CCIMI PPDE Courses: Advanced stochastic analysis (Hairer 2016)
Component 3: Introduction to SDE and Stochastic Integrals

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Brownian motion

Definition (BM)

A continuous[a] stochastic process $W : [0, \infty) \to \mathbb{R}$ is called a Brownian Motion / Wiener Process if

1. $W(0) = 0 \quad a.s.$
2. $W(t) - W(s) \sim N(0, t-s)$ for all $t \geq s$
3. $(W(t_2) - W(t_1)) \perp (W(t_4) - W(t_3)) \quad \forall t_1 \leq t_2 \leq t_3 \leq t_4$

[a] In the sense that $W(\cdot, \omega)$ is continuous for all $\omega \in \Omega$ a.s.

Definition (Paley-Wiener-Zygmund)

Consider a deterministic $g \in C^1([0, T]; \mathbb{R})$ with $g(0) = g(T) = 0$, we define

$$\int_0^T g dW := - \int_0^T g' W dt$$

Riemann sum approximation

A partition P of $[0, T]$ is a finite collection of distinct points in $[0, T]$, denoted orderly

$$P := \{0 = t_0 < t_1 < \dots < t_m = T\}$$

The mesh size

$$|P| := \max_{0 \leq k \leq m-1} |t_{k+1} - t_k|$$

Consider a point $\tau_k := (1-\lambda)t_k + \lambda t_{k+1}$ with $\lambda \in [0,1]$, usually fixed, then we have the following definition.

Definition (Riemann sum approximation of $\int_0^T WdW$)

$$R(P, \lambda) = \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k))$$

Essential result:

$$\lim_{n\to\infty} R_n = \frac{W(T)^2}{2} + (\lambda - \frac{1}{2})T$$

Way towards the Ito stochastic integral:
Denote $\mathbb{L}^2(0, T)$ as the space of all real-valued, progressively measurable stochastic processes $G(\cdot)$ s.t. $\mathbb{E}[\int_0^T G^2 dt] < \infty$
$G \in \mathbb{L}^2(0, T)$ is a step process if
$\exists P := \{0 = t_0 < t_1 < \ldots < t_m = T\}$ s.t.

$$G(t) = G_k \quad \forall t \in [t_k, t_{k+1}) \quad \forall k$$

Definition (Ito stochastic integral)

Let G be described as above. Then

$$\int_0^T G dW := \sum_{k=0}^{m-1} G_k (W(t_{k+1}) - W(t_k))$$

Ito's integral in $\mathbb{L}^2(0, T)$

Definition (Approximation by step processes)

Let $G \in \mathbb{L}^2(0, T)$ and let[a] $G^n \in \mathbb{L}^2(0, T)$ be a sequence of bounded step processes such that

$$\mathbb{E}\left[\int_0^T |G - G^n|^2 dt\right] \to 0$$

Then

$$\int_0^T G dW := \lim_{n\to\infty} \int_0^T G^n dW$$

[a] The existence is guaranteed.

Introduction to SDE

Definition (Stochastic differential, introductory version)

Let $F \in \mathbb{L}^1(0,T)$, $G \in \mathbb{L}^2(0,T)$. We say $X(\cdot)$ to have the stochastic differential

$$dX = Fdt + GdW \quad \forall t \in [0,T]$$

if $\forall 0 \leq s \leq r \leq T$,

$$X(r) = X(s) + \int_s^r Fdt + \int_s^r GdW$$

Theorem (Ito's chain rule)

Let X be above and assume[a] $u \in C^2(\mathbb{R} \times [0, T]; \mathbb{R})$, *then* $Y(t) := u(X(t), t)$ *has the stochastic differential*[b]

$$du = (u_t + u_x F + \frac{1}{2} u_{xx} G^2)dt + u_x G dW$$

[a] The actual assumption could be even weaker

[b] All variables are applied to arguments $(X(t), t)$

Set Up

Given deterministic functions:

- $b : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$
- $B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}$

Given a n dimensional r.v. X_0, independent of an m dimensional BM $W(\cdot)$

Definition (Stochastic differential equation (SDE))

An $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ is a solution of the Ito stochastic differential equation with $X(0) = X_0$ and

$$dX = b(X, t)dt + B(X, t)dW$$

if $\forall t \in [0, T]$,

$$X(t) = X_0 + \int_0^t b(X(s), s)ds + \int_0^t b(X(s), s)dW \quad \text{a.s.}$$

Example: stock prices

With strictly 1D positive constants μ, σ and $X(0) = x_0$, we try to solve

$$dX = \mu X dt + \sigma X dW \tag{1}$$

We use Ito's chain rule with $u(X) = \log(X)$ to get

$$du = \frac{dX}{X} - \frac{\sigma^2 dt}{2} = (\mu - \frac{\sigma^2}{2})dt + \sigma dW =: bdt + \sigma dW$$

Then by definition, $\forall t$,

$$u(t) = u(0) + bt + \sigma W(t)$$

One may take exponential transform and get to the "famous expression"

$$X = x_0 \exp(bt + \sigma W(t)) \tag{2}$$

An application to PDE

$U \subset \mathbb{R}^n$ bounded and open, $u \in C^2(\mathbb{R}^d; \mathbb{R})$, c, f smooth with $c \geq 0$. Consider a system

$$\begin{cases} -\frac{\Delta u}{2} + cu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Theorem (Feynman-Kac formula)

The unique solution to the above PDE is $\forall x \in U$,

$$X := W + x, \quad u(x) = \mathbb{E}\left[\int_0^{\tau_x} f(X(t)) \exp\left(-\int_0^t c(X)ds\right) dt\right]$$

where τ_x *is the first hitting time of* ∂U

Recall the Reimann sum in Foundations.
In Stratonovich Integral, we define

$$\int_0^T B(W,t) \circ dW$$

by

$$\lim_{|P^n| \to 0} \sum_{k=0}^{m_n - 1} B\left(\frac{W(t_{k+1}^n) - W(t_k^n)}{2}, t_k^n\right)\left(W(t_{k+1}^n) - W(t_k^n)\right)$$

S' Chain Rule: Ito's Chain Rule holds for Stratonovich differentials