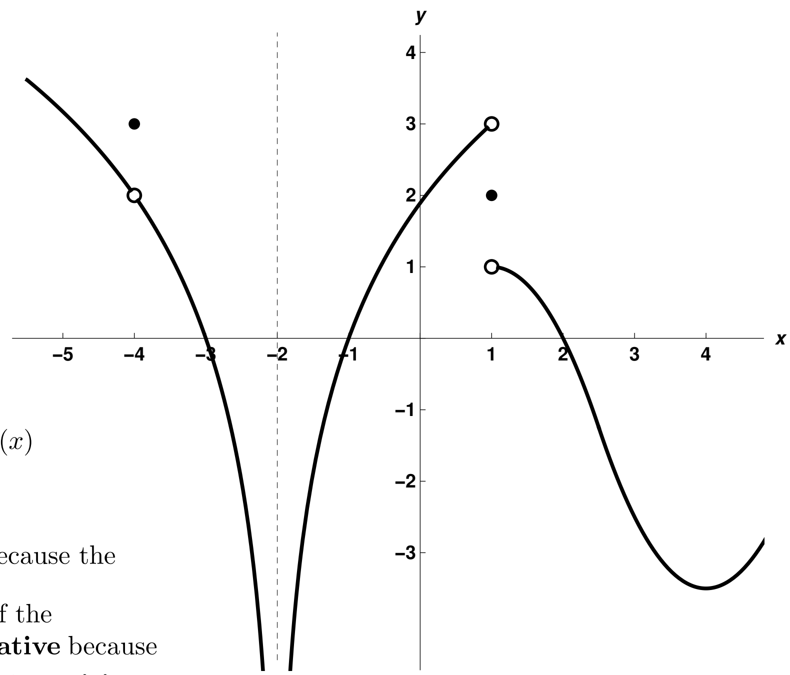
$\boxed{\text{13pt}}$ **1.** The graph of a function $f(x)$ is shown at right below.

Part I. State the value of each limit. If a limit is infinite, state whether it is ∞ or $-\infty$. If a limit does not exist (but is not infinite), **explain why not**.



(a) $\lim\limits_{x\to -4} f(x) = \mathbf{2}$

(b) $\lim\limits_{x\to -2} f(x) = -\infty$

(c) $\lim\limits_{x\to 1^-} f(x) = \mathbf{3}$

(d) $\lim\limits_{x\to 1^+} f(x) = \mathbf{1}$

(e) $\lim\limits_{x\to 1} f(x)$ **d.n.e.** because $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$

(f) $\lim\limits_{x\to 3} \dfrac{f(x)}{(x-3)^2} = -\infty$ (the limit is **infinite** because the limit of the denominator is zero and the limit of the numerator is a nonzero number; the limit is **negative** because the numerator is **negative** and the denominator is **positive**.

(g) $\lim\limits_{x\to 1^-} (x-1)f(x) = \lim\limits_{x\to 1^-} (x-1) \cdot \lim\limits_{x\to 1^-} f(x) = 0 \cdot 3 = \mathbf{0}$

(h) $\lim\limits_{x\to 1^+} (x-1)f(x) = \lim\limits_{x\to 1^+} (x-1) \cdot \lim\limits_{x\to 1^+} f(x) = 0 \cdot 1 = \mathbf{0}$

(i) $\lim\limits_{x\to 1} (x-1)f(x) = \mathbf{0}$ because $\lim\limits_{x\to 1^-} (x-1)f(x) = \lim\limits_{x\to 1^+} (x-1)f(x) = \mathbf{0}$

Part II.

(a) At what number $x = a$ does f have a **removable** discontinuity? What value $f(a)$ should be assigned to f at $x = a$ in order to make f continuous at a?

$a = \mathbf{-4}$ $\qquad\qquad$ $f(a) = \mathbf{2}$

(b) At what number $x = b$ does f have a **jump** discontinuity? What value $f(b)$ should be assigned to f at $x = b$ to make f **continuous from the right** at $x = b$?

$b = \mathbf{1}$ $\qquad\qquad$ $f(b) = \mathbf{1}$

$\boxed{\text{6pt}}$ **2.** Find the limits. Circle your answer. You do not need to justify your answers.

$\lim_{x\to-\infty} e^{-x} =$	0	1	∞	$-\infty$
$\lim_{x\to\infty} e^{-x} =$	0	1	∞	$-\infty$
$\lim_{x\to 0} e^{-x} =$	0	1	∞	$-\infty$
$\lim_{x\to\infty} \ln x =$	0	1	∞	$-\infty$
$\lim_{x\to 0^+} \ln x =$	0	1	∞	$-\infty$
$\lim_{x\to 1} \ln x =$	0	1	∞	$-\infty$

$\boxed{\text{10pt}}$ **3.** Suppose $\dfrac{x^2+3x+2}{x+1} \leq g(x) \leq e^{x+1}$ for all $x \neq -1$.

(**a**) Find the following limits:

$$\lim_{x\to-1} \frac{x^2+3x+2}{x+1} = \lim_{x\to-1} \frac{(x+1)(x+2)}{x+1} = \lim_{x\to-1} (x+2) = -1+2 = \mathbf{1}$$

$$\lim_{x\to-1} e^{x+1} = e^{-1+1} = e^0 = \mathbf{1}$$

(**b**) Are the limits found in part (a) **equal** to each other? Circle your answer.

Yes **No**

(**c**) Can the Squeeze Theorem be applied to find $\lim_{x\to-1} g(x)$?

Yes $\boldsymbol{\lim_{x\to-1} g(x) = 1}$ **No**

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$\boxed{\text{6pt}}$ **4.** (**A**) Find the limit

$$\lim_{x\to 2}\frac{\frac{1}{3x+1}-\frac{1}{7}}{x-2}=\lim_{x\to 2}\frac{\frac{7-3x-1}{7(3x+1)}}{x-2}=\lim_{x\to 2}\frac{6-3x}{(x-2)7(3x+1)}=\lim_{x\to 2}\frac{-3(x-2)}{(x-2)7(3x+1)}=\lim_{x\to 2}\frac{-3}{7(3x+1)}$$

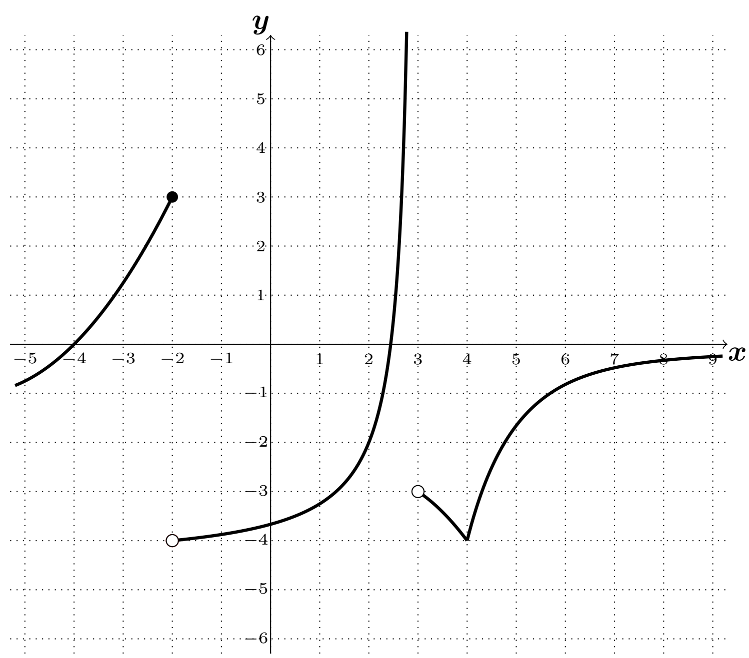
$$=\frac{-3}{7(3\cdot 2+1)}=-\frac{\mathbf{3}}{\mathbf{49}}$$

(**B**) The limit in part (A) represents the derivative of some function f at some number a. Find such an f and a

$$\boldsymbol{f(x)=\frac{1}{3x+1}} \qquad\qquad \boldsymbol{a=2}$$

$\boxed{\text{9pt}}$ **5.** Use the grid below to sketch the graph of a function $f(x)$ that satisfies all of the given conditions.

- f is continuous on $(-\infty,\infty)$ except at -2 and 3
- $\lim_{x\to -2^-} f(x)=3,\quad \lim_{x\to -2^+} f(x)=-4$
- $f(x)$ is continuous from the left at $x=-2$
- $\lim_{x\to 3^-} f(x)=\infty,\quad \lim_{x\to 3^+} f(x)=-3$
- $f(4)=-4$
- $f(x)$ is continuous but **not differentiable** at $x=4$
- $\lim_{x\to\infty} f(x)=0$



The function whose graph is shown is not defined at $x=3$. Since there is no information about the value $f(3)$, you have a choice to show the point $(3,-3)$ as an open circle or as a bold point...

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$\boxed{\text{9pt}}$ **6.** Let $f(x) = \begin{cases} \dfrac{\sqrt{x}-2}{x-4} + b & \text{if } \quad 0 \le x < 4 \\ a & \text{if } \quad x = 4 \\ \tan\left(\dfrac{\pi}{x}\right) & \text{if } \quad x > 4 \end{cases}$

where a and b are constants.

(**a**) Find $\lim\limits_{x\to 4^-} f(x)$ and $\lim\limits_{x\to 4^+} f(x)$ (please note: your answer may depend on b).

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}\left[\frac{\sqrt{x}-2}{x-4} + b\right] = \lim_{x\to 4^-}\left[\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} + b\right] = \lim_{x\to 4^-}\left[\frac{x-4}{(x-4)(\sqrt{x}+2)} + b\right]$$

$$= \lim_{x\to 4^-}\left[\frac{1}{\sqrt{x}+2} + b\right] = \frac{1}{\sqrt{4}+2} + b = \boldsymbol{\frac{1}{4} + b}$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \tan\left(\frac{\pi}{x}\right) = \tan\frac{\pi}{4} = \mathbf{1}$$

(**b**) For what value of b does the $\lim\limits_{x\to 4} f(x)$ exist?

$\lim\limits_{x\to 4} f(x)$ exists if and only if $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x) \;\Rightarrow\; \dfrac{1}{4} + b = 1 \Rightarrow \boldsymbol{b = \dfrac{3}{4}}$.

(**c**) Find the values of a and b (if any) for which $f(x)$ is **continuous** at $x = 4$.

$f(x)$ is continuous at $x = 4$ if and only if the $\lim\limits_{x\to 4} f(x)$ **exists** and **is equal** to $f(4)$. Therefore,

$f(x)$ is continuous at $x = 4$ when $\boldsymbol{b = \dfrac{3}{4}}$ and $\boldsymbol{a = \lim\limits_{x\to 4} f(x) = 1}$.

$\boxed{\text{3pt}}$ **7.** Let $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } \quad x \ne 1 \\ 2 & \text{if } \quad x = 1 \end{cases}$

(**a**) Is there a solution to the equation $f(x) = 0$ in the interval $(0, 2)$? Circle your answer.

Yes **No**

(**b**) Explain **clearly** why the Intermediate Value Theorem may not be applied to the function $f(x)$ on $[0, 2]$.

The Intermediate Value Theorem may not be applied to the function $f(x)$ on $[0, 2]$ because it is **not continuous on $[0, 2]$** (it is discontinuous at $x = 1$).

$\boxed{\text{12pt}}$ **8.** Let $f(x) = \dfrac{3x^3+3x^2}{(x-2)^2(x+1)} = \dfrac{3x^3+3x^2}{x^3-3x^2+4}$.

(**a**) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Show your work.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^3+3x^2}{x^3-3x^2+4} = \lim_{x\to\infty} \frac{\frac{3x^3}{x^3}+\frac{3x^2}{x^3}}{\frac{x^3}{x^3}-\frac{3x^2}{x^3}+\frac{4}{x^3}} = \lim_{x\to\infty} \frac{3+\frac{3}{x}}{1-\frac{3}{x}+\frac{4}{x^3}} = \frac{3+0}{1-0+0} = \mathbf{3}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^3+3x^2}{x^3-3x^2+4} = \lim_{x\to-\infty} \frac{\frac{3x^3}{x^3}+\frac{3x^2}{x^3}}{\frac{x^3}{x^3}-\frac{3x^2}{x^3}+\frac{4}{x^3}} = \lim_{x\to-\infty} \frac{3+\frac{3}{x}}{1-\frac{3}{x}+\frac{4}{x^3}} = \frac{3+0}{1-0+0} = \mathbf{3}$$

(**b**) Find the limit if it exists. Show your work.

(**i**) $\lim\limits_{x\to2} f(x) = \lim\limits_{x\to2} \dfrac{3x^3+3x^2}{(x-2)^2(x+1)}$. Since $\lim\limits_{x\to2}[(x-2)^2(x+1)] = 0$ and $\lim\limits_{x\to2}(3x^3+3x^2) = 36 \neq 0$, $\lim\limits_{x\to2} f(x)$ is **infinite**. Furthermore, since both the numerator and denominator near $x=2$ are positive, $\lim\limits_{x\to2} f(x) = \boldsymbol{\infty}$.

(**ii**) $\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \dfrac{3x^3+3x^2}{(x-2)^2(x+1)} = \lim\limits_{x\to-1} \dfrac{3x^2(x+1)}{(x-2)^2(x+1)} = \lim\limits_{x\to-1} \dfrac{3x^2}{(x-2)^2} = \dfrac{3(-1)^2}{(-1-2)^2} = \mathbf{\dfrac{1}{3}}$

(**c**) Find equations of the **horizontal** and **vertical** asymptotes of the graph of $f(x)$

HA: $\boldsymbol{y=3}$ **VA:** $\boldsymbol{x=2}$

$\boxed{\text{6pt}}$ **9.** Find the derivative $f'(x)$ of the function $f(x) = \sqrt{x+4}$ using the limit definition of derivative. Please note: you **may NOT use differentiation rules here**.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+4}-\sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4}+\sqrt{x+4}}{\sqrt{x+h+4}+\sqrt{x+4}}$$

$$= \lim_{h\to 0} \frac{x+h+4-(x+4)}{h(\sqrt{x+h+4}+\sqrt{x+4})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+4}+\sqrt{x+4})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+4}+\sqrt{x+4}}$$

$$= \frac{1}{\sqrt{x+0+4}+\sqrt{x+4}} = \boldsymbol{\frac{1}{2\sqrt{x+4}}}$$

IN PROBLEMS 10–12 YOU MAY USE DIFFERENTIATION RULES.

$\boxed{\text{9pt}}$ **10.** Differentiate. Show your work.

(**a**) $f(t) = 5t^{2/3} + \dfrac{2}{t^3} + 4e^3 = 5t^{2/3} + 2t^{-3} + 4e^3$

$$f'(x) = 5 \cdot \frac{2}{3}\, t^{2/3-1} + 2(-3)t^{-3-1} + 0 = \boldsymbol{\frac{10}{3}\, t^{-1/3} - 6t^{-4}}$$

(**b**) $g(x) = x^3\, e^x$

$$g'(x) = x^3 \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x^3) = \boldsymbol{x^3 e^x + 3x^2 e^x}$$

IN PROBLEMS 10–12 YOU MAY USE DIFFERENTIATION RULES.

$\boxed{\text{10pt}}$ **11.** Let $f(x) = \dfrac{x}{x^2+2}$.

(**a**) Find $f'(x)$.

$$f'(x) = \frac{(x^2+2)\cdot 1 - x\cdot(2x+0)}{(x^2+2)^2} = \frac{x^2+2-2x^2}{(x^2+2)^2} = \boldsymbol{\frac{2-x^2}{(x^2+2)^2}}$$

(**b**) Find an equation of the tangent line to the curve $y = f(x)$ at the point where $x = 1$. Give your answer in the slope-intercept form.

The point-slope form is: $y - f(1) = f'(1)(x-1)$.

Here $f(1) = \dfrac{1}{1^2+2} = \dfrac{1}{3}$ and $f'(1) = \dfrac{2-1^2}{(1^2+2)^2} = \dfrac{1}{9}$. Then $y - \dfrac{1}{3} = \dfrac{1}{9}(x-1)$

Therefore, the slope-intercept form is $\boldsymbol{y = \frac{1}{9}x + \frac{2}{9}}$

(**c**) Find all values of x at which the curve $y = f(x)$ has a horizontal tangent.

The slope of a horizontal tangent is 0, so we need to find all values of x such that $f'(x) = 0$.

$$f'(x) = \frac{2-x^2}{(x^2+2)^2} = 0 \Rightarrow 2 - x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow \boldsymbol{x = \sqrt{2}} \text{ and } \boldsymbol{x = -\sqrt{2}}$$

IN PROBLEMS 10–12 YOU MAY USE DIFFERENTIATION RULES.

$\boxed{\text{11pt}}$ **12.** The equation of motion of a particle is $s(t) = \dfrac{t^3+3}{t}$, $t > 0$, where s is in meters and t is in seconds.

(**a**) Find the average velocity of the particle over the time interval $1 \le t \le 3$. Please include units of measurement in your answer.

$$v_{ave} = \frac{s(3)-s(1)}{3-1} = \frac{\frac{27+3}{3} - \frac{1+3}{1}}{2} = \frac{10-4}{2} = \mathbf{3}\text{ m/s}$$

(**b**) Find the instantaneous velocity v and acceleration a of the particle at time t (*hint: simplify* $s(t)$ *first*).

$$v(t) = s'(t) = \frac{d}{dt}\left(\frac{t^3+3}{t}\right) = \frac{d}{dt}\left(t^2 + 3t^{-1}\right) = 2t - 3t^{-2}$$

$$a(t) = v'(t) = \frac{d}{dt}\left(2t - 3t^{-2}\right) = 2 + 6t^{-3}$$

(**c**) Find $v(1)$ and $a(1)$. Please include units of measurement in your answers.

$$v(1) = 2 \cdot 1 - 3(1)^{-2} = 2 - 3 = -1\text{ m/s}$$

$$a(1) = 2 + 6(1)^{-3} = 2 + 6 = 8\text{ m/s}^2$$