ICCS310: Assignment 5

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1: Reject TM

$$\mathsf{REJECT_{TM}} = \{\langle M, x\rangle |\ \text{M is a TM that rejects input}\}$$

Show directly (i.e., without resorting to reduction) that $\mathsf{REJECT_{TM}}$ is undecidable.

Proof:

We want to show that $\mathsf{REJECT_{TM}}$ is undecidable by showing the contradiction.

Assume for the sake of contradiction that the TM $W(\langle M, w\rangle)$ decides $\mathsf{REJECT_{TM}}$.

$$W(\langle M, w\rangle) = \begin{cases} \text{reject} & \text{if } M \text{ accepts } w \\ \text{accept} & \text{if } M \text{ rejects } w \\ \text{reject} & \text{if } M \text{ loops on } w \end{cases}$$

Also, define TM $Y(\langle X\rangle)$ where X is a TM. On input $\langle X\rangle$,

1. Run W on $\langle X, \langle X\rangle\rangle$.

2. If W accepts, we accept. If W rejects, we rejects.

So we have,

$$Y(\langle X\rangle) = \begin{cases} \text{reject} & \text{if } X \text{ accepts } \langle X\rangle \\ \text{accept} & \text{if } X \text{ rejects } \langle X\rangle \\ \text{reject} & \text{if } X \text{ loops on } \langle X\rangle \end{cases}$$

Then,

$$Y(\langle Y\rangle) = \begin{cases} \text{reject} & \text{if } Y \text{ accepts } Y \\ \text{accept} & \text{if } Y \text{ rejects } Y \\ \text{reject} & \text{if } Y \text{ loops on } Y \end{cases}$$

We have that whatever Y try to decides on $\langle Y\rangle$ turns out to be that there is no certain answer. Since Y cannot decides, then W also cannot decides. Hence, it contradicts to our assumption that W decides $\mathsf{REJECT_{TM}}$. Therefore, $\mathsf{REJECT_{TM}}$ is undecidable. $\square$

2: Accept vs. Reject

$$\mathsf{ACCEPT_{TM}} = \{\langle M, x\rangle |\ \text{M is a TM that accepts input x}\}$$

(i) Prove that $\mathsf{ACCEPT_{TM}} \leq \mathsf{REJECT_{TM}}$

Proof:

Suppose that TM M_{REJECT} decides $\mathsf{REJECT_{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT_{TM}}$, we want to show how to decide $\mathsf{ACCEPT_{TM}}$ using M_{REJECT}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M by reversing the accept and reject states.

2. Run M_{REJECT} with $\langle M', w\rangle$.

3. If M_{REJECT} rejects, we accept. If M_{REJECT} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w. Hence, M_{ACCEPT} can correctly decide $\mathsf{ACCEPT}_{\mathsf{TM}}$ provided that there is a TM M_{REJECT}. Therefore, $\mathsf{ACCEPT}_{\mathsf{TM}} \leq \mathsf{REJECT}_{\mathsf{TM}}$. $\square$

(ii)) Prove that $\mathsf{REJECT}_{\mathsf{TM}} \leq \mathsf{ACCEPT}_{\mathsf{TM}}$

Proof:

Suppose that TM M_{REJECT} decides $\mathsf{REJECT}_{\mathsf{TM}}$ and TM M_{ACCEPT} decides $\mathsf{ACCEPT}_{\mathsf{TM}}$, we want to show how to decide $\mathsf{REJECT}_{\mathsf{TM}}$ using M_{ACCEPT}.

Given $\langle M, w\rangle$ as input:

1. Make TM M' from M by reversing the accept and reject states.

2. Run M_{ACCEPT} with $\langle M', w\rangle$.

3. If M_{ACCEPT} accepts, we accept. If M_{ACCEPT} rejects, we reject.

Notice that this mechanism accepts if and only if M accepts w and rejects if and only if M' rejects w. Hence, M_{REJECT} can correctly decide $\mathsf{REJECT}_{\mathsf{TM}}$ provided that there is a TM M_{ACCEPT}. Therefore, $\mathsf{REJECT}_{\mathsf{TM}} \leq \mathsf{ACCEPT}_{\mathsf{TM}}$. $\square$

3: Reverse on TM

$$\mathsf{T} = \{\langle M\rangle |\ \text{M is a TM that accepts } \mathbf{rev}(w) \text{ whenever it accepts}\}$$

where $\mathbf{rev}(w)$ is the reverse of the string w. Show that T is undecidable.

Proof:

4: Undecidability

(i) Show that

$$\mathsf{TOTAL} = \{\langle M\rangle |\ \text{M is a Turing machine that halts on every input}\}$$

is undecidable

Proof:

(ii) Show that

$$\mathsf{FINITE} = \{\langle M\rangle |\ \text{M is a Turing machine and } L(M) \text{ is a finite set}\}$$

is undecidable

Proof:

(iii) Show that

$$\mathsf{REGULAR} = \{\langle M\rangle |\ \text{M is a Turing machine and } L(M) \text{ is regular}\}$$

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where $\mathbf{rev}(w)$ is the reverse of the string w. Show that T is undecidable.

Proof:

5: Total Is No Harder Than Finite

Prove that

$$\mathsf{TOTAL} \leq_T \mathsf{FINITE}$$

is undecidable

Proof:

6: Finite Is No Harder Than Total

Prove that

$$\mathsf{FINITE} \leq_T \mathsf{TOTAL}$$

is undecidable

Proof:

7: Extra: Undecidability of Nontrivial Properties

Proof: It is non trivial. How to decide on it though?