On the generalized derangement problem of n-card decks and variations

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The Historical Problem

Rencontres

- Old French card game
- 52 cards are laid out in a row
- A second deck of 52 cards is placed on top (randomly)
- Score is determined by counting the number of matching pairs
- 1708 - Pierre Raymond de Mort poses a question: what is the probability that no matches take place?

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Rencontres Restated

Rencontres: *Restated*

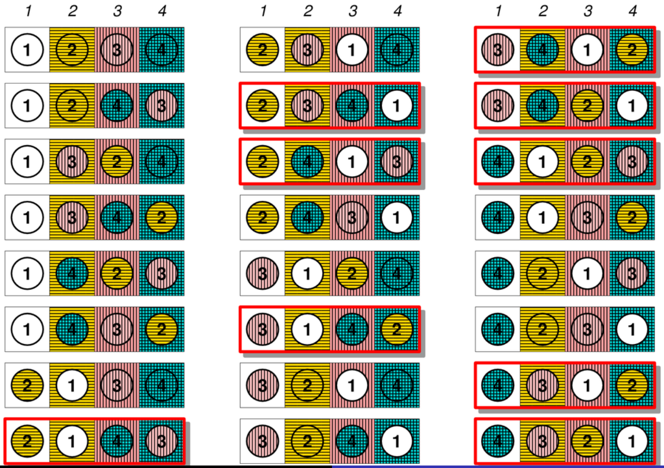
- Two decks A and B of cards are given
- Cards in A are laid out in a row
- Cards in B are placed at random on top of each card of A
- What is the probability that no 2 cards are the same in each pair?

What is a derangement?

Definition

A *derangement* of $\{1, 2, \ldots, n\}$ in which the location of each of the integers is a permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ such that $i_1 \neq 1, i_2 \neq 2, \ldots, i_n \neq n$. Thus a derangement of $\{1, 2, \ldots, n\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that no integer is in its natural position.

Derangement Example



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Inclusion-Exclusion Principle

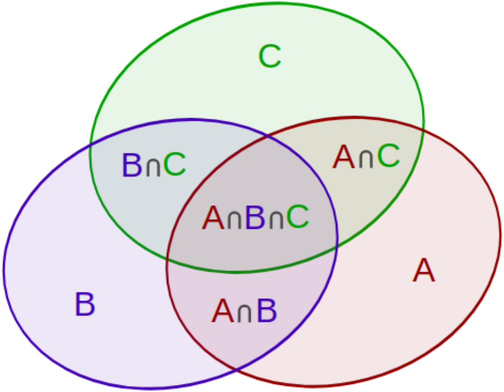
Theorem

For finite sets $A_1, A_2, \ldots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^n \left|A_1 \cap \cdots \cap A_n\right|.$$

where $|A|$ is the cardinality of a set A.

Inclusion-Exclusion Principle



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A Formula for Calculating Derangements

Theorem

$$\begin{aligned}
\mathcal{D}_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\
&= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\
&= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).
\end{aligned}$$

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A Closed Form for Calculating Derangements

The Taylor Series expansion of e^x for $x = -1$ gives us

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{k!} = \frac{1}{e}.$$

Therefore, the theorem follows.

Theorem

$$\mathcal{D}_n = \left[\frac{n!}{e}\right]$$

for $n \geq 1$ where $[x]$ represents the nearest integer to x and e is the base of the natural logarithm.

Why the nearest integer function?

Our alternating series is bounded above by the first term and bounded below by the sum of the first two terms. That is,

Bounded Series

$$\frac{1}{n+2} \leq |n! \cdot e^{-1} - \mathcal{D}_n| \leq \frac{1}{n+1}.$$

This motivates our use of the nearest integer function in our closed form.

The 52-Card Example

Given that we have found the closed form for $\mathcal{D}_n$ we can conclude that there are

$$\mathcal{D}_{52} = \left[\frac{52!}{e}\right]$$

ways to arrange a 52 card deck such that no two cards pair up when one deck is randomly placed on top of another. Because we know there are 52! ways to arrange 52 cards, the probability of not forming a pair when one deck of 52 cards is randomly placed on top of another is

$$\frac{\left[\frac{52!}{e}\right]}{52!} \approx \frac{1}{e}.$$

The 52-Card Example cont'd.

Note that for larger decks, the probability of obtaining no matches approaches $\frac{1}{e}$. In particular, in the 52-card case, the probability of obtaining no matches is accurate to about 70 decimal points. In fact,

$$\mathcal{D}_{52} - \frac{1}{e} = 2.29669960035474825809106 \cdots \times 10^{-70}.$$

The Case of n-decks

n-decks

What if we have n decks of k cards instead of two decks of 52 cards?

To solve this problem, we utilized the concept of normalized Latin rectangles.

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Latin Rectangles Defined

Definition

A Latin rectangle is a $k \times n$ matrix where the entries in each row and column are distinct.

- Our original problem of n decks can be interpreted as a Latin rectangle of size $52 \times n$, ensuring that no cards with the same "value" fall in the same row when placed on top of each other.
- For example, the case when $n = 2$ brings us back to our original problem.

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Latin Rectangle Example

LATIN RECTANGLE

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

NOT A LATIN RECTANGLE

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Normalized and Reduced Latin Rectangles

Definition

A normalized Latin rectangle has first row $\{1, \ldots, n\}$.

Definition

A Latin rectangle is *reduced* when the first row is $\{1, 2, \ldots, n\}$ and the first column is $\{1, 2, \ldots, k\}^T$.

The number of $k \times n$ normalized Latin rectangles is given by

$$K_{k,n} = \frac{(n-1)!}{(n-k)!} R_{k,n}.$$

where $K_{k,n}$ and $R_{k,n}$ represent the number of normalized and reduced Latin rectangles, respectively.

Normalized and Reduced Latin Rectangles cont'd.

Determining Probability of No Matches

We now conclude that the probability of having no matches given k n-card decks is

$$\frac{K_{k,n}}{L_{k,n}}$$

where $L_{k,n}$ represents all Latin rectangles of size $k \times n$.

Research Questions for Further Exploration

Research Questions

Given decks A and B with n-cards, respectively, how many (possibly empty) partitions of B are there such that $i \notin X_i$ for $i \in \{1, \ldots, n\}$ where X_i represents the ith partition of B?

Partition Example

Table: Partition Example when $n = 4$ and $k = 2$

B	1	2	3	4
X_i	{2,3}	$\emptyset$	{4}	{1}

Conclusion

In our research, we were able to determine the number of derangements in a 52-card deck and generalized our findings to an n-card deck using the concept of normalized Latin rectangles. We were also able to determine the probability of obtaining no matches for n decks of k cards. Lastly, we proposed two research questions and an initial example of the first question.

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