Problem Set 2

Tuesday, October 4, 2022 10:10 PM

Problem 1 (5 points)

Consider the one-link robot in Figure 1. The Jacobian linearization of the robot about the equilibrium point $z^* = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$ and $u^* = 0$ was derived in PSET1 as:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mg}{I} & -\frac{b}{I} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix}$$

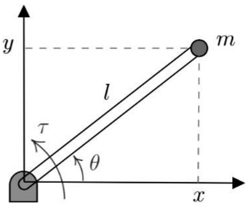

Figure 1: Single-link robot

a) Assume $m = 0.1$, $I = 1$, $b = 2$, and $g = 10$, and all the states are individually measured. Design a state-feedback controller (i.e. determine the gain matrix K) such that the closed-loop system has eigenvalues at $\{-3 \pm j\}$.

b) For this part, assume only the state associated with the angular velocity of the robot is measured, that is, the output matrix C is given by $C = \begin{bmatrix} 0 & 2 \end{bmatrix}$. Design an observer (i.e. determine the observer gain L) that places the observer eigenvalues at $\{-1, -2\}$. Explicitly write down the resulting observer dynamics.

c) Use the estimated states to implement the control signal $u = -K\hat{x}$. Form the state-space representation for the overall closed loop system, including the actual states x and the error e. Discuss the stability properties of the overall system.

A.) (a) $\forall$ $m = 0.1$, $I = 1$, $b = 2$ & $g = 10$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assume $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$

** Check Controllability:

$C = \begin{bmatrix} B & AB \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

$\text{rank}(C) = 2 = \text{rank}(A)$

$\therefore$ System is controllable.

$$\therefore A' = A - BK = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} 0 & 1 \\ 1-K_1 & -2-K_2 \end{bmatrix}$$

Eigenvalue of $A' = \lambda_1, \lambda_2$

$\therefore \det.(A' - \lambda I) = 0$

$$\left| \begin{bmatrix} 0 & 1 \\ 1-K_1 & -2-K_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1-K_1 & -2-K_2-\lambda \end{vmatrix} = 0$$

$\therefore \lambda(\lambda + K_2 + 2) - (1 - K_1) = 0$

$\underbrace{\lambda^2 + (K_2+2)\lambda + (K_1 - 1) = 0}_{A}$ $\Rightarrow$ Characteristic Polynomial.

$\lambda = -3 \pm j$

$\therefore (\lambda - (-3+j))(\lambda - (-3-j)) = 0$

$\lambda^2 - (-3 + j - 3 - j)\lambda + (-3+j)(-3-j) = 0$

$\lambda^2 + 6\lambda + 9 + 1 = 0$

$\underbrace{\lambda^2 + 6\lambda + 10 = 0}_{B}$ $\Rightarrow$ Characteristic Polynomial

Comparing A & B

$K_2 + 2 = 6 \Rightarrow K_2 = 4$

$K_1 - 1 = 10 \Rightarrow K_1 = 11$

$\therefore \boxed{K = \begin{bmatrix} 11 & 4 \end{bmatrix}}$

(b) $C = \begin{bmatrix} 0 & 2 \end{bmatrix}$

1) Checking Observability

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}$$

Rank $(O) = 2 = \text{rank}(A)$ $\therefore$ Observable.

Assume $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$

$$A' = A - LC = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 2l_1 \\ 0 & 2l_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1-2l_1 \\ 1 & -2-2l_2 \end{bmatrix}$$

Characteristic Polynomial:

$$|A' - \lambda I| = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1-2l_1 \\ 1 & -2-2l_2-\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda + 2l_2 + 2) + (2l_1 - 1) = 0$$

$$\lambda^2 + (2l_2+2)\lambda + (2l_1 - 1) = 0 \quad \text{—} 1$$

Given $\lambda = -1, -2$.

$$\therefore (\lambda+1)(\lambda+2) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0 \quad \text{—} 2$$

Comparing 1 & 2

$$2l_2 + 2 = 3 \Rightarrow l_2 = 1/2$$

$$2l_1 - 1 = 2 \Rightarrow l_1 = 3/2$$

$$\therefore L = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\hat{y} - y)$$

$$\hat{y} = C\hat{x} + \underbrace{\cancel{Du}}_{0}$$

$$\dot{\hat{x}} = A\hat{x} - BK\hat{x} - L(C\hat{x} - y)$$

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly$$

$$\dot{\hat{x}} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 11 & 4 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} \right\} \hat{x} + \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} y$$

$$\dot{\hat{x}} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 11 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \right\} \hat{x} + \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} y$$

$$\boxed{\dot{\hat{x}} = \begin{bmatrix} 0 & -2 \\ -10 & -7 \end{bmatrix} \hat{x} + \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} y}$$

c) $u = -K\hat{x}$

$$K = \begin{bmatrix} 11 & 4 \end{bmatrix}$$

$$u = \begin{bmatrix} -11 & -4 \end{bmatrix} \hat{x}$$

$$\dot{\hat{x}} = \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$BK = \begin{bmatrix} 0 & 0 \\ 11 & 4 \end{bmatrix} \qquad LC = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{\dot{\hat{x}} = \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10 & -6 & -11 & -4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}}$$

Problem 2 (6 points)

Consider the nonlinear system

$$m\ddot{y} = mg - c\frac{u^2}{y^2}$$

which models a steel ball suspended in a magnetic field, as shown in Figure 2. The control input is the current fed to the filed coil ($u := i(t) \geq 0$). The physical parameters are assumed to be $m = g = c = 1$.

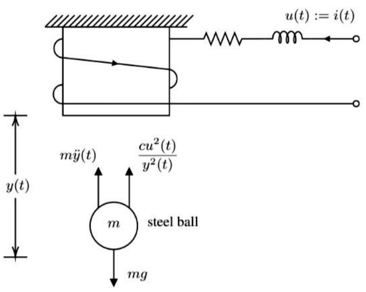

Figure 2: Magnetic levitation system

a) Let $x_1 = y$ and $x_2 = \dot{y}$. Write a *nonlinear state-space model* for the system.

b) Determine what equilibrium input u^* is required to maintain an output of $y = 1$. Specify the equilibrium point (x^*, u^*) associated with this equilibrium input.

c) Determine the linearized state-space system, representing the deviation from this equilibrium. Assume y is the output.

d) Is the origin of the linearized system stable? What does this tell you about the stability of the equilibrium of the original nonlinear system?

e) Is the linearized system controllable? Is it observable?

f) Design a state-feedback controller to place the closed-loop eigenvalues of the linearized system at -2, -2.

g) Design a full-order observer so that the state estimator dynamics has eigenvalues at -4, -4.

) a) $m\ddot{y} = mg - c\frac{u^2}{y^2}$

$m = g = c = 1 \qquad u(t) = i(t)$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ g - \frac{c}{m} u^2/y^2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 - u^2/x_1^2 \end{bmatrix}$$

$$\therefore \boxed{\dot{x} = \begin{bmatrix} x_2 \\ 1 - u^2/x_1^2 \end{bmatrix}}$$

(b) Find Equ.^{m} pts. for $y=1 \Rightarrow x_1 = 1$

$$\dot{x} = 0 \quad \Rightarrow \begin{cases} x_2^* = 0 \\ 1 - u^{*2}/x_1^{*2} = 0 \end{cases}$$

$$u^{*2} = x_1^{*2} = (1)^2$$
$$u^* = 1$$

$$\therefore \text{Equ.}^m \text{ pts.} : (x^*, u^*) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1 \right)$$

(c) Linearized System:

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 2u^{*2}/x_1^{*3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ -\frac{1}{x_1^{*2}}(2u^*) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\therefore \bar{x} = x - x^* = \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$\bar{u} = u - u^* = [u - 1]$$

$$\therefore \boxed{\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} [u - 1]} \Rightarrow \text{State Space format}$$

$$\dot{\bar{x}} = \begin{bmatrix} x_2 - 1 \\ 2x_1 - 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 - u \end{bmatrix} =$$

$$\boxed{\dot{\bar{x}} = \begin{bmatrix} x_2 - 1 \\ 2x_1 - u \end{bmatrix}} \Rightarrow \text{Super Simplified format.}$$

(d) Finding Stability:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2 = 0$$

$$\lambda = \pm\sqrt{2}$$

↳ System is unstable since all $Re(\lambda_i) \not< 0$

This tells us that the original nonlinearized system is unstable.

(e) Controllability: $C = [B \quad AB] = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

$\det(C) \neq 0$

$\therefore$ System is controllable.

Observability: $y = x_1$

$\therefore \; y = [1 \quad 0]x$

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\det(O) \neq 0$

$\therefore$ System is observable.

(f) $K = [K_1 \quad K_2] \qquad u = -Kx$

$$A' = A - BK = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -2K_1 & -2K_2 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 1 \\ 2+2K_1 & 2K_2 \end{bmatrix}$$

Characteristic Polynomial of A':

$$|A' - \lambda I| = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1 \\ 2+2K_1 & 2K_2 - \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda - 2K_2) - (2 + 2K_1) = 0$$

$$\lambda^2 + (-2K_2)\lambda + (-2 - 2K_1) = 0 \quad -(1)$$

Given Eigen Values: $\lambda = -2, -2$

$$\therefore (\lambda + 2)^2 = 0$$

$$\lambda^2 + 4\lambda + 4 = 0 \quad -(2)$$

Comparing (1) & (2)

$$\begin{cases} -2K_2 = 4 \quad \Rightarrow K_2 = -2 \\ -2 - 2K_1 = 4 \quad \Rightarrow K_1 = -3 \end{cases}$$

$$\therefore K = [-3 \quad -2]$$

$$\boxed{u = [3 \quad 2]x}$$

(9) Assume $L=\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$

$c=\begin{bmatrix} 1 & 0 \end{bmatrix}$

$\therefore A' = A-LC = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$

Q.2 $A' = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$

$A' = \begin{bmatrix} -l_1 & 1 \\ 2-l_2 & 0 \end{bmatrix}$

Characteristic Polynomial of A':

$|A'-\lambda I| = 0$

$\begin{vmatrix} -l_1-\lambda & 1 \\ 2-l_2 & -\lambda \end{vmatrix} = 0$

$\lambda(\lambda+l_1) + (l_2-2) = 0$

$\lambda^2 + l_1\lambda + (l_2-2) = 0$

Given eigen Values:

$\lambda = -4, -4$

$\therefore (\lambda+4)^2 = 0$

$\lambda^2 + 8\lambda + 16 = 0$ — (2)

From (1) & (2)

$\begin{cases} l_1 = 8 \\ l_2 - 2 = 16 \rightarrow l_2 = 18 \end{cases}$

$L = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$

$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x}) - L(C\hat{x} - y)$

$= (A-BK-LC)\hat{x} + Ly$

$= \left\{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\begin{bmatrix} -3 & -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 18 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right\}\hat{x} + \begin{bmatrix} 8 \\ 18 \end{bmatrix} y$

$= \left\{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 18 & 0 \end{bmatrix}\right\}\hat{x} + \begin{bmatrix} 8 \\ 18 \end{bmatrix} y$

$= \begin{bmatrix} -8 & 1 \\ -22 & -4 \end{bmatrix}\hat{x} + \begin{bmatrix} 8 \\ 18 \end{bmatrix} y$

$\boxed{\dot{\hat{x}} = \begin{bmatrix} -8 & 1 \\ -22 & -4 \end{bmatrix}\hat{x} + \begin{bmatrix} 8 \\ 18 \end{bmatrix} y}$

Problem 3 (4 points)

Consider the same single-link robot in Figure 1, with $m = 0.1$, $I = 1$, and $g = 10$, but no friction ($b = 0$). We define the system states as $x = [\theta \quad \dot{\theta}]^T$ and output as $y = \theta$.

The desired characteristic polynomial for the system is given by $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$, with ω_n and ξ as the natural frequency and damping ratio of the system, respectively.

Design a state-feedback control law with feedforward gain in the form of $u = -Kx + k_r r$ to stabilize the output of the system to $y = r$.

Hint: The final control law will be in terms of r, ω_n, and ξ.

Q.3) $m=0.1$, $I=1$, $g=10$, $b=0$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad y = [\theta] = [x_1]$

$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{mg}{I}\theta - \frac{b}{I}\dot{\theta} + \frac{1}{I}\tau \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \theta + \tau \end{bmatrix}$

$\therefore \dot{x} = \begin{bmatrix} x_2 \\ x_1 + u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

Characteristic Polynomial: $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$ — (1)

$A' = A - BK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix}$

$|A' - \lambda I| = 0$

$\therefore \begin{vmatrix} -\lambda & 1 \\ 1-k_1 & -k_2-\lambda \end{vmatrix} = 0$

$\lambda(\lambda + k_2) + (k_1 - 1) = 0$

$\lambda^2 + k_2\lambda + (k_1 - 1) = 0$ — (2)

From (1) & (2)

$\begin{cases} k_2 = 2\xi\omega_n \\ k_1 = \omega_n^2 + 1 \end{cases}$

$\therefore K = \begin{bmatrix} \omega_n^2 + 1 & 2\xi\omega_n \end{bmatrix}$

$k_r = -\left(C(A-BK)^{-1}B\right)^{-1}$

(1) $A - BK = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}$

(2) $(A-BK)^{-1} = \frac{1}{\omega_n^2}\begin{bmatrix} -2\xi\omega_n & -1 \\ \omega_n^2 & 0 \end{bmatrix} = \begin{bmatrix} -2\xi/\omega_n & -1/\omega_n^2 \\ 1 & 0 \end{bmatrix}$

(3) $C(A-BK)^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2\xi/\omega_n^2 & -1/\omega_n^2 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -2\xi/\omega_n^2 & -1/\omega_n^2 \end{bmatrix}$

④ $C(A-BK)^{-1}\cdot B = \begin{bmatrix} -2\xi/\omega_n^2 & -1/\omega_n^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= -1/\omega_n^2$

⑤ $\left(C(A-BK)^{-1}B\right)^{-1} = -\omega_n^2$

⑥ $-\left(C(A-BK)^{-1}\cdot B\right)^{-1} = \omega_n^2$

$K_r = -\left(C(A-BK)^{-1}\cdot B\right)^{-1} = \omega_n^2$

$u = -Kx + K_r\cdot r$

$u = -\begin{bmatrix} \omega_n^2+1 & 2\xi\omega_n \end{bmatrix} x + \omega_n^2 r$

$$\boxed{u = -(\omega_n^2+1)\theta - 2\xi\omega_n\dot{\theta} + \omega_n^2 r}$$