Ground Rules:

- This assignment contains only written problems. You should first attempt the problems by yourself and only start working in groups after a few days of thinking.
- Typeset your solution (it's a good excuse to learn LaTeX) or write legibly. You are handing in your work electronically. We accept only PDF files (name them whatever you want as long as it's a single PDF file).
- Exercise common sense when collaborating with others. Even if you work together on a problem, the writeup should be your own. This is the only way I know for you to master this kind of subject.

Task 1: Review: Something About Sets (2 points)

In class, we saw a proof of a simple version of DeMorgan's theorem. Here are two more forms that you will explore. Remember that if A is a set from a universe $\mathscr{U}$, the complement of A, written $\overline{A}$, is the set that contains everything from the universe excluding what is present in A. In other words, $\overline{A} = \mathscr{U} \setminus A$.

1. Let A_1, A_2, A_3 be any sets from a universe $\mathscr{U}$. Prove that $\overline{A_1 \cup A_2 \cup A_3} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.
2. Let A and B be any sets from a universe $\mathscr{U}$. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Task 2: Prime and Irrational (4 points)

This problem contains two parts:

1. Let $p \geq 2$ be a prime and a be a positive integer. Prove that if p divides a^2, then p divides a. (*Hint:* Contrapositive.)
2. Prove that if p is any positive prime number, then $\sqrt{p}$ is irrational.

Task 3: Spacing (2 points)

Prove that in any set of $n+1$ numbers from $\{1, \ldots, 2n\}$, there are always two numbers that are consecutive.

Task 4: Curious Fact about Graphs (2 points)

Let $G = (V, E)$ be an undirected graph. Show that G contains two nodes that have equal degrees.

Task 5: Basic DFAs (2 points)

Let $\Sigma = \{a, b, c\}$. Draw DFAs for the following languages. Briefly justify why your DFAs recognize the correct language by explaining the "meaning" of each state.

1. The language of strings on Σ whose length is divisible by 5.
2. The language of strings on Σ whose length is either even or divisible by 5 (or both).
3. The language of strings on Σ that has at least one a and contains an even number of bs.

Task 6: Penultimate (4 points)

Consider the alphabet $\Sigma = \{0, 1\}$.

1. Let $L_2 = \{x \in \Sigma^* : \text{the 2nd-to-last symbol of } x \text{ is } 1\}$. Note that to have a chance to be in L_2, a string x must be at least 2 characters long. Draw a DFA with 4 states that accepts L_2. Also, justify why it works by explaining what each state "means."
2. Show that *any* DFA that correctly recognizes L_2 must have at least 4 states. (*Hint:* Pigeonhole. A high-level reasoning is that if it had 3 states, there are two strings that ended up in the same state but should not end up in the same state.)

Task 7: Digit Sum (2 points)

Consider the alphabet $\Sigma = \{0, 1, 2, 3\}$. For each $s \in \Sigma^*$, let $\text{DIGITSUM}(s)$ be the sum of its digits. For example, $\text{DIGITSUM}(123) = 6$ and $\text{DIGITSUM}(1001) = 2$. Let L be the language

$$L = \{x \in \Sigma^* : \text{DIGITSUM}(x) \text{ is divisible by } 3\}.$$

Draw a DFA that recognizes L. Explain what each of your states represents.