Homework 3 solutions

S520

Due at the beginning of class, Thursday 4th February

Trosset question numbers refer to the hardcover textbook. Show all working.

1. Trosset exercise 4.5.10

   Each of the $n = 12$ attendees represents a Bernoulli trial. The possible outcomes are attendance and nonattendance. If we designate attendance as success and nonattendance as failure, then the probability of success is $p = 0.5 \cdot 0.8 = 0.4$. Let Y denote the observed number of successes, so that $Y \sim \text{Binomial}(12, 0.4)$. Then

$$\begin{aligned} P(Y > 7) &= 1 - P(Y \leq 7) \\ &= \texttt{1 - pbinom(7, 12, 0.4)} \\ &= 0.057 = 5.7\%. \end{aligned}$$

2. Trosset exercise 4.5.14

   (a) Each attempt to send/receive a symbol is a Bernoulli trial. There are 5 symbols, so the probability of success is $p = 0.2$. There are $n = 25$ trials, so the expected number of success is $np = 5$.

   (b) Let Y denote the number of successes. Then $Y \sim \text{Binomial}(25, 0.2)$ and

$$\begin{aligned} P(Y > 7) &= 1 - P(Y \leq 7) \\ &= \texttt{1 - pbinom(7, 25, 0.2)} \\ &= 0.1091 = 10.9\% \end{aligned}$$

   (c) Let E denote the event that at least one of the 20 receivers will attain a score indicative of ESP. Then E^c is the event all of the receivers attain scores of ≤ 7 matches, so

$$\begin{aligned} P(E) &= 1 - P(E^c) \\ &= 1 - (1 - 0.1091)^{20} \\ &= 0.901 = 90.1\% \end{aligned}$$

3. Trosset exercise 5.6.2

   (b) $f(x) \geq 0$ for all x and the area of the triangle is 1, so f is a pdf.

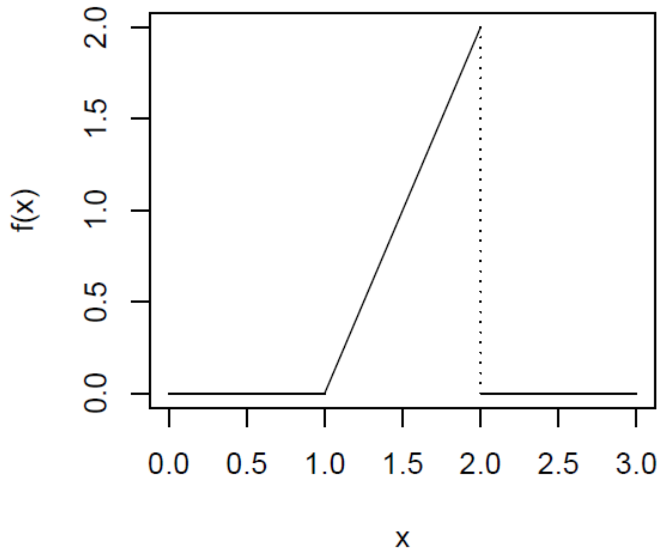
(a) 

Figure 1: pdf for Exercise 5.6.2.

(c) This is the difference between the areas of two triangles, one with base 0.5 and height 1, and one with base 0.75 and height 1.5.

$$\begin{aligned} P(1.5 < X < 1.75) &= P(X < 1.75) - P(X < 1.5) \\ &= (0.5 \times 0.75 \times 1.5) - (0.5 \times 0.5 \times 1) \\ &= \frac{5}{16} = 0.3125. \end{aligned}$$

4. Trosset exercise 5.6.3

(a) The pdf is plotted in Figure 2. c must be nonnegative for f to be a pdf. The total area under f is the sum of two triangles:

$$(0.5 \times 1.5 \times 1.5c) + (0.5 \times 1.5 \times 1.5c) = \frac{9}{4}c.$$

This has to equal 1 for a pdf, so

$$\begin{aligned} \frac{9}{4}c &= 1 \\ c &= \frac{4}{9}. \end{aligned}$$

(b) Looking at Figure 2, it's evident the f is symmetric about $x = 1.5$. So the expected value of X must be 1.5.

(c) $P(X > 2)$ is the area under the pdf between 2 and 3, which is the area of a triangle. The base of the triangle is $3 - 2 = 1$ and the height of the triangle is $f(2) = c = 4/9$. The area is $1/2 \times 1 \times 4/9 = 2/9$.

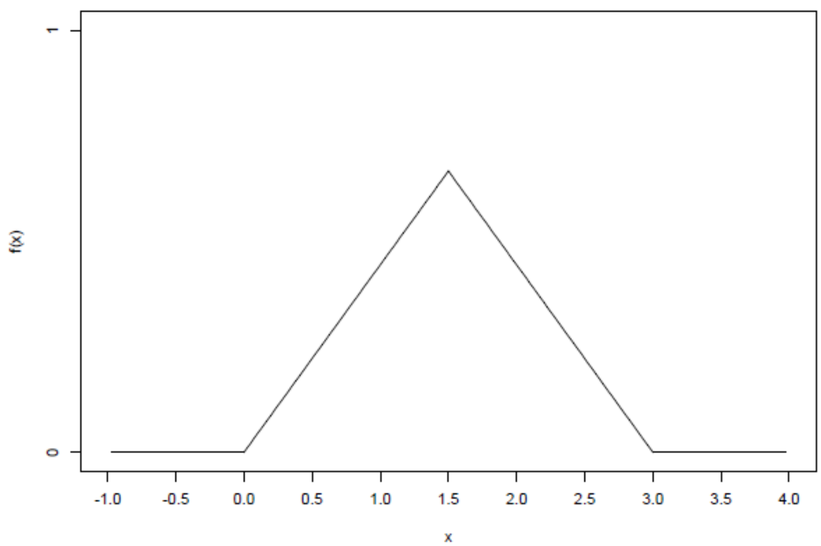


Figure 2: pdf for Exercise 5.6.3.

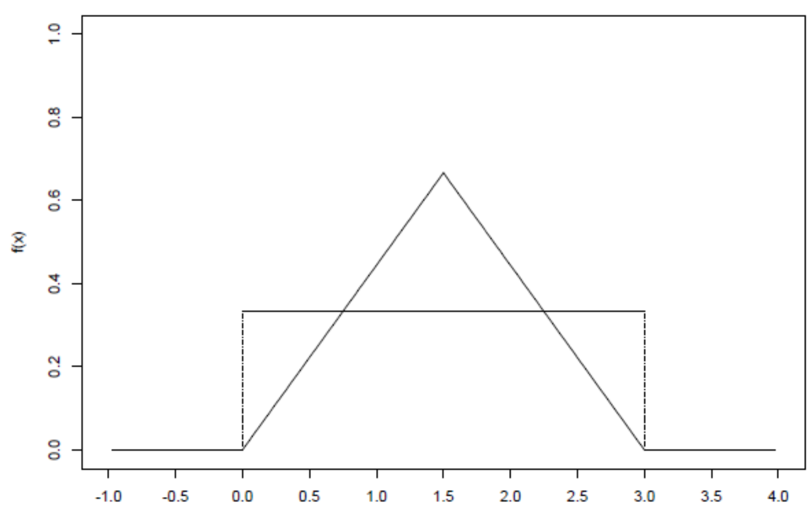


Figure 3: pdfs for Exercise 5.6.3(d).

(d) Figure 3 plots the two pdfs on top of each other. Both have the same expected value: $EX = EY = 1.5$. However, the values of X tend to cluster a bit nearer 1.5 than do the values of X. So Y has the larger variance.

(e) Firstly, if $y < 0$, then $F(y) = 0$, and if $y > 3$, then $F(y) = 1$.
If $0 \le y \le 1.5$, then $F(y)$ is the area of a triangle:

$$\begin{aligned} F(y) &= P(X \le y) \\ &= \frac{1}{2} \cdot y \cdot cy \\ &= \frac{2y^2}{9}. \end{aligned}$$

If $1.5 \le y \le 3$, then $F(y)$ is one minus the area of a triangle. The base of the triangle is $3 - y$ and the height is $c(3 - y)$.

$$\begin{aligned} F(y) &= 1 - P(X > y) \\ &= 1 - \frac{1}{2} \cdot (3 - y) \cdot c(3 - y) \\ &= 1 - \frac{c}{2}(3 - y)^2 \\ &= 1 - \frac{2}{9}(3 - y)^2 \end{aligned}$$

One way of writing all of this down formally is:

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{2y^2}{9} & 0 \le y < 1.5 \\ 1 - \frac{2}{9}(3 - y)^2 & 1.5 \le y < 3 \\ 1 & y \ge 3 \end{cases}.$$

5. *Consider an unfair six-sided die. Let X be a discrete random variable representing the result of a roll of the die. The probability mass function of X is*

$$f(x) = \begin{cases} 0.1 & x = 1 \\ 0.1 & x = 2 \\ 0.3 & x = 3 \\ 0.3 & x = 4 \\ 0.1 & x = 5 \\ 0.1 & x = 6 \\ 0 & \text{otherwise.} \end{cases}$$

(a) *Find $F(x)$, the cumulative distribution function of X, for all $x \in (-\infty, \infty)$.*

$$F(y) = \begin{cases} 0 & x < 1 \\ 0.1 & 1 \leq y < 2 \\ 0.2 & 2 \leq y < 3 \\ 0.5 & 3 \leq y < 4 \\ 0.8 & 4 \leq y < 5 \\ 0.9 & 5 \leq y < 6 \\ 1 & x \geq 6 \end{cases}$$

(b) *Find the expected value and the variance of X.*
$EX = 1 \cdot 0.1 + 2 \cdot 0.1 + 3 \cdot 0.3 + 4 \cdot 0.3 + 5 \cdot 0.1 + 6 \cdot 0.1 = 3.5$
$EX^2 = 1^2 \cdot 0.1 + 2^2 \cdot 0.1 + 3^2 \cdot 0.3 + 4^2 \cdot 0.3 + 5^2 \cdot 0.1 + 6^2 \cdot 0.1 = 14.1$
$\operatorname{Var} X = EX^2 - (EX)^2 = 14.1 - 3.5^2 = 1.85$

(c) *Suppose I roll the die ten times (all independently.) Let Y be the sum of the ten die rolls. What are the expected value and the variance of Y?*
$EY = EX_1 + EX_2 + \cdots + EX_{10} = (10 \times 3.5) = 35$
$\operatorname{Var} Y = \operatorname{Var} X_1 + \operatorname{Var} X_2 + \cdots + \operatorname{Var} X_{10} = (10 \times \operatorname{Var} X) = 18.5$

6. Let X be a continuous random variable with probability density function (PDF)

$$f(x) = \begin{cases} 2k & 0 \leq x < 3 \\ 3k & 3 \leq x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant.

(a) Find k.

The pdf consists of two rectangles, one with area $6k$ and the other also with area $6k$. The total area is 1, so $12k = 1$, so $k = 1/12$.

(b) Find $F(4)$, the cumulative distribution function at $x = 4$.

We wish to find $P(X \leq 4)$. Split this up into two parts: $P(0 < X < 3)$ and $P(3 < X < 4)$. $P(0 < X < 3)$ is the area of a rectangle with width 3 and height $2k$, giving area $6k$. $P(3 < X < 4)$ is the area of a rectangle with width 1 and height $3k$, giving area $3k$. So the total probability is $9k$, which is $3/4$.

(c) Find the expected value of X.

Half the time we have a variable uniform on $[0, 3)$, and the other half the time we have a variable uniform on $[3, 5)$. A variable uniform on $[0, 3)$ has an expected value of 1.5, while a variable uniform on $[3, 5)$ has an expected value of 4. The overall expected value is halfway in between: $(1.5 + 4)/2 = 2.75$.