \left(D_d^2 + D_{ds}^2\right)^2 - D_s^2 \\ &= (D_d + D_{ds} - D_s)(D_d + D_{ds} + D_s) \end{aligned} \tag{11.04}$$

If we now use the law of sines,

$$\frac{\sin\alpha}{D_s} = \frac{\sin(\theta - \beta)}{D_{ds}} \tag{11.05}$$

and consider that all the angles are small, so that $\sin\alpha \approx \alpha$, $\sin(\theta - \beta) \approx \theta - \beta$, $\sin^2 \alpha/2 \approx \alpha^2/4$, and $D_d + D_{ds} + D_s \approx 2D_s$, then

$$D_d + D_{ds} - D_s = \frac{4D_dD_{ds}\sin^2\left(\alpha^2/2\right)}{D_d + D_{ds} + D_s}$$
$$= \frac{D_dD_{ds}\alpha^2}{2D_s} \tag{11.06}$$

Finally, by substituting for α using (10.07), we have

$$D_d + D_{ds} - D_s = \frac{D_dD_s}{2D_{ds}}\left(\vec{\theta} - \vec{\beta}\right)^2 = c\,\Delta t \tag{11.07}$$

Since there are two values of $\vec{\theta}$, there are two values for the geometrical time delay. So the delay of one image with respect to the other (which is all we can observe), is

$$\Delta T_g = \frac{D_dD_s}{2cD_{ds}}\left\{(\theta_1 - \beta)^2 - (\theta_2 - \beta)^2\right\} \tag{11.08}$$

If you now recall that

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2}\frac{D_{ds}}{D_dD_s}\right)^{1/2} \tag{10.09}$$

and

$$\beta = \theta - \frac{\alpha_0^2}{\theta} \tag{10.10}$$

then, we can substitute these in to get

$$\Delta T_g = \frac{1}{2c}\frac{4G\mathcal{M}}{c^2\alpha_0^2}\left\{\frac{\alpha_0^4}{\theta_1^2} - \frac{\alpha_0^4}{\theta_2^2}\right\}$$
$$= \frac{2G\mathcal{M}}{c^3}\alpha_0^2\left\{\frac{\theta_2^2 - \theta_1^2}{\theta_1^2\theta_2^2}\right\} \tag{11.09}$$

But, we also have the relation

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{10.12}$$

which gives

$$\theta_1\theta_2 = -\alpha_0^2 \tag{10.20}$$

So, with this substitution, we get the final expression for the geometrical time delay from one image to the other.

$$\Delta T_g = \frac{2G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{|\theta_1\theta_2|}\right\} \tag{11.10}$$

The second source of signal time delay arises from the passage of the photons through a potential. Any time light passes through a gravitational potential, it experiences the Shapiro time delay

$$\Delta t = -\frac{2}{c^3}\int \Phi dl \tag{11.11}$$

where dl is the photon's path through the potential well. For the case of a weak potential and small deflection, the delay a photon experiences going from source S to point I is

$$\begin{aligned}\Delta t_1 &= -\frac{2}{c^3}\int_0^{D_{ds}} \frac{G\mathcal{M}}{r} dl = -\frac{2G\mathcal{M}}{c^3}\int_0^{D_{ds}} \frac{dx}{(x^2+\xi^2)^{1/2}} \\ &= -\frac{2G\mathcal{M}}{c^3}\ln\left\{x + \sqrt{x^2+\xi^2}\right\}\Bigg]_0^{D_{ds}} = \frac{2G\mathcal{M}}{c^3}\ln\left(\frac{\xi}{2D_{ds}}\right)\end{aligned} \tag{11.12}$$

Similarly, the delay in going from point I to the observer is

$$\Delta t_2 = \frac{2G\mathcal{M}}{c^3}\ln\left(\frac{\xi}{2D_d}\right) \tag{11.13}$$

So the total time delay for the potential is

$$\Delta t_p = \Delta t_1 + \Delta t_2 = \frac{2G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi}{2D_{ds}}\right) + \ln\left(\frac{\xi}{2D_d}\right)\right\} \quad (11.14)$$

We can now calculate the difference in the time delay from one image to the other, by realizing that at very large distances from the lense, the potential delay is negigible. So, if we pick some distance, $D' \gg \xi$, but $D' \ll D_{ds}, D_s$, then we can write (11.14) as

$$\Delta t_p = \frac{2G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi}{2D'}\right) + \ln\left(\frac{D'}{D_{ds}}\right) + \ln\left(\frac{\xi}{2D'}\right) + \ln\left(\frac{D'}{D_d}\right)\right\} \quad (11.15)$$

The second and fourth terms do not depend on ξ and are the same for both images. So, when we take the difference of the two time delays we get

$$\Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi_i}{2D'}\right) - \ln\left(\frac{\xi_2}{2D'}\right)\right\} \quad (11.16)$$

If we then substitute for ξ using $\theta = \xi/D_d$ and note that D' is the same for both rays, we get the final expression for the difference between the two Shapiro delays

$$\Delta T_p = \frac{4G\mathcal{M}}{c^3}\ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right) \quad (11.17)$$

The total time delay of one image with respect to the other is then

$$\Delta T = \Delta T_g + \Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{2|\theta_1\theta_2|} + \ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right)\right\} \quad (11.18)$$

Note, however, that this equation neglects two important effects.

1) The universe is expanding an possibly non-Euclidean; thus, the variables D_s, D_s, and D_{ds} are complex functions of z, q_0, and Λ. Specifically, D_d and D_s are angular distances, which are related to the proper distances by

$$D_{ang} = D_p(1+z)^{-1} \tag{2.48}$$

However, D_{ds} is the angular distance of z_s, *as seen by an observer at the redshift of the lens, z_d.* For those of you who are too sane to work it out on your own,

$$D_{ds} = \frac{c(1+z_d)}{H_0 q_0^2 (1+z_d)^2 (1+z_s)^2} \left\{ (2q_0 z_d + 1)^{1/2} (1 + q_0 z_s - q_0) - \right.$$

$$\left. (2q_0 z_s + 1)^{1/2} (1 + q_0 z_d - q_0) \right\} \tag{11.19}$$

2) Cosmological lenses are not point sources. The best lenses are elliptical galaxies (which are fairly concentrated collections of stars), but these galaxies are located inside of clusters. So real lenses include the effects of both the galaxy and the cluster (which, of course, add vectorially).

The Uses of Gravitational Lenses

Gravitational lenses are extremely interesting for four reasons.

1) Because the magnification properties of lenses depend on the mass of the lensing object, microlensing experiments, such as the MACHO experiment of monitoring millions of stars in the Large Magellanic Cloud, can probe the properties of halo stars (and other objects) that are two faint to see. In fact, by studying the light curve of a gravitational microlens, one can deduce not only the mass of the intervening lens (with some assumption about the lens' distance), but also whether the lens is a single star or something else. Microlensing experiments have measured light curves that must be due to binary-star lenses, and, in theory, it is even possible to deduce the presence of an earth-mass planet circling a lens.

2) Because of their high magnification, gravitational lenses can be used to observe high-redshift galaxies that would normally be much too faint to observe otherwise. There is a slight problem with this, in that the image of the galaxy is usually distorted (spread out) over many pixels (into an arc). However, if the light can be collected, you can use it to study a representative sample of normal (not exceptionally bright) high-redshift galaxies.

3) Because the time delay of a gravitational lens depends on distance, these objects can be used to measure the Hubble Constant. Unfortunately, this method has two major drawbacks. First, the time delay depends not only on the distance to the lens, but its mass as well. This mass is not known *a priori:* for a typical elliptical galaxy lens, one must try to infer the lens mass from the velocity dispersion of its stars. Second, the geometry of the lens is usually complicated. Elliptical galaxies are in clusters, and the cluster's mass will also contribute to the time delay. One must therefore measure the velocity dispersion of the galaxies in the cluster in order to determine the cluster's mass (while, of course, hoping that the cluster is virialized, and that none of the objects you are measuring are foreground or background objects). Moreover, for distant sources, it is likely that the calculations will be further complicated by the presence of multiple lenses along the line-of-sight.

4) The area of a shell at redshift z_s that is lensed by an object at redshift z_l is $\pi D_s^2 \beta^2$. Thus, the fraction of the universe at z_s that is lensed depends on the angular diameter distance, D_s, and D_s depends on the geometry of the universe. Thus, by counting the number of gravitational lenses versus redshift, one can constrain the geometry of the universe, and, in particular, Λ.

General Comments on Real Lenses

As we have seen, the bending angle for a lens is

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{10.05}$$

This implies that as $\xi \longrightarrow 0$, the bending angle (and magnification) becomes infinite. This doesn't happen with cosmological lenses, because the lenses are never point masses. Instead, the mass is distributed, *i.e.*, $\mathcal{M}(\xi)$, which goes to zero as ξ goes to zero. So the equations do not diverge.

In practice, the gravitational deflection due to a galaxy's cluster is almost as important as that of the galaxy itself. Consequently, one must model the lens with a galaxy plus cluster potential, and this breaks the symmetry of the problem. Hence, the equation to solve is

$$\vec{\alpha} = \int \frac{4G \sum \vec{\xi}'}{c^2} \, \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \, d\xi' \tag{11.20}$$

In other words, the bending angle is the vector sum of the gravitational deflections caused by all the mass in the two-dimensional cluster.

The best lenses are massive and compact objects. For cosmological lenses, this means elliptical galaxies. (They are the most compact of all galaxies.) Consequently, even though the majority of galaxies are spirals, it is generally reasonable to assume that all gravitatational lenses are caused by ellipticals. Ideally, to use a gravitational lens as a cosmological probe, you would like

1) Optical identifications for both the lens and the source. (Occasionally, the lens is too faint to be studied.)

2) A measured velocity dispersion for the lens (to estimate its mass).

3) A variable source (to measure the time delay for the two images).

4) An angular separation between the two images of $1'' < \Delta\theta < 2''$. If the separation is any smaller, it is hard to resolve the images; if the separation is any larger, then the cluster's mass begins to dominate the problem. Also, recall that

$$\Delta T = \Delta T_g + \Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{2|\theta_1\theta_2|} + \ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right)\right\} \quad (11.18)$$

hence the gravitational time delay $\Delta T \propto \Delta\theta^2$. Small separations keep the time delay manageable. (For instance, the lens 0957+561 has a separation of $6''$, and a time delay of between 1 and 2 years.)

5) VLBI radio structure, so that the shape of the images can be used to constrain the mass distribution of the lens.

6) Multiple images to constrain the geometry. Four images are better than two, and an Einstein ring is best.

Statistical Microlensing

Consider a Schwartzschild (point source) lens at redshift z in front of a source at redshift z_s. The magnification of the two images is given by

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{10.15}$$

If the two images are unresolved (*i.e.*, the case of microlensing), then the total magnification is

$$\mu_T = \mu_+ + \mu_- = \frac{1}{2}\left\{\frac{2\alpha_0^2+\beta^2}{\beta\left(4\alpha_0^2+\beta^2\right)^{1/2}}\right\} \tag{11.21}$$

With a little bit of math, this equation can be inverted to give the angle β, which yields a total magnification of μ_T

$$\beta^2 = 2\alpha_0^2\left\{\frac{\mu_T}{\left(\mu_T^2-1\right)^{1/2}} - 1\right\} \tag{11.22}$$

The cross-sectional area for a magnification of greater than μ_T to occur is therefore

$$\sigma(\mu_T, z, z_s) = \pi\beta^2 \tag{11.23}$$

This means that redshift z_s, any object located in an area of

$$\sigma(\mu_T, z, z_s) = \pi D_s^2 \beta^2 \tag{11.24}$$

will be microlensed.

Now, let's calculate the probability that a source at redshift z_s will be microlensed. To begin, let's assume that the density of lenses is given by η. (Since the best lenses are elliptical galaxies, you consider η to be the density of elliptical galaxies.) For simplicity, let's also assume that η does not evolve with time (there are as many elliptical galaxies today as there were at redshift z), and that each lens has identical lensing properties (same mass, same mass concentration, *etc.*)

If η_0 is the density of lenses today, then the density of lenses at redshift z is given by

$$\eta(z) = \eta_0(1+z)^3 \tag{11.25}$$

and the total number of lenses between z and $z + dz$ is

$$dN(z) = \eta dV = \eta_0(1+z)^3 dV \tag{11.26}$$

where dV is the cosmological volume element,

$$dV = D_{ang}^2 \frac{c}{H_0\,(2q_0 z + 1)^{1/2}\,(1+z)^2} \sin\theta\, dz\, d\theta\, d\phi \tag{2.59}$$

and D_{ang} the angular diameter distance of redshift z.

Now if the density of lenses is small enough so that the probability of two lenses overlapping each other is negligible, then the total area at redshift z_s that is microlensed up to at least a magnification of μ_T is just the sum of the cross-sections of all the lenses, *i.e.*,

$$\sigma_t(z_s) = \int \sigma(\mu_T, z, z_s) dN = \int_0^{z_s} \sigma(\mu_T, z, z_s)\eta_0(1+z)^3 dV \tag{11.27}$$

The probability of a microlens is then just this area, normalized to the total surface area of the universe at redshift z_s

$$A_t = 4\pi D_{ang}^2 \tag{11.28}$$

This "statistical microlensing" is extremely important. In fact, with typical numbers for the space density of elliptical galaxy lenses, you find that by $z = 2$, half the objects in the sky are magnified by at least 10%!