Homework 6

Computation Theory (Term II/2020–21)

built on 2021/03/18 at 21:51:02 **due:** *mon mar 29 @ 11:59pm*

Ground Rules:

- This assignment contains only written problems. You should first attempt the problems by yourself and start working in groups after a few days of thinking.
- Typeset your solution (a good excuse to learn LaTeX) or write legibly. You're handing in your work electronically. We only accept PDF files. Name the file whatever you want as long as it's a single PDF.
- Exercise common sense when collaborating with others or looking things up online. Even if you work together on a problem, the writeup should be your own. This is the only way I know for you to master this kind of subject.

Task 1: The Meaning of Things (2 points)

In your own words:

(1) Give a definition of the class NP.
(2) Explain how one can prove that a problem belongs to the class NP.
(3) What is NP-complete?
(4) Describe a strategy for showing that a problem is NP-complete.

Task 2: Closure of NP (4 points)

Let $A, B \in \mathsf{NP}$. Prove or disprove:

(i) $A \cap B$ must be in NP.
(ii) $A \cup B$ must be in NP.

Task 3: This is NP (2 points)

Remember that a *coloring* of a graph is an assignment of colors to its vertices so that no two adjacent vertices are given the same color. A graph G is said to be k-colorable if there is a coloring of G that uses at most k colors. Let

$$\mathsf{5COLOR} = \left\{ \langle G \rangle \mid G \text{ is 5-colorable} \right\}.$$

Prove that $\mathsf{5COLOR} \in \mathsf{NP}$.

Task 4: NP-Complete (4 points)

Consider the following problems:

- HAM-CYCLE is, given an <u>directed</u> graph, is there a cycle that visits every vertex exactly once?
- HAM-PATH is, given a <u>directed</u> graph and two vertices $s \neq t$, is there a path starting at s and ending at t that visits each and every vertex exactly once?
- UNDIRECTED-HAM-PATH is, given an <u>undirected</u> graph and two vertices $s \neq t$, is there a path starting at s and ending at t that visits each and every vertex exactly once?

In class, we showed that HAM-CYCLE is NP-complete. Your task here is to prove the following:

(1) Prove that HAM-PATH is NP-complete by showing either a reduction HAM-CYCLE $\leq_m$ HAM-PATH or a reduction 3-SAT $\leq_m$ HAM-PATH.

(2) Prove that UNDIRECTED-HAM-PATH is NP-complete by showing a reduction HAM-PATH $\leq_m$ UNDIRECTED-HAM-PATH. (*Hint:* Turn each node in the original directed graph into 3 nodes connected in such a way that will simulate direction.)

Task 5: Silver Lining If P = NP (2 points)

Given (an encoding of) a logic circuit C (consisting of only AND, OR, and NOT), the *smallest-possible circuit* (SPC) problem is to check whether C is the smallest-possible circuit with the exact same behavior as C. In other words, C is the smallest-possible circuit if there exist no circuits with fewer gates than C that behave identically to C.

Show that if P = NP, then SPC $\in$ P. (*Hint:* If P = NP, what do we know about coNP?)

Task 6: Longest-probe Bound For Hashing [Extra] (0 points)

Say we have hash table that uses open-addressing. The table has size m and we're to store $n \leq \frac{m}{2}$ keys.

(i) When sufficiently many keys are inserted, collisions are bound to happen. Assuming uniform hashing, show that for $i \geq 1$, the probability that the i-th insertion requires strictly more than k probes is at most $1/2^k$.

(ii) Use this to prove that for $i \geq 1$, the probability is $O(1/n^2)$ that the i-th insertion requires more than $2\log_2 n$ probes.

Task 7: Prime Density [Extra] (0 points)

We'll show that primes are not too rare, providing evidence that if one picks a number at random, it doesn't take all that long to find a prime. In this problem, you will prove that for $N \geq 1$, the number of primes smaller than $2N$ is at least $N/\log_2(2N)$.

1. Let $C_n = \binom{2n}{n}$. Show that for $n \geq 1$, $C_n \geq 2^n$.
2. Consider a positive integer n and a prime p. Let $\xi(a, p)$ be the number of times the prime p divides a—that is, the largest k such that p^k divides a.
   Prove that the number of times the prime p divides $n!$, denoted by $\xi(n!, p)$, is
   $$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$
3. For $N \geq 1$, let p be a prime and $k = \xi(C_N, p)$. Prove that
   $$k = \sum_{i=1}^{\infty} \left(\left\lfloor \frac{2N}{p^i} \right\rfloor - 2 \left\lfloor \frac{N}{p^i} \right\rfloor \right).$$
   Use this to conclude that $p^k \leq 2N$.
4. Let $t(n)$ be the number of primes less than n. Using the ingredients developed so far, prove that $t(2N) \geq \frac{N}{\log_2(2N)}$.