FALL 2018

Test 2

MATH 1042

Department of Mathematics
Temple University

November 13, 2018

Name: Solutions

Instructor/Section: ______________________

This exam consists of 12 questions. Show all your work. **No work, no credit.** Good Luck!

Question	Points	Out of
1		**5**
2		**10**
3		**10**
4		**11**
5 a		**16**
6		**6**
7		**6**
8		**7**
9		**7**
10		**7**
11ab		**10**
11cd		**10**
Total		**105**

5pt **1.** Make a trigonometric substitution in the integral and simplify the integrand.
DO NOT EVALUATE the integral.

$$\int \frac{x^4}{\sqrt{9+x^2}}\,dx = \int \frac{x^4}{3\sqrt{1+(\frac{x}{3})^2}}\,dx$$

$\frac{x}{3} = \tan\theta$

$x = 3\tan\theta \longleftrightarrow x^4 = 81\tan^4\theta$

$dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{81\tan^4\theta(\cancel{3}\sec^2\theta)\,d\theta}{\cancel{3}\sqrt{1+\tan^2\theta}}$$

$$= \int \frac{81\tan^4\theta\sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sec}\theta} = \boxed{81\int\tan^4\theta\sec\theta\,d\theta}$$

10pt **2.** Evaluate the integral.

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{2x^2\sqrt{1-(\frac{x}{2})^2}}\,dx$$

$\frac{x}{2} = \sin\theta$

$x = 2\sin\theta,\quad x^2 = 4\sin^2\theta$

$dx = 2\cos\theta\,d\theta$

$$= \int \frac{\cancel{2}\cos\theta\,d\theta}{\cancel{2}(4\sin^2\theta)\sqrt{1-\sin^2\theta}}$$

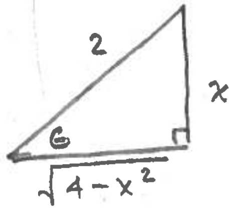
$$= \int \frac{\cancel{\cos}\theta\,d\theta}{4\sin^2\theta\sqrt{\cancel{\cos^2}\theta}}$$

$$= \frac{1}{4}\int\csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x}$$

$$= \boxed{-\frac{\sqrt{4-x^2}}{4x} + C}$$



10pt **3.** Write out the form of the partial fraction decomposition of the functions below.

DO NOT determine the numerical values of the coefficients.

(a) $f(x) = \frac{x^3}{\underbrace{x^2-1}_{\text{Improper}}}$

$$\begin{array}{r} x \\ x^2-1 \,\big)\, x^3 \\ \underline{x^3 - x} \\ x \end{array}$$

$$f(x) = x + \frac{x}{x^2-1}$$

$$= x + \frac{A}{x+1} + \frac{B}{x-1}$$

(b) $g(x) = \frac{3x^3 - x - 2}{(x+2)^2(x^2+3)^2}$. $= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+3} + \frac{Ex+F}{(x^2+3)^2}$

11pt **4.** Evaluate the integral

$$\int \frac{x^2+17}{(x+3)(x^2+4)}\,dx \quad = I$$

$$\frac{x^2+17}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$$

$$x^2+17 = A(x^2+4) + (Bx+c)(x+3)$$

$x=-3:$ $\quad 9+17 = 13A + 0 \quad \longleftrightarrow \quad 13A = 26 \longleftrightarrow \boxed{A=2}$

$$x^2+17 = Ax^2 + 4A + Bx^2 + (3B+C)x + 3C$$

$$x^2+17 = (A+B)x^2 + (3B+c)x + 4A + 3C$$

$A+B=1$

$2+B=1$

$\boxed{B=-1}$

$3B+C=0$

$-3+c=0$

$\boxed{c=3}$

$$I = \int \frac{2\,dx}{x+3} + \int \frac{3-x}{x^2+4}\,dx$$

$$= 2\int \frac{dx}{x+3} + 3\int \frac{dx}{x^2+4} - \int \frac{x}{x^2+4}\,dx$$

$$= 2\ln|x+3| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln(x^2+4) + C$$

$$\int \frac{dx}{x^2+4}$$

$$= \frac{1}{4}\int \frac{dx}{(\frac{x}{2})^2+1}$$

$u = \frac{x}{2}$

$du = \frac{1}{2}dx$

$$= \frac{1}{2}\int \frac{du}{u^2+1}$$

$$= \frac{1}{2}\tan^{-1}u$$

$$\int \frac{x\,dx}{x^2+4}$$

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u|$$

$\boxed{\text{16pt}}$ **5.** Write each improper integral below as a limit of proper integrals. Determine whether each integral is convergent or divergent. Evaluate the integral if it is convergent.

(a) $\displaystyle\int_{-1}^{8} \frac{2}{\sqrt[3]{x+1}}\,dx$

$$= \lim_{a\to -1^+} \int_a^8 \frac{2}{\sqrt[3]{x+1}}\,dx$$

$$\int \frac{dx}{\sqrt[3]{x+1}} = \int (x+1)^{-1/3}\,dx$$

$$= \frac{3}{2}(x+1)^{2/3} + c$$

$$= 2\lim_{a\to -1^+} \int_a^8 \frac{dx}{\sqrt[3]{x+1}}$$

$$= 3\lim_{a\to -1^+} (x+1)^{2/3}\Big|_a^8 = 3\left[(8+1)^{2/3} - \lim_{a\to -1^+}(a+1)^{2/3}\right]$$

$$= 3\left(9^{2/3} - 0\right)$$

$$= 3\sqrt[3]{81}$$ Converges!

(b) $\displaystyle\int_{-\infty}^{0} t e^{-t}\,dt$

$$= \lim_{a\to -\infty} \int_a^0 te^{-t}\,dt$$

$$\int te^{-t}\,dt$$

$$\begin{array}{ccc} \underline{u} & & \underline{dv} \\ & & e^{-t} \\ t & \xrightarrow{+} & -e^{-t} \\ 1 & \xrightarrow{-} & e^{-t} \end{array}$$

$$= -te^{-t} - e^{-t}$$

$$= -e^{-t}(t+1)$$

$$= -\lim_{a\to -\infty} e^{-t}(t+1)\Big|_a^0$$

$$= -\left[e^0(0+1) - \lim_{a\to -\infty} e^{-a}(a+1)\right]$$

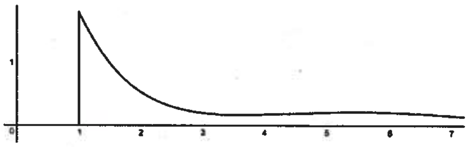
$$= -1 + \underbrace{\lim_{a\to -\infty} e^{-a}(a+1)}_{=\infty(-\infty)}$$

$$= -\infty$$

Diverges to $-\infty$.

6pt **6.** The region shown below lies above the x-axis and below the curve $y = \frac{2+\cos(x)}{\sqrt{x^3+1}}$ for $x \geq 1$.

(a) Express the area of the enclosed region as an improper integral.



$$A = \int_1^{\infty} \frac{2+\cos(x)}{\sqrt{x^3+1}}\,dx$$

$$\frac{2+\cos x}{\sqrt{x^3+1}} \leq \frac{3}{\sqrt{x^3}} = \frac{3}{x^{3/2}}$$

$$\int_1^{\infty} \frac{3}{x^{3/2}}\,dx$$

converges

(b) Use the **Comparison Theorem** to determine whether the area of the shaded region is finite or infinite. Circle your conclusion. (No work needs to be shown.)

[✓ **finite**] **infinite** **no conclusion can be made**

(c) You reached your conclusion because (circle the inequality that is needed to justify your conclusion)

[✓ $f(x) \leq \frac{3}{x^{3/2}}$] $f(x) \geq \frac{1}{x^{3/2}}$

6pt **7.** Consider the sequence $\{a_n\}$, where $a_n = \left(1+\frac{1}{n}\right)^{2n}$.

(a) Is the sequence $\{a_n\}$ convergent or divergent? Circle your answer. If it is convergent, find its limit.

[✓ **Convergent**] **Divergent**

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^2$$

$$= e^2$$

(b) Is the series $\sum_{n=1}^{\infty} a_n$ convergent or divergent? If it is convergent, find its sum. If it is divergent, explain why (i.e., according to what test).

$$\lim_{n\to\infty} a_n = e^2 \neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence

7pt **8.** Consider the telescoping series $\sum_{n=1}^{\infty}(3^{1/n} - 3^{1/(n+1)})$ $= \sum_{n=1}^{\infty} a_n$

(a) Find s_n, the n-th partial sum of the series.

$$s_n = (3 - \cancel{3^{1/2}}) + (\cancel{3^{1/2}} - \cancel{3^{1/3}}) + (\cancel{3^{1/3}} - \cancel{3^{1/4}}) + \dots + (\cancel{3^{1/n}} - 3^{1/(n+1)})$$

$$= 3 - 3^{1/(n+1)}$$

(b) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(3 - 3^{1/(n+1)})$$

$$= 3 - \lim_{n\to\infty} 3^{1/(n+1)}$$

$$= 3 - 3^0$$

$$= 3 - 1$$

$$= 2$$

Series converges to the sum, 2.

7pt **9.** Consider the geometric series $\sum_{n=0}^{\infty} \frac{2(\pi)^{n-1}}{3^{2n}}$. $= \sum_{n=0}^{\infty} \frac{2\pi^{n}\cdot\pi^{-1}}{9^{n}} = \sum_{n=0}^{\infty} \frac{2}{\pi}\left(\frac{\pi}{9}\right)^{n}$

(a) Find r, the common ratio of the series.

$$r = \frac{\pi}{9}$$

(b) Is the series convergent or divergent (justify your answer)? If it is convergent, find its sum.

This is a convergent geometric series with $|r| = \frac{\pi}{9} < 1$

$$\sum_{n=0}^{\infty} \frac{2(\pi)^{n-1}}{3^{2n}} = \frac{a}{1-r} = \frac{\frac{2}{\pi}}{1-\frac{\pi}{9}} \cdot \frac{9\pi}{9\pi}$$

$$= \frac{18}{9\pi - \pi^{2}}$$

7pt **10.** Consider the geometric series $\sum_{n=0}^{\infty} \frac{3x^{n}}{2^{n}}$. $= \sum_{n=0}^{\infty} 3\left(\frac{x}{2}\right)^{n}$

(a) Find r, the common ratio of the series.

$$r = \frac{x}{2}$$

(b) Determine all values of x for which the series is convergent.

Converges for $|r| = \left|\frac{x}{2}\right| < 1$

$$\frac{|x|}{2} < 1 \longleftrightarrow |x| < 2$$

$$-2 < x < 2$$

(c) For the values of x which you found in part (b), what is the sum of the series?

For any x on $(-2, 2)$,

$$\sum_{n=0}^{\infty} \frac{3x^{n}}{2^{n}} = \frac{a}{1-r}$$

$$= \frac{3}{1-\frac{x}{2}} \cdot \frac{2}{2}$$

$$= \frac{6}{2-x}$$

$\boxed{\text{20pt}}$ **11.** Determine the convergence or divergence of the series by appying the appropriate test. Name the test and justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{4^n}{3^n - 2^n}}_{a_n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^n}{3^n - 2^n} = \lim_{n\to\infty} \frac{\left(\frac{4}{3}\right)^n}{1 - \cancel{\left(\frac{2}{3}\right)^n}^{\to 0}}$$

$$= \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty$$

$\therefore \displaystyle\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence

Also, $3^n - 2^n < 3^n$, $\dfrac{1}{3^n - 2^n} > \dfrac{1}{3^n}$, and $\dfrac{4^n}{3^n - 2^n} > \dfrac{4^n}{3^n} = \left(\dfrac{4}{3}\right)^n$. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is a divergent geometric series with $|r| = \frac{4}{3} \geq 1$.

$\therefore \sum a_n$ also diverges, by Direct Comparison.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n}{(n^2-3)^2} = \sum_{n=1}^{\infty} a_n$, a series of positive terms

Let $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$, also a series of positive terms.

$\displaystyle\sum_{n=1}^{\infty} b_n$ is a convergent p-series with $p = 3 > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+2n}{(n^2-3)^2} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{n^3 + 2n^4}{n^4 - 6n^2 + 9} = \lim_{n\to\infty} \frac{\frac{1}{n} + 2}{1 - \frac{6}{n^2} + \frac{9}{n^4}} = \frac{0+2}{1-0+0} = 2 > 0$$

$\therefore \displaystyle\sum_{n=1}^{\infty} a_n$ converges by the L.C.T.

(c) $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ $= \sum_{n=3}^{\infty} a_n$

On $[3, \infty)$, $\ln n > 1$, so $\frac{\ln n}{n} > \frac{1}{n}$.

$\sum_{n=3}^{\infty} \frac{1}{n}$ is the divergent harmonic series with $p = 1 \leq 1$.

$\therefore$ by The Comparism Test, $\sum_{n=3}^{\infty} a_n$ also diverges.

$a_n = \frac{\ln n}{n} = f(n)$. Let $f(x) = \frac{\ln x}{x}$, which is positive and continuous on $[3, \infty)$.

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} < 0 \Rightarrow 1 - \ln x < 0$ or $\ln x > 1$, which is true for all $x > e$. $\therefore f \downarrow$ on $[3, \infty)$

$$\int_3^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_3^b \frac{\ln x}{x}\,dx = \frac{1}{2}\lim_{b\to\infty} (\ln x)^2 \Big|_3^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\left(\cancelto{\infty}{\ln b}\right)^2 - (\ln 3)^2\right] = \infty,$$ which diverges

$\int \frac{\ln x}{x}\,dx = \int u\,du$
$= \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2$

$u = \ln x$
$du = \frac{1}{x}dx$

$\therefore$ $\sum a_n$ also diverges by The Integral Test.

(d) $\underbrace{\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+\sqrt{n}}}$

$\Downarrow$

$\sum a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+\sqrt{n}}$ is an alternating series.

$u_n = \frac{\sqrt{n}}{n+\sqrt{n}}$ is a sequence of positive terms.

$= \frac{1}{\sqrt{n}+1}$. $\sqrt{n}+1$ is increasing, so $\frac{1}{\sqrt{n}+1} = \frac{\sqrt{n}}{n+\sqrt{n}}$ is decreasing for all $n \geq 1$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{\cancelto{0}{\frac{1}{\sqrt{n}}}}{1+\cancelto{0}{\frac{1}{\sqrt{n}}}} = 0.$$

$\therefore$ by AST, $\sum_{n=1}^{\infty} a_n$ converges.