Optimization Algorithm Design via Electric Circuits

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Motivation

- classical optimization algorithm design: based on worst-case guarantees
  - provably convergent but often conservative and slow in practice
- modern ML optimizers: tuned for fast empirical performance
  - often lack convergence guarantees and may not converge under convexity

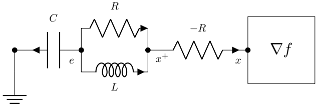
goal

- principled way to design algorithms that are fast and provably convergent

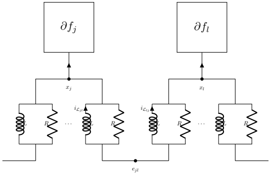
Principled optimization algorithm design

- reframe optimization algorithms as circuit diagrams: RLC circuits composed with nonlinear resistor ∇f
- replace heavy differential relations with modular diagrams governed by local rules
  - similar to how Feynman diagrams make computation in QFT easier
- convergent by construction due to energy dissipation
- equilibrium state representing the primal-dual solution

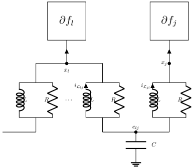
Nesterov Acceleration



Decentralized ADMM



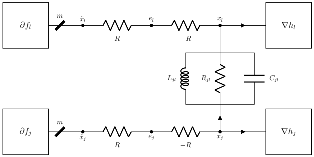
DADMM+C (new!)

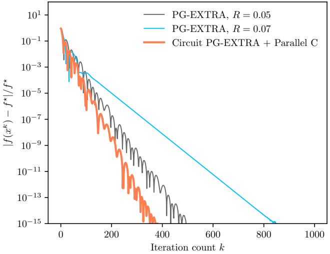


Principled optimization algorithm design

- vary RLC circuits to create new algorithms with convergence guarantees
- use proof preserving automatic discretization to get implementable algorithm

PG-EXTRA + Parallel C (new!)





S. Boyd, T. Parshakova, E. Ryu, and J. Suh. Optimization algorithm design via electric circuits. *NeurIPS Spotlight*, 2024.

Outline

Continuous-time optimization with circuits

Circuits for classical algorithms

Automatic discretization

Distributed convex optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{R}(E^\mathsf{T}) \qquad\qquad (\textit{i.e.}, x = E^\mathsf{T} z \text{ for some } z) \end{array}$$

- $f : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ is closed, convex, and proper
- n nets $N_1, \dots, N_n$ forming a partition of $\{1, \dots, m\}$
- $E \in \mathbf{R}^{n \times m}$ is a selection matrix

$$E_{ij} = \begin{cases} +1 & \text{if } j \in N_i \\ 0 & \text{otherwise} \end{cases}$$

$x \in \mathcal{R}(E^\mathsf{T})$ ensures consistency among components: $x_j = x_{j'}$ if $j, j' \in N_i$

Simple example

$$\begin{array}{ll} \underset{x\in\mathbf{R}^5}{\text{minimize}} & f(x) \\ \text{subject to} & x_1 = x_3 \\ & x_2 = x_4 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \underset{x\in\mathbf{R}^5, z\in\mathbf{R}^3}{\text{minimize}} & f(x) \\ \text{subject to} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \underset{x\in\mathbf{R}^5}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{R}(E^\intercal) \end{array}$$

- $N_1 = \{1, 3\}$, $N_2 = \{2, 4\}$, $N_3 = \{5\}$, and $E^\intercal = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: Unconstrained problem

$$\underset{x \in \mathbf{R}^m}{\text{minimize}} \quad f(x) \qquad \Leftrightarrow \qquad \begin{array}{ll} \underset{x,z \in \mathbf{R}^m}{\text{minimize}} & f(x) \\ \text{subject to} & x = z \end{array} \qquad \Leftrightarrow \qquad \begin{array}{ll} \underset{x \in \mathbf{R}^m}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{R}(E^\mathsf{T}) \end{array}$$

- $x \in \mathbf{R}^m$ is the decision variable
- $N_i = \{i\}$ for all $i = 1, \ldots, m$
- $E^\mathsf{T} = I \in \mathbf{R}^{m \times m}$

Example: Consensus problem

$$\begin{array}{ll} \underset{x_1,\ldots,x_N \in \mathbf{R}^{m/N}}{\text{minimize}} & \sum_{i=1}^N f_i(x_i) \\ \text{subject to} & x_1 = \cdots = x_N \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \underset{x_1,\ldots,x_N \in \mathbf{R}^{m/N}, z \in \mathbf{R}^{m/N}}{\text{minimize}} & \sum_{i=1}^N f_i(x_i) \\ \text{subject to} & \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} z \\ \vdots \\ z \end{bmatrix} \end{array}$$

$$\Leftrightarrow \quad \begin{array}{ll} \underset{x \in \mathbf{R}^m}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{R}(E^\intercal) \end{array}$$

- $f(x) = f_1(x_1) + \cdots + f_N(x_N)$ is block-separable, and $E^\intercal = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \in \mathbf{R}^{m \times m/N}$

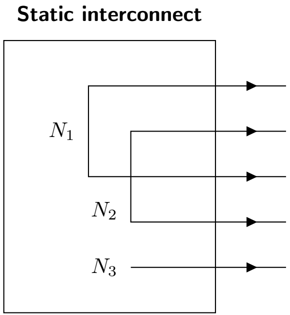
Our goal

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{R}(E^\intercal) \end{array}$$

our goal is to find a primal-dual solution satisfying the KKT conditions

$$y \in \partial f(x), \qquad x \in \mathcal{R}(E^\intercal), \qquad y \in \mathcal{N}(E)$$

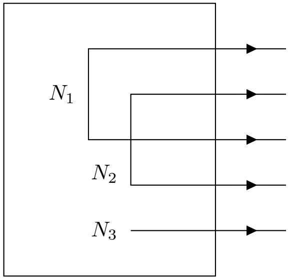
Static interconnect



$$m = 5, \quad n = 3$$

- static interconnect is a set of (ideal) wires connecting m terminals and forming n nets
- $x \in \mathbf{R}^m$ is terminal potentials, and $y \in \mathbf{R}^m$ is vector of currents leaving the terminals
- Kirchhoff's voltage law (KVL) is $x \in \mathcal{R}(E^\mathsf{T})$
- Kirchhoff's current law (KCL) is $y \in \mathcal{N}(E)$
- static interconnect enforces the V-I relationship $(x, y) \in \mathcal{R}(E^\mathsf{T}) \times \mathcal{N}(E)$

Static interconnect for simple example



$N_1 = \{1, 3\}$, $N_2 = \{2, 4\}$,
$N_3 = \{5\}$

$$E^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

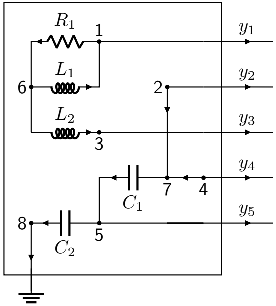
$$x \in \mathcal{R}(E^{\mathsf{T}}) \quad \Leftrightarrow \quad x_1 = x_3, \; x_2 = x_4$$

$$y \in \mathcal{N}(E) \quad \Leftrightarrow \quad y_1 + y_3 = 0, \; y_2 + y_4 = 0, \; y_5 = 0$$

Dynamic interconnect

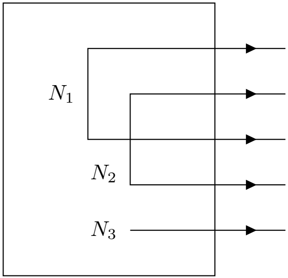
- dynamic interconnect is an RLC circuit with m terminals and 1 ground node

Dynamic interconnect



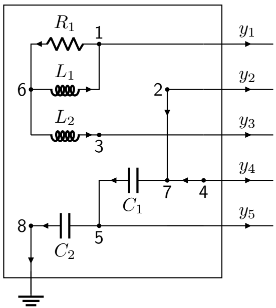
Admissible dynamic interconnect

Static interconnect



$\Longleftarrow$

Dynamic interconnect



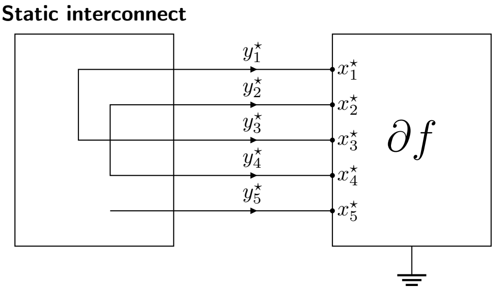
- dynamic interconnect is admissible if it relaxes to the static interconnect at equilibrium

$$\left\{(x, y) \,\middle|\, Ai = \begin{bmatrix} -y \\ 0 \end{bmatrix}, v = A^\mathsf{T} \begin{bmatrix} x \\ e \end{bmatrix}, v_\mathcal{R} = D_\mathcal{R} i_\mathcal{R}, v_\mathcal{L} = 0, i_\mathcal{C} = 0 \right\} = \mathcal{R}(E^\mathsf{T}) \times \mathcal{N}(E)$$

Composing static interconnect with ∂f

- ∂f is interpreted as m-terminal grounded electric device
- ∂f enforces the V-I relation $y \in \partial f(x)$

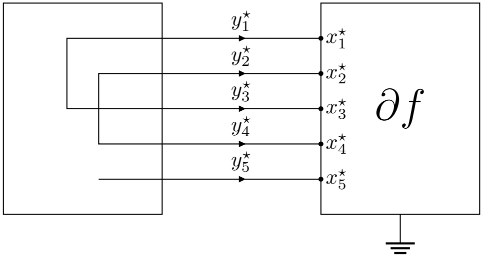
$$
\begin{aligned}
y &\in \partial f(x) \quad \text{(nonlinear resistor)} \\
x &\in \mathcal{R}(E^\mathsf{T}) \quad \text{(KVL)} \\
y &\in \mathcal{N}(E) \quad \text{(KCL)}
\end{aligned}
$$



Composing static interconnect with ∂f

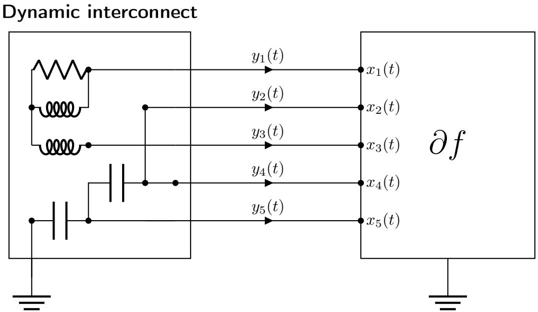
$$
\begin{aligned}
y &\in \partial f(x) && \text{(stationarity)}\\
x &\in \mathcal{R}(E^{\mathsf{T}}) && \text{(primal feasibility)}\\
y &\in \mathcal{N}(E) && \text{(dual feasibility)}
\end{aligned}
$$

Static interconnect



Composing dynamic interconnect with ∂f

$$
\begin{array}{rcll}
y(t) & \in & \partial f(x(t)) & \text{(nonlinear resistor)} \\
v(t) & = & A^\intercal \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} & \text{(KVL)} \\
Ai(t) & = & \begin{bmatrix} -y(t) \\ 0 \end{bmatrix} & \text{(KCL)} \\
v_{\mathcal{R}}(t) & = & D_{\mathcal{R}} i_{\mathcal{R}}(t) & \text{(resistor)} \\
v_{\mathcal{L}}(t) & = & D_{\mathcal{L}} \dfrac{d}{dt} i_{\mathcal{L}}(t) & \text{(inductor)} \\
i_{\mathcal{C}}(t) & = & D_{\mathcal{C}} \dfrac{d}{dt} v_{\mathcal{C}}(t) & \text{(capacitor)}
\end{array}
$$



Energy dissipation

- $(v^\star, i^\star, x^\star, y^\star)$ is equilibrium of admissible dynamic interconnect composed with ∂f
- define the energy of the circuit at time t as

$$\mathcal{E}(t) = \frac{1}{2}\|v_\mathcal{C}(t) - v_\mathcal{C}^\star\|_{D_\mathcal{C}}^2 + \frac{1}{2}\|i_\mathcal{L}(t) - i_\mathcal{L}^\star\|_{D_\mathcal{L}}^2$$

- convexity of f is incremental passivity of ∂f

$$\langle x - x', y - y'\rangle \geq 0, \quad y \in \partial f(x),\ y' \in \partial f(x')$$

- energy is a dissipative (non-increasing) quantity

$$\frac{d}{dt}\mathcal{E} = \langle v_\mathcal{C} - v_\mathcal{C}^\star, i_\mathcal{C} - i_\mathcal{C}^\star\rangle + \langle i_\mathcal{L} - i_\mathcal{L}^\star v_\mathcal{L} - v_\mathcal{L}^\star\rangle = -\|i_\mathcal{R}\|_{D_\mathcal{R}}^2 - \underbrace{\langle x - x^\star, y - y^\star\rangle}_{\geq 0} \leq 0$$

Energy dissipation

Theorem

Assume $f : \mathbf{R}^m \to \mathbf{R}$ is strongly convex and smooth. Assume the dynamic interconnect is admissible, and let $(x^\star, y^\star)$ be a primal-dual solution pair. Let $(v(t), i(t), x(t), y(t))$ be a curve satisfying dynamic interconnect differential algebraic inclusion. Then,

$$\lim_{t \to \infty} (x(t), y(t)) = (x^\star, y^\star).$$

Outline

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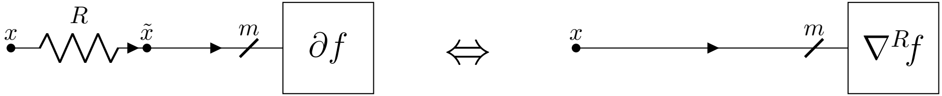
Automatic discretization

Moreau envelope

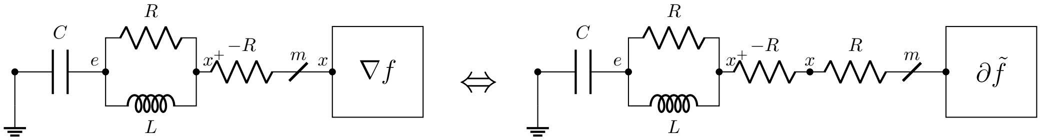
- for $R > 0$, the Moreau envelope of $f : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ with parameter R is

$$^{R}f(x) = \inf_{z \in \mathbf{R}^m} \left(f(z) + \frac{1}{2R} \|z - x\|_2^2 \right)$$

- ∂f composed with a resistor is equivalent to $\nabla {}^{R}f(x)$, *i.e.*, same V-I relation on the m pins of x



Nesterov acceleration



- $f : \mathbf{R}^m \to \mathbf{R}$ is a $1/R$-smooth convex function
- V-I relations

$$y = \nabla f(x), \qquad \frac{x^+ - x}{-R} = \nabla f(x), \qquad v_C = e,$$

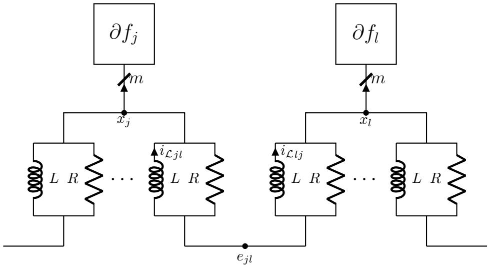
$$\frac{d}{dt} i_{\mathcal{L}} = \frac{1}{L}(v_C - x^+), \qquad \frac{d}{dt} v_C = -\frac{1}{C} \nabla f(x)$$

- high-resolution ODE of Nesterov acceleration

$$\frac{d^2}{dt^2} x + \frac{R}{L} \frac{d}{dt} x + \left(\frac{1}{C} - \frac{R^2}{L} \right) \frac{d}{dt} \nabla f(x) + \frac{R}{LC} \nabla f(x) = 0$$

Decentralized ADMM

- $f_1, \dots, f_N : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ be CCP functions
- communication graph G over computation nodes
- for every edge (j, l) in graph G the circuit is on the right



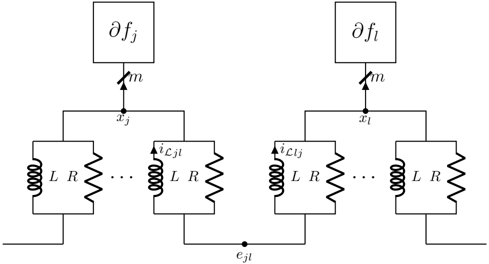
Decentralized ADMM

- V-I relations for $j \in \Gamma_l$

$$
\begin{aligned}
x_j &= \mathbf{prox}_{(R/|\Gamma_j|)f_j}\left(\frac{1}{|\Gamma_j|}\sum_{l\in\Gamma_j}(Ri_{\mathcal{L}jl}+e_{jl})\right)\\
e_{jl} &= \frac{1}{2}(x_j+x_l)\\
\frac{d}{dt}i_{\mathcal{L}jl} &= \frac{1}{L}(e_{jl}-x_j)
\end{aligned}
$$

- L/R stepsize Euler discretization

$$
\begin{aligned}
x_j^{k+1} &= \mathbf{prox}_{(R/|\Gamma_j|)f_j}\left(\frac{1}{|\Gamma_j|}\sum_{l\in\Gamma_j}(Ri_{\mathcal{L}jl}^k+e_{jl}^k)\right)\\
e_{jl}^{k+1} &= \frac{1}{2}(x_j^{k+1}+x_l^{k+1})\\
{i_\mathcal{L}}_{jl}^{k+1} &= {i_\mathcal{L}}_{jl}^k+\frac{1}{R}(e_{jl}^{k+1}-x_j^{k+1})
\end{aligned}
$$

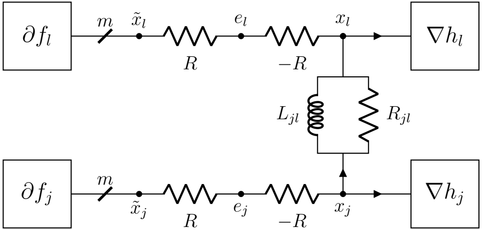


PG-EXTRA

- $f_1, \ldots, f_N : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ are CCP functions and $h_1, \ldots, h_N : \mathbf{R}^m \to \mathbf{R}$ are convex M-smooth functions

- define mixing matrix $W \in \mathbf{R}^{N \times N}$

$$W_{jl} = \begin{cases} 1 - \sum_{l \in \Gamma_j} \frac{R}{R_{jl}} & \text{if } j = l \\ \frac{R}{R_{jl}} & \text{if } j \neq l, \quad l \in \Gamma_j \\ 0 & \text{otherwise} \end{cases}$$



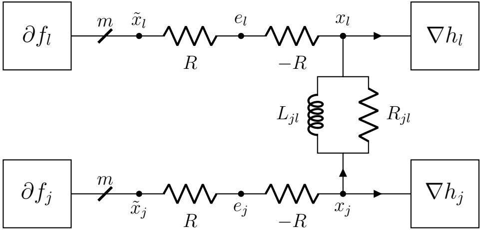
PG-EXTRA

- V-I relations for $j \in \Gamma_l$

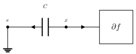
$$
\begin{aligned}
x_j &= \mathbf{prox}_{Rf_j}\left(\sum_{l=1}^{N} W_{jl}x_l - R\nabla h_j(x_j) - w_j\right)\\
\frac{d}{dt}w_j &= x_j - \sum_{l=1}^{N} W_{jl}x_l
\end{aligned}
$$

- $1/2$ stepsize Euler discretization

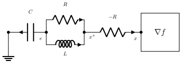
$$
\begin{aligned}
x^{k+1} &= \mathbf{prox}_{Rf}\left(Wx^k - R\nabla h(x^k) - w^k\right)\\
w^{k+1} &= w^k + \frac{1}{2}(I-W)x^k
\end{aligned}
$$



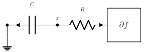
Gradient Descent



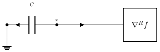
Nesterov Acceleration



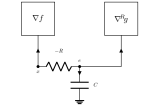
Proximal Point Method



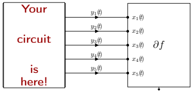
Proximal (Moreau)



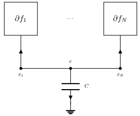
Prox-Grad Method



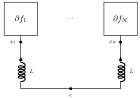
Your Algorithm



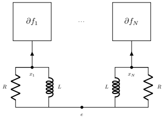
Primal Decomposition



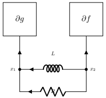
Dual Decomposition



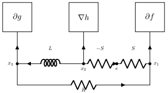
Prox Decomposition



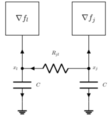
Douglas–Rachford Splitting



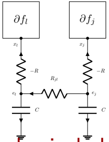
Davis–Yin Splitting



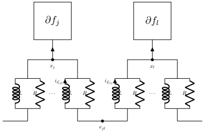
Decentralized GD



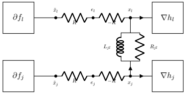
Adapt-then-Combine



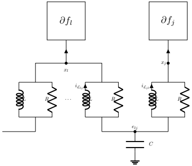
Decentralized ADMM



PG-EXTRA



DADMM+C (new!)



Outline

Continuous-time optimization with circuits

Circuits for classical algorithms

Automatic discretization

Challenges

not every dynamics discretization leads to a convergent algorithm

- in numerical analysis we focus on convergence of x^k to the solution trajectory $x(kh)$ as $h \to 0$
- in optimization we are interested in convergence

$$\lim_{k \to \infty} x^k = x^\star, \quad \lim_{k \to \infty} f(x^k) = f(x^\star)$$

we provide a novel discretization methodology

- that *preserves the proof structure*
- is automatic, based on computer-assisted approach

Sufficiently dissipative discretization

- discretize the dynamics of an admissible dynamic interconnect using a two-stage Runge–Kutta method with parameters α, β and stepsize $h > 0$

- $\{(v^k, i^k, x^k, y^k)\}_{k=1}^{\infty}$ the iterates of discretized algorithm

$$(v^{k+1}, i^{k+1}, x^{k+1}, y^{k+1}) = T_{f,\alpha,\beta,h}(v^k, i^k, x^k, y^k)$$

- energy stored in the circuit at time $t = kh$ is

$$\mathcal{E}_k = \frac{1}{2}\|v_{\mathcal{C}}^k - v_{\mathcal{C}}^\star\|_{D_{\mathcal{C}}}^2 + \frac{1}{2}\|i_{\mathcal{L}}^k - i_{\mathcal{L}}^\star\|_{D_{\mathcal{L}}}^2$$

- discretization is *sufficiently dissipative* if there is an $\eta > 0$ such that for all $k = 1, 2, \ldots$

$$\mathcal{E}_{k+1} - \mathcal{E}_k + \eta \underbrace{\langle x^k - x^\star, y^k - y^\star \rangle}_{\geq 0} \leq 0$$

Sufficiently dissipative discretization

Lemma

Assume $f\colon \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ *is a strictly convex function and the dynamic interconnect is admissible. If the two-stage Runge–Kutta discretization generates a discrete-time sequence* $\{(v^k, i^k, x^k, y^k)\}_{k=1}^{\infty}$ *that is sufficiently dissipative, then*

$$\lim_{k\to\infty} x^k = x^\star.$$

Automatic discretization

- discretization, characterized by (α, β, h), is dissipative for a given $\eta > 0$ if the objective of the worst-case optimization problem is non-positive

$$
\begin{array}{ll}
\text{maximize} & \mathcal{E}_2 - \mathcal{E}_1 + \eta\langle x^1 - x^\star, y^1 - y^\star\rangle \\
\text{subject to} & \mathcal{E}_1 = \frac{1}{2}\|v_\mathcal{C}^1 - v_\mathcal{C}^\star\|_{D_\mathcal{C}}^2 + \frac{1}{2}\|i_\mathcal{L}^1 - i_\mathcal{L}^\star\|_{D_\mathcal{L}}^2 \\
& \mathcal{E}_2 = \frac{1}{2}\|v_\mathcal{C}^2 - v_\mathcal{C}^\star\|_{D_\mathcal{C}}^2 + \frac{1}{2}\|i_\mathcal{L}^2 - i_\mathcal{L}^\star\|_{D_\mathcal{L}}^2 \\
& (v^2, i^2, x^2, y^2) = T_{f,\alpha,\beta,h}(v^1, i^1, x^1, y^1) \\
& (v^1, i^1, x^1, y^1) \text{ feasible initial point} \\
& (v^\star, i^\star, x^\star, y^\star) \text{ values at equilibrium} \\
& f \in \mathcal{F}
\end{array}
$$

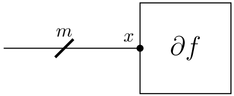
  - $f, v^1, i^1, x^1, y^1, v^\star, i^\star, x^\star, y^\star$ are the decision variables
  - $\mathcal{F}$ is a family of functions (*e.g.*, L-smooth convex)

- can be solved as finite dimensional semidefinite program (SDP) using ideas from performance estimation problem (PEP)

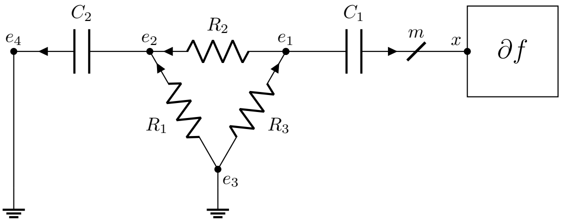
Package `ciropt`

step 1: create the static interconnect representing the optimality conditions of your problem

$$\text{minimize} \quad f(x)$$



step 2: design your algorithm: design RLC circuit that relaxes to the static interconnect in equilibrium



Package `ciropt`

step 3: write the V-I relations: a convergent dynamics by the construction

$$\begin{aligned} x &= \mathbf{prox}_{(R/2)f}\left(z\right) \\ y &= \frac{2}{R}(z-x) \\ \frac{d}{dt}e_2 &= -\frac{1}{2CR}(Ry+3e_2) \\ \frac{d}{dt}z &= -\frac{1}{4CR}(5Ry+3e_2) \end{aligned} \qquad \Bigg| \qquad \begin{aligned} x^k &= \mathbf{prox}_{(R/2)f}\left(z^k\right) \\ y^k &= \frac{2}{R}(z^k-x^k) \\ e_2^{k+1} &= e_2^k - \frac{h}{2CR}(Ry^k+3e_2^k) \\ z^{k+1} &= z^k - \frac{h}{4CR}(5Ry^k+3e_2^k). \end{aligned}$$

Package `ciropt`

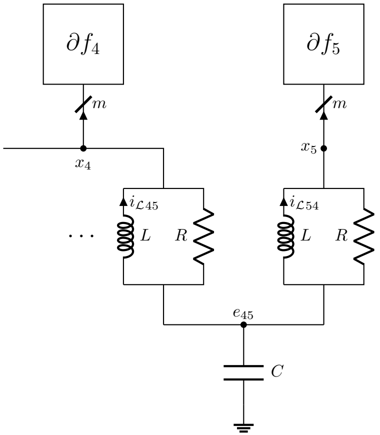
step 4: leverage our PEP-based automatic discretization package `ciropt` and obtain algorithm

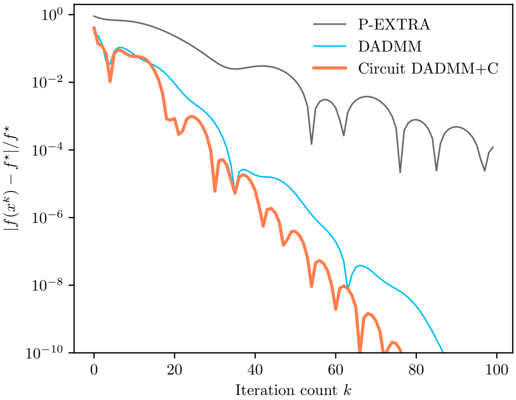
```
import ciropt as co
problem = co.CircuitOpt()
f = co.def_function(problem, mu=0, M=np.inf)
...
In [2]: params["eta"], params["h"]
Out[2]: (6.66, 6.66)
```

step 5: your algorithm is ready to use!

$$\begin{aligned} x^k &= \mathbf{prox}_{(1/2)f}(z^k) \\ y^k &= 2(z^k - x^k) \\ w^{k+1} &= w^k - 0.33(y^k + 3w^k) \\ z^{k+1} &= z^k - 0.16(5y^k + 3w^k) \end{aligned}$$

Numerical results: DADMM+C





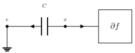
Conclusions

- introduce a framework for designing optimization algorithms via RLC circuits
  - convergent due to energy dissipation
- electric circuits for standard methods
  - Nesterov acceleration, proximal point method, prox-gradient, primal decomposition, dual decomposition, DYS, DRS, decentralized gradient descent, diffusion, DADMM and PG-EXTRA
- PEP-based automated discretization that preserves proof structure
  - https://github.com/cvxgrp/optimization_via_circuits

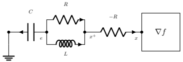
Future directions

- extending the framework for stochastic programming
- extracting convergence rates
  - energy dissipation over multiple steps
- include methods with time dependent step sizes
  - using time dependent electric components
- key question: how to automate the development of accelerated algorithms?
  - search over admissible circuit designs
  - find circuits which result in fast relaxation, *e.g.*, critical damping
  - guidance on how to design fast optimization method from circuit architecture

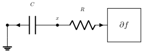
Gradient Descent



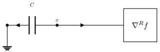
Nesterov Acceleration



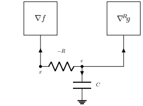
Proximal Point Method



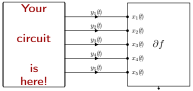
Proximal (Moreau)



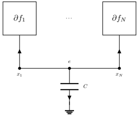
Prox-Grad Method



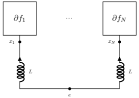
Your Algorithm



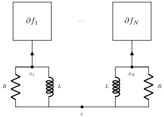
Primal Decomposition



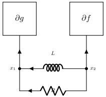
Dual Decomposition



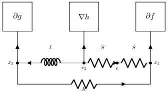
Prox Decomposition



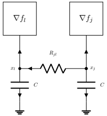
Douglas–Rachford Splitting



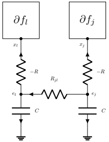
Davis–Yin Splitting



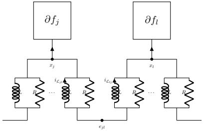
Decentralized GD



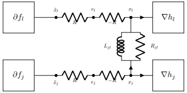
Adapt-then-Combine



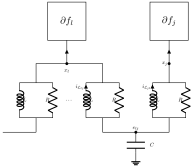
Decentralized ADMM



PG-EXTRA



DADMM+C (new!)



$O(1/K)$ convergence rate

- Lagrangian $L(x, z, y) = f(x) + y^T(x - Ez)$
- by dissipativity,

$$0 \le L(x^k, z^\star, y^\star) - f(x^\star) \le \underbrace{-\langle y^k, x^\star - x^k\rangle}_{\ge f(x^k) - f(x^\star)} + \langle y^\star, x^k - \underbrace{Ez^\star}_{=x^\star}\rangle$$

$$= \langle y^k - y^\star, x^k - x^\star\rangle \le \frac{1}{\eta}(\mathcal{E}_k - \mathcal{E}_{k+1})$$

- by summability,

$$\min_{k\in\{0,\ldots,K\}} L(x^k, z^\star, y^\star) - f(x^\star) \le \frac{1}{\eta(K+1)}\mathcal{E}_0 = O\left(\frac{1}{K}\right)$$

Automatic discretization

- capacitor and inductor ODEs are of the form

$$\frac{d}{dt}x(t) = F(x(t))$$

- two-stage Runge–Kutta method for discretization, with coefficients α, β, and stepsize h

$$\begin{aligned} x^{k+1/2} &= x^k + \alpha h F(x^k) \\ x^{k+1} &= x^k + \beta h F(x^k) + (1-\beta) h F(x^{k+1/2}) \end{aligned}$$

- interpolation lemma states $f_l \in \mathcal{F}_{\mu_l, M_l}(\mathbf{R}^n)$ if and only if

$$\begin{aligned} I_K &= \{1, 1.5, 2, \star\} \\ 0 &\geq f_l^j - f_l^i + \langle g_l^j, x^i - x^j \rangle + \frac{1}{2M_l}\|g_l^i - g_l^j\|_2^2 \\ &\quad + \frac{\mu_l}{2(1-\mu_l/M_l)}\|x^i - x^j - 1/M_l(g_l^i - g_l^j)\|_2^2, \quad i, j \in I_K \end{aligned}$$

Automatic discretization

- decentralized optimization problem

$$\begin{array}{ll} \underset{x_1,\ldots,x_N \in \mathbf{R}^{m/N}}{\text{minimize}} & f_1(x_1) + \cdots + f_N(x_N) \\ \text{subject to} & x_j = x_l, \quad j = 1, \ldots, N, \quad l \in \Gamma_j \end{array}$$

- computer-assisted discretization

$$\begin{array}{ll} \text{maximize} & \eta \\ \text{subject to} & \mathcal{E}_2 - \mathcal{E}_1 + \eta \langle x^1 - x^\star, y^1 - y^\star \rangle \leq 0 \\ & \mathcal{E}_1 = \frac{1}{2}\|v_{\mathcal{C}}^1 - v_{\mathcal{C}}^\star\|_{D_{\mathcal{C}}}^2 + \frac{1}{2}\|i_{\mathcal{L}}^1 - i_{\mathcal{L}}^\star\|_{D_{\mathcal{L}}}^2 \\ & \mathcal{E}_2 = \frac{1}{2}\|v_{\mathcal{C}}^2 - v_{\mathcal{C}}^\star\|_{D_{\mathcal{C}}}^2 + \frac{1}{2}\|i_{\mathcal{L}}^2 - i_{\mathcal{L}}^\star\|_{D_{\mathcal{L}}}^2 \\ & (v^2, i^2, x^2, y^2) = T_{f,\alpha,\beta,h}(v^1, i^1, x^1, y^1) \\ & (v^1, i^1, x^1, y^1) \text{ feasible initial point} \\ & (v^\star, i^\star, x^\star, y^\star) \text{ values at equilibrium} \\ & f_l \in \mathcal{F}_{\mu_l, M_l}(\mathbf{R}^n) \\ & \alpha, \beta \geq 0, h, \eta > 0 \end{array}$$