

## RBE 502 — ROBOT CONTROL

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## Problem Set 1

Due: September 20, 2022 at 11:59 pm

Please show your work. Correct answers not accompanied by sufficient explanations will receive little or no credit.

## Problem 1 (5 points)

The equations of motion for a quadrotor moving in a 2D plane are given by:

$$\begin{aligned} m\ddot{x} &= -(F_1 + F_2) \sin \theta \\ m\ddot{y} &= (F_1 + F_2) \cos \theta - mg \\ I\ddot{\theta} &= L(F_2 - F_1) \end{aligned}$$

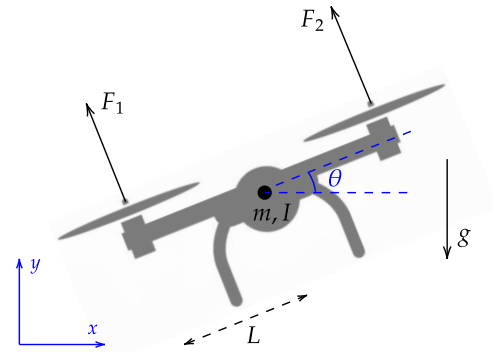


Figure 1: A planar quadrotor

The coordinates  $(x, y, \theta)$  represent the position and orientation of the quadrotor,  $m$  and  $I$  are the mass and moment of inertia,  $L$  denotes the length of the quadrotor's arm, and  $g$  is the gravitational acceleration.  $F_1$  and  $F_2$  denote the thrust forces generated by the propellers.

- Defining the control inputs  $u_1 := (F_1 + F_2)$  and  $u_2 := (F_2 - F_1)$  and denoting the horizontal and vertical positions of the quadrotor as the outputs  $w$ , construct a state-space model for this system in the form of  $\dot{f}(z, u)$  and  $w = h(z, u)$  (we have denoted the state vector by  $z$  and the output vector by  $w$  to avoid confusion with the horizontal position  $x$  and the vertical position  $y$ ).
- Find the equilibrium point of the system associated with  $x^* = x_e$ ,  $y^* = y_e$ ,  $u_1^* = mg$  and  $u_2^* = 0$ .
- Linearize the system around the equilibrium point found in part (b), and explicitly write out the state and output equations corresponding to the linearized dynamics.

## Problem 2 (5 points)

Consider the one-link robot in Figure 2, where  $\theta$  denotes the angle of the link with the horizontal,  $\tau$  the torque applied at the joint,  $(x, y)$  the position of the tip,  $l$  the length of the link,  $I$  its moment of inertia,  $m$  the mass at tip,  $g$  gravity's acceleration, and  $b$  the friction coefficient. This system evolves according to the following equation of motion:

$$I\ddot{\theta} = -b\dot{\theta} - mg \cos \theta + \tau$$

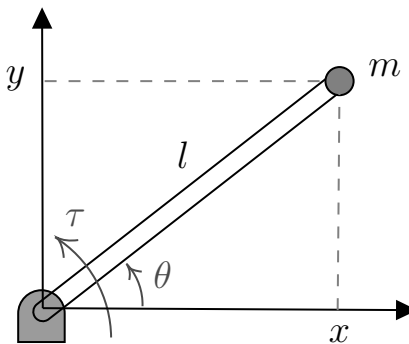


Figure 2: Single-link robot

- a) Compute the state-space model for the system with  $u = \tau$  as the input and the vertical position of the tip  $y$  as the output. Please denote the state vector by  $z$  to avoid confusion with the horizontal position of the tip  $x$ , and write the model in the form:

$$\dot{z} = f(z, u), \quad y = h(z, u)$$

for appropriate functions  $f$  and  $h$ . Do not forget the output equation!

- b) Find the equilibrium points of the system when  $\tau(t) = 0$ .
- c) Compute the linearization of the system around the equilibrium point with “upward” configuration. From your answer, can you predict if there will be problems when one wants to control the tip position close to this configuration just using feedback from  $y$ ?
- d) Assume  $m = 0.1$ ,  $I = 1$ ,  $b = 0$ , and  $g = 10$ . Determine the stability properties of the equilibrium point in part (c). Is the linearized system around that equilibrium point controllable?

### Problem 3 (5 points)

A unicycle robot moving on the plane with linear velocity  $v$  and angular velocity  $\omega$  can be modeled by the nonlinear system:

$$\dot{p}_x = v \cos \theta, \quad \dot{p}_y = v \sin \theta, \quad \dot{\theta} = \omega,$$

where  $(p_x, p_y)$  denote the Cartesian coordinates of the wheel and  $\theta$  its orientation. Regard this as a system with input  $u := [v \ \omega]^T \in \mathbb{R}^2$ .

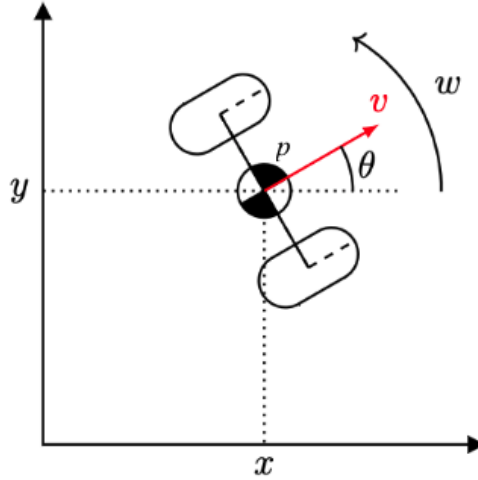


Figure 3: Unicycle model

- a) Construct a state-space model for this system in the form of  $f(x, u)$  with state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x \cos \theta + (p_y - 1) \sin \theta \\ -p_x \sin \theta + (p_y - 1) \cos \theta \\ \theta \end{bmatrix}$$

and output  $y := [x_1 \ x_2]^T \in \mathbb{R}^2$

- b) Compute a local linearization for this system around the equilibrium point  $x^* = 0$ ,  $u^* = 0$ .
- c) It is given that the trajectory  $\omega(t) = v(t) = 1$ ,  $p_x(t) = \sin t$ ,  $p_y(t) = 1 - \cos t$ ,  $\theta(t) = t$ ,  $\forall t \geq 0$  is a solution to the system. Show that a local linearization of the system around this trajectory results in an LTI system.

**Problem 4 (5 points)**

a) The second-order system

$$\ddot{\omega} + g(\omega)\dot{\omega} + \omega = 0 \quad (*)$$

with equilibrium point  $\omega = \dot{\omega} = 0$ . Construct a state-space model for the system. Determine for which values of  $g(0)$  we can guarantee convergence to the origin based on the local linearization.

**Note.** Equation  $(*)$  is called the Liénard equation and can be used to model several different mechanical systems, for distinct choice of the function  $g(\cdot)$

b) Consider the nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - x_2^2)\end{aligned}$$

Linearize the system around the trajectory  $(x_1, x_2) = (\sin t, \cos t)$ .

Is the linearized system time-invariant or time-varying?