سوال ۱)

الف)

$$H(e^{j\omega})e^{j\omega n} = be^{j\omega n} + 0.8\,H(e^{j\omega})e^{j\omega(n-1)}$$
$$-0.81\,H(e^{j\omega})\,e^{j\omega(n-2)}$$

$$\rightarrow H(e^{j\omega})\left[e^{j\omega n} - 0.8\,e^{j\omega(n-1)} + 0.81e^{j\omega(n-2)}\right] = be^{j\omega n}$$

$$\rightarrow H(e^{j\omega}) = \frac{b}{1-0.8e^{-j\omega}+0.81e^{-j\omega 2}}$$

ب)

$$|H(e^{j\omega})| = \frac{|b|}{|1-0.8e^{-j\omega}+0.81e^{-j\omega 2}|}$$

$$|1-0.8e^{-j\omega}+0.81e^{-j\omega 2}| = \Big|\left[1-0.8\cos(\omega)+0.81\cos(2\omega)\right]$$
$$+j\left[0.8\sin(\omega)-0.81\sin(2\omega)\right]\Big|$$

$$\overset{*}{=} \sqrt{\left[1-0.8\cos(\omega)+0.81\cos(2\omega)\right]^2+\left[0.8\sin(\omega)-0.81\sin(2\omega)\right]^2}$$

مقدار مینیمم این ترم را می‌خواهیم $\Leftarrow$ مشتق می‌گیریم و برابر صفر قرار می‌دهیم (نسبت به ω)

$$\rightarrow \text{حاصل مشتق} = \frac{1}{\cancel{2}\times *} \times \Big[\cancel{2}\times\big(1-0.8\cos(\omega)+0.81\cos(2\omega)\big)\times\big(0.8\sin(\omega)-1.62\sin(2\omega)\big)$$
$$+\cancel{2}\times\big(0.8\sin(\omega)-0.81\sin(2\omega)\big)\times\big(0.8\cos(\omega)-1.62\cos(2\omega)\big)\Big]$$

حاصل مشتق را برابر صفر قرار می‌دهیم.

$$0.8\sin w - 1.62\sin 2w - 0.64\cancel{\cos w\sin w} + 1.286\cos w\sin 2w$$

$$+\,0.648\cos 2w\sin w - 1.3122\cancel{\cos 2w\sin 2w}$$

$$+\,0.64\cancel{\sin w\cos w} - 1.286\sin w\cos 2w - 0.648\sin 2w\cos w$$

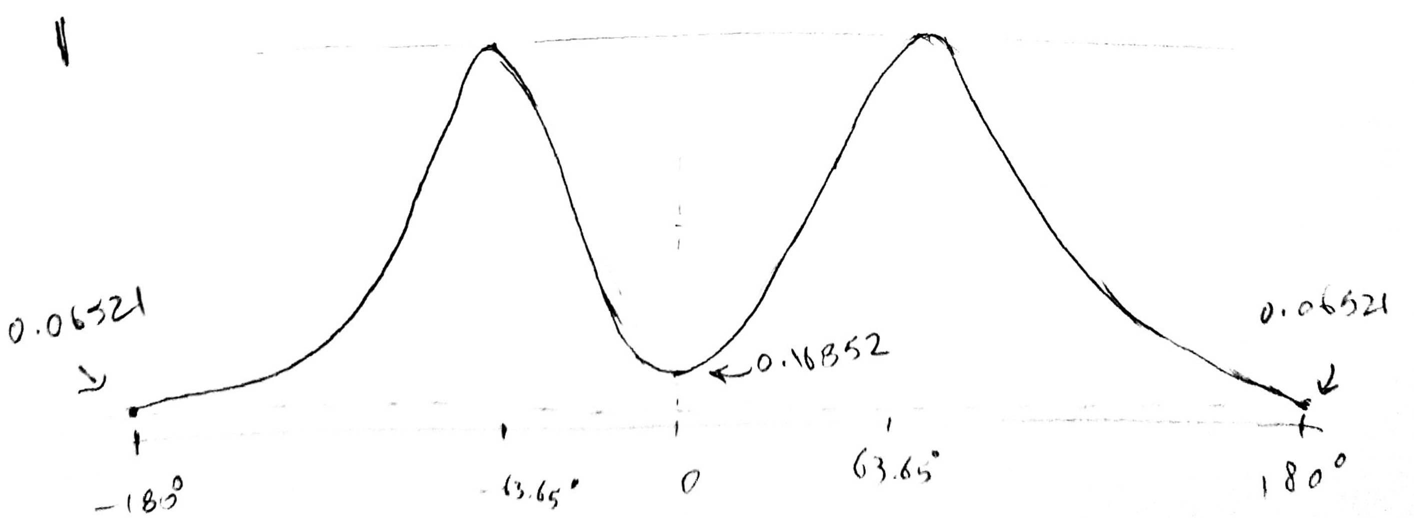
$$+\,1.3122\cancel{\cos 2w\sin 2w} = 0$$

$$\xrightarrow[\cos 2w = 2\cos^2 w - 1]{\sin 2w = 2\sin w\cos w} \sin w \times \Big[0.8 - 3.24\cos w + \cancel{2.572\cos^2 w} - 0.648$$

$$+\,\cancel{1.296\cos^2 w} + 1.286 - \cancel{2.572\cos^2 w} - \cancel{1.296\cos^2 w}\Big] = 0$$

$$\Rightarrow \sin w \times \big[-3.24\cos w + 1.438\big] = 0 \begin{cases} \sin w = 0 \rightarrow \underline{w = 0} \\ \cos w = \dfrac{1.438}{3.24} \rightarrow \begin{cases} w = 63.65^\circ \\ w = -63.65^\circ \end{cases} \end{cases}$$

← مقدار در $w=0$ یک مینیمم محلی است و برای $w=\pm 63.65^\circ$ ماکزیمم است



مقادیر مورد نظر را:

قبل از آن، مقدار b را بدست می‌آوریم:

$$|b| = \left|1 - 0.8e^{-j(63.65^\circ)} + 0.81e^{-j(2\times 63.65^\circ)}\right|$$

$$= 0.170205$$

مقدار در $\omega = 0$:

$$|H(j\omega)|_{\omega=0} = \frac{0.170205}{\left|1-0.8e^{-j0}+0.81e^{-j0}\right|} = \frac{0.170205}{|1-0.8+0.81|} = 0.16852$$

مقدار در $\omega = \pm 180^\circ$:

$$|H(j\omega)|_{\omega=180^\circ} = \frac{0.170205}{\left|1-0.8e^{-j180^\circ}+0.81e^{-j360}\right|} = \frac{0.170205}{|1+0.8+0.81|} = 0.06521$$

ب)

$$x[n] = 2\cos\left(\frac{n\pi}{3} + 45^\circ\right)$$

$$\longrightarrow A_x = 2,\ \omega = \frac{\pi}{3},\ \phi_x = 45^\circ = \frac{\pi}{4}$$

$$\xrightarrow{(5-12)} y[n] = A_x \left|H(e^{j\omega})\right| \cos\left[\omega n + \phi_x + \measuredangle H(e^{j\omega})\right]$$

$$= 2 \times \frac{0.170205}{\left|1-0.8e^{-j\frac{\pi}{3}}+0.81e^{-j\frac{2\pi}{3}}\right|} \cos\left[\frac{\pi}{3}n + \frac{\pi}{4} + \measuredangle H(e^{j\omega})\right]$$

$$\measuredangle H(e^{j\omega}) = -\measuredangle\left(1-0.8e^{-j\frac{\pi}{3}}+0.81e^{-j\frac{2\pi}{3}}\right) = -\tan^{-1}\left(\frac{0.8\sin\frac{\pi}{3} - 0.81\sin\frac{2\pi}{3}}{1-0.8\cos\frac{\pi}{3}+0.81\cos\frac{2\pi}{3}}\right)$$

$$= -\tan^{-1}\left(\frac{0.01\times\frac{\sqrt{3}}{2}}{1-0.01\times\frac{1}{2}}\right) = -\tan^{-1}\left(\frac{\sqrt{3}}{199}\right) \simeq -0.5^\circ$$

$$y[n] = 0.08645 \cos\left(\frac{\pi}{3} n + 44.5^\circ\right)$$

جواب ۲)

الف)

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) \frac{1}{1-(0.1)e^{-j\omega}} e^{j\omega n} \, d\omega$$

همچنین $y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega}) e^{j\omega n} \, d\omega$

$$= (0.25^n + 0.2^n) u[n]$$

$$\rightarrow Y(e^{j\omega}) = \frac{1}{1-(0.25)e^{-j\omega}} + \frac{1}{1-(0.2)e^{-j\omega}}$$

$$\rightarrow H(e^{j\omega}) = \frac{1}{1-(0.1)e^{-j\omega}} \times \frac{(1-0.25)e^{-j\omega})(1-(0.2)e^{-j\omega})}{1-(0.25)e^{-j\omega} + 1-(0.2)e^{-j\omega}}$$

$$= \frac{1}{1-(0.1)e^{-j\omega}} \times \frac{(1-0.25e^{-j\omega})(1-0.2e^{-j\omega})}{2-(0.45)e^{-j\omega}}$$

$$= \frac{(1-0.25e^{-j\omega})(1-0.2e^{-j\omega})}{(1-0.1e^{-j\omega})(2-0.45e^{-j\omega})}$$

Linearity: $a x_1[n] + b x_2[n] \xrightarrow{\mathcal{H}} a y_1[n] + b y_2[n]$ ✓

Time-Invariance: $x[n-n_0] = (0.25^{n-n_0} + 0.2^{n-n_0}) u[n-n_0] \xrightarrow{\mathcal{H}} 0.1^{n-n_0} u[n-n_0] = y[n-n_0]$ ✓

Stability: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$H(e^{j\omega}) = 1 + \frac{\frac{245}{900} e^{-j\omega} + \frac{11}{9}}{(1-0.1e^{-j\omega})(2-0.45e^{-j\omega})}$$

$$H(e^{j\omega}) = 1 + \frac{5.4}{1-0.1e^{-j\omega}} + \frac{0.1}{1-0.225e^{-j\omega}}$$

$$\Rightarrow h[n] = \delta[n] + (5.4)(0.1)^n u[n] + (0.1)(0.225)^n u[n]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = 1 + (5.4)\times\sum_{n=0}^{\infty}(0.1)^n + (0.1)\times\sum_{n=0}^{\infty}(0.225)^n$$

$$= 1 + 5.4\times\frac{1}{1-0.1} + 0.1\times\frac{1}{1-0.225}$$

$$= 1 + \frac{54}{9} + \frac{100}{775} < \infty$$

از آنجا که فاز و اندازه پاسخ فرکانسی به صورت یکتا بدست آمده‌اند، جواب یکتا است.

ب) چون تغییر فرکانس داریم، یک سیستم LTI نمی‌تواند وجود داشته باشد.

پ) طبق نتیجه کتاب در صفحه 203 یک سیستم LTI پایدار به صورت یکتا وجود دارد که دامنه و فاز آن به صورت زیر خواهد بود:

$$y[n] = A_x |H(e^{j\omega})| \cos[\omega n + \phi_x + \angle H(e^{j\omega})]$$

که: $x[n] = A_x \cos[\omega n + \phi_x]$

ابتدا $x[n]$ را به cos تبدیل می‌کنیم.

$$x[n] = \cos\left[\frac{\pi n}{3} - \frac{\pi}{2}\right]$$

$$\Rightarrow |H(e^{j\omega})| = 5, \quad \angle H(e^{j\omega}) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

$$\rightarrow H(e^{j\omega}) = 5e^{-j\frac{\pi}{4}}$$

جواب ۳) الف)

چند تعریف انجام می‌دهیم:

$$H_1(e^{j\omega}) = 1 \quad \text{for } 0<|\omega|<\pi$$

$$H_2(e^{j\omega}) = \begin{cases} 1 & , 0<|\omega|<\frac{7\pi}{8} \\ 0 & , \text{o.w.} \end{cases}$$

$$H_3(e^{j\omega}) = \begin{cases} 1 & , 0<|\omega|<\frac{5\pi}{8} \\ 0 & , \text{o.w.} \end{cases}$$

$$H_4(e^{j\omega}) = \begin{cases} 1 & , 0<|\omega|<\frac{3\pi}{8} \\ 0 & , \text{o.w.} \end{cases}$$

$$H_5(e^{j\omega}) = \begin{cases} 1 & , 0<|\omega|<\frac{\pi}{8} \\ 0 & , \text{o.w.} \end{cases}$$

حال $H(e^{j\omega})$ را بازنویسی می‌کنیم:

$$H(e^{j\omega}) = \left[\frac{1}{3} H_1(e^{j\omega}) - \frac{1}{3} H_2(e^{j\omega}) + \frac{2}{3} H_3(e^{j\omega}) - \frac{2}{3} H_4(e^{j\omega}) + H_5(e^{j\omega})\right] e^{-j\omega n_d}$$

$$\rightarrow h[n] = \delta[n-n_d] * \left[\frac{1}{3}\delta[n] - \frac{1}{3}\frac{\sin\frac{7\pi}{8}n}{\pi n} + \frac{2}{3}\frac{\sin\frac{5\pi}{8}n}{\pi n} - \frac{2}{3}\frac{\sin\frac{3\pi}{8}n}{\pi n} + \frac{\sin\frac{\pi}{8}n}{\pi n}\right]$$

$$h[n]=\frac{1}{3}\delta[n-n_d]-\frac{1}{3}\,\frac{\sin\frac{7\pi}{8}(n-n_d)}{\pi(n-n_d)}+\frac{2}{3}\,\frac{\sin\frac{5\pi}{8}(n-n_d)}{\pi(n-n_d)}$$

$$-\frac{2}{3}\,\frac{\sin\frac{3\pi}{8}(n-n_d)}{\pi(n-n_d)}+\frac{\sin\frac{\pi}{8}(n-n_d)}{\pi(n-n_d)}$$

HW9_Ans3B.py (ب

$$h[n]=\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\frac{1}{3}e^{j\omega n}\,d\omega+\int_{-\frac{7\pi}{8}}^{\frac{7\pi}{8}}-\frac{1}{3}e^{j\omega n}\,d\omega+\int_{-\frac{5\pi}{8}}^{\frac{5\pi}{8}}\frac{2}{3}e^{j\omega n}\,d\omega\right.$$

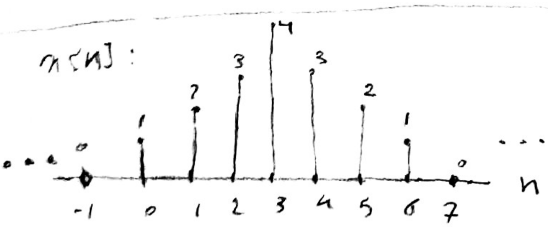
$$\left.+\int_{-\frac{3\pi}{8}}^{\frac{3\pi}{8}}-\frac{2}{3}e^{j\omega n}\,d\omega+\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}e^{j\omega n}\,d\omega\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{3jn}\left(e^{j\pi n}-e^{-j\pi n}\right)-\frac{1}{3jn}\left(e^{j\frac{7\pi}{8}n}-e^{-j\frac{7\pi}{8}n}\right)\right.$$

$$+\frac{2}{3jn}\left(e^{j\frac{5\pi}{8}n}-e^{-j\frac{5\pi}{8}n}\right)-\frac{2}{3jn}\left(e^{j3\frac{\pi}{8}n}-e^{-j3\frac{\pi}{8}n}\right)$$

$$\left.+\frac{1}{jn}\left(e^{j\frac{\pi}{8}n}-e^{-j\frac{\pi}{8}n}\right)\right]$$

جواب ج)



$$c_k(x)=\frac{1}{N}\sum_{n=0}^{N-1}x[n]\,e^{-j\frac{2\pi}{N}kn}$$

$$=\frac{1}{8}\left[1+2e^{-j\frac{2\pi}{8}k}+3e^{-j\frac{4\pi}{8}k}+4e^{-j\frac{6\pi}{8}k}+3e^{-j\pi k}+2e^{-j\frac{10\pi}{8}k}+e^{-j\frac{12\pi}{8}k}\right]$$

الف)

$$H_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$H_1(e^{j\frac{2\pi}{N}k}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\frac{2\pi}{N}kn} = 4 - e^{-j\frac{2\pi}{8}k} + e^{-j\frac{4\pi}{8}k} + 3e^{-j\frac{6\pi}{8}k} + 6e^{-j\pi k}$$
$$+ 3e^{-j\frac{10\pi}{8}k} + e^{-j\frac{12\pi}{8}k} - e^{-j\frac{14\pi}{8}k}$$

$$\rightarrow c_k(y) = H_1(e^{j\frac{2\pi}{N}k})\, c_k(x)$$

$$c_0 = 32,\quad c_1 = 6.54e^{-j\frac{\pi}{4}},\quad c_2 = 0,\quad c_3 = 0.54e^{j\frac{\pi}{4}}$$

$$c_4 = 0,\quad c_5 = 0.54e^{-j\frac{\pi}{4}},\quad c_6 = 0,\quad c_7 = 6.54e^{j\frac{\pi}{4}}$$

$$y_{ss}[n] = \sum_{k=0}^{7} c_k^{(y)}\, e^{j\frac{2\pi}{8}kn} \rightarrow \text{Stemed in HW9-Ans4.py}$$

← با مقایسه $c_k(x)$ و $c_k(y)$ مشاهده می‌شود که هم اعوجاج دامنه داریم و هم اعوجاج فاز

ب)

$$H_2(e^{j\omega}) = 5e^{j\frac{\pi}{4}} \rightarrow H_2(e^{j\frac{2\pi}{N}k}) = 5e^{j\frac{\pi}{4}}$$

$$c_k(y) = H_2(e^{j\frac{2\pi}{N}k})\, c_k(x)$$

$$c_0 = 10e^{j\frac{\pi}{4}},\quad c_1 = 4.27e^{j\pi},\quad c_2 = 0,\quad c_3 = 0.73e^{j\frac{\pi}{2}}$$

$$c_4 = 0,\quad c_5 = 0.73,\quad c_6 = 0,\quad c_7 = 4.27e^{-j\frac{\pi}{2}}$$

$$y_{ss}[n] = \sum_{k=0}^{7} c_k(y)\, e^{j\frac{2\pi}{8}kn} \rightarrow \text{Stemed in HW9-Ans4.py}$$

اعوجاج فاز و دامنه نداریم ← بدون اعوجاج

پ)

$$H_3(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$H_3(e^{j\frac{2\pi}{N}k}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\frac{2\pi}{N}kn} = 1 + \frac{1}{2}e^{-j\frac{2\pi}{8}k} + \frac{1}{4}e^{-j\frac{4\pi}{8}k} + \frac{1}{8}e^{-j\frac{6\pi}{8}k}$$
$$+ \frac{1}{16}e^{-j\pi k}$$

$$C_k(y) = H_3\left(e^{j\frac{2\pi}{N}k}\right) C_k(x)$$

$$C_0 = 1.4375,\ C_1 = 0.51e^{j0.65\pi},\ C_2 = 0,\ C_3 = 0.21e^{j0.15\pi}$$

$$C_4 = 0,\ C_5 = 0.21e^{-j0.15\pi},\ C_6 = 0,\ C_7 = 0.51e^{-j0.65\pi}$$

$$y_{ss}[n] = \sum_{k=0}^{7} C_k^{(y)} e^{j\frac{2\pi}{8}kn}$$ → Stemed in HW9-Ans4.py

← با مقایسه $C_k^{(y)}$ و $C_k^{(x)}$ ها مشاهده می‌شود که هم اعوجاج فاز داریم و هم اعوجاج دامنه

در این سیستم $C_k^{(x)}$ برابر بود با:

$$C_0 = 2,\ C_1 = 0.85e^{j\frac{3\pi}{4}},\ C_2 = 0,\ C_3 = 0.15e^{j\frac{\pi}{4}}$$

$$C_4 = 0,\ C_5 = 0.15e^{-j\frac{\pi}{4}},\ C_6 = 0,\ C_7 = 0.85e^{-j\frac{3\pi}{4}}$$