Sliding Mode Control: Controller Derivation

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$$s_1 = \dot{e} + \lambda e = (\dot{z} - \dot{z}_d) + \lambda_1 (z - z_d)$$

$$\dot{s}_1 = (\ddot{z} - \ddot{z}_d) + \lambda_1 (\dot{z} - \dot{z}_d)$$

$$\ddot{z} = \frac{1}{m}(c\phi\, c\theta) u_1 - g$$

$$s_1 \dot{s}_1 = s_1 \left[\frac{1}{m}(c\phi c\theta) u_1 - g - \ddot{z}_d + \lambda_1 (\dot{z} - \dot{z}_d) \right]$$

$$\therefore s_1 \dot{s}_1 = \frac{c\phi c\theta s_1}{m} \left\{ u_1 + \frac{m}{\phi c\theta} \left[-g - \ddot{z}_d + \lambda (\dot{z} - \dot{z}_d) \right] \right\}$$

$$u_1 = \frac{-m}{c\phi c\theta} \left[-g - \ddot{z}_d + \lambda (\dot{z} - \dot{z}_d) + u_{r_1} \right]$$

$$\therefore s_1 \dot{s}_1 = \frac{c\phi c\theta s_1}{m} \left[\frac{-m}{c\phi c\theta} \left[\cancel{-g - \ddot{z}_d + \lambda (\dot{z} - \dot{z}_d)} + u_{r_1} \right] + \frac{m}{c\theta c\phi} \left[\cancel{-g - \ddot{z}_d + \lambda (\dot{z} - \dot{z}_d)} \right] \right]$$

$$s_1 \dot{s}_1 = \frac{\cancel{c\phi c\theta}}{\cancel{m}} s_1 \cdot \left(\frac{-\cancel{m}}{\cancel{c\phi c\theta}} \cdot u_r \right) = -s_1 u_r \qquad u_r = K_1 \,\mathrm{sat}(s_1)$$

$$\therefore s_1 \dot{s}_1 = -s_1 K_1 \,\mathrm{sat}(s_1)$$

$$\therefore u = \frac{m}{c\theta c\phi} \left[g + \ddot{z}_d - \lambda (\dot{z} - \dot{z}_d) - K_1 \,\mathrm{sat}(s_1) \right]$$

$$s_2 = \dot{e} + \lambda e = (\dot{\phi} - \cancel{\dot{\phi}_d}) + \lambda_2 (\phi - \phi_d) = \dot{\phi} + \lambda_2 (\phi - \phi_d)$$

$$\dot{s}_2 = (\ddot{\phi} - \cancel{\ddot{\phi}_d}) + \lambda_2 (\dot{\phi} - \cancel{\dot{\phi}_d}) = \ddot{\phi} + \lambda \dot{\phi}$$

$$\ddot{\phi} = \dot{\theta}\dot{\psi} \left[\frac{I_y - I_z}{I_x} \right] - \frac{I_p}{I_x} \Omega \dot{\theta} + \frac{1}{I_x} u_2$$

$$s_2 \dot{s}_2 = s_2 \left[\frac{1}{I_x} \left[\dot{\theta}\dot{\psi} [I_y - I_z] - I_p \Omega \dot{\theta} + u_2 \right] + \lambda \dot{\phi} \right]$$

$$s_2 \dot{s}_2 = \frac{s_2}{I_x} \left[u_2 + I_x \left[\dot{\theta}\dot{\psi} \left(\frac{I_y - I_z}{I_x} \right) - \frac{I_p}{I_x} \Omega \dot{\theta} + \lambda \dot{\phi} \right] \right]$$

$$u_2 = -I_x \left[\dot{\theta}\dot{\psi} \left(\frac{I_y - I_z}{I_x} \right) - \frac{I_p}{I_x} \Omega \dot{\theta} + \lambda \dot{\phi} + u_r \right]$$

$$s_2 \dot{s}_2 = \frac{s_2}{I_x} \left[-I_x \left[\cancel{\dot{\theta}\dot{\psi} \left(\frac{I_y - I_z}{I_x} \right) - \frac{I_p}{I_x} \Omega \dot{\theta} + \lambda \dot{\phi}} + u_r \right] + I_x \left[\cancel{\dot{\theta}\dot{\psi} \left(\frac{I_y - I_z}{I_x} \right) - \frac{I_p}{I_x} \Omega \dot{\theta} + \lambda \dot{\phi}} \right] \right]$$

$$s_2 \dot{s}_2 = \frac{s_2}{\cancel{I_x}} \left[-\cancel{I_x} u_r \right] = -s_2 u_r$$

$$u_r = K_2 \,\mathrm{sat}(s_2)$$

$$\therefore s_2 \dot{s}_2 = -s_2 K_2 \,\mathrm{sat}(s_2)$$

$$u_2 = -\left[\dot{\theta}\dot{\psi} (I_y - I_z) - I_p \Omega \dot{\theta} - \lambda I_x \dot{\phi} + I_x K_2 \,\mathrm{sat}(s_2) \right]$$

$$s_3 = \dot{e} + \lambda e = (\dot{\theta} - \cancel{\dot{\theta}_d}) + \lambda_3 (\theta - \theta_d) = \dot{\theta} + \lambda_3 (\theta - \theta_d)$$

$$\dot{s}_3 = \ddot{\theta} + \lambda_3 (\dot{\theta} - \cancel{\dot{\theta}_d}) = \ddot{\theta} + \lambda_3 \dot{\theta}$$

$$\ddot{\theta} = \dot{\phi}\dot{\psi} \left(\frac{I_z - I_x}{I_y} \right) + \frac{I_p}{I_y} \Omega \dot{\phi} + \frac{1}{I_y} u_3$$

$$s_3 \cdot \dot{s}_3 = s_3 \left[\dot{\phi}\dot{\psi} \left(\frac{I_z - I_x}{I_y} \right) + \frac{I_p}{I_y} \Omega \dot{\phi} + \frac{1}{I_y} u_3 + \lambda \dot{\theta} \right]$$

$$= \frac{s_3}{I_y} \left[\dot{\phi}\dot{\psi} (I_z - I_x) + I_p \Omega \dot{\phi} + u_3 + I_y \lambda \dot{\theta} \right]$$

$$= \frac{s_3}{I_y} \left[u_3 + \left[\dot{\phi}\dot{\psi} (I_z - I_x) + I_p \Omega \dot{\phi} + I_y \lambda \dot{\theta} \right] \right]$$

$$u_3 = -\left[\dot{\phi}\dot{\psi} (I_z - I_x) + I_p \Omega \dot{\phi} + I_y \lambda \dot{\theta} + I_y u_r \right]$$

$$\therefore s_3 \dot{s}_3 = \frac{s_3}{I_y} \left[-\left[\cancel{\dot{\phi}\dot{\psi} (I_z - I_x) + I_p \Omega \dot{\phi} + I_y \lambda \dot{\theta}} + I_y u_r \right] + \left[\cancel{\dot{\phi}\dot{\psi} (I_z - I_x) + I_p \Omega \dot{\phi} + I_y \lambda \dot{\theta}} \right] \right]$$

$$s_3 \dot{s}_3 = \frac{s_3}{\cancel{I_y}} \cdot \left(-\cancel{I_y} u_r \right) = -s_3 u_r$$

$$u_r = K_3 \,\mathrm{sat}(s_3)$$

$$\therefore s_3 \dot{s}_3 = -s_3 K_3 \,\mathrm{sat}(s_3)$$

$$u = -\left[\dot{\phi}\dot{\psi} (I_y - I_x) + I_p \Omega \dot{\phi} + I_y \lambda \dot{\theta} + I_y K_3 \,\mathrm{sat}(s_3) \right]$$

$$s_4 = \dot{e} + \lambda e = (\dot{\psi} - \cancel{\dot{\psi}_d}) + \lambda_4 (\psi - \cancel{\psi_d}) = \dot{\psi} + \lambda_4 \psi$$

$$\dot{s}_4 = \ddot{\psi} + \lambda_4 \dot{\psi}$$

$$\ddot{\psi} = \dot{\phi}\dot{\theta} \left(\frac{I_x - I_y}{I_z} \right) + \frac{1}{I_z} u_4$$

$$s_4 \dot{s}_4 = s_4 \left[\dot{\phi}\dot{\theta} \left(\frac{I_x - I_y}{I_z} \right) + \frac{1}{I_z} u_4 + \lambda \dot{\psi} \right]$$

$$= \frac{s_4}{I_z} \left[\dot{\phi}\dot{\theta} (I_x - I_y) + u_4 + \lambda I_z \dot{\psi} \right]$$

$$= \frac{s_4}{I_z} \left[u_4 + \left[\dot{\phi}\dot{\theta} (I_x - I_y) + \lambda I_z \dot{\psi} \right] \right]$$

$$U_4 = -\left[\dot{\phi}\dot{\theta}(I_x - I_y) + \lambda I_z \dot{\psi} + I_z U_r\right]$$

$$S_4 \dot{S}_4 = \frac{S_4}{I_z}\left[-\left[\cancel{\dot{\phi}\dot{\theta}(I_y - I_z)} + \cancel{\lambda I_z \dot{\psi}} + I_z U_r\right] + \left[\cancel{\dot{\phi}\dot{\theta}(I_x - I_y)} + \cancel{\lambda I_z \dot{\psi}}\right]\right]$$

$$S_4 \dot{S}_4 = \frac{S_4}{\cancel{I_z}}\left(-\cancel{I_z} U_r\right) = -S_4 U_r$$

$$U_r = K_4 \operatorname{sat}(S_4)$$

$$S_4 \dot{S}_4 = -S_4 K_4 \operatorname{sat}(S_4)$$

$$U_4 = -\left[\dot{\phi}\dot{\theta}\left[I_x - I_y\right] + \lambda I_z \dot{\psi} + I_z K_4 \operatorname{sat}(S_4)\right]$$