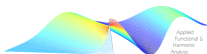


Instabilities in statistical modelling  
and  
a proposal of the adaptive learning method in time series

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The latest version can be found on my website.

- 1 Stability and Instability
- 2 Time series and adaptive learning
- 3 Additional research remark

# Statistical contemporaries: what is regression?

## (Vapnik 1999)

Let  $y$  be the true value of response to a given input  $x$ , and  $f(x, \alpha)$  be the response provided by the learning machine, with  $\alpha \in A$ , where  $A$  denotes the set of parameters. Then, given some specification of the loss function  $L(\cdot, \cdot)$ , we aim to minimise the risk functional

$$R(\alpha) = \mathbb{E}_{x,y}[L(y, f(x, \alpha))]$$

by finding over the class of functions  $\alpha \in A$ . Write  $\alpha^* := \arg \min_{\alpha \in A} R(\alpha)$

### Theorem (Regression estimation (Vapnik 1999))

*Let the loss to take the form  $L(a, b) = (a - b)^2$ . Define the regression function as*

$$f(x, \alpha_0) := \mathbb{E}[y|x]$$

*Then*

$$\alpha^* = \arg \max_{\alpha \in A} \mathbb{E}_x[(f(x, \alpha) - f(x, \alpha_0))^2]$$

*In particular, if  $\alpha_0 \in A$ , then  $\alpha^* = \alpha_0$*

# Overview

- Deterministic models:

- Stability:

$$\forall \varepsilon > 0, \forall v \in B(\mathcal{D}_x, \varepsilon), \quad |g(v; \hat{\theta}) - f(v)| \leq \text{UpperBound}(f, \hat{\theta}, \varepsilon)$$

- Instability:

$$\forall \varepsilon > 0, \exists v \in B(\mathcal{D}_x, \varepsilon) \text{ such that } |g(v; \hat{\theta}) - f(v)| \geq \text{LowerBound}(f, \hat{\theta}, \varepsilon)$$

- Probabilistic models: depend on distributions and the underlying Data Generating Process (DGP).

- Here we present two of the four analyses:

- Deterministic instability in Single Layer Neural Network (SLNN)
  - Probabilistic stability in linear model with correct specification

# Deterministic instability in SLNN [Whiteboard]

$$g(\cdot; \theta) \in \mathcal{G} \quad \Longleftrightarrow \quad g(x; \theta) = W_2(\rho(W_1(x)))$$

Here the parameter  $\theta = (W_1, W_2)$ , which is two affine transformations. Write  $\rho(x) = (\tilde{\rho}(x_1), \dots, \tilde{\rho}(x_{N_1}))$  and in this example, we focus on twice continuously differentiable activation function with bounded derivatives.

## Corollary (Instability by perturbation)

Let  $\varepsilon > 0$ . For all  $M > 0$ , there exists  $\theta \in \Theta$  such that  $\exists v \in B(\mathcal{D}_x; \varepsilon)$  satisfying

$$|g(v; \theta) - f(v)| \geq M\varepsilon$$

# Probabilistic stability: linear model with correct specification [Whiteboard]

- DGP  $y_i|x_i \sim N(\langle x_i, \beta \rangle, \sigma^2)$  independently  $\forall i \in [n]$
- Functional class

$$g(\cdot; \beta) \in \mathcal{G} \quad \Longleftrightarrow \quad g(x; \beta) = \langle x, \beta \rangle$$

- Estimation:  $\hat{\beta}$  by MLE

## Lemma (Stability of linear model under correct specification)

Pick any positive real constant  $c$ .  $\forall v \in \mathbb{R}^N$ ,

$$\mathbb{P} \left[ |g(v; \hat{\beta}) - f(v)| < \sqrt{N}c \|v\| \mid \mathbf{X} \right] > 1 - \sqrt{\frac{2}{\pi}} c^{-1} \sum_{i \in [N]} \exp \left( -\frac{c^2}{2s_i(\mathbf{X}, \sigma^2)} \right)$$

# End of Part I

```
if Presentation.time.length() <=10:  
    Presentation.Q_&_A.open()  
else:  
    Presentation.Continue()
```

# Time series introduction: by plots

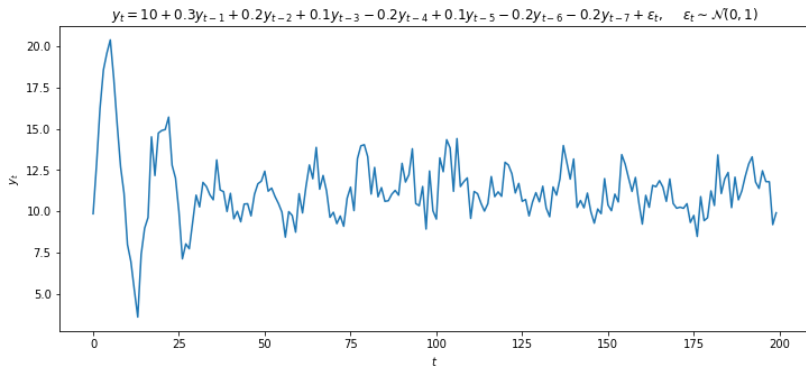


Figure: An example of simulated time series



## Stationarity Check

```
In [10]: np.random.seed(0)
random.seed(0)
all_ar_coef_7 = [0.3, 0.2, 0.1, -0.2, 0.1, -0.2, -0.2]
all_ar_coef_10 = [0.7, 0.1, -0.4, 0.3, 0.1, 0.05, -0.2, 0.1, 0.8, -0.7]

change_list_0 = [7, 0, all_ar_coef_7, [], 150, 10, 1]
change_list_1 = [10, 0, all_ar_coef_10, [], 150, 6, 1]

plt.figure(figsize = (17,6))
plt.plot(gen_arma_change([change_list_0, change_list_1], 10))
```

```
Out[10]: [matplotlib.lines.Line2D at 0x23af983bf88]
```

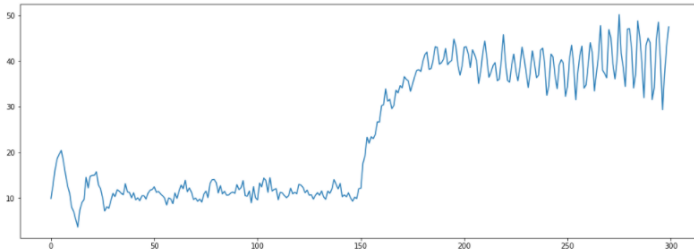


Figure: Another example of simulated time series



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Yield Curve &  
VIX Term Structure Inversion  
27 Feb 2020

Spot VIX: 39.16  
VIX Futures Front Month: 26.60  
Treasury, 1 Month: 1.56  
Treasury, 10 Year: 1.30



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Market Wide Circuit-Breaker  
16 Mar 2020 - Level 1

Spot VIX: 82.69 (All-Time Closing High)  
VIX Futures: 72.05 (March 18<sup>th</sup> Expiry)  
Treasury, 1 Month: 0.25

Figure: An example of empirical time series

# Time series introduction: by definitions

- Hamilton (1994): Dynamic consequences of events over time.
- Prado and West (2010): A set of observations collected sequentially in time.
- Tsay (2010): Financial time series analysis is concerned with the theory and practice of asset valuation over time.
- Tsay and Chen (2018): Time series analysis is concerned with understanding the dynamic dependence of real-world phenomena.

# Time series introduction: mathematical frameworks

## [Whiteboard]

- $y_t \in \mathbb{Y}$ 
  - Usually  $\mathbb{Y} \subset \mathbb{R}^d$  for some  $d \geq 1$
- ARMA modelling: [Assume  $\mathbb{Y} = \mathbb{R}$ ]
  - An autoregressive moving average (ARMA) model of order  $p, q$  can be written as

$$\Phi_p(L)(y_t - \mu) = \Psi_q(L)\varepsilon_t \quad \text{where} \quad \varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2) \quad \forall t$$

- We denote an ARMA( $p, 0$ ) model as AR( $p$ ), and an ARMA( $0, q$ ) model as MA( $q$ ).
- Forecasting:
  - $\Phi_t := \{y_\tau : \tau \in [t]\}$
  - let  $k \geq 1$ , denote the conditional forecast as

$$y_{t+k|t} = \mathbb{E}[y_{t+k} | \Phi_t]$$

- Evaluation:
  - AIC
  - Forecasting evaluation: MAE and RMSE

# Time series introduction: forecasting example

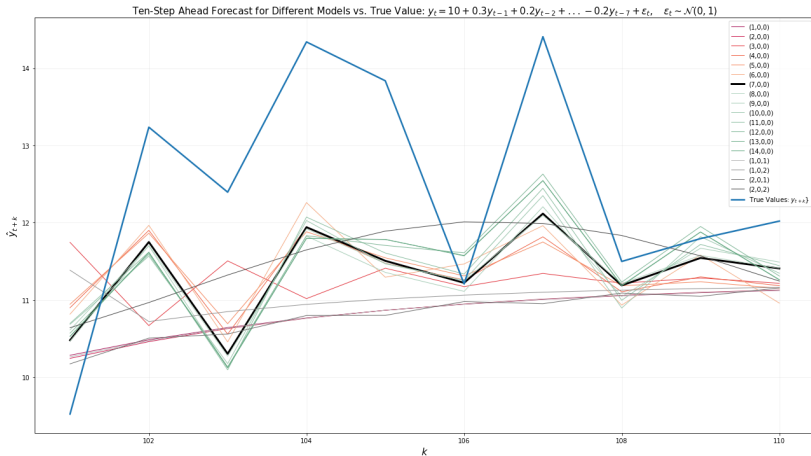
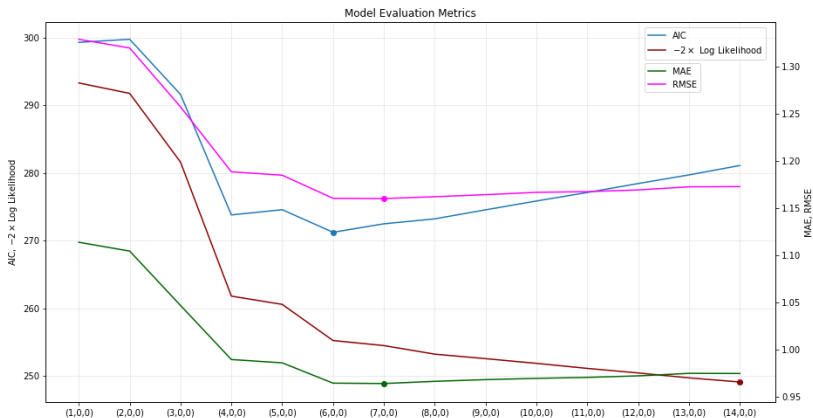


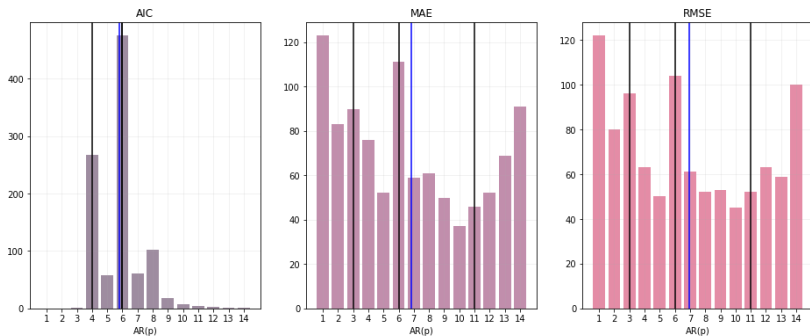
Figure: An example of forecasts by different models

# Time series introduction: evaluation example



**Figure:** 1000 repetitions of an AR(7) model: AIC under-fits and MAE and RMSE evaluates appropriately

# Time series introduction: evaluation example



**Figure:** Count of the best models out of the 1000 repetitions induced by the in-sample AIC and out-of-sample MAE and RMSE. The three black lines indicates 25%, 50%, and 75% quantiles, and the blue line indicates the mean.

# Time series introduction: stationarity [Whiteboard]

## Definition ((Weak) stationarity)

Let  $(x_t)_{t \in \mathbb{N}}$  be a real-valued time series. It is weakly stationary and denoted  $x_t \sim I(0)$  if the first two moments of  $(x_t)_{t \in \mathbb{N}}$  exists and time invariant. That is, for all  $t, l \in \mathbb{N}$ ,

$$\mu := \mathbb{E}[x_t] < \infty$$

$$\gamma_l := \text{Cov}(x_t, x_{t+l}) < \infty$$

And note that both  $\mu, \gamma_l$  are independent of  $t$ .

For a positive integer  $d$ , we denote  $x_t \sim I(d)$  if

$$\Delta^d x_t := (1 - L)^d x_t \sim I(0).$$

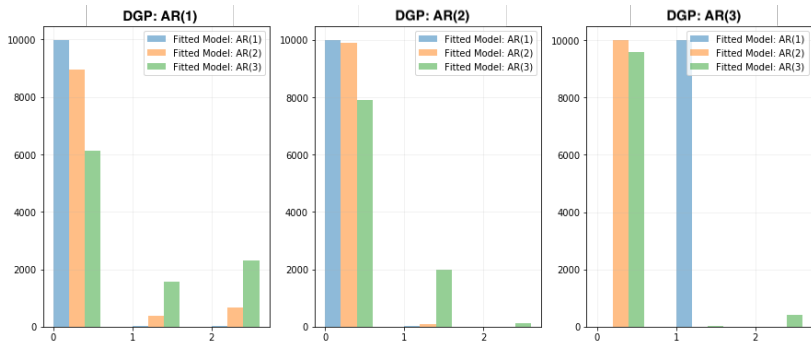
## Example (Stationarity in ARMA(p,q))

Denote  $\Phi_p^{-1}(L)\Psi_p(L) =: \Theta(L) := \sum_{i \in \mathbb{N}} \theta_i L^i$  Then:

$$\text{ARMA}(p,q) \text{ stationary} \Leftrightarrow (\theta_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{N}) \quad (1)$$



# Where stationarity tests fail



**Figure:** Model misspecification leads to false recommendation for non-stationarity

# Inference on AR models: stability [Whiteboard]

## Lemma (The consequence of correct specification)

Suppose  $y_t$  to have a true stationary AR(1) process

$$y_t = \phi_1 y_{t-1} + \varepsilon_t \quad (2)$$

And that we use an AR(1) model to fit the data  $\{y_0, \dots, y_T\}$ , noted

$$y_t = \rho y_{t-1} + e_t, \quad e_t \sim N(0, \sigma^2) \quad \forall t \in \{0, 1, \dots, T\} \quad (3)$$

Then, for the OLS estimate  $\hat{\rho}$ ,

$$\hat{\rho} \xrightarrow[T \rightarrow \infty]{P} \phi_1 \quad (4)$$

$$T^{\frac{1}{2}}(\hat{\rho} - \phi_1) \xrightarrow[T \rightarrow \infty]{D} N(0, 1 - \phi_1^2) \quad (5)$$

Hence

$$\hat{y}_{T+1|T} - y_{T+1|T} \xrightarrow[T \rightarrow \infty]{P} 0$$

## Inference on AR models: instability [Whiteboard]

## Lemma (The consequence of misfit)

Suppose  $y_t$  to have a true stationary AR(2) process

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \quad (6)$$

Suppose further that  $v_1 := \text{Var}(y_t y_{t-1})$  is finite and time-independent.

Denote  $v_2 := \text{Var}(y_t) = (1 - \phi_2)\sigma^2((1 + \phi_2)((1 - \phi_2)^2 - \phi_1^2))^{-1}$ .

Suppose that we use an AR(1) model to fit the data  $\{y_0, \dots, y_T\}$  as noted in Equation 3, then

$$\hat{\rho} - \phi_1 \xrightarrow[T \rightarrow \infty]{P} \frac{\phi_1 \phi_2}{1 - \phi_2} \quad (7)$$

$$T^{\frac{1}{2}}(\hat{\rho} - \phi_1 - \frac{\phi_1 \phi_2}{1 - \phi_2}) \xrightarrow[T \rightarrow \infty]{D} N(0, \frac{v_1 \phi_2^2 + v_2 \sigma^2}{v_2^2}) \quad (8)$$

Hence

$$\hat{y}_{T+1|T} - y_{T+1|T} - (\phi_2(\phi_1 y_T (1 - \phi_2)^{-1} - y_{T-1})) \xrightarrow[T \rightarrow \infty]{P} 0$$

# Adaptive learning: introduction

- Exploit the ordered nature of the time series, that because it is ordered, its pasts can verify forecasts that were produced based on previous observations, hence dynamically offer information which can be helpful for model determination.
- This can be similar to some basic statistical learning concepts including cross validation and can overlap with methods including model ensemble.
- First proposed in Yang (2019).

# Adaptive learning: a sample algorithm

## B.2 A sample algorithm for implementing adaptive learning

### Input:

- Data:  $\{y_t : t \in T\}$ .
- Model specifications:  $H$ .
- Forecasting horizon  $k$ .
- Window size for the model estimations  $w$ .
- Specification of loss function:  $l(\cdot)$ . Default choice can be  $\text{ARMSE}(\bar{T})$ .

$$l(\{e_\tau\}_{\tau \in \bar{T}}) = \sum_{\tau \in \bar{T}} \|e_\tau\|_2 \quad (21)$$

At time  $t$ ,  $\bar{T} = \{t - w - k + 1, \dots, t - k - 1, t - k\}$

### Output:

- $T^{\text{test}}$  the index of testing forecasts.
- Induced-optimal functional forms  $(h_t^*)_{t \in T^{\text{test}}}$
- Induced-optimal forecasts for testing purposes:  $\{\hat{y}_{t+k|t}(h_t^*)\}_{t \in T^{\text{test}}}$
- General evaluation of MAE and MSE at time  $t \in T^{\text{test}}$  and the overall evaluations.

Figure: Input and output

## Computing steps:

1. Call  $T^{max} := \max(T)$ . Define  $T^{test} = \{w + v + k, \dots, T^{max} - k - 1, T^{max} - k\}$
2. For all  $t \in \{w + 1, \dots, T^{max} - k\}$ :
  - (a) For all  $h \in H$ :
    - i. Train  $h$  over the data  $\{y_\tau : \tau \in [t] \setminus [t - w - 1]\}$
    - ii. Obtain vector  $\hat{y}_{t+k|t}(h)$
    - iii. Obtain vector  $e_t(h)$
3. For all  $t \in T^{test}$ :
  - (a) Declare  $\tilde{T}$ .
  - (b) For all  $h \in H$ :
    - i. Collect the set  $\{e_\tau(h)\}_{\tau \in \tilde{T}}$
    - ii. Evaluate  $\ell(h) = l(\{e_\tau(h)\}_{\tau \in \tilde{T}})$
  - (c) Find  $h^* := \arg \min\{\ell(h) : h \in H\}$ .
  - (d) Hence save this  $h_t^*$  and make and save the associated forecast  $\hat{y}_{t+k|t}(h_t^*)$
4. Produce general evaluation for  $\{\hat{y}_{t+k|t}(h_t^*)\}_{t \in T^{test}}$ . Return the output.

Figure: Computational steps

# Adaptive learning: formal definition

## Definition (Adaptive learning)

An adaptive learning on a time series dataset  $\mathcal{D}$  and forecasting horizon  $k$ , is a learning method in which there is a learning function  $I$ , a set of models  $H$ , and estimation techniques

$$\Lambda_h : \Theta(h) \mapsto \theta(h) \quad \forall h \in H$$

where  $\Theta(h)$  is the domain of the parameter and  $\theta(h)$  is the estimated parameter, such that for all time  $t \in T^{\text{validation}} \cup T^{\text{test}}$ , we are able to obtain the followings:

- ❶  $\forall h \in H$ , an estimated statistic  $\theta_t(h)$
- ❷  $\forall h \in H$ , an estimated forecast  $y_{t+k|t}(h)$
- ❸ an optimal model  $h_t^* \in H$  selected by  $I$ , with an induced optimal forecast  $y_{t+k|t}(h_t^*)$

The design of  $(I, H, \{\Lambda_h\}_{h \in H})$  should be made on the basis of theoretical validity and tuned in the validation data. Their ability to improve forecasts can then be demonstrated in the testing data.

# Dynamic model selection (Yang 2020; Yang 2021)

Let  $H$  be finite. Each  $h \in H$  gives  $f_k(\cdot; \cdot; h)$  a functional specification. For instance, it could be  $f_k(\Phi_t; \theta_t; h) = \theta_{t,1} + \theta_{t,2}x_t$ .  $\Theta(h) = \mathbb{R}^2$  in this case, and the training of  $\theta(h)$  can be done by MLE over a  $w$  windowed training set at time  $t$ .

The above specifies  $H$  and  $\{\Lambda_h\}_{h \in H}$ . Now, we introduce an MSE-styled loss function:

$$l_t(h) = \sum_{\tau \in \tilde{T}} RMSE_{\tau}(h)$$

At time  $t$ ,  $\tilde{T} = \{t - v - k + 1, \dots, t - k - 1, t - k\}$  indicating that we equally evaluate the loss over the  $v$  windowed history up to time  $t - k$ . Then, we can obtain an optimal model  $h_t^* \in H$  by executing a finite minimisation operation:

$$h_t^* = \arg \min_{h \in H} l_t(h)$$



# Contribution

- Section 1:
  - Provided a framework for analysing stability and instability of statistical modelling.
  - Engaged with the analysis of NN and adversarial attacks from a different dimension.
- Section 2:
  - Extended the framework to time series analysis.
  - Demonstrated the distributional results of autoregressive misfit, and proposed a new learning regime with potential extensions.

# Further work

- Section 1:
  - Further diggings in to more distributions, analysis of NN, and adversarial attacks.
  - Relevant work-in-progress in AFHA: Bastounis, Hansen, and Vlacic (2021).
- Section 2:
  - More inference and computational experiments. Then upload to Arxiv.
  - Combine empirical results (collaboratively with OPRG) to submit to the BoE annual conference and RSSC.

# Nobel Laureates

## Robert F. Engle III Facts



Photo from the Nobel Foundation archive.

Robert F. Engle III  
The Sveriges Riksbank Prize in Economic Sciences in  
Memory of Alfred Nobel 2003

Born: 10 November 1942, Syracuse, NY, USA

Affiliation at the time of the award: New York University, New  
York, NY, USA

Prize motivation: "for methods of analyzing economic time  
series with time-varying volatility (ARCH)."

Contribution: Developed methods to study the volatility  
properties of time series in economics, particular in financial  
markets. His method (ARCH) could, in particular, clarify  
market developments where turbulent periods, with large  
fluctuations, are followed by calmer periods, with modest  
fluctuations.

Prize share: 1/2

## Clive W.J. Granger Facts



Photo from the Nobel Foundation archive.

Clive W.J. Granger  
The Sveriges Riksbank Prize in Economic Sciences in  
Memory of Alfred Nobel 2003

Born: 4 September 1934, Swansea, United Kingdom

Died: 27 May 2009, San Diego, CA, USA

Affiliation at the time of the award: University of California,  
San Diego, CA, USA










Prize motivation: "for methods of analyzing economic time  
series with common trends (cointegration)."

Contribution: Developed and applied new statistical methods,  
based on so-called "cointegration", to differentiate between,  
and combine the analysis of, short-term fluctuations and  
long-term trends.

Prize share: 1/2

**Figure:** The last time the Noble Prize in Economic Sciences was awarded to people who made landmark contribution to Time Series

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