On the derangement of n-card decks and their subsets

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February 20th, 2018

Abstract

This paper addresses the generalization of the French card game *rencontres*. Specifically, we solve the problem of determining the probability that no two pairs of cards in two 52-card decks will match when both decks are laid out on top of each other. We show that this problem can be solved using the combinatorial action of permuting a set such that no element is in its original position. We go on to find the solution for an n-card deck and show that there exists a closed form

for our solution. Finally, we explore the possibility where we have n decks instead of two.

1 Introduction

In an old French card game known as *rencontres* (matching), 52 cards are laid out in a row. The cards of a second deck are laid out on top of the cards of the first deck. The score is determined by counting the matching number of pairs in the two decks. In 1708, Pierre Raymond de Monte Mort posed *le problem de rencontres*: what is the probability that no matches take place in the game of *rencontres*?

We can restate the problem as follows. Suppose two decks A and B of cards are given. The cards of A are first laid out in a row, and those of B are then placed at random, one at the top on each other card of A so that 52 pairs of cards are formed. What is the probability that no 2 cards are the same in each pair? In other words, how many ways can we arrange a 52-card deck such that no pairs arise?

2 The Closed Form of Calculating the Derangements of an n-card Deck

We will show the closed form for the number of derangements of an n-card deck along with its proof. We start by introducing the two following definitions.

Definition 1. [5] A *derangement* of $\{1, 2, \ldots, n\}$ is a permutation $(i_1 i_2 \ldots i_n)$ of $\{1, 2, \ldots, n\}$ such that $i_1 \neq 1, i_2 \neq 2, \ldots, i_n \neq n$. Thus a derangement of $\{1, 2, \ldots, n\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that no integer is in its natural position.

Theorem 2. *[5] (Inclusion-Exclusion Principle) For finite sets $A_1, A_2, \ldots, A_n$,*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \tag{1}$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n| \tag{2}$$

where $|A|$ denotes the cardinality of a finite set A.

Now we give a closed form for the number of a derangements of a finite set.

Theorem 3. *[5] The number $\mathcal{D}_n$ of derangements of a set $A = \{1, \ldots, n\}$. Given by,*

$$\mathcal{D}_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Proof. Let us denote by S the set of all all $n!$-permutations of $A = \{1, \ldots, n\}$. Let P_k be the property that for a permutation k all $i_k = k$ for $k \in C$. A *derangement* is a permutation such that $i_k \neq k$ for all $k \in A$. We denote by A_k the set of all permutations of C with property P_k; thus, the derangements of A are equivalent to the permutations in $\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$ (here, $\overline{A_x}$ is the complement of the set $A_x \in S$). Therefore, we see that $\mathcal{D}(n) = \left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right|$, and we can use the inclusion-exclusion principle to determine $\mathcal{D}(n)$. The permutations of A_1 are of the form $1, i_2, \ldots, i_n$ where $i_2 \ldots i_n$ is a permutation of $\{2, \ldots, n\}$. Therefore, $|A_1| = (n-1)!$ and it is clear that

$$|A_k| = (n-1)! \tag{3}$$

for all $k \in \{1, \ldots, n\}$. The permutations of $A_1 \cap A_2$ are of the form $1, 2i_3, \ldots, i_n$ where $i_3, \ldots, i_n$ is a permutation of $\{3, \ldots, n\}$. Therefore $|A_1 \cap A_2| = (n-2)!$, and it is clear that

$$|A_j \cap A_j| = (n-2)! \tag{4}$$

for all $i \neq j$ where $i, j \in A$. More generally,

$$|A_{i_1} \cap A_{i_2} \cap A_{i_k} \cap \cdots| = (n-k)! \tag{5}$$

for $\{i_1, i_2, \ldots, i_k\}$ which is a subset of $\{1, 2, \ldots, n\}$. Because we have $\binom{n}{k}$ subsets of $\{1, 2, \ldots, n\}$, we find via the inclusion-exclusion principle that

$$\begin{aligned}
\mathcal{D}_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! && (6)\\
&= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} && (7)\\
&= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right). && (8)
\end{aligned}$$

□

Theorem 4. *If $[x]$ is the nearest integer function, Theorem 3 implies that*

$$\mathcal{D}_n = \left[\frac{n!}{e}\right]$$

for $n \geq 1$.

Proof. Note that

$$\mathcal{D}_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{9}$$

And

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}. \tag{10}$$

Thus, multiplying both sides of (10) by $n!$ yields

$$n! \cdot e^{-1} = n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}. \tag{11}$$

Now, let us consider the difference of (11) and (9), which is

$$n! \cdot e^{-1} - \mathcal{D}_n = n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \tag{12}$$

Now, let us rewrite the summation as an infinite sum

$$n! \cdot e^{-1} - \mathcal{D}_n = n! \left(\frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \frac{(-1)^{n+3}}{(n+3)!} + \cdots \right). \tag{13}$$

Distribute the $n!$ to get

$$n! \cdot e^{-1} - \mathcal{D}_n = \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} + \frac{(-1)^{n+3}}{(n+1)(n+2)(n+3)} + \cdots. \tag{14}$$

Applying exponent properties, we see that $(-1)^{n+2} = -(-1)^{n+1}$, allowing us to rewrite (16) as

$$n! \cdot e^{-1} - \mathcal{D}_n = \frac{(-n)^{n+1}}{n+1} \left(1 - \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} - \ldots \right). \tag{15}$$

Notice that the right hand side is an alternating series. This series converges by the alternating series test. We say that an alternating series $\sum(-1)^n a_n$ converges if (a_n) converges to 0 monotonically. In this instance, we have

$$\begin{aligned} a_{n+1} &= \frac{1}{n+1} \\ a_{n+2} &= \frac{1}{(n+1)(n+2)} \\ &\vdots \\ a_{n+k} &= \frac{1}{(n+1)(n+2)+\dots(n+k)}. \end{aligned}$$

Clearly, $a_{n+k} \leq a_{n+k-1}$ which implies that the series is monotonically decreasing. In fact, $(a_m) \to 0$ as $m \to \infty$. Thus we can say

$$\left|\sum_{k=0}^{\infty}(-1)^k a_k - \sum_{k=0}^{n}(-1)^k a_k\right| = \left|\sum_{k+1}^{\infty}(-1)^k a_k\right| \leq \left|a_{n+1}\right|. \tag{16}$$

Our series is upper bounded by the first term and lower bounded by the sum of the first two terms. That is,

$$\frac{1}{n+2} \leq |n! \cdot e^{-1} - \mathcal{D}_n| \leq \frac{1}{n+1}. \tag{17}$$

For all $n \geq 1$, the difference is bounded by at most $\frac{1}{2}$. Since $\mathcal{D}_n$ is an integer and its difference from $n! \cdot e^{-1}$ less than $\frac{1}{2}$, rounding $n! \cdot e^{-1}$ to the nearest integer gives $\mathcal{D}_n$. Therefore,

$$\mathcal{D}_n = \left[\frac{n!}{e}\right]. \tag{18}$$

□

Corollary. The probability that there are no two pairs when one deck of n cards is randomly placed on top of another deck is

$$\frac{\mathcal{D}_n}{n!} = \frac{\left[\frac{n!}{e}\right]}{n!} \approx \frac{1}{e}. \tag{19}$$

3 The 52-Card Example

Given that we have found the closed form for $\mathcal{D}_n$ with (21), we can conclude that there are

$$\mathcal{D}_{52} = \left[\frac{52!}{e}\right] \tag{20}$$

ways to arrange a 52 card deck such that no two cards pair up when one deck is randomly placed on top of another. Because we know there are 52! ways to arrange 52 cards, the probability of not forming a pair when one deck of 52 cards is randomly placed on top of another is

$$\frac{\left[\frac{52!}{e}\right]}{52!} \approx \frac{1}{e}. \tag{21}$$

In fact, as the number of cards in the deck increases, the probability of no pairs forming approximates $\frac{1}{e}$ more accurately. In particular, in the 52-card case, the probability of obtaining no matches is accurate to about 70 decimal points. That is,

$$\frac{\mathcal{D}_{52}}{52!} - \frac{1}{e} = 2.2966996003547482580910\,6 \cdots \times 10^{-70}. \tag{22}$$

4 The Case of Multiple Decks of n Cards

A natural question would be to ask what the generalization of our problem would be with more than two decks. That is, if we laid out k shuffled decks as we did the two in the original *recontres* problem, what would the possibility be that no cards in the same position have the same value as each other? To investigate this problem, we utilized the concept of Latin rectangles.

Definition 5. [4] A Latin rectangle is a $k \times n$ matrix where the entries in each row and column are distinct.

We can represent our laid out decks of cards as a $k \times n$ matrix such that the ith row is the ith deck, the jth column is all the cards in the jth position, and each element is a card. By doing this, we can see our research question is essentially asking what the probability is that a deck configuration is a Latin rectangle. For example, our original *recontres* problem is the same as the Latin rectangle problem for the $n = 2, k = 52$ case.

Since the first deck in our problem is not shuffled, we are looking for a very specific type of Latin rectangle known as a normalized Latin rectangle.

Definition 6. [4] A normalized Latin rectangle has first row $[1, \ldots, n]$.

Additionally, for the purposes of our paper, we introduce a specific type of normalized Latin rectangle.

Definition 7. [4] A Latin rectangle is *reduced* when the first row is $[1, 2, \ldots, n]$ and the first column is $[1, 2, \ldots, k]^T$ (where X^T is the transpose of the matrix X).

We can use these concepts to find an answer to our question. Letting $K_{k,n}$ and $R_{k,n}$ represent the number of normalized and reduced Latin rectangles for n k-card decks, respectively, we have by Stones et al. (4) that

Theorem 8. [4].

$$K_{k,n} = \frac{(n-1)!}{(n-k)!} R_{k,n}. \tag{23}$$

Therefore, using this equation, we can see that the probability that an arrangement of n k-card decks would be

$$\frac{K_{k,n}}{M_{kn}} = \frac{(n-1)!}{M_{kn}(n-k)!} R_{k,n}.$$

5 Further Research Questions

Let us assume that A and B are two decks of 52 cards, where $X_1, X_2, \ldots, X_{52}$ are all (possibly empty) subsets of B. We want to determine the probability that the cards in B are placed on top of the cards of A such that $X_i \neq i$ for the ith card of B where $i = 1, 2, \ldots, 52$.

Initial Thoughts

We can rephrase the scenario in terms of having 52 piles in which we can put up to 52 cards in each pile. We are, in effect, placing each pile on top of each card in A. We want to ensure that no pile has the same card as the card it is currently laying on. For example, if we have a pile containing the queen of hearts from B, we want to make sure that this pile is not placed on top of the queen of hearts in A.

Given these conditions, let us first establish that we have 51 choices for each pile, as we must remove the matching card. It is also important to note that some of our piles can be empty. For example, we can have a pile with no cards on top of the queen of hearts, which automatically guarantees that we do not have a pair. Because of this, empty piles are optimal for not forming matches.

Therefore, the probability of not having pairs given that B is composed of 52 possibly empty subsets is

$$\left(\frac{51}{52}\right)^{52}.$$

6 Research Questions

What happens if we want to find pairs, triplets, etc. given n decks? For example, what if we want a triplet of cards given 5 decks?

7 Conclusion

In our research, we were able to determine the number of derangements in a 52-card deck and generalized our findings to an n-card deck using the concept of normalized Latin rectangles. We were also able to determine the probability of obtaining no matches for n decks of k cards. Lastly, we proposed two research questions and an initial example of the first question.

References

[1] D.P. Apte. *Probability and Combinatorics.* Excel Books India, 2007.

[2] E. Gaughan. *Introduction to Analysis.* Brooks/Cole Publishing Co., 5th ed., 1998.

[3] H.J. Ryser. "Latin Rectangles." §3.3 in *Combinatorial Mathematics.* Buffalo, NY; MAA., pp. 35-37, 1963.

[4] Rebecca J. Stones et al. "On Computing the Number of Latin Rectangles." *Graphs and Combinatorics.* May 2016, Volume 32, Issue 3, pp 11871202.

[5] Richard A. Brualdi. *Introductory Combinatorics.* North-Holland, New York, New York, 1977.