

ICCS310: Assignment 1

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1: Review: Something About Sets

(1) Let A_1, A_2, A_3 be any sets from a universe \mathcal{U} . Prove that $\overline{A_1 \cup A_2 \cup A_3} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

Proof: We want to show that $\overline{A_1 \cup A_2 \cup A_3} \subseteq \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ and $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \subseteq \overline{A_1 \cup A_2 \cup A_3}$.
Let A_1, A_2, A_3 be any three given sets. We'll first prove that $\overline{A_1 \cup A_2 \cup A_3} \subseteq \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$. Let $x \in \overline{A_1 \cup A_2 \cup A_3}$. Then, $x \notin A_1 \cup A_2 \cup A_3$ by the definition of complement, so then $x \notin A_1$, $x \notin A_2$ and $x \notin A_3$, by the definition of union. This means that $x \in \overline{A_1}$, $x \in \overline{A_2}$, and $x \in \overline{A_3}$, by the definition of complement. Hence, $x \in \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ since x is in $\overline{A_1}$, $\overline{A_2}$, and $\overline{A_3}$.

Also, we will show that $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \subseteq \overline{A_1 \cup A_2 \cup A_3}$. Let $y \in \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$, so y is in $\overline{A_1}$, $\overline{A_2}$, and $\overline{A_3}$, by the definition of intersection. This means $y \notin A_1$, $y \notin A_2$, and $y \notin A_3$, by the definition of complement. It follows that $y \notin A_1 \cup A_2 \cup A_3$, and so $y \in \overline{A_1 \cup A_2 \cup A_3}$.

In conclusion, $\overline{A_1 \cup A_2 \cup A_3} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

(2) Let A and B be any sets from a universe \mathcal{U} . Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof: We want to show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Let A and B be any two given sets. We'll first prove that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$. Then, $x \notin A \cup B$ by the definition of complement, so then $x \notin A$, and $x \notin B$, by the definition of union. This means that $x \in \overline{A}$, and $x \in \overline{B}$, by the definition of complement. Hence, $x \in \overline{A} \cap \overline{B}$ since x is in \overline{A} and \overline{B} .

Also, we will show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. Let $y \in \overline{A} \cap \overline{B}$, so y is in \overline{A} and \overline{B} , by the definition of intersection. This means $y \notin A$ and $y \notin B$, by the definition of complement. It follows that $y \notin A \cup B$, and so $y \in \overline{A \cup B}$.

In conclusion, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

2: Prime and Irrational

(1) Let $p \geq 2$ be a prime and a be a positive integer. Prove that if p divides a^2 , then p divides a .

Proof: Using contraposition, we can prove that if p does not divide a , then p does not divide a^2 instead.

Let a be any number that cannot be divided by p , so $a = (p * q) + r$ where $r < p$ and $r, q \in \mathbb{I}^+$. So, we have $a^2 = ((p * q) + r)^2 = (p * q)^2 + 2 * (p * q * r) + r^2$. We can see that r^2 cannot be divided by p . Hence, p does not divide a^2 .

Therefore, if p divides a^2 , then p divides a .

(2) Prove that if p is any positive prime number, then \sqrt{p} is irrational.

Proof: Assume for the sake of contradiction, let \sqrt{p} be a rational number and p is any positive prime number. Then, $\sqrt{p} = a/b$ where a and b are integers and $b \neq 0$.

$$\sqrt{p} = a/b \quad (1)$$

$$p = a^2/b^2 \quad (2)$$

$$p * b^2 = a^2 \quad (3)$$

From observation, p divides a^2 and that means p divide a (Lemma 2.1). Let $a = bq$ for some $q \in \mathbb{I}^+$ and plug it back into equation 3.

$$p * b^2 = p^2 q^2 \quad (4)$$

$$b^2 = p q^2 \quad (5)$$

From observation, p divides b^2 and that means p divide b (Lemma 2.1). Since we showed that $p = a^2/b^2$ but p is a common factor of a and b . We can conclude that the equation is false and our assumption is contradicting. Hence, \sqrt{p} is irrational. Therefore, if p is any positive prime number, then \sqrt{p} is irrational.

3: Spacing

Prove that in any set of $n + 1$ numbers from $\{1, \dots, 2n\}$, there are always two numbers that are consecutive.

Proof: Without loss of generality,

4: Curious Fact about Graphs

Let $G = (V, E)$ be an undirected graph. Show that G contains two nodes that have equal degrees.

Proof: Without loss of generality,

5: Basic DFAs

(1) Prove *Theorem*: for any $n \geq 0$, `Solve_Hanoi` (n , `From_Peg`, `To_Peg`, `Aux_Peg`) generates exactly $2^n - 1$ lines of instruction

Predicate : $P(x) \equiv$ for any $x \geq 0$, `Solve_Hanoi` (x , ...) generates exactly $2^x - 1$ lines of instruction

Base case : $P(0) \equiv$ `Solve_Hanoi` (n , ...) generates exactly 0 lines of instruction which is true

Inductive Steps : Assume that if $P(x)$ is true then $P(x+1)$ is true

$P(x) \equiv$ `Solve_Hanoi` (x , ...) generates exactly $2^x - 1$ lines of instruction

$P(x+1) \equiv$ `Solve_Hanoi` (x , ...) generates exactly $2^{x+1} - 1$ lines of instruction

To show that this is true in mathematically way, $T(x)$ is the number of line generated from the function using recurrence.

$$T(x) = 2T(x-1) + 1; T(1) = 1; T(0) = 0 \quad (6)$$

$$T(x) = 2^{x-1} + \dots + 2 + 1 \quad (7)$$

$$T(x) = 2^x - 1 \quad (8)$$

$$LHS = 2^x - 1; T(x) \quad (9)$$

$$RHS = 2^x - 1 \quad (10)$$

$$LHS \equiv RHS \quad (11)$$

So, $P(x)$ is true, this time we will prove the $P(x+1)$ by using the equations above.

$$T(x+1) = 2T(x) + 1; T(1) = 1; T(0) = 0 \quad (12)$$

$$T(x+1) = 2^x + \dots + 2 + 1 \quad (13)$$

$$T(x+1) = 2^x + 2^x - 1; \quad (14)$$

Using $T(x)$ to solve the equation below

$$T(x+1) = 2^{x+1} - 1 \quad (15)$$

$$LHS = 2^{x+1} - 1; T(x+1) \quad (16)$$

$$RHS = 2^{x+1} - 1; \quad (17)$$

$$LHS \equiv RHS \quad (18)$$

From the induction hypothesis, $P(x-1) \implies P(x)$ and $P(x)$ holds for any $x \geq 0$, `Solve_Hanoi` (x, \dots) generates exactly $2^x - 1$ lines of instruction. Q.E.D.

(3) Prove *printRuler*.

These are the equations we know from this problem

$f(n) = 2f(n-1) + 1, f(0) = 0$ is number of lines

$g(n) = 2g(n-1) + n, g(0) = 0$ is number of dashes

$g(n) = a * f(n) + b * n + c$

1. Basically, I just followed the hint

$$g(0) = a * f(0) + b * 0 + c \quad (19)$$

$$g(0) = c \quad (20)$$

$$g(0) = 0 \quad (21)$$

So, $c = 0$

2. We will find a and b

$$g(n) = 2g(n-1) + n \quad (22)$$

$$a * f(n) + b * n = 2(a * f(n-1) + b * (n-1)) + n \quad (23)$$

$$a * f(n) + b * n = 2a * f(n-1) + 2b * (n-1) + n \quad (24)$$

$$a * f(n) = 2a * f(n-1) + b * n - 2b + n \quad (25)$$

$$a * f(n) - 2a * f(n-1) = b * n - 2b + n \quad (26)$$

Let's do it side by side

$$a(f(n) - 2f(n-1)) = b * n + n - 2b \quad (27)$$

$$a(1) = n * (b+1) - 2b \quad (28)$$

$$a + 2b = n * (b+1) \quad (29)$$

$$a + 2b - n * (b+1) = 0 \quad (30)$$

To find a and b, we know that substitute P and Q = 0 will solve this equation

$$P + Qn = 0 \quad (31)$$

$$a + 2b = P \quad (32)$$

$$(b+1) = Q \quad (33)$$

$$b = -1 \quad (34)$$

$$a = 2 \quad (35)$$

$$g(n) = a * f(n) + b * n \quad (36)$$

$$g(n) = 2 * f(n) - n \quad (37)$$

3. Previously, we got $g(n) = 2 * f(n) - n$

Also, $f(n) = 2^n - 1$. In fact, $g(n) = 2^{n+1} - n - 2$.

4. Theorem : $g(n) = 2^{n+1} - n - 2$ works for all $n \geq 0$

Predicate : $P(x) \equiv g(x) = 2^{x+1} - x - 2$ works for all $x \geq 0$

Base case : $P(0) \equiv g(0) = 0$ is true

Inductive Steps : Assume that if $P(x)$ is true then $P(x+1)$ is true

$P(x) \equiv g(x) = 2^{x+1} - x - 2$

$P(x+1) \equiv g(x+1) = 2^{x+2} - x - 3$

Actually, we know that $g(n) = 2g(n-1) + n$ has a close form of $g(x) = 2^{x+1} - x - 2$ according to what we have done on part 2.

So, $P(x)$ is true, this time we will prove the $P(x+1)$ by using the equations above.

$$g(x+1) = 2^{x+2} - x - 3 \quad (38)$$

$$g(x+1) = 2 * 2^{x+1} - x - 3 \quad (39)$$

$$g(x+1) = 2(g(x) + x + 2) - x - 3 \quad (40)$$

$$g(x+1) = 2g(x) + (x+1) \quad (41)$$

$$LHS = 2^{x+2} - x - 3 \quad (42)$$

$$RHS = 2g(x) + (x+1) \quad (43)$$

$$LHS \equiv RHS \quad (44)$$

From the induction hypothesis, $P(x) \implies P(x+1)$ and $P(x)$ holds for any $x \geq 0$ which will make $g(x) = 2^{x+1} - x - 2$ true. So, $g(x) = 2^{x+1} - x - 2$ works. Q.E.D.

(1) Prove *Theorem*: for any $n \geq 0$, `Solve_Hanoi` (n , `From_Peg`, `To_Peg`, `Aux_Peg`) generates exactly $2^n - 1$ lines of instruction

Predicate : $P(x) \equiv$ for any $x \geq 0$, `Solve_Hanoi` (x , ...) generates exactly $2^x - 1$ lines of instruction

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$P(x) \equiv$ `Solve_Hanoi` (x , ...) generates exactly $2^x - 1$ lines of instruction

$P(x+1) \equiv$ `Solve_Hanoi` (x , ...) generates exactly $2^{x+1} - 1$ lines of instruction

7: Digit Sum

(1) Prove *Theorem*: for any $n \geq 0$, `Solve_Hanoi` (n , `From_Peg`, `To_Peg`, `Aux_Peg`) generates exactly $2^n - 1$ lines of instruction

Predicate : $P(x) \equiv$ for any $x \geq 0$, `Solve_Hanoi` (x , ...) generates exactly $2^x - 1$ lines of instruction

Base case : $P(0) \equiv$ `Solve_Hanoi` (n , ...) generates exactly 0 lines of instruction which is true

Inductive Steps : Assume that if $P(x)$ is true then $P(x+1)$ is true

$P(x) \equiv$ `Solve_Hanoi` (x , ...) generates exactly $2^x - 1$ lines of instruction

$P(x+1) \equiv$ `Solve_Hanoi` (x , ...) generates exactly $2^{x+1} - 1$ lines of instruction