ICCS310: Assignment 1

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1: Review: Something About Sets

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(1) Let A_1, A_2, A_3 be any sets from a universe $\mathcal{U}$. *Prove that* $\overline{A_1 \cup A_2 \cup A_3} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

Proof: We want to show that $\overline{A_1 \cup A_2 \cup A_3} \subseteq \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ and $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \subseteq \overline{A_1 \cup A_2 \cup A_3}$

Let A_1, A_2, A_3 be any three given sets. We'll first prove that $\overline{A_1 \cup A_2 \cup A_3} \subseteq \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$. Let $x \in \overline{A_1 \cup A_2 \cup A_3}$. Then, $x \notin A_1 \cup A_2 \cup A_3$ by the definition of complement, so then $x \notin A_1$, $x \notin A_2$ and $x \notin A_3$, by the definition of union. This means that $x \in \overline{A_1}$, $x \in \overline{A_2}$, and $x \in \overline{A_3}$, by the definition of complement. Hence, $x \in \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ since x is in $\overline{A_1}$, $\overline{A_2}$, and $\overline{A_3}$.

Also, we will show that $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \subseteq \overline{A_1 \cup A_2 \cup A_3}$. Let $y \in \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$, so y is in $\overline{A_1}$, $\overline{A_2}$, and $\overline{A_3}$, by the definition of intersection. This means $y \notin A_1$, $y \notin A_2$, and $y \notin A_3$, by the definition of complement. It follows that $y \notin A_1 \cup A_2 \cup A_3$, and so $y \in \overline{A_1 \cup A_2 \cup A_3}$.

In conclusion, $\overline{A_1 \cup A_2 \cup A_3} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

(2) Let A and B be any sets from a universe $\mathcal{U}$. *Prove that* $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof: We want to show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Let A and B be any two given sets. We'll first prove that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$. Then, $x \notin A \cup B$ by the definition of complement, so then $x \notin A$, and $x \notin B$, by the definition of union. This means that $x \in \overline{A}$, and $x \in \overline{B}$, by the definition of complement. Hence, $x \in \overline{A} \cap \overline{B}$ since x is in $\overline{A}$ and $\overline{B}$.

Also, we will show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. Let $y \in \overline{A} \cap \overline{B}$, so y is in $\overline{A}$ and $\overline{B}$, by the definition of intersection. This means $y \notin A$ and $y \notin B$, by the definition of complement. It follows that $y \notin A \cup B$, and so $y \in \overline{A \cup B}$.

In conclusion, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

2: Prime and Irrational

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(1) Let $p \geq 2$ be a prime and a be a positive integer. Prove that if p divides a^2, then p divides a.

Proof: Using contraposition, we can prove that if p does not divides a, then p does not divides a^2 instead.

Let a be any number that cannot be divided by p, so $a = (p * q) + r$ where $r < p$ and $r, q \in \mathbb{I}^+$. So, we have $a^2 = ((p * q) + r)^2 = (p * q)^2 + 2 * (p * q * r) + r^2$. We can see that r^2 cannot be divided by p. Hence, p does not divides a^2.

Therefore, if p divides a^2, then p divides a.

(2) Prove that if p is any positive prime number, then $\sqrt{p}$ is irrational.

Proof: Using contraposition, we can prove that if p does not divides a, then p does not divides a^2 instead.

Let a be any number that cannot be divided by p, so $a = (p * q) + r$ where $r < p$ and $r, q \in \mathbb{I}^+$. So, we have $a^2 = ((p * q) + r)^2 = (p * q)^2 + 2 * (p * q * r) + r^2$. We can see that r^2 cannot be divided by p. Hence, p does not divides a^2.

Therefore, if p divides a^2, then p divides a.

3: Spacing

Prove that in any set of $n + 1$ numbers from $\{1, ..., 2n\}$, there are always two numbers that are consecutive.

Proof: Without loss of generality,

4: Curious Fact about Graphs

Let $G = (V, E)$ be an undirected graph. Show that G contains two nodes that have equal degrees.

5: Basic DFAs

(1) Prove *Theorem*: for any $n \geq 0$, `Solve_Hanoi` (n, From_Peg, To_Peg, Aux_Peg) generates exactly $2^n - 1$ lines of instruction

Predicate : P(x) $\equiv$ for any $x \geq 0$, `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction

Base case : P(0) $\equiv$ `Solve_Hanoi` (n, ...) generates exactly 0 lines of instruction which is true

Inductive Steps : Assume that if P(x) is true then P(x+1) is true
P(x) $\equiv$ `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction
P(x+1) $\equiv$ `Solve_Hanoi` (x, ...) generates exactly $2^{x+1} - 1$ lines of instruction

To show that this is true in mathematically way, T(x) is the number of line generated from the function using recurrence.

$$
\begin{aligned}
T(x) &= 2T(x-1) + 1; T(1) = 1; T(0) = 0 && (1)\\
T(x) &= 2^{x-1} + ... + 2 + 1 && (2)\\
T(x) &= 2^x - 1 && (3)\\
LHS &= 2^x - 1; T(x) && (4)\\
RHS &= 2^x - 1 && (5)\\
LHS &\equiv RHS && (6)
\end{aligned}
$$

So, P(x) is true, this time we will prove the P(x+1) by using the equations above.

$$
\begin{aligned}
T(x+1) &= 2T(x) + 1; T(1) = 1; T(0) = 0 && (7)\\
T(x+1) &= 2^x + ... + 2 + 1 && (8)\\
T(x+1) &= 2^x + 2^x - 1; && (9)
\end{aligned}
$$

Using T(x) to solve the equation below

$$
\begin{aligned}
T(x+1) &= 2^{x+1}-1 && (10)\\
LHS &= 2^{x+1}-1; T(x+1) && (11)\\
RHS &= 2^{x+1}-1; && (12)\\
LHS &\equiv RHS && (13)
\end{aligned}
$$

From the induction hypothesis, P(x-1) $\implies$ P(x) and P(x) holds for any $x \geq 0$, `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction. Q.E.D.

(3) Prove *printRuler*:

These are the equations we know from this problem
$f(n) = 2f(n-1) + 1, f(0) = 0$ is number of lines
$g(n) = 2g(n-1) + n, g(0) = 0$ is number of dashes
$g(n) = a * f(n) + b * n + c$

1. Basically, I just followed the hint

$$
\begin{aligned}
g(0) &= a * f(0) + b * 0 + c && (14)\\
g(0) &= c && (15)\\
g(0) &= 0 && (16)
\end{aligned}
$$

   So, c = 0

2. We will find a and b

$$
\begin{aligned}
g(n) &= 2g(n-1) + n && (17)\\
a * f(n) + b * n &= 2(a * f(n-1) + b * (n-1)) + n && (18)\\
a * f(n) + b * n &= 2a * f(n-1) + 2b * (n-1) + n && (19)\\
a * f(n) &= 2a * f(n-1) + b * n - 2b + n && (20)\\
a * f(n) - 2a * f(n-1) &= b * n - 2b + n && (21)
\end{aligned}
$$

   Let's do it side by side

$$
\begin{aligned}
a(f(n) - 2f(n-1)) &= b * n + n - 2b && (22)\\
a(1) &= n * (b+1) - 2b && (23)\\
a + 2b &= n * (b+1) && (24)
\end{aligned}
$$

$$
a + 2b - n * (b+1) = 0 \qquad (25)
$$

   To find a and b, we know that substitute P and Q = 0 will solve this equation

$$
\begin{aligned}
P + Qn &= 0 && (26)\\
a + 2b &= P && (27)\\
(b+1) &= Q && (28)\\
b &= -1 && (29)\\
a &= 2 && (30)
\end{aligned}
$$

$$
\begin{aligned}
g(n) &= a * f(n) + b * n && (31)\\
g(n) &= 2 * f(n) - n && (32)
\end{aligned}
$$

3. Previously, we got $g(n) = 2 * f(n) - n$
   Also, $f(n) = 2^n - 1$. In fact, $g(n) = 2^{n+1} - n - 2$.

4. Theorem : $g(n) = 2^{n+1} - n - 2$ works for all $n \geq 0$
   Predicate : P(x) $\equiv g(x) = 2^{x+1} - x - 2$ works for all $x \geq 0$
   Base case : P(0) $\equiv g(0) = 0$ is true
   Inductive Steps : Assume that if P(x) is true then P(x+1) is true
   P(x) $\equiv g(x) = 2^{x+1} - x - 2$
   P(x+1) $\equiv g(x+1) = 2^{x+2} - x - 3$
   Actually, we know that $g(n) = 2g(n-1) + n$ has a close form of $g(x) = 2^{x+1} - x - 2$ according to what we have done on part 2.

   So, P(x) is true, this time we will prove the P(x+1) by using the equations above.

$$
\begin{aligned}
g(x+1) &= 2^{x+2} - x - 3 && (33)\\
g(x+1) &= 2 * 2^{x+1} - x - 3 && (34)\\
g(x+1) &= 2(g(x) + x + 2) - x - 3 && (35)\\
g(x+1) &= 2g(x) + (x+1) && (36)\\
LHS &= 2^{x+2} - x - 3 && (37)\\
RHS &= 2g(x) + (x+1) && (38)\\
LHS &\equiv RHS && (39)
\end{aligned}
$$

   From the induction hypothesis, P(x) $\implies$ P(x+1) and P(x) holds for any $x \geq 0$ which will make $g(x) = 2^{x+1} - x - 2$ true. So, $g(x) = 2^{x+1} - x - 2$ works. Q.E.D.

6: Penultimate

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(1) Prove *Theorem*: for any $n \geq 0$, `Solve_Hanoi` (n, From_Peg, To_Peg, Aux_Peg) generates exactly $2^n - 1$ lines of instruction

Predicate : P(x) $\equiv$ for any $x \geq 0$, `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction

Base case : P(0) $\equiv$ `Solve_Hanoi` (n, ...) generates exactly 0 lines of instruction which is true

Inductive Steps : Assume that if P(x) is true then P(x+1) is true
P(x) $\equiv$ `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction
P(x+1) $\equiv$ `Solve_Hanoi` (x, ...) generates exactly $2^{x+1} - 1$ lines of instruction

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7: Digit Sum

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(1) Prove *Theorem*: for any $n \geq 0$, `Solve_Hanoi` (n, From_Peg, To_Peg, Aux_Peg) generates exactly $2^n - 1$ lines of instruction

Predicate : P(x) $\equiv$ for any $x \geq 0$, `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction

Base case : P(0) $\equiv$ `Solve_Hanoi` (n, ...) generates exactly 0 lines of instruction which is true

Inductive Steps : Assume that if P(x) is true then P(x+1) is true
P(x) $\equiv$ `Solve_Hanoi` (x, ...) generates exactly $2^x - 1$ lines of instruction
P(x+1) $\equiv$ `Solve_Hanoi` (x, ...) generates exactly $2^{x+1} - 1$ lines of instruction