FALL 2018

Test 1

MATH 1042

Department of Mathematics
Temple University

October 2, 2018

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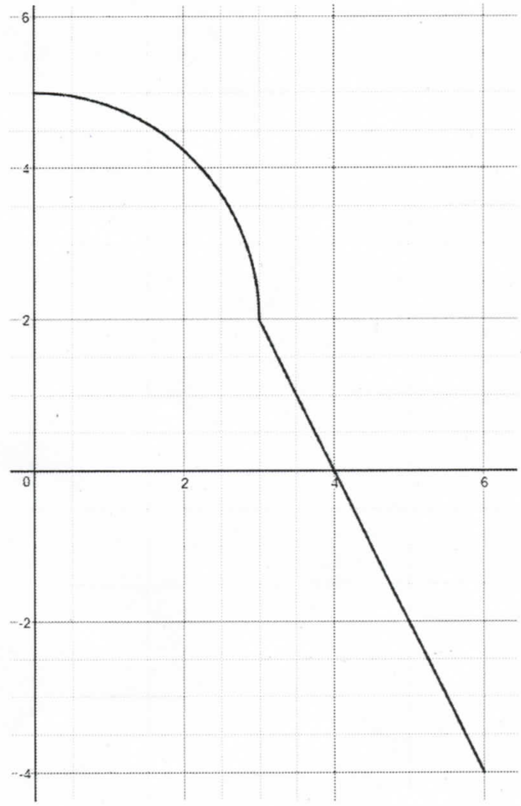
Instructor/Section: ______________________

This exam consists of 8 questions. Show all your work. **No work, no credit.** Good Luck!

Question	Points	Out of
1		**12**
2		**6**
3		**8**
4		**14**
5 a b		**13**
5 c d		**12**
6		**8**
7		**8**
8		**12**
9		**12**
Total		**105**

12pt **1.** Let $g(x) = \int_0^x f(t)\,dt$, $0 \le x \le 6$, where $f(t)$ is the function whose graph is to the right. This graph consists of a quarter of a circle and a straight line.



(a) Find the following values:

$g(0) = \boxed{0}$ $\quad g(3) = 3\cdot 2 + \frac{1}{4}\pi\cdot 3^2 = \boxed{\frac{9\pi}{4} + 6}$

$g(4) = g(3) + \frac{1\cdot 2}{1} = \boxed{\frac{9\pi}{4} + 7}$ $\quad g(6) = g(4) - \frac{2\cdot 4}{2} = \boxed{\frac{9\pi}{4} + 3}$

(b) Find $g'(4)$ and $g''(4)$

$g'(4) = f(4) = \boxed{0}$

$g''(4) = f'(4) = m_{\text{line}\ldots} = \frac{\Delta y}{\Delta x} = \frac{-2-2}{5-3} = \boxed{-2}$

(c) On what interval(s) is $g(x)$ **decreasing**?

g decreasing where $g' = f < 0$, that is on $\boxed{(4;6)}$

(d) Find all **critical points** of $g(x)$ on the interval $(0, 6)$ and classify each of them as a local minimum, local maximum, or neither.

$g'(x) = f(x) = 0 \implies x = 4$, which is a local max for g since g' changes its sign through $x=4$

(e) On what interval(s) is the graph of $g(x)$ concave **down**?

Graph of g is concave down where $g'' = f' < 0$, that is on the entire $(0;6)$ where f is decreasing

6pt **2.** Let $F(x) = \displaystyle\int_0^{\sin(e^x)} \sqrt{\arcsin t}\, dt, \qquad x < \ln(\pi/2).$

Use the Fundamental Theorem of Calculus to find the derivative $F'(x)$. Simplify your answer.

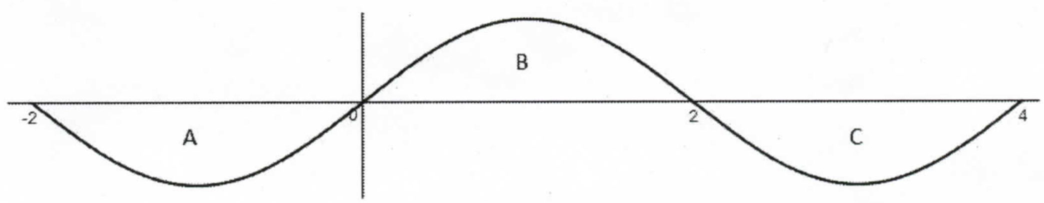
$$F(u) = \int_0^u \sqrt{\arcsin t}\, dt \quad \text{where} \quad u(x) = \sin(e^x)$$

Use Chain Rule: $\dfrac{dF}{dx} = \left.\dfrac{dF}{du}\right|_{u=u(x)} \cdot \dfrac{du}{dx} = \underbrace{\sqrt{\arcsin(\sin(e^x))}}_{\sqrt{e^x}} \cdot \cos(e^x) \cdot e^x$

if using the cancellation relation $\arcsin(\sin) = $ identity function

$$\Rightarrow \boxed{F'(x) = e^{\frac{3x}{2}} \cdot \cos(e^x)}$$

8pt **3.** Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 4. Find the value of $\displaystyle\int_{-2}^{4} \left[f(x) - \sin\left(\frac{\pi x}{4}\right)\right] dx.$



$$\int_{-2}^{4} \left[f(x) - \sin\left(\frac{\pi x}{4}\right)\right] dx = \int_{-2}^{4} f(x)\, dx - \int_{-2}^{4} \sin\left(\frac{\pi x}{4}\right) dx$$

$$= (-4+4-4) - \left[-\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_{-2}^{4} = -4 + \frac{4}{\pi}\left[\cos\pi - \cancel{\cos\left(-\frac{\pi}{2}\right)}\right]$$

$$= -4 - \frac{4}{\pi} = \boxed{(-4)\cdot\frac{\pi+1}{\pi}}$$

14pt **4.** Let $v(t) = (t+1)\cos t$ be the velocity function (in meters per second) of a particle moving along a line. Please include units of measurement in your answers.

(a) Find the **displacement** of the particle over the time interval $0 \le t \le \pi$.

$$\text{displacement after } \pi \text{ seconds} = \int_0^{\pi} (t+1)\cos t\,dt = (t+1)\sin t\Big|_0^{\pi} - \int_0^{\pi} \sin t\,dt$$

$u = t+1$	$dv = \cos t\,dt$
$du = dt$	$v = \sin t$

$$= \cancel{(\pi+1)\sin\pi} - \cancel{1\cdot\sin 0} - (-\cos t)\Big|_0^{\pi} = \cos\pi - \cos 0 = -1-1 = \boxed{-2} \text{ (meters)}$$

(b) When is the particle at rest in the interval $0 \le t \le \pi$?

$$v(t) = 0 \Rightarrow (t+1)\cos t = 0 \text{ where } t \in [0;\pi]$$

$$\cancel{t = -1} \qquad \boxed{t = \frac{\pi}{2}} \text{ (seconds)}$$

since -1 not in $[0;\pi]$

(c) Find the **total distance traveled** by the particle during the time interval $0 \le t \le \pi$.

$$\text{total distance traveled, } t \in [0;\pi] = \int_0^{\pi} |v(t)|\,dt = \int_0^{\pi/2} (t+1)\cos t\,dt - \int_{\pi/2}^{\pi} (t+1)\cos t\,dt$$

sign of $v(t) = (t+1)\cos t$

0, $\pi/6$, $\pi/2$, $\frac{5\pi}{6}$, π

$\oplus$ $\ominus$

use a)

$$= \left[(t+1)\sin t\Big|_0^{\pi/2} - (-\cos t)\Big|_0^{\frac{\pi}{2}}\right] - \left[(t+1)\sin t\Big|_{\pi/2}^{\pi} - (-\cos t)\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \left[\left(\frac{\pi}{2}+1\right) - 1\right] - \left[-\left(\frac{\pi}{2}+1\right) + (-1)\right] = \frac{\pi}{2} + \frac{\pi}{2} + 2 = \boxed{\pi+2} \text{ (meters)}$$

$\boxed{\text{25pt}}$ **5.** Evaluate the integrals. **Show your work.**

(a) $\int x^{5/2} \ln x \, dx = \frac{2}{7} x^{7/2} \ln x - \int \frac{2}{7} x^{\frac{7}{2}} \cdot \frac{1}{x} \, dx$

$u = \ln x$	$dv = x^{5/2} dx$
$du = \frac{dx}{x}$	$v = \frac{x^{7/2}}{7/2}$

$$\frac{2}{7} \int x^{\frac{5}{2}} dx = \frac{2}{7} \cdot \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + C$$

$$= \frac{2}{7} x^{7/2} \ln x - \frac{4}{49} x^{7/2} + C = \boxed{\frac{2}{7} x^{7/2} \left(\ln x - \frac{2}{7} \right) + C}$$

(b) $\int_0^2 t\, e^{-t^2} \, dt = \left(-\frac{1}{2}\right) \int_0^{-4} e^u \, du = \left(-\frac{1}{2}\right) e^u \Big|_0^{-4} = \left(-\frac{1}{2}\right)\left(e^{-4} - e^0\right) =$

$\boxed{u = -t^2} \Rightarrow du = -2t\,dt \Rightarrow t\,dt = -\frac{1}{2} du$

and $\begin{cases} t=0 \Rightarrow u=0 \\ t=2 \Rightarrow u=-4 \end{cases}$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{e^4} - 1\right) = \boxed{\frac{1}{2}\left(1 - \frac{1}{e^4}\right)}$$

(c) $\int \arctan(2x)\,dx = x\arctan(2x) - \int \frac{2x\,dx}{1+4x^2}$

$u=\arctan(2x)$	$dv = dx$
$du = \frac{1}{1+(2x)^2}\cdot 2dx$	$v = x$

$v = 1+4x^2 \Rightarrow$

$dv = 8x\,dx$

$\Rightarrow 2x\,dx = \frac{1}{4}dv$

$\int \frac{\frac{1}{4}dv}{v} = \frac{1}{4}\ln|v| = \frac{1}{4}\ln(\underbrace{1+4x^2}_{>0})$

$$\Rightarrow \int \arctan(2x)\,dx = \boxed{x\arctan(2x) - \frac{1}{4}\ln(1+4x^2) + C}$$

(d) $\int \tan^3\theta\sec^6\theta\,d\theta = \int \tan^3\theta\cdot\sec^4\theta\cdot\sec^2\theta\,d\theta =$

- $u = \tan\theta$

$\Rightarrow du = \sec^2\theta\,d\theta$

- $\sec^4\theta = (\sec^2\theta)^2 = (1+u^2)^2$

if using $1+\tan^2\theta = \sec^2\theta$

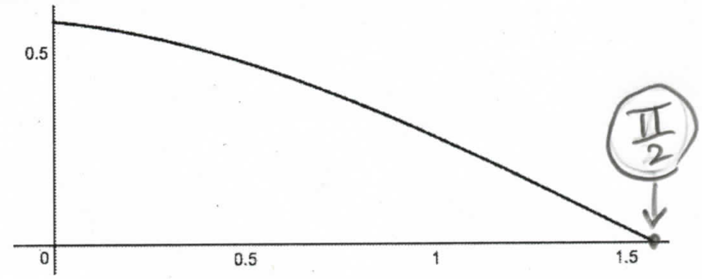
$= \int u^3(1+u^2)^2\,du$

$= \int u^3(1+2u^2+u^4)\,du$

$= \int (u^3+2u^5+u^7)\,du$

$= \frac{1}{8}u^8 + 2\cdot\frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \boxed{\frac{1}{8}\tan^8\theta + \frac{1}{3}\tan^6\theta + \frac{1}{4}\tan^4\theta + C}$

8pt **6.** Find the area of the region above the x-axis and below the curve $y = \dfrac{\cos x}{\sqrt{3 + \sin x}}$ between $x = 0$ and $x = \pi/2$.



$$\text{Area} = \int_0^{\pi/2} \frac{\cos x \, dx}{\sqrt{3 + \sin x}} = \int_3^4 \frac{du}{\sqrt{u}} = \int_3^4 u^{-1/2} \, du = \left. \frac{u^{1/2}}{1/2} \right|_3^4$$

$u = 3 + \sin x \Rightarrow du = \cos x \, dx$

$$\begin{cases} x = \frac{\pi}{2} \Rightarrow u = 3 + 1 = 4 \\ x = 0 \Rightarrow u = 3 + 0 = 3 \end{cases}$$

$$= 2\left(\sqrt{4} - \sqrt{3}\right)$$

$$= \boxed{2\left(2 - \sqrt{3}\right)}$$

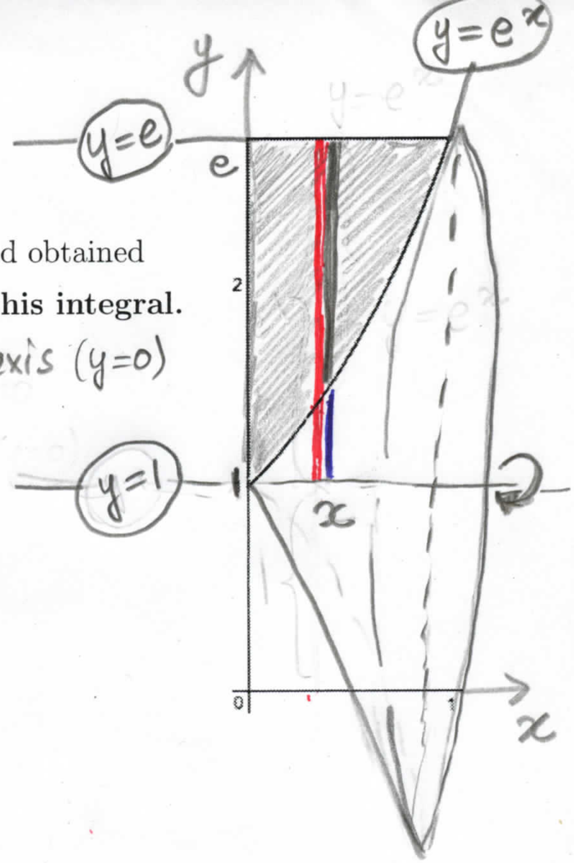
8pt **7.** Consider the region bounded by $x = 0$, $y = e^x$, and $y = e$.
This region is pictured to the right.

(a) Set up the integral that represents the volume of the solid obtained by rotating this region around $y = 1$. **DO NOT evaluate this integral.**

parallel to x-axis $(y=0)$

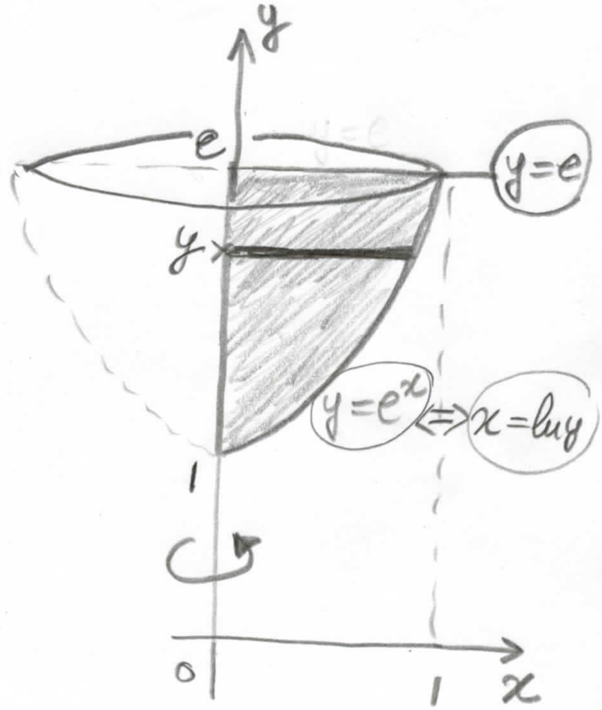
$$\begin{cases} \text{outer radius} = e-1 \\ \text{inner radius} = e^x - 1 \end{cases}$$

$$\boxed{V = \pi \int_0^1 \left[(e-1)^2 - (e^x-1)^2\right] dx}$$

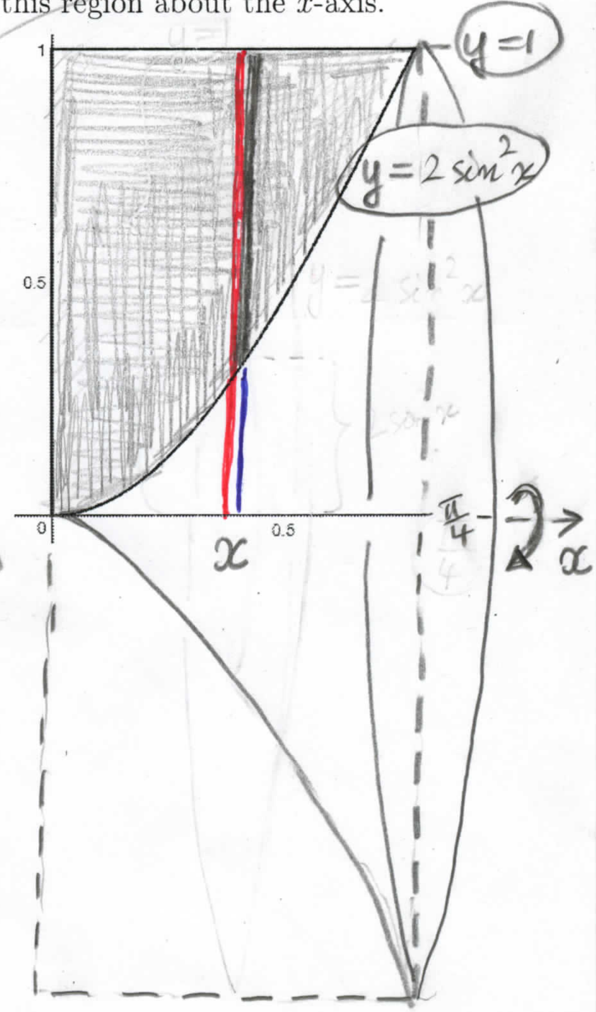


(b) Set up the integral that represents the volume of the solid obtained by rotating this region about the y-axis. **DO NOT evaluate this integral.**

$$\boxed{V = \pi \int_1^e (\ln y)^2 \, dy}$$



12pt **8.** Consider the region enclosed by the curves $y = 1$, $y = 2\sin^2 x$, $x = 0$, and $x = \pi/4$ that is pictured. Find the volume of the solid obtained by rotating this region about the x-axis.



This is the planar region that is rotated about the x-axis.

- Intersections: $y = 1 = 2\sin^2 x$

$$\Rightarrow \sin^2 x = \frac{1}{2} \Rightarrow \sin x = \pm\frac{\sqrt{2}}{2}$$

$\Rightarrow \left(\frac{\pi}{4}, 1\right)$ is one of these intersections

$$\begin{cases} \text{outer radius} = 1 \\ \text{inner radius} = 2\sin^2 x \end{cases}$$

$$V = \pi \int_0^{\pi/4} \left[1^2 - (2\sin^2 x)^2\right] dx$$

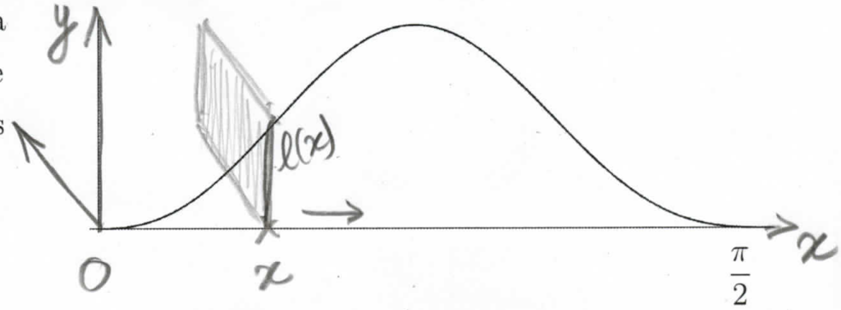
$$4\int \sin^2 x\, dx = 4\int \frac{1-\cos 2x}{2}\, dx = 2\left[x - \frac{\sin(2x)}{2}\right] + C$$

$$V = \pi \int_0^{\pi/4} \left[1 - 2x + \sin(2x)\right] dx = \pi\left[x - x^2 - \frac{\cos(2x)}{2}\right]_0^{\pi/4}$$

$$= \pi\left[\left(\frac{\pi}{4} - \frac{\pi^2}{16} - 0\right) - \left(0 - 0 - \frac{1}{2}\right)\right] = \pi\left(\frac{\pi}{4} - \frac{\pi^2}{16} + \frac{1}{2}\right)$$

$$\boxed{V = \frac{\pi}{16}\left(-\pi^2 + 4\pi + 8\right)}$$

12pt **9.** Consider the region bounded by the curve $y = 3\sin^{5/2}x\cos^3 x$ and the x-axis between $x = 0$ and $x = \pi/2$. Find the volume of a solid whose base is this region and whose cross-sections perpendicular to the x-axis are squares.



$$V = \int_0^{\pi/2} A(x)\,dx \quad \text{where } A(x) = [\ell(x)]^2 = \left(3\sin^{5/2}x \cdot \cos^3 x\right)^2$$

$$\Rightarrow V = \int_0^{\pi/2} 9\sin^5 x \cdot \cos^6 x\,dx = 9\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \cdot \sin x\,dx$$

$$V = 9\int_1^0 (1 - 2u^2 + u^4)\,u^6\,(-1)\,du = (+9)\int_0^1 (u^{10} - 2u^8 + u^6)\,du$$

- $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\begin{cases} x = \frac{\pi}{2} \Rightarrow u = \cos\frac{\pi}{2} = 0 \\ x = 0 \Rightarrow u = \cos 0 = 1 \end{cases}$$

and $\sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2 = (1-u^2)^2 = 1 - 2u^2 + u^4$

$$\Rightarrow V = 9\left(\frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7}\right)\Big|_0^1 = 9\cdot\left(\frac{1}{11} - \frac{2}{9} + \frac{1}{7}\right)$$

$$\Rightarrow V = \cancel{9}\,\frac{63 - 154 + 99}{11\cdot\cancel{9}\cdot 7} = \boxed{\frac{8}{77}}$$