Thermal Convection Chebyshev Code

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1 Governing Equations

We will solve for the vertical velocity $v\left(x,y\right)$, the horizontal velocity $u\left(x,y\right) = \overline{u}\left(y\right) + u\left(x,y\right)$ (decomposed into the horizontal mean $\overline{u}\left(y\right)$ and the fluctuations about the mean $u\left(x,y\right)$. We will also ultimately solve for the temperature field $T\left(x,y\right)$ which is the sum of the conduction solution and the departure from the conduction solution (see below). The nondimensional equations are

$$\frac{\partial \nabla^2 v}{\partial t} + \partial_x \left(u\nabla^2 v - v\nabla^2 u\right) = \nu_0 \nabla^2\nabla^2 v \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa_0 \nabla^2 T \tag{2}$$

$$\partial_x u + \partial_y v = 0 \tag{3}$$

$$\frac{\partial \overline{u}}{\partial t} + \partial_y \overline{uv} = \nu_0 {\partial_y}^2 \overline{u}. \tag{4}$$

with boundary conditions

$$v = \partial_y v = 0, \qquad y = \pm 1 \tag{5}$$

$$u\left(\pm 1\right) = 0, \quad T\left(\pm 1\right) = \mp 1. \tag{6}$$

We will solve for the temperature departure from the conduction state. In dimensionless units the conduction solution is $T_c = -y$. The departure from the conduction state is $\theta = T + y$. The equation and boundary conditions governing θ are

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta - v = \kappa_0 \nabla^2\theta \tag{7}$$

$$\theta\left(\pm 1\right) = 0. \tag{8}$$

We shall use $\mathcal{N}$ to denote the nonlinear terms in the temperature equation.

2 Spatial Discretization

We use a Fourier basis in the x-direction and a Chebyshev basis in the y-direction.

2.1 x-Discretization

Taking the Fourier transform in the x-direction of the temperature equation gives

$$\partial_t \widehat{\theta}_l + \widehat{\mathcal{N}}_l - \widehat{v}_l = \kappa_0 \left(- (\alpha k_x)^2 \widehat{\theta}_l + \partial_y^2 \widehat{\theta}_l \right) \tag{9}$$

where αk_x is the wavenumber.

The continuity equation becomes,

$$i (\alpha k_x) \widehat{u}_l + \partial_y \widehat{v} = 0. \tag{10}$$

Note that we do not take the Fourier transform of the mean flow equation because it is has no horizontal component.

2.2 y-Discretization

We use Chebyshev expansions in the y-direction following very closely Wally's Chebyshev approach. We let

$$\partial_y^2 \widehat{\theta}_l = \sum_{m=0}^{N_C - 1} \widehat{b}_{ml} T_m (y) = \mathbf{T}_N (y) \widehat{\mathbf{b}}_l. \tag{11}$$

Note that $\mathbf{P}_T$ is the projection matrix from physical to Chebyshev space. We have $\mathbf{P}_T \mathbf{T}_N = \mathbf{I}$. Following Wally's Chebyshev "integration" method (using the hybrid approach) we have

$$\partial_y \widehat{\theta}_l = D\mathbf{\Psi} (y) \widehat{\mathbf{b}}_l \tag{12}$$

$$\widehat{\theta}_l = \mathbf{\Psi}_N (y) \widehat{\mathbf{b}}_l. \tag{13}$$

Note that the matrices $\mathbf{\Psi} (y)$ have the boundary conditions built in automatically (check their form!). These are the Galerkin modes in the code.

Similarly, for the velocity fields we have

$$\partial_y^4 \widehat{v}_l = \mathbf{T}_N (y) \widehat{\mathbf{a}}_l \tag{14}$$

$$\partial_y^2 \overline{u} = \mathbf{T}_N (y) \mathbf{c}. \tag{15}$$

This gives,

$$\partial_y^3 \widehat{v}_l = D3\mathbf{V} (y) \widehat{\mathbf{a}}_l \tag{16}$$

$$\partial_y^2 \widehat{v}_l = D2\mathbf{V} (y) \widehat{\mathbf{a}}_l \tag{17}$$

$$\partial_y \widehat{v}_l = D\mathbf{V} (y) \widehat{\mathbf{a}}_l \tag{18}$$

$$\widehat{v}_l = \mathbf{V} (y) \widehat{\mathbf{a}}_l \tag{19}$$

and

$$\partial_y \overline{u} = D\mathbf{U} (y) \mathbf{c} \tag{20}$$

$$\overline{u} = \mathbf{U} (y) \mathbf{c}. \tag{21}$$

where, once again, the Galerkin modes $\mathbf{V}$ and $\mathbf{U}$ have the appropriate boundary conditions built in.

Using the Chebyshev expansions in (9) gives

$$\mathbf{\Psi}\left(y\right)\partial_t\widehat{\mathbf{b}}_l+\widehat{\mathcal{N}}_l-\mathbf{V}\left(y\right)\widehat{\mathbf{a}}_l=\kappa_0\left(-\left(\alpha k_x\right)^2\mathbf{\Psi}\left(y\right)\widehat{\mathbf{b}}_l+\mathbf{T}\left(y\right)\widehat{\mathbf{b}}_l\right). \tag{22}$$

Finally, we project into Chebyshev space,

$$\mathbf{P}_T\mathbf{\Psi}\left(y\right)\partial_t\widehat{\mathbf{b}}_l+\mathbf{P}_T\widehat{\mathcal{N}}_l-\mathbf{P}_T\mathbf{V}\left(y\right)\widehat{\mathbf{a}}_l=\kappa_0\left(-\left(\alpha k_x\right)^2\mathbf{P}_T\mathbf{\Psi}\left(y\right)+\mathbf{I}\right)\widehat{\mathbf{b}}_l. \tag{23}$$

Note that we will only use the continuity equation to solve for the horizontal velocity perturbation. Further note that the horizontal velocity perturbation only appears in the nonlinear terms in each equation. The nonlinear terms are computed in physical space (since we're using a pseudospectral method) and so we never actually need the horizontal velocity perturbation in Chebyshev space. With this in mind, we proceed to compute the relevant derivatives of the horizontal velocity perturbation that will be used in the nonlinear terms in each of the main evolution equations (temperature, vertical velocity, and mean flow). We have,

$$\partial_x u+\partial_y v=0. \tag{24}$$

Taking the Fourier transform gives

$$\widehat{u}_l\left(y\right)=\frac{i}{\alpha k_x}\partial_y\widehat{v}_l\left(y\right). \tag{25}$$

Expanding the y-derivative of v in Chebyshev modes leads to,

$$\widehat{u}_l\left(y\right)=\frac{i}{\alpha k_x}D\mathbf{V}\left(y\right)\widehat{\mathbf{a}}_l. \tag{26}$$

We will also need the first derivative in y of the horizontal velocity perturbation. We begin by taking the y-derivative of the continuity equation,

$$\partial_x\partial_y u+\partial_y^2 v=0. \tag{27}$$

After taking the Fourier transform we get,

$$\partial_y\widehat{u}_l=\frac{i}{\alpha k_x}\partial_y^2\widehat{v}_l. \tag{28}$$

Expanding in Chebyshev modes results in

$$\partial_y\widehat{u}_l\left(y\right)=\frac{i}{\alpha k_x}D2\mathbf{V}\left(y\right)\widehat{\mathbf{a}}_l. \tag{29}$$

The mean flow equation (4) using the Chebyshev expansions becomes,

$$\mathbf{U}\left(y\right)\partial_t\mathbf{c}+\mathcal{N}_u=\nu_0\mathbf{T}_N\left(y\right)\mathbf{c} \tag{30}$$

where $\mathcal{N}_u=\partial_y\overline{uv}$.

After projection into Chebyshev space, we have,

$$\mathbf{P}_T\mathbf{U}\left(y\right)\partial_t\mathbf{c}+\mathbf{P}_T\mathcal{N}_u=\nu_0\mathbf{I}\mathbf{c}. \tag{31}$$

Our notation regarding the nonlinear term is a bit strange. However, we shall discuss how this term is computed which will shed some light on our notation of choice. First, we rewrite the nonlinear term as

$$\partial_y\overline{uv}=\overline{u\partial_y v}+\overline{v\partial_y u}. \tag{32}$$

Note that in the code each term is in Fourier space. Here are some steps for computing the nonlinear term:

1. The first step is to inverse transform from wavenumber space to x-space.

2. Then, using (26) and (29) we can bring each term into physical space.

3. Next compute the average in the x-direction.

4. At this point, we have computed the nonlinear term in physical space. The last step is to bring it back into Chebyshev space. Note that it is not necessary to do a Fourier transform because we have averaged the horizontal direction out.

3 Temporal Discretization

4 Verification and Validation (V&V)

In this section we summarize some of the tests that we performed to verify our code. We primarily focus on code verification at this point rather than validation. At some point, we would need to do some validation testing as well.

4.1 Mean-Flow Implementation

We ran a simulation with $\overline{u}\left(y, t=0\right) = 0$ and turned off the nonlinear term. The test here is to ensure that no mean flow develops. If a mean flow does develop then we have inadvertently introduced a source into the mean flow equation. When running this test, the mean flow remained zero for all time.

The next set of tests involved running a simulation with $\overline{u}\left(y, t=0\right) = \overline{u}_0\left(y\right)$ and no nonlinear term. In this situation, we have an analytical solution and we can compare our analytical solution to the numerical solution. We also do a convergence study. First we did a mesh refinement by factors of 2 in the y-direction and then we did a mesh refinement by factors of 2 in the x-direction. In each case we plotted the L_2 error as a function of mesh parameter to make sure that we achieved the correct convergence rate.