Homework 4 solutions

S520

Due at the beginning of class, Thursday 11th February

Trosset question numbers refer to the hardcover textbook. Show all working and give R code where appropriate.

1. *Let X be a random variable with PDF*

$$f(x) = \begin{cases} \frac{1}{30} & 0 \le x < 20 \\ \frac{1}{60} & 20 \le x < 40 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) *Find the CDF of X, $F(y)$, for all y.*

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{30} & 0 \le y < 20 \\ \frac{y+20}{60} & 20 \le y < 40 \\ 1 & y \ge 40. \end{cases}$$

   (b) *Find y such that $F(y) = 0.5$. (That is, set the correct piece of $F(y)$ equal to zero, and solve for y.) Is this larger than, smaller than, or the same as EX?*

   To meet this condition, $y/30 = 0.5$, so $y = 15$. Intuitively, the expected value will be pulled upwards by the possibility of large values, so 15 will be smaller than the expected value. To (optionally) show this formally:

$$\begin{aligned} EX &= \int_0^{40} x \cdot f(x)\, dx \\ &= \int_0^{20} \frac{1}{30} x\, dx + \int_{20}^{40} \frac{1}{60} x\, dx \\ &= \left[x^2/60\right]_0^{20} + \left[x^2/120\right]_{20}^{40} \\ &= (20/3 - 0) + (40/3 - 10/3) \\ &= 50/3. \end{aligned}$$

   In any case, y is smaller.

2. (a) *Suppose that buses go past my stop exactly 30 minutes apart. I arrive at the stop at a completely random time during the day. What is the expected length of time I will have to wait for a bus?*

   The length of time has a Uniform$[0, 30)$ distribution, so the expected value is 15 minutes.

(b) *Suppose that buses go past my father's stop at exactly ten minutes past the hour and thirty minutes past the hour (e.g. 9:10, 9:30) every hour. My father arrives at his stop at a completely random time during the day. What is the expected length of time he will have to wait for a bus?*

Thinking about the PDF, there's a 2/3 chance the waiting time will be between 0 and 20 minutes, and a 1/3 chance the time will be between 20 and 40 minutes. So the expected waiting time is $(2/3 \times 10) + (1/3 \times 30) = 16\ 2/3$ minutes. (Alternatively, note that the distribution here is the same as in the previous question, so $EX = 50/3$ as before.)

3. Trosset exercise 5.6.4. (Part (e) is worth one point of extra credit.)

   (a) X measures the distance that a point in a disc of radius 1 lies from its center, so $0 \leq X \leq 1$.

   (b) Let A_1 denote the concentric disc of radius 0.5. Then

   $$P(X \leq 0.5) = P(A_1) = \frac{\text{area}(A_1)}{\pi} = \frac{\pi 0.5^2}{\pi} = 0.25.$$

   (c) Let A_1 denote the concentric disc of radius 0.7. Then

   $$P(X \leq 0.7) = P(A_2) = \frac{\text{area}(A_2)}{\pi} = \frac{\pi 0.7^2}{\pi} = 0.49$$

   and

   $$P(0.5 < X \leq 0.7) = P(A_2) - P(A_1) = 0.49 = 0.25 = 0.24.$$

   (d)

   $$F(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

   (e) The pdf is the derivative of the cdf.

   $$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

4. *Trosset exercise 5.6.7 (use R and give code)*

   (a) `pnorm(0, mean=-5, sd=10)` = 69%

   (b) `1 - pnorm(5, mean=-5, sd=10)` = 16%

   (c) `pnorm(7, mean=-5, sd=10) - pnorm(-3, mean=-5, sd=10)` = 31%

   (d) `pnorm(5, mean=-5, sd=10) - pnorm(-15, mean=-5, sd=10)` = 68%

   (e) `pnorm(1, mean=-5, sd=10) + 1 - pnorm(5, mean=-5, sd=10)` = 88%

5. Trosset exercise 5.6.8

   (a) Expected value 4, variance 25

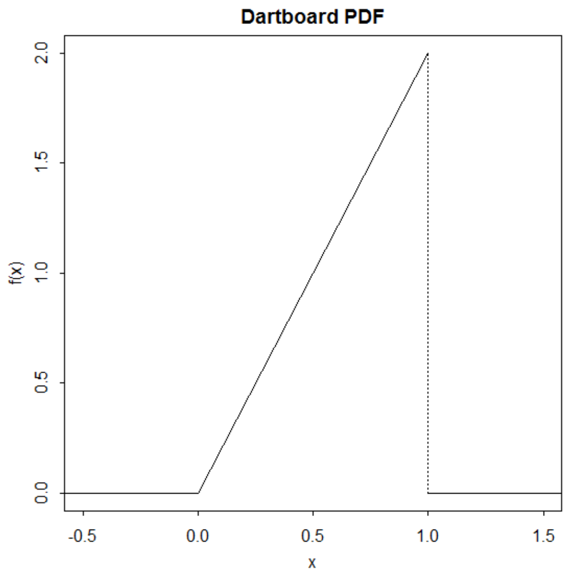

Figure 1: PDF for Trosset exercise 5.6.4.

(b) Expected value -3, variance 16

(c) Expected value -2, variance 25

(d) Expected value 2, variance 36

(e) Expected value -4, variance 100

6. *Let X_1, X_2, X_3, and X_4 be independent standard normal random variables. Let $\bar{X}$ be the mean of X_1 to X_4:*

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4}{4}.$$

Note that because $X_1, \ldots, X_4$ are random, $\bar{X}$ is also a normal random variable.

(a) *Using R, find $P(X_1 > 1.96)$.*

`1 - pnorm(1.96)` $= 0.025$.

(b) *Using the binomial in conjunction with your answer in (a), find the probability that at least two of the random variables X_1, X_2, X_3, X_4 are greater than 1.96.*

```
> p = 1 - pnorm(1.96)
> 1 - pbinom(1, 4, p)
[1] 0.003625572
```

(c) *Find $P(\bar{X} > 1.96)$.*

Let S be the sum of the four random variables. $P(\bar{X} > 1.96)$ is the same as $P(S > 7.84)$. S has a normal distribution with mean 0 and variance $4(= 2^2)$, so the required probability is

```
> 1 - pnorm(7.84, sd=2)
[1] 4.427448e-05
```

i.e. pretty unlikely.