ICCS310: Assignment 5

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1: Reject TM

$$\mathsf{REJECT_{TM}} = \{\langle M, x\rangle |\text{ M is a TM that rejects input}\}$$

Show directly (i.e., without resorting to reduction) that $\mathsf{REJECT_{TM}}$ is undecidable.

Proof:

We want to show that $\mathsf{REJECT_{TM}}$ is undecidable by showing the contradiction.

Assume for the sake of contradiction that the TM M_{REJECT} decides $\mathsf{REJECT_{TM}}$.

On input $\langle M, w\rangle$:

$$M_{\mathsf{REJECT}} = \begin{cases} \text{reject} & \text{if } M \text{ accepts } w \\ \text{accept} & \text{if } M \text{ rejects } w \\ \text{reject} & \text{if } M \text{ loops on } w \end{cases}$$

Also, define TM $Y(\langle X\rangle)$ where X is a TM. On input $\langle X\rangle$,

1. Run M_{REJECT} on $\langle X, \langle X\rangle\rangle$.

2. If M_{REJECT} accepts, we accept. If M_{REJECT} rejects, we rejects.

So we have,

$$Y(\langle X\rangle) = \begin{cases} \text{reject} & \text{if } X \text{ accepts } \langle X\rangle \\ \text{accept} & \text{if } X \text{ rejects } \langle X\rangle \\ \text{reject} & \text{if } X \text{ loops on } \langle X\rangle \end{cases}$$

Then,

$$Y(\langle Y\rangle) = \begin{cases} \text{reject} & \text{if } Y \text{ accepts } Y \\ \text{accept} & \text{if } Y \text{ rejects } Y \\ \text{reject} & \text{if } Y \text{ loops on } Y \end{cases}$$

We have that whatever Y try to decides on $\langle Y\rangle$ turns out to be that there is no certain answer and possibly failed to predict the outcome. Since Y cannot decides, then W also cannot decides. Hence, it contradicts to our assumption that M_{REJECT} decides $\mathsf{REJECT_{TM}}$. Therefore, $\mathsf{REJECT_{TM}}$ is undecidable. □

2: Accept vs. Reject

$$\mathsf{ACCEPT_{TM}} = \{\langle M, x\rangle |\text{ M is a TM that accepts input x}\}$$

(i) Prove that $\mathsf{ACCEPT_{TM}} \leq \mathsf{REJECT_{TM}}$

Proof:

Suppose that TM $M_{\textsf{REJECT}}$ decides $\textsf{REJECT}_{\textsf{TM}}$ and TM $M_{\textsf{ACCEPT}}$ decides $\textsf{ACCEPT}_{\textsf{TM}}$, we want to show how to decide $\textsf{ACCEPT}_{\textsf{TM}}$ using $M_{\textsf{REJECT}}$.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M by reversing the accept and reject states.

2. Run $M_{\textsf{REJECT}}$ with $\langle M', w\rangle$.

3. If $M_{\textsf{REJECT}}$ accepts, we accept. If $M_{\textsf{REJECT}}$ rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\textsf{REJECT}} \text{ accepts } \langle M', w\rangle \iff M_{\textsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, $M_{\textsf{ACCEPT}}$ can correctly decide $\textsf{ACCEPT}_{\textsf{TM}}$ provided that there is a TM $M_{\textsf{REJECT}}$. Therefore, $\textsf{ACCEPT}_{\textsf{TM}} \leq \textsf{REJECT}_{\textsf{TM}}$. □

(ii)) Prove that $\textsf{REJECT}_{\textsf{TM}} \leq \textsf{ACCEPT}_{\textsf{TM}}$

Proof:

Suppose that TM $M_{\textsf{REJECT}}$ decides $\textsf{REJECT}_{\textsf{TM}}$ and TM $M_{\textsf{ACCEPT}}$ decides $\textsf{ACCEPT}_{\textsf{TM}}$, we want to show how to decide $\textsf{REJECT}_{\textsf{TM}}$ using $M_{\textsf{ACCEPT}}$.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M by reversing the accept and reject states.

2. Run $M_{\textsf{ACCEPT}}$ with $\langle M', w\rangle$.

3. If $M_{\textsf{ACCEPT}}$ accepts, we accept. If $M_{\textsf{ACCEPT}}$ rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\textsf{REJECT}} \text{ accepts } \langle M', w\rangle \iff M_{\textsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, $M_{\textsf{REJECT}}$ can correctly decide $\textsf{REJECT}_{\textsf{TM}}$ provided that there is a TM $M_{\textsf{ACCEPT}}$. Therefore, $\textsf{REJECT}_{\textsf{TM}} \leq \textsf{ACCEPT}_{\textsf{TM}}$. □

3: Reverse on TM

$$\textsf{T} = \{\langle M\rangle |\text{ M is a TM that accepts } \mathbf{rev}(w) \text{ whenever it accepts } w\}$$

where $\mathbf{rev}(w)$ is the reverse of the string w. Show that $\textsf{T}$ is undecidable.

Proof: $(\textsf{ACCEPT}_{\textsf{TM}} \leq \textsf{T}_{\textsf{TM}})$

Suppose that TM $M_{\textsf{T}}$ decides $\textsf{T}_{\textsf{TM}}$ and TM $M_{\textsf{ACCEPT}}$ decides $\textsf{ACCEPT}_{\textsf{TM}}$, we want to show how to decide $\textsf{ACCEPT}_{\textsf{TM}}$ using $M_{\textsf{T}}$.

On input $\langle M, w\rangle$:

1. Make TM M' that accepts a string w, then it also accepts w^R. On input x:

1.1 If $x = 01$, then accept x.

1.2 If $x \neq 01$, then run M on input w and accept if M accepts.

2. Run $M_{\textsf{T}}$ on $\langle M', w\rangle$.

3. If M_{T} accepts, we accept. If M_{T} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{T}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

From observation, if M accepts w, then M' accepts every string, so $\langle M'\rangle \in \mathsf{T}_{\mathsf{TM}}$. If M does not accept w, then only 01 will be accepted which means $\langle M'\rangle \notin \mathsf{T}_{\mathsf{TM}}$

Then, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{T}.

So, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{T}_{\mathsf{TM}}$. Therefore, T is undecidable. □

4: Undecidability

(i) Show that

$$\mathsf{TOTAL} = \{\langle M\rangle |\text{ M is a Turing machine that halts on every input}\}$$

is undecidable

Proof: ($\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{TOTAL}_{\mathsf{TM}}$)

Suppose that TM M_{TOTAL} decides $\mathsf{TOTAL}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ using M_{TOTAL}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M where if M accepts, we accept and enter loops when M rejects.
2. Run M_{TOTAL} with $\langle M', w\rangle$.
3. If M_{TOTAL} accepts, we accept. If M_{TOTAL} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{TOTAL}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{TOTAL}. So, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{TOTAL}_{\mathsf{TM}}$. Therefore, TOTAL is undecidable. □

(ii) Show that

$$\mathsf{FINITE} = \{\langle M\rangle |\text{ M is a Turing machine and } L(M) \text{ is a finite set}\}$$

is undecidable

Proof: ($\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{FINITE}_{\mathsf{TM}}$)

Suppose that TM M_{FINITE} decides $\mathsf{FINITE}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ using M_{FINITE}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M where if M accepts, we accept and enter loops when M rejects.
2. Run M_{FINITE} with $\langle M', w\rangle$.
3. If M_{FINITE} accepts, we accept. If M_{FINITE} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{FINITE}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT_{TM}}$ provided that there is a TM M_{FINITE}. So, $\mathsf{ACCEPT_{TM}} \leq \mathsf{FINITE_{TM}}$. Therefore, FINITE is undecidable. □

(iii) Show that

$$\mathsf{REGULAR} = \{\langle M\rangle | \text{ M is a Turing machine and } L(M) \text{ is regular}\}$$

is undecidable

Proof: ($\mathsf{ACCEPT_{TM}} \leq \mathsf{REGULAR_{TM}}$)

Suppose that TM M_{REGULAR} decides $\mathsf{REGULAR_{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT_{TM}}$, we want to show how to decide $\mathsf{ACCEPT_{TM}}$ using M_{REGULAR}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M. On input x:

1.1 If x has the form 0^n1^n, accepts.

1.2 If x does not have the form 0^n1^n, run M on input w and accept if M accepts w.

2. Run M_{REGULAR} with $\langle M', w\rangle$.

3. If M_{REGULAR} accepts, we accept. If M_{REGULAR} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{REGULAR}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT_{TM}}$ provided that there is a TM M_{REGULAR}. So, $\mathsf{ACCEPT_{TM}} \leq \mathsf{REGULAR_{TM}}$. Therefore, $\mathsf{REGULAR}$ is undecidable. □

5: Total Is No Harder Than Finite

Prove that

$$\mathsf{TOTAL} \leq_T \mathsf{FINITE}$$

Proof:

Suppose that TM M_{TOTAL} decides $\mathsf{TOTAL_{TM}}$ and TM M_{FINITE} decides $\mathsf{FINITE_{TM}}$, we want to show how to decide $\mathsf{TOTAL_{TM}}$ using M_{FINITE}.

Given $\langle M\rangle$ as input:

1. Run M_{FINITE} on $\langle M\rangle$.

2. If M_{FINITE} accepts, we accept. If M_{FINITE} rejects, we rejects.

Refer to the fact that M_{FINITE} can determine whether M has finite set of $L(M)$ or not, M will halt on every input only if $L(M)$ is a finite set. If $L(M)$ is not a finite set, we would not be able to determine that it will halt on every input since there would be at least one input that would not halt.

Hence, M_{TOTAL} can correctly decide $\mathsf{TOTAL}_{\mathsf{TM}}$ provided that there is a TM M_{FINITE}. Therefore, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REGULAR}_{\mathsf{TM}}$. □

6: Finite Is No Harder Than Total

Prove that

$$\mathsf{FINITE} \leq_T \mathsf{TOTAL}$$

Proof:

Suppose that TM M_{TOTAL} decides $\mathsf{TOTAL}_{\mathsf{TM}}$ and TM M_{FINITE} decides $\mathsf{FINITE}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{FINITE}_{\mathsf{TM}}$ using M_{TOTAL}.

Given $\langle M \rangle$ as input:

1. Run M_{TOTAL} on $\langle M \rangle$.
2. If M_{TOTAL} accepts, we accept. If M_{TOTAL} rejects, we rejects.

Refer to the fact that M_{TOTAL} can determine whether M halt on every input or not, $L(M)$ has to be finite set if M halt on every input. If M does not halt on every input, we can tell that $L(M)$ is not finite set since there would be another input to be execute.

Hence, M_{TOTAL} can correctly decide $\mathsf{TOTAL}_{\mathsf{TM}}$ provided that there is a TM M_{FINITE}. Therefore, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REGULAR}_{\mathsf{TM}}$. □

7: Extra: Undecidability of Nontrivial Properties

Proof: It is non trivial. How to decide on it though?