ICCS310: Assignment 6

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March 24, 2021

1: The Meaning of Things

(1) Class NP is the problems that can be solved within polynomial time using a NFA. Besides, we can solve it in polynomial time using a machine that compute all possibilities at once.

(2) Show that

(3) Show that

(4) Show that

2: Closure of NP

$$\mathsf{ACCEPT}_{\mathsf{TM}} = \{\langle M, x\rangle | \text{ M is a TM that accepts input x}\}$$

(i) Prove that $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REJECT}_{\mathsf{TM}}$

Proof:

Suppose that TM M_{REJECT} decides $\mathsf{REJECT}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ using M_{REJECT}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M by reversing the accept and reject states.

2. Run M_{REJECT} with $\langle M', w\rangle$.

3. If M_{REJECT} accepts, we accept. If M_{REJECT} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{REJECT}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{REJECT}. Therefore, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REJECT}_{\mathsf{TM}}$. □

(ii)) Prove that $\mathsf{REJECT}_{\mathsf{TM}} \leq \mathsf{ACCEPT}_{\mathsf{TM}}$

Proof:

Suppose that TM M_{REJECT} decides $\mathsf{REJECT}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{REJECT}_{\mathsf{TM}}$ using M_{ACCEPT}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M by reversing the accept and reject states.

2. Run M_{ACCEPT} with $\langle M', w\rangle$.

3. If M_{ACCEPT} accepts, we accept. If M_{ACCEPT} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{REJECT}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{REJECT} can correctly decide $\mathsf{REJECT_{TM}}$ provided that there is a TM M_{ACCEPT}. Therefore, $\mathsf{REJECT_{TM}} \leq \mathsf{ACCEPT_{TM}}$. □

3: This is NP

$$\mathsf{T} = \{\langle M\rangle \mid \text{M is a TM that accepts } \mathbf{rev}(w) \text{ whenever it accepts } w\}$$

where $\mathbf{rev}(w)$ is the reverse of the string w. Show that T is undecidable.

Proof: $(\mathsf{ACCEPT_{TM}} \leq \mathsf{T_{TM}})$

Suppose that TM M_{T} decides $\mathsf{T_{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT_{TM}}$, we want to show how to decide $\mathsf{ACCEPT_{TM}}$ using M_{T}.

On input $\langle M, w\rangle$:

1. Make TM M' that accepts a string w, then it also accepts w^R. On input x:

1.1 If $x = 01$, then accept x.

1.2 If $x \neq 01$, then run M on input w and accept if M accepts.

2. Run M_{T} on $\langle M', w\rangle$.

3. If M_{T} accepts, we accept. If M_{T} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{T}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

From observation, if M accepts w, then M' accepts every string, so $\langle M'\rangle \in \mathsf{T_{TM}}$. If M does not accept w, then only 01 will be accepted which means $\langle M'\rangle \notin \mathsf{T_{TM}}$

Then, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT_{TM}}$ provided that there is a TM M_{T}.

So, $\mathsf{ACCEPT_{TM}} \leq \mathsf{T_{TM}}$. Therefore, T is undecidable. □

4: NP-Complete

(i) Show that

$$\mathsf{TOTAL} = \{\langle M\rangle \mid \text{M is a Turing machine that halts on every input}\}$$

is undecidable

Proof: $(\mathsf{ACCEPT_{TM}} \leq \mathsf{TOTAL_{TM}})$

Suppose that TM M_{TOTAL} decides $\mathsf{TOTAL_{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT_{TM}}$, we want to show how to decide $\mathsf{ACCEPT_{TM}}$ using M_{TOTAL}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M where if M accepts, we accept and enter loops when M rejects.

2. Run M_{TOTAL} with $\langle M', w\rangle$.

3. If M_{TOTAL} accepts, we accept. If M_{TOTAL} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{TOTAL}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{TOTAL}. So, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{TOTAL}_{\mathsf{TM}}$. Therefore, TOTAL is undecidable. $\square$

(ii) Show that

$$\mathsf{FINITE} = \{\langle M\rangle |\text{ M is a Turing machine and } L(M) \text{ is a finite set}\}$$

is undecidable

Proof: $(\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{FINITE}_{\mathsf{TM}})$

Suppose that TM M_{FINITE} decides $\mathsf{FINITE}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ using M_{FINITE}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M where if M accepts, we accept and enter loops when M rejects.

2. Run M_{FINITE} with $\langle M', w\rangle$.

3. If M_{FINITE} accepts, we accept. If M_{FINITE} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{FINITE}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{FINITE}. So, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{FINITE}_{\mathsf{TM}}$. Therefore, FINITE is undecidable. $\square$

(iii) Show that

$$\mathsf{REGULAR} = \{\langle M\rangle |\text{ M is a Turing machine and } L(M) \text{ is regular}\}$$

is undecidable

Proof: $(\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REGULAR}_{\mathsf{TM}})$

Suppose that TM M_{REGULAR} decides $\mathsf{REGULAR}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ using M_{REGULAR}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M. On input x:

1.1 If x has the form 0^n1^n, accepts.

1.2 If x does not have the form 0^n1^n, run M on input w and accept if M accepts w.

2. Run M_{REGULAR} with $\langle M', w\rangle$.

3. If M_{REGULAR} accepts, we accept. If M_{REGULAR} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w.

$$M_{\mathsf{REGULAR}} \text{ accepts } \langle M', w\rangle \iff M_{\mathsf{ACCEPT}} \text{ accepts } \langle M, w\rangle$$

Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{REGULAR}. So, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REGULAR}_{\mathsf{TM}}$. Therefore, $\mathsf{REGULAR}$ is undecidable. □

5: Silver Lining If P = NP

Prove that

$$\mathsf{TOTAL} \leq_T \mathsf{FINITE}$$

Proof:

Suppose that TM M_{TOTAL} decides $\mathsf{TOTAL}_{\mathsf{TM}}$ and TM M_{FINITE} decides $\mathsf{FINITE}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{TOTAL}_{\mathsf{TM}}$ using M_{FINITE}.

Given $\langle M\rangle$ as input:

1. Run M_{FINITE} on $\langle M\rangle$.

2. If M_{FINITE} accepts, we accept. If M_{FINITE} rejects, we rejects.

Refer to the fact that M_{FINITE} can determine whether M has finite set of $L(M)$ or not, M will halt on every input only if $L(M)$ is a finite set. If $L(M)$ is not a finite set, we would not be able to determine that it will halt on every input since there would be at least one input that would not halt.

Hence, M_{TOTAL} can correctly decide $\mathsf{TOTAL}_{\mathsf{TM}}$ provided that there is a TM M_{FINITE}. Therefore, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REGULAR}_{\mathsf{TM}}$. □