$\boxed{\text{12pt}}$ **1.** Consider the equation $e^{x+y} = xy + 10$.

(**a**) Use implicit differentiation to find $\dfrac{dy}{dx}$.

$$\frac{d}{dx}(e^{x+y}) = \frac{d}{dx}(xy+10) \;\Rightarrow\; e^{x+y}\left(1+\frac{dy}{dx}\right) = y + x\,\frac{dy}{dx} + 0 \;\Rightarrow\; = e^{x+y} + e^{x+y}\,\frac{dy}{dx} = y + x\,\frac{dy}{dx} \;\Rightarrow$$

$$e^{x+y}\,\frac{dy}{dx} - x\,\frac{dy}{dx} = y - e^{x+y} \;\Longrightarrow\; \frac{dy}{dx}\,(e^{x+y} - x) = y - e^{x+y} \;\Longrightarrow\; \boldsymbol{\frac{dy}{dx} = \frac{y - e^{x+y}}{e^{x+y} - x}}$$

(**b**) Find an equation of the tangent line to the graph $e^{x+y} = xy + 10$ at the point $(3, -3)$.

The slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{(3,-3)} = \dfrac{-3 - e^0}{e^0 - 3} = \dfrac{-4}{-2} = 2$. Then an equation of the tangent line can be written in the form: $y - (-3) = 2(x-3)$ or $\boldsymbol{y = 2x - 9}$.

$\boxed{\text{8pt}}$ **2.** Use Logarithmic Differentiation to find y' if $y = x^{\sqrt{x}}$.

$$\ln y = \ln(x^{\sqrt{x}}) = \sqrt{x}\,\ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sqrt{x}\,\ln x) = \sqrt{x}\,\frac{1}{x} + \ln x\,\frac{1}{2\sqrt{x}} = \frac{2 + \ln x}{2\sqrt{x}}$$

$$\frac{y'}{y} = \frac{2 + \ln x}{2\sqrt{x}}$$

$$y' = y \cdot \frac{2 + \ln x}{2\sqrt{x}} \;\Longrightarrow\; \boldsymbol{y' = x^{\sqrt{x}}\,\frac{2 + \ln x}{2\sqrt{x}}}$$

15pt **3.** A particle moves in a straight line with its position function $s(t)$ given by the equation $s(t) = te^{1-t}$, $t \geq 0$ (where s is measured in meters and t is measured in seconds).

(a) Find $v(t)$, the velocity of the particle at time t.

$v(t) = s'(t) = t\,e^{1-t}(-1) + 1 \cdot e^{1-t} = \boldsymbol{e^{1-t}(1-t)}$ (m/s)

(b) When is the particle at rest?

$v(t) = 0 \;\Rightarrow\; e^{1-t}(1-t) = 0 \;\Rightarrow\; 1 - t = 0$ $(e^{1-t} > 0$ for all $t)$. Therefore, the particle is at rest when $\boldsymbol{t = 1}$ (s).

(c) When is the particle moving in the positive direction?

$v(t) > 0 \;\Rightarrow\; e^{1-t}(1-t) > 0 \;\Rightarrow\; 1 - t > 0$ (since $e^{1-t} > 0$ for all t). Therefore, the particle is moving in the positive direction when $\boldsymbol{0 \leq t < 1}$.

(d) Find the total distance traveled by the particle during the first 2 seconds. Show your work.

Distance $= |s(1) - s(0)| + |s(2) - s(1)| = |1e^0 - 0| + |2e^{-1} - e^0| = 1 + 1 - \dfrac{2}{e} = \boldsymbol{2 - \dfrac{2}{e}}$ (m)

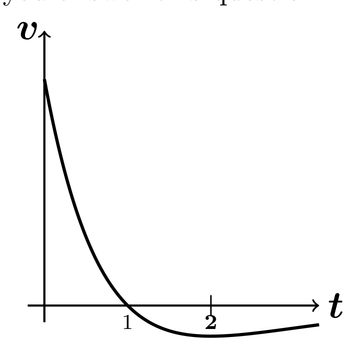
(e) Find the acceleration $a(t)$ of the particle at time t.

$a(t) = v'(t) = \dfrac{d}{dt}(e^{1-t}(1-t)) = e^{1-t}(0-1) + e^{1-t}(-1)(1-t) = \boldsymbol{e^{1-t}(t-2)}$ (m/s^2)

(f) When during the first 2 seconds is the particle speeding up and when is it slowing down? The graph of the velocity $v(t)$ of the particle, shown at right below, will help you answer this question.

Speeding up on the interval $\boldsymbol{(1, 2)}$ (where $v < 0$ and $a < 0$)

Slowing down on the interval $\boldsymbol{(0, 1)}$ (where $v > 0$ and $a < 0$)



$\boxed{\text{7pt}}$ **4.** Let $f(x) = 4\sqrt[4]{x}$.

(**a**) Find the linearization $L(x)$ of $f(x)$ at $a = 16$.

$$L(x) = f(16) + f'(16)(x - 16)$$

$$f(16) = 4\sqrt[4]{16} = 4 \cdot 2 = 8 \quad \text{and} \quad f'(x) = 4 \cdot \tfrac{1}{4}\, x^{-3/4} = \frac{1}{x^{3/4}}, \quad \text{so} \quad f'(16) = \frac{1}{16^{3/4}} = \frac{1}{2^3} = \frac{1}{8}$$

Therefore, $\boldsymbol{L(x) = 8 + \frac{1}{8}(x - 16)}$.

(**b**) Use $L(x)$ from part (a) to estimate the value of $f(14)$.

$$f(14) \approx L(14) = 8 + \frac{1}{8}(14 - 16) = 8 - \frac{1}{4} = \boldsymbol{\frac{31}{4} = 7.75}$$

$\boxed{\text{4pt}}$ **5.** Find $f'(\theta)$ and $f''(\theta)$ of the function $f(\theta) = \ln|\sec(2\theta) + \tan(2\theta)|$. Simplify your answers.

$$f'(\theta) = \frac{\sec(2\theta)\tan(2\theta) \cdot 2 + \sec^2(2\theta) \cdot 2}{\sec(2\theta) + \tan(2\theta)} = \frac{2\sec(2\theta)(\tan(2\theta) + \sec(2\theta))}{\sec(2\theta) + \tan(2\theta)} = \boldsymbol{2\sec(2\theta)}$$

$$f''(\theta) = 2\sec(2\theta)\tan(2\theta) \cdot 2 = \boldsymbol{4\sec(2\theta)\tan(2\theta)}$$

$\boxed{\text{8pt}}$ **6.** Find the absolute maximum and absolute minimum values of the function $f(x) = x + \cos x$ on the interval $[0, \pi]$.

The absolute maximum and absolute minimum values can only occur at the critical numbers of f in the interval $(0, \pi)$ or at the endpoints of that interval.

$f'(x) = 1 - \sin x$. If $f'(x) = 0$, then $\sin x = 1$. There is only one solution of this equation in $(0, \pi)$: it is $x = \dfrac{\pi}{2}$. Thus, $\dfrac{\pi}{2}$ is the only critical number of f in $(0, \pi)$.

We have $f(\frac{\pi}{2}) = \frac{\pi}{2} + \cos(\frac{\pi}{2}) = \frac{\pi}{2}$, $\quad f(0) = 0 + \cos 0 = 1$, and $f(\pi) = \pi + \cos \pi = \pi - 1$.

By comparing these values, we conclude that the absolute minimum of f on $[0, \pi]$ is $\boldsymbol{f(0) = 1}$ and the absolute maximum value is $\boldsymbol{f(\pi) = \pi - 1}$.

$\boxed{\text{8pt}}$ **7.** Let $f(x) = \dfrac{1}{x^2}$.

(**a**) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval $[\mathbf{2, 6}]$ by filling the blanks.

f is <u>**continuous**</u> on the interval $[2, 6]$;

f is <u>**differentiable**</u> on the interval $(2, 6)$.

(**b**) Find all numbers c that satisfy the conclusion of the Mean Value Theorem on the interval $[\mathbf{2, 6}]$.

$$f'(c) = -2c^{-3} = -\frac{2}{c^3}$$

$$f'(c) = \frac{f(6) - f(2)}{6 - 2} = \frac{\frac{1}{36} - \frac{1}{4}}{4} = \frac{1 - 9}{36 \cdot 4} = \frac{-8}{36 \cdot 4} = -\frac{1}{18}$$

$$-\frac{2}{c^3} = -\frac{1}{18} \quad \Rightarrow \quad c^3 = 36 \quad \Rightarrow \quad \boldsymbol{c = \sqrt[3]{36}}$$

$\boxed{\text{13pt}}$ **8.** In each part indicate whether the ratio represents an indeterminate form $0/0$ or ∞/∞ or neither of them. Then find the limit.

(**a**) $\displaystyle\lim_{x\to 1}\frac{x^5-1}{x^6-1}$. Circle one: $\mathbf{\frac{0}{0}}$ $\quad \mathbf{\frac{\infty}{\infty}}$ $\quad$ **neither** . $\qquad$ Find the limit.

$$\lim_{x\to 1}\frac{x^5-1}{x^6-1} \overset{\text{L'H}}{=} \lim_{x\to 1}\frac{5x^4}{6x^5} = \mathbf{\frac{5}{6}}$$

(**b**) $\displaystyle\lim_{x\to \infty}\frac{x^5-1}{x^6-1}$. Circle one: $\mathbf{\frac{0}{0}}$ $\quad \mathbf{\frac{\infty}{\infty}}$ $\quad$ **neither** . $\qquad$ Find the limit.

$$\lim_{x\to \infty}\frac{x^5-1}{x^6-1} \overset{\text{L'H}}{=} \lim_{x\to \infty}\frac{5x^4}{6x^5} = \lim_{x\to \infty}\frac{5}{6x} = \mathbf{0}$$

(**c**) $\displaystyle\lim_{x\to \infty}\frac{x^{-5}-1}{x^{-6}-1}$. Circle one: $\mathbf{\frac{0}{0}}$ $\quad \mathbf{\frac{\infty}{\infty}}$ $\quad$ **neither** . $\qquad$ Find the limit.

$$\lim_{x\to \infty}\frac{x^{-5}-1}{x^{-6}-1} = \frac{0-1}{0-1} = \mathbf{1}$$

(**d**) $\displaystyle\lim_{x\to 1}\frac{e^x-e}{\sin(\pi x)}$. Circle one: $\mathbf{\frac{0}{0}}$ $\quad \mathbf{\frac{\infty}{\infty}}$ $\quad$ **neither** . $\qquad$ Find the limit.

$$\lim_{x\to 1}\frac{e^x-e}{\sin(\pi x)} \overset{\text{L'H}}{=} \lim_{x\to 1}\frac{e^x}{\pi\cos(\pi x)} = \frac{e}{\pi\cos\pi} = -\boldsymbol{\frac{e}{\pi}}$$

20pt **9.** Consider the function $f(x) = \ln x + \dfrac{2}{x^2}$, $x > 0$.

(**a**) Find $f'(x)$.

$$f'(x) = \frac{1}{x} + 2(-2)x^{-3} = \boldsymbol{\frac{1}{x} - \frac{4}{x^3}}$$

(**b**) Find the critical number(s) of f (remember, $x > 0$).

$$f'(x) = 0 \Rightarrow \frac{1}{x} - \frac{4}{x^3} = 0 \Rightarrow \frac{x^2-4}{x^3} = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow \boldsymbol{x = 2} \ (x > 0)$$

(**c**) Determine the intervals of increase and decrease of f. Show your work.

For $x > 0$, $f'(x) = \dfrac{x^2-4}{x^3} > 0$ when $x > 2$ and $f'(x) = \dfrac{x^2-4}{x^3} < 0$ when $0 < x < 2$.

f **increases** on the interval(s) $\boldsymbol{(2, \infty)}$	f **decreases** on the interval(s) $\boldsymbol{(0, 2)}$

(**d**) For each critical number of f found in part (b), state whether f has a local maximum, local minimum, or neither at that number.

f has a local **minimum** at $\boldsymbol{x = 2}$

(**e**) Find $f''(x)$ and the value(s) of x, $x > 0$, such that $f''(x) = 0$.

$$f''(x) = \frac{d}{dx}\left(\frac{1}{x} - \frac{4}{x^3}\right) = (-1)x^{-2} - 4(-3)x^{-4} = \boldsymbol{-\frac{1}{x^2} + \frac{12}{x^4} = \frac{12 - x^2}{x^4}}$$

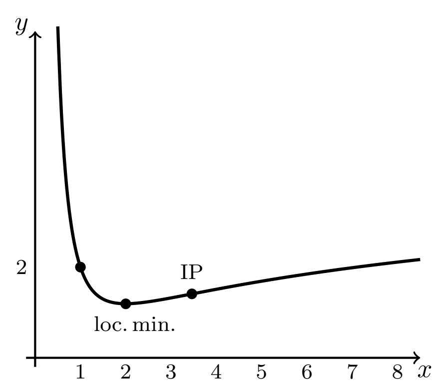
$$f''(x) = 0 \Rightarrow \frac{12 - x^2}{x^4} = 0 \Rightarrow x^2 = 12 \Rightarrow x = \pm\sqrt{12} \Rightarrow x = \boldsymbol{2\sqrt{3}} \ (x > 0)$$

(**f**) Find the intervals of concavity and the number(s) x at which f has its inflection point(s).

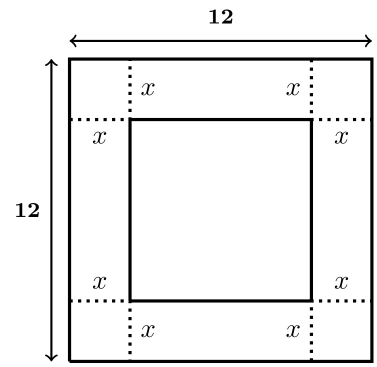
For $x > 0$, $f''(x) = \dfrac{12 - x^2}{x^4} > 0$ when $0 < x < 2\sqrt{3}$ and $f''(x) = \dfrac{12 - x^2}{x^4} < 0$ when $x > 2\sqrt{3}$.

concave **up** on the interval $\boldsymbol{(0, 2\sqrt{3})}$	concave **down** on the interval $\boldsymbol{(2\sqrt{3}, \infty)}$
Inflection point at $x = \boldsymbol{2\sqrt{3}}$	

(**g**) Sketch the graph of $f(x)$ assuming that $\lim_{x \to 0^+} f(x) = \infty$. Plot the point $(1, f(1))$ and label the points of local maximum/minimum and inflection points.



$\boxed{\text{10pt}}$ **10.** A box with an open top is to be constructed from a square piece of cardboard with dimensions 12 in. by 12 in. by cutting out equal squares of side x at each corner and then folding up the sides.



(**a**) Express the volume V of the box as a function of x.

$\boldsymbol{V(x) = (12 - 2x)^2\, x} = 4x(6-x)^2 = 4x(36 - 12x + x^2)$

or $\boldsymbol{V(x) = 4(x^3 - 12x^2 + 36x)}$

(**b**) Find the value of x that maximizes the volume of the box.

Be sure to justify your answer.

Here $0 \leq x \leq 6$. Since $V(x)$ is continuous on the closed interval $[0,6]$ (as a polynomial, $V(x)$ is continuous everywhere), $V(x)$ attains its absolute maximum value either at its critical numbers in $(0,6)$ or at the endpoints of that interval.

$V'(x) = 4(3x^2 - 24x + 36) = 12(x^2 - 8x + 12)$

$V'(x) = 0 \;\Rightarrow\; x^2 - 8x + 12 = 0 \;\Rightarrow\; (x-2)(x-6) = 0 \;\Rightarrow\; x = 2$ is the only critical number of V in $(0,6)$.

Comparing the values $V(2) = (12 - 2\cdot 2)^2 \cdot 2 = 128$, $V(0) = 0$, and $V(6) = 0$, we conclude that the absolute maximum of V occurs at $\boldsymbol{x = 2}$.

(Another way to justify that $V(x)$ attains its absolute maximum value at $x = 2$ is to use the First or the Second Derivative Test to show that $V(x)$ has a **local** maximum at $x = 2$.)