Computer Vision

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1 Prove the following property of k dimensional Gaussian distribution:

$Norm_x[\mu, \Sigma] : \int Norm_x[a, A]Norm_x[b, B]dx = Norm_a[b, A + B] \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b, \Sigma_*]dx$

where $\Sigma_* = (A^{-1} + B^{-1})^{-1}$

Answer:

Definitions and Notations

The PDF of a multivariate Gaussian distribution is given by:

$$Norm_x[\mu, \Sigma] = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right),$$

where $x \in \mathbb{R}^k$, $\mu \in \mathbb{R}^k$ is the mean, and $\Sigma \in \mathbb{R}^{k\times k}$ is the covariance matrix.

We need to evaluate the integral:

$$\int Norm_x[a, A]Norm_x[b, B]\, dx.$$

Combine the Gaussian Distributions

Substituting the expressions for $Norm_x[a, A]$ and $Norm_x[b, B]$, we have:

$$\int Norm_x[a, A]Norm_x[b, B]\, dx = \int \frac{1}{(2\pi)^{k/2}|A|^{1/2}} e^{-\frac{1}{2}(x-a)^T A^{-1}(x-a)} \cdot \frac{1}{(2\pi)^{k/2}|B|^{1/2}} e^{-\frac{1}{2}(x-b)^T B^{-1}(x-b)}\, dx.$$

Combine the exponents into a single quadratic form:

$$-\frac{1}{2}(x-a)^T A^{-1}(x-a) - \frac{1}{2}(x-b)^T B^{-1}(x-b) = -\frac{1}{2}x^T(A^{-1}+B^{-1})x + x^T(A^{-1}a+B^{-1}b) + \text{constant terms.}$$

Let $\Sigma_* = \left(A^{-1} + B^{-1}\right)^{-1}$ and $\mu_* = \Sigma_* \left(A^{-1}a + B^{-1}b\right)$. The quadratic form becomes:

$$-\frac{1}{2}x^T\Sigma_*^{-1}x + x^T\Sigma_*^{-1}\mu_* - \frac{1}{2}\mu_*^T\Sigma_*^{-1}\mu_*.$$

The integral now becomes:

$$\int \frac{1}{(2\pi)^{k/2}|A|^{1/2}} \frac{1}{(2\pi)^{k/2}|B|^{1/2}} e^{-\frac{1}{2}x^T\Sigma_*^{-1}x + x^T\Sigma_*^{-1}\mu_* - \frac{1}{2}\mu_*^T\Sigma_*^{-1}\mu_*}\, dx.$$

Separate Terms

Factorize the Gaussian normalization:

$$Norm_a[b, A + B] = \frac{1}{(2\pi)^{k/2}|A + B|^{1/2}} e^{-\frac{1}{2}(a-b)^T(A+B)^{-1}(a-b)}.$$

The remaining integral is over a Gaussian distribution centered at μ_* with covariance Σ_*:

$$\int Norm_x\left[\Sigma_*\left(A^{-1}a + B^{-1}b\right), \Sigma_*\right] dx.$$

The integral over the full domain of a normalized Gaussian distribution evaluates to 1:

$$\int Norm_x \left[\Sigma_* \left(A^{-1}a + B^{-1}b \right), \Sigma_* \right] dx = 1.$$

Conclusion

Substituting everything back, we verify that:

$$\int Norm_x[a, A] Norm_x[b, B] \, dx = Norm_a[b, A + B].$$