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Closed-form expressions for perpetual and finite-maturity American binary options

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Introduction

In this post, the reader is shown how to derive closed-form expressions for perpetual and finite-maturity American binary (a.k.a. digital) options written on a stock following geometric Brownian motion. These expressions are closed-form in that they consist of a finite number of arithmetic operations and well-known functions (e.g. erf). These follow immediately from expressions for the Laplace transform of the distribution of the conditional first passage time of Brownian motion to a particular level; derived herein.

A binary option is a type of option in which the payoff can take only two possible outcomes, either some fixed monetary amount of some asset or nothing at all. The two main types of binary options are the cash-or-nothing binary option and the asset-or-nothing binary option. The cash-or-nothing binary option pays some fixed amount of cash if the option expires in-the-money while the asset-or-nothing pays the value of the underlying security. They are also called all-or-nothing options, digital options (more common in forex/interest rate markets), and fixed return options (FROs) (on the American Stock Exchange).

A European option may be exercised only at the expiration date of the option (i.e. at a single pre-defined point in time). An American option, on the other hand, may be exercised at any time before the expiration date. For an American binary put (resp. call), it is sufficient to consider the cash-or-nothing case with an option paying off a single unit in the numéraire. The asset-or-nothing case is a simple scaling (by the strike price) of the cash-or-nothing case.

Option prices

Closed-form expressions for the prices of American binary options written on an asset following geometric Brownian motion as per

$$S_t = x \exp\left(\left(r - \delta - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)$$

are presented in this section. Both the *perpetual* and *finite-maturity* cases are considered. It is assumed that r is real, $\delta \geq 0$, and $\sigma > 0$. $K > 0$ is used to denote the strike price of an option. It will be helpful to define the following symbols: (it is readily verified that $\xi^2 + 2r \geq 0$; i.e. b is real)

$$a = \frac{1}{\sigma}\log\frac{K}{x},\ \xi = \frac{r-\delta}{\sigma} - \frac{\sigma}{2},\ \text{and}\ b = \sqrt{\xi^2 + 2r}.$$

Because a binary option is exercised as soon as it is in the money, it is assumed $K < x$ (resp. $K > x$) for a put (resp. call). Otherwise, the option is exercised upon inception and worth exactly one unit. The expressions are summarized below:

Theorem. *The price of a cash-or-nothing American binary put (resp. call) with strike $K < x$ (resp. $K > x$) and time-to-expiry T is*

$$\frac{1}{2}e^{a(\xi-b)}\left\{1 + \operatorname{sgn}(a)\operatorname{erf}\left(\frac{bT-a}{\sqrt{2T}}\right) + e^{2ab}\left[1 - \operatorname{sgn}(a)\operatorname{erf}\left(\frac{bT+a}{\sqrt{2T}}\right)\right]\right\}.$$

Corollary. *The price of a cash-or-nothing, perpetual American binary put (resp. call) with strike $K < x$ (resp. $K > x$) is $e^{a\xi - |a|b}$.*

These expressions follow immediately from the Laplace transform of the distribution of the conditional first passage time of Brownian motion to a particular level (see below).

For the case of an (ordinary) perpetual American call, when the interest rate is nonnegative (more generally, $2r \geq -\sigma^2$) and the dividend rate is zero, it is never optimal to exercise and the option is worth the initial stock price. A corresponding result for the perpetual American binary call is as follows:

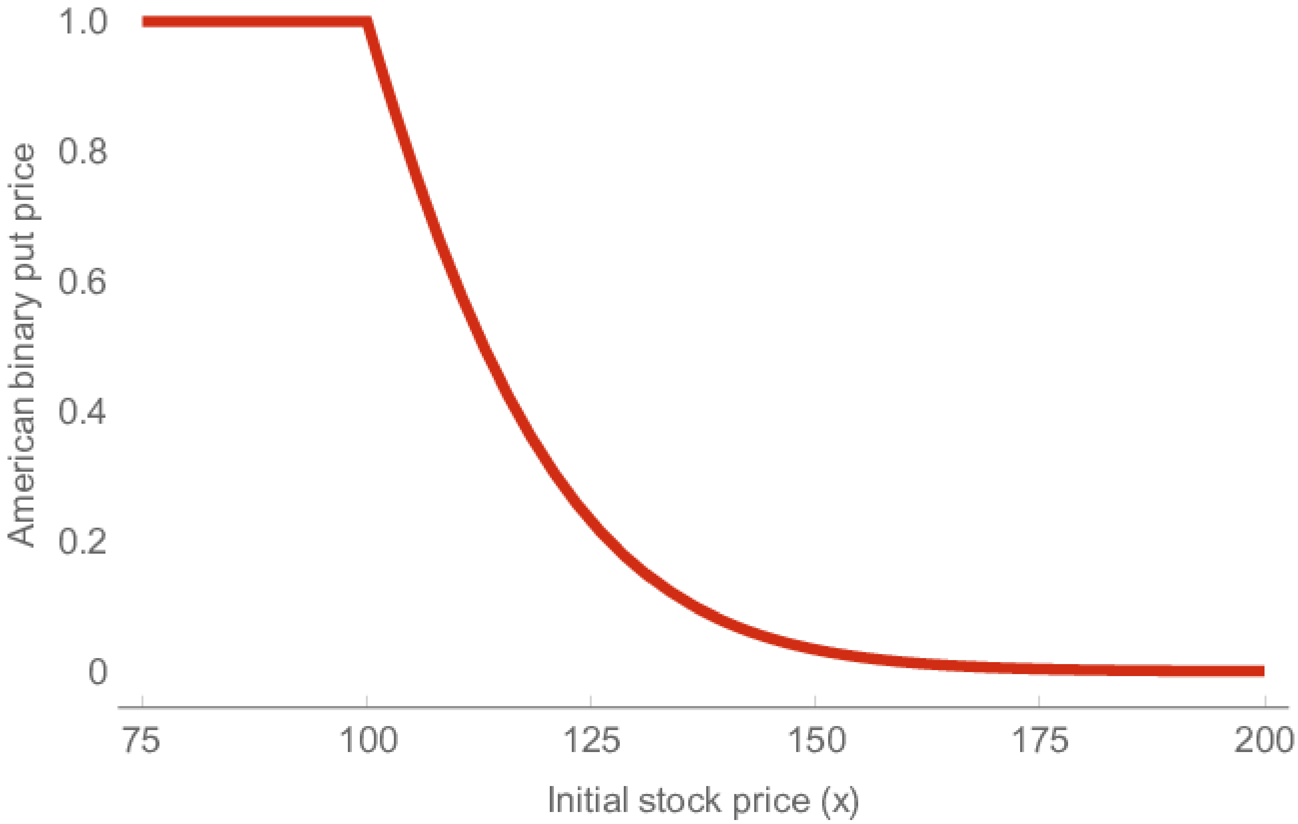
Corollary. *When $2r \geq -\sigma^2$ (resp. $2r \leq -\sigma^2$) and $\delta = 0$, the price of a cash-or-nothing, perpetual American binary call (resp. put) with strike $K > x$ (resp. $K < x$) is x/K.*

Proof. It is sufficient to consider the case of the call; the put is handled similarly. Since $a > 0$,

$$e^{a\xi - |a|b} = e^{a(\xi-b)} = e^{-a\sigma} = x/K.$$

■

Below, we graph the value of an American binary put with $K = 100$, $r = 0.04$, $\sigma = 0.2$, and $T = 1$.



GNU Octave/MATLAB code to generate a graph similar to the above is given below.

```
K = 100.;
x = K:1:K*2;
r = 0.04;
sigma = 0.2;
delta = 0.01;
T = 1.;

a = 1. / sigma * log (K ./ x);
xi = (r - delta) / sigma - sigma / 2.;
b = sqrt (xi * xi + 2. * r);

v = 0.5 * exp (a * (xi - b)) .* ( ...
        1 + sign (a) .* erf ((b * T - a) / sqrt (2 * T)) ...
    + exp (2 * a * b) ...
    .* (1 - sign (a) .* erf ((b * T + a) / sqrt (2 * T))) ...
);

plot ([0 1 x], [1 1 v], 'linewidth', 2);
axis ([K/2 K*2 0 1+2^(-5)]);
xlabel ('Initial stock price (x)');
ylabel ('American binary put value (P)');
```

First passage times

For real numbers a and ξ let

$$\tau_{a\xi} = \inf\{t \geq 0 : \xi t + W_t = a\}$$

be the random variable corresponding to the first time the Brownian motion with drift ξ reaches level a. The following is well-known (see, e.g., [1]):

Lemma. *The density of $\tau_{a\xi}$ is*

$$f_{\tau_{a\xi}}(t)=\frac{|a|}{\sqrt{2\pi t^3}}\exp\left(-\frac{(a-\xi t)^2}{2t}\right).$$

In deriving the above, it is sufficient to consider the case of $a\geq 0$ and to use the reflection principle to derive the density for $a<0$.

A lengthy computation yields an expression for the Laplace transform of the distribution of the conditional first passage time of Brownian motion to a particular level, which is of separate interest:

Lemma. *Let c be real, $b=\sqrt{\xi^2+2c}$ and $0<T<\infty$. If $\xi^2+2c\geq 0$,*

$$\begin{aligned}E\left[e^{-c\tau_{a\xi}}\mathbf{1}_{\tau_{a\xi}\leq T}\right]&=\int_0^T e^{-ct}f_{\tau_{a\xi}}(t)\,dt\\&=\frac{1}{2}e^{a(\xi-b)}\left\{1+\operatorname{sgn}(a)\operatorname{erf}\left(\frac{bT-a}{\sqrt{2T}}\right)+e^{2ab}\left[1-\operatorname{sgn}(a)\operatorname{erf}\left(\frac{bT+a}{\sqrt{2T}}\right)\right]\right\}.\end{aligned}$$

Corollary. *Let c be real and $b=\sqrt{\xi^2+2c}$. If $\xi^2+2c\geq 0$, $E\left[e^{-c\tau_{a\xi}}\right]=e^{a\xi-|a|b}$.*

To arrive at the results of the previous section, it suffices to note that the price of an American binary put (resp. call) with strike $K<x$ (resp. $K>x$) and time-to-expiry T is exactly

$$E\left[e^{-r\tau^\star}\mathbf{1}_{\tau^\star\leq T}\right]$$

where

$$\tau^\star=\inf\{t\geq 0: S_t=K\}=\inf\left\{t\geq 0:\left(\frac{r-\delta}{\sigma}-\frac{\sigma}{2}\right)t+W_t=\frac{1}{\sigma}\log\frac{K}{x}\right\}.$$

The results now follow immediately from the fact that $\tau^\star=\tau_{a\xi}$ with a and ξ defined as in the previous section.

Bibliography

1. Shreve, Steven E. *Stochastic calculus for finance II: Continuous-time models*. Vol. 11. Springer Science & Business Media, 2004.

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