Problem Set 4

Tuesday, October 25, 2022 2:19 PM

Problem 1 (5 points)

Investigate the local and/or global asymptotic stability properties of the origin of the following systems using Lyapunov Theorem. If the origin is *locally* asymptotically stable, specify the domain D for your Lyapunov argument. *(answer any two out of the four parts)*

a)

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2$$

b)

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

c)

$$\dot{x}_1 = x_2(1 - x_1^2)$$
$$\dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$$

d)

$$\dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4$$
$$\dot{x}_2 = -x_1 - x_2 + x_1 x_2$$

A.1) 1.) $\dot{x}_1 = -x_1 + x_1 x_2$

$\dot{x}_2 = -x_2$

Let Lyapunov function candidate be:

$$V(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2}(x_1^2 + x_2^2)$$

$V(0) = 0$

$V(x) > 0 \quad \forall \ x \in R - \{0\}$

$\dot{V}(x) = x_1 \cdot \dot{x}_1 + x_2 \dot{x}_2$

$= x_1(-x_1 + x_1 x_2) + x_2(-x_2)$

$= -x_1^2 + x_1^2 x_2 - x_2^2$

$\dot{V}(0) = 0 \quad \forall \ x = 0$

$\dot{V}(x) < 0 \quad \forall \ x \in D$

$D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R}^- \end{cases}$

$\therefore$ Lyapunov Candidate $V(x) = \frac{1}{2}\|x\|^2$ is asymptotically stable for $D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R}^- + \{0\} \end{cases}$

2.) $\dot{x}_1 = x_2(1 - x_1^2)$

$\dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$

Assume Lyapunov Candidate to be:

$$V(x) = \frac{1}{2}(ax_1^2 + bx_2^2)$$

$V(0) = 0 \quad , V(x) > 0 \ \forall \ x \in R - \{0\} \rightarrow$ P.D.

$\dot{V}(x) = ax_1 \cdot \dot{x}_1 + bx_2 \cdot \dot{x}_2$

$= ax_1 x_2(1 - x_1^2) - bx_2(x_1 + x_2)(1 - x_1^2)$

$= ax_1 x_2 - ax_1^3 x_2 - (bx_1 x_2 + bx_2^2)(1 - x_1^2)$

$= ax_1 x_2 - ax_1^3 x_2 - bx_1 x_2 + bx_1^3 x_2 + bx_2^2 - bx_1^2 x_2^2$

$= (a-b)x_1 x_2 - (a-b)x_1^3 x_2 + bx_2^2(1 - x_1^2)$

for $\dot{V}(x)$ to be N.D. , $(a-b)x_1 x_2$ & $(a-b)x_1^3 x_2 \le 0$

$\therefore a = b$

$\therefore \dot{V}(x) = bx_2^2(1 - x_1^2)$

for $\dot{V}(x) < 0 \longrightarrow$ N.D.

$b \cdot x_2^2 \cdot (1 - x_1^2)$

$\hookrightarrow$ always +ve

$\rightarrow$ less than zero only when $|x_1| > 1$

$\hookrightarrow$ Can be positive or negative

$\therefore$ If $b < 0 \longrightarrow D \in \begin{cases} x_2 \in \mathbb{R} \\ x_1 \in (-1, 1) \end{cases}$

If $b > 0 \longrightarrow D \in \begin{cases} x_2 \in \mathbb{R} \\ x_1 \in \mathbb{R} / (-1, 1) \end{cases}$

Problem 2 (5 points)

The differential equation below describes a phase lock loop (PLL) in communication networks:

$$\ddot{y} + (a + b\cos y)\dot{y} + c\sin y = 0$$

with the constant $c > 0$.

(Remark: PLL is a control system that generates an output signal whose phase is related to the phase of an input signal. It is a very useful building block, particularly for radio frequency applications.)

a) Using a Lyapunov analysis, show that $y = 0$, $\dot{y} = 0$ is a locally stable equilibrium if $a \geq b \geq 0$. Define the domain D for the Lyapunov analysis.
Hint: You may consider the Lyapunov candidate $V = c(1 - \cos y) + 0.5\dot{y}^2$ (don't forget to show that this is a valid Lyapunov function.)

b) Using a Lyapunov (and possibly LaSalle) analysis, show that $y = 0$, $\dot{y} = 0$ is locally asymptotically stable if $a > b \geq 0$. Define the domain D for the Lyapunov analysis.

A.2) (a) Assume $y = x_1 \longrightarrow \dot{x}_1 = \dot{y} = x_2$

$\dot{y} = x_2$

$\therefore \dot{x}_2 + (a + b\cos x_1)x_2 + c\sin x_1 = 0 \longrightarrow \dot{x}_2 = -(a + b\cos x_1)x_2 - c\sin x_1$

$V(x) = c(1 - \cos x_1) + 0.5x_2^2$

$\cos x_1 \le 0 \quad \forall \ x_1 \in \mathbb{R}$

$\therefore 1 - \cos x_1 \geq 0 \quad \forall \ x_1 \in \mathbb{R}$

$x_2^2 \geq 0 \quad \forall \ x_2 \in \mathbb{R}$

$\therefore c > 0 \quad \forall \quad V(x)$ to be N.D.

$\dot{V}(x) = c(\sin x_1) \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2$

$= c(\sin x_1)(x_2) + x_2[-(a + b\cos x_1)x_2 - c\sin x_1]$

$= \cancel{cx_2 \sin x_1} - (a + b\cos x_1)x_2^2 - \cancel{cx_2 \sin x_1}$

$= -(a + b\cos x_1)x_2^2$

$\dot{V}(x) = 0 \quad \forall \ x_2 = 0$ & $x_1 \in \mathbb{R}$

Since system is N.S.D. for $x_2 = 0$ & $x_1 = \mathbb{R}$; System is stable:

when $x_2 \neq 0$;

$(a + b\cos x_1) \geq 0$

$\cos x_1 \geq -a/b$.

① If $b \neq 0$

$\rightarrow a \geq b \rightarrow D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \end{cases}$

$\rightarrow a < b \rightarrow D = \begin{cases} x_1 \in \mathbb{R} - \{\cos x_1 < -a/b\} \\ x_2 \in \mathbb{R} \end{cases}$

$a + b\cos x_1 > 0$

$\cos x_1 > -a/b$

② If $b = 0 \longrightarrow D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \end{cases}$

Testing Local Stability at $y = 0$ & $\dot{y} = 0$

$x_1 = 0 \quad x_2 = 0$

In Both Stated Domain ① & ②, $(x_1, x_2) = (0, 0)$ lies in range of domain. Thus, System is locally stable for the pts. $(0, 0)$.

(b) Applying Lasalle :

$S = \{x \mid \dot{V}(x) = 0\} = \{(x_1, x_2) \mid x_2 = 0\}$

$x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0$

$\therefore \cancel{\dot{x}_2} = -(a + b\cos x_1)\cancel{x_2} - c\sin x_1 = 0$

$-c\sin x_1 = 0$

$\sin x_1 = 0 \Rightarrow x_1 = n\pi$

$\therefore M = \{0, n\pi\} \quad n \in \mathbb{R}$

from (a) ① If $b \neq 0$ → $a \geq b \rightarrow D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \end{cases}$

→ $a < b \rightarrow D = \begin{cases} x_1 \in \mathbb{R} - \{\cos x_1 < -a/b\} \\ x_2 \in \mathbb{R} \end{cases}$

② If $b = 0 \rightarrow D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \end{cases}$

If Domain D is restricted from $\mathbb{R}$ to $(-\pi, \pi)$ for x_1

Then: ① If $b \neq 0$ → $a \geq b \rightarrow D = \begin{cases} x_1 \in (-\pi, \pi) \\ x_2 \in \mathbb{R} \end{cases}$ $M = \{0, 0\}$ $n = 0$

→ $a < b \rightarrow D = \begin{cases} x_1 \in (-\pi, \pi) - \{\cos x_1 < -a/b\} \\ x_2 \in \mathbb{R} \end{cases}$ $M = \{0, 0\}$ $n = 0$

② If $b = 0 \rightarrow D = \begin{cases} x_1 \in (-\pi, \pi) \\ x_2 \in \mathbb{R} \end{cases}$ $M = \{0, 0\}$ $n = 0$

Thus, proved local Asymptotical stability at $(0,0)$

Problem 3 (5 points)

Consider the mass-spring system shown in Figure 1. In presence of viscous damping, the equation of motion are derived as:

$$m\ddot{y} = mg - ky - b_1\dot{y} - b_2\dot{y}\,|\dot{y}|$$

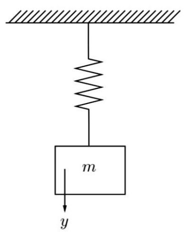


Figure 1: Mass-spring system with viscous damping.

a) Determine the equilibrium point of the system.

b) Use Lyapunov theorem (and possibly LaSalle's, if needed) to determine whether the equilibrium point is stable, locally asymptotically stable, or globally asymptotically stable.

A:) (a) At Equ^m pt. $\ddot{y} = 0 \quad \dot{y} = 0$

$\therefore m\cancel{\ddot{y}} = mg - ky - b_1\cancel{\dot{y}} - b_2\cancel{\dot{y}}|\cancel{\dot{y}}|$

$mg = ky$

$y_{eq.} = \frac{mg}{k} \rightarrow$ Equ^m pt.

(b) $y - \frac{mg}{k} = 0 = x_1 \rightarrow$ Shifted coordinates

$\dot{y} = x_2 \rightarrow$ Shifted coordinates. → Stability pt.

$x_1 = 0 \rightarrow y = mg/k$

$x_2 = 0 \rightarrow \dot{y} = 0$

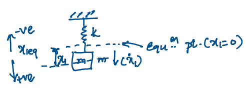
$\therefore m\dot{x}_2 = mg - k\left(x_1 + \frac{mg}{k}\right) - b_1 x_2 - b_2 x_2 |x_2|$

$\therefore m\dot{x}_2 = -kx_1 - b_1 x_2 - b_2 x_2 |x_2|$ — (*) Shifted coordinates Equ^n

Designing Lyapunov Candidate via System Energy.

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}_1)^2$

$PE = mgx_1 + \frac{1}{2}kx_1^2$



$\therefore TE = mgx_1 + \frac{1}{2}kx_1^2 + \frac{1}{2}m\dot{x}_1^2$

$\therefore V(x) = \frac{mg}{2}x_1^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}m\dot{x}_1^2$

$= \left(\frac{mg + k}{2}\right)x_1^2 + \frac{1}{2}mx_2^2 \qquad \boxed{x_2 = \dot{x}_1}$

$V(x) = 0 \quad \forall x = 0$

$V(x) > 0 \quad \forall x \in \mathbb{R} - \{0\}$ } P.D.

$\therefore$ $V(x)$ is a suitable Lyapunov Candidate.

$\dot{V}(x) = (mg + k)x_1 \cdot \dot{x}_1 + mx_2 \cdot \dot{x}_2$

$= (mg + k)x_1 \cdot x_2 + mx_2(-kx_1 - b_1x_2 - b_2x_2|x_2|)$

$= mgx_1x_2 + kx_1x_2 - mkx_1x_2 - mb_1x_2^2 - mb_2x_2^2|x_2|$

$\dot{V}(x) = (\underbrace{mg}_{\text{Always +ve}} + k - \underbrace{mk}_{\text{Always +ve}})x_1x_2 - \underbrace{\underbrace{(mb_1 + mb_2|x_2|)}_{\text{always +ve}}\underbrace{x_2^2}_{\text{always +ve}}}_{\text{always +ve}}$ — ①

always -ve

$-(mb_1 + mb_2|x_2|)x_2^2 \rightarrow$ Never +ve

$(mg + k - mk)x_1x_2 \rightarrow$ can be +ve if → $(mg + k - mk) > 0$ & $x_1 \cdot x_2 > 0$

→ $(mg + k - mk) < 0$ & $x_1 \cdot x_2 < 0$

$\therefore$ We choose system such that $mg + k - mk = 0$

$\therefore mg + k(1 - m) = 0$

$k = \frac{-mg}{1 - m}$

$\therefore \boxed{K = \frac{mg}{m - 1}} \Rightarrow \boxed{m > 1}$

If the required conditions are met;

$\dot{V}(x) = -(mb_1 + mb_2|x_2|)x_2^2$ — ②

$\dot{V}(x) = 0 \quad \forall x_2 = 0$ & $x_1 = \mathbb{R}$

$\dot{V}(x) < 0 \quad \forall \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} - \{0\} \end{cases}$

$\therefore$ System is N.S.D. & hence Stable

Applying Lasalle: ① → $\dot{V}(x) = (mg + k - mk)x_1x_2 - (mb_1 + mb_2|x_2|)x_2^2$ } Only if Domain of x_1, x_2 is restricted.

$S = \{x \mid \dot{V}(x) = 0\} = \{(x_1, x_2) \mid x_2 = 0\}$

$x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0$

$\cancel{\dot{x}_2} = -\frac{k}{m}x_1 - \frac{b}{m}\cancel{x_2} - \frac{b}{m}\cancel{x_2}|x_2| = 0$

$-\frac{k}{m}x_1 = 0$

$\boxed{x_1 = 0} \longrightarrow$ Local Asymptotical Stability

① If $(mg + k - mk) \geq 0 \qquad D = \begin{cases} x_1 \in \mathbb{R}^- + \{0\} \\ x_2 \in \mathbb{R}^+ + \{0\} \end{cases}$ or $D = \begin{cases} x_1 \in \mathbb{R}^+ + \{0\} \\ x_2 \in \mathbb{R} + \{0\} \end{cases}$

② If $(mg + k - mk) < 0 \qquad D = \begin{cases} x_1 \in \mathbb{R}^+ + \{0\} \\ x_2 \in \mathbb{R}^+ + \{0\} \end{cases}$ or $D = \begin{cases} x_1 \in \mathbb{R}^- + \{0\} \\ x_2 \in \mathbb{R}^- + \{0\} \end{cases}$

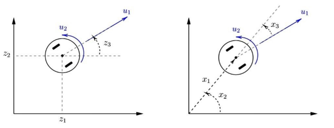
No GAS Since Domain $\neq \mathbb{R}$

Problem 4 (5 points)

Consider the wheeled robot shown in Figure 2 (a), for which the dynamics are derived as:

$$\dot{z}_1 = u_1 \cos z_3$$
$$\dot{z}_2 = u_1 \sin z_3$$
$$\dot{z}_3 = u_2$$

In the equations above, z_1 and z_2 represent the Cartesian position of the robot, and z_3 is the orientation with respect to the horizontal axis. The control inputs u_1 and u_2 are the linear and angular velocities of the robot, respectively.



(a) (b)

Figure 2: Wheeled robot coordinates and control inputs.

The objective is to design a feedback control law $u = \alpha(x)$ to derive the robot to the origin $z = 0$ from any initial condition. As explained in [1] *, the problem would be carried out easier in polar coordinates by defining x_1 and x_2 to be the radial and angular coordinates of the robot, respectively, and $x_3 = x_2 - z_3$, as shown in Figure 2 (b).

The dynamics can be re-written in terms of the new coordinates as:

$$\dot{x}_1 = u_1 \cos x_3$$
$$\dot{x}_2 = -\frac{u_1 \sin x_3}{x_1}$$
$$\dot{x}_3 = -\frac{u_1 \sin x_3}{x_1} - u_2$$

a) A candidate Lyapunov function is suggested as

$$V(x) = V_1(x_1) + V_2(x_2, x_3),$$
$$\text{with} \quad V_1(x_1) = \frac{1}{2}x_1^2 \quad \text{and} \quad V_2(x_2, x_3) = \frac{1}{2}(x_2^2 + x_3^2).$$

Calculate the time derivatives $\dot{V}_1(x_1, u_1)$ and $\dot{V}_2(x, u)$.

b) Show that if we design the control laws to be:

$$u_1 = -x_1 \cos x_3$$
$$u_2 = x_3 + \frac{(x_2 + x_3)\cos x_3 \sin x_3}{x_3}$$

the time derivative of the Lyapunov functions would be rendered negative-semidefinite, that is, $\dot{V}_1(x, u_1) \leq 0$ and $\dot{V}_2(x, u_2) \leq 0$.

*[1] M. Aicardi, G. Casalino, A. Bicchi and A. Balestrino, "Closed loop steering of unicycle like vehicles via Lyapunov techniques," in IEEE Robotics & Automation Magazine, vol. 2, no. 1, pp. 27-35, March 1995

c) Plug in the control laws in the equations of motion to derive the closed-loop dynamics $\dot{x} = f(x, u)$. Use the Lyapunov and LaSalle's theorem to show asymptotic stability of the closed-loop system. Determine whether the stability is local or global.

d) **(Optional)** Implement a simulation of the system in MATLAB, and verify the performance of the system under the proposed control law starting from different initial conditions.

a) $\dot{V}_1(x, u_1) = x_1 \dot{x}_1 = x_1(u_1 \cos x_3) = x_1 u_1 \cos x_3$

$\dot{V}_2(x, u_2) = x_2 \cdot \dot{x}_2 + x_3 \cdot \dot{x}_3$

$= x_2\left(-\frac{u_1 \sin x_3}{x_1}\right) + x_3\left(-\frac{u_1 \sin x_3}{x_1} - u_2\right)$

$= -u_1 \frac{x_2}{x_1} s x_3 - u_1 \frac{x_3}{x_1} s x_3 - u_2 x_3$

$= -\frac{u_1}{x_1}(x_2 + x_3) s x_3 - u_2 x_3$

b) After placing control laws:

$\dot{V}_1(x, u_1) = x_1(-x_1 c x_3) \cdot c x_3 = -x_1^2 c^2 x_3$

$\dot{V}_2(x, u_2) = -\frac{(-\cancel{x_1} c x_3)}{\cancel{x_1}}(x_2 + x_3) s x_3 - \left(x_3 + \left(\frac{x_2 + x_3}{x_3}\right) c x_3 s x_3\right) x_3$

$= \cancel{(x_2 + x_3) c x_3 s x_3} - x_3^2 \cancel{- (x_2 + x_3) c x_3 s x_3}$

$\dot{V}_2(x, u_2) = -x_3^2$

$\dot{V}_1(x_1, u_1) = 0 \quad \forall \begin{cases} x_1 = 0, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \\ x_3 = \{\pi/2, 3\pi/2\} \\ \quad \hookrightarrow x_1 \in \mathbb{R}, x_2 \in \mathbb{R} \end{cases}$

$\dot{V}_2(x, u_2) = 0 \quad \forall \begin{cases} x_3 = 0, x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \end{cases}$

$\dot{V}_1(x, u_1) < 0 \quad \forall \begin{cases} x_1 \in \mathbb{R} - \{0\}. \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} - \{\pi/2, 3\pi/2\} \end{cases}$

$\dot{V}_2(x, u_2) < 0 \quad \forall \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} - \{0\} \end{cases}$

Since $V_1(x, u_1) = 0$ & $V_2(x, u_2) = 0$ $\forall$ more than one pts &
$V_1(x, u_1) < 0$ & $V_2(x, u_2) < 0$ $\forall$ rest region

$V_1(x, u_1)$ & $V_2(x, u_2)$ are N.S.D.

c) $\dot{x}_1 = u_1 \cos x_3 \qquad \dot{x}_2 = -\frac{u_1}{x_1}\sin x_3 \qquad \dot{x}_3 = -\frac{u_1}{x_1}\sin x_3 - u_2$

$u_1 = -x_1 \cos x_3 \qquad u_2 = x_3 + \frac{(x_2 + x_3)}{x_3}\cos x_3 \sin x_3$

Substituting values:

$\dot{x}_1 = -x_1 \cos^2 x_3$

$\dot{x}_2 = \cos x_3 \sin x_3$

$\dot{x}_3 = \cos x_3 \sin x_3 - x_3 - \left(\frac{x_2 + x_3}{x_3}\right)\cos x_3 \sin x_3$

$= \cancel{\cos x_3 \sin x_3} - x_3 - \frac{x_2}{x_3}\cos x_3 \sin x_3 \cancel{- \cos x_3 \sin x_3}$

$\dot{x}_3 = -x_3 - \frac{x_2}{x_3}\cos x_3 \sin x_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 \cos^2 x_3 \\ \cos x_3 \sin x_3 \\ -x_3 - \frac{x_2}{x_3}\cos x_3 \sin x_3 \end{bmatrix} \Bigg\} \text{Closed Loop Dynamics}$$

Assume Lyapunov Candidate: $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

$V(0) = 0, \quad V(x) > 0 \ \forall x \in \mathbb{R}$

$\therefore \dot{V}(x) = x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2 + x_3 \cdot \dot{x}_3$

$\dot{V}(x) = -x_1^2 \cos^2 x_3 \cancel{+ x_2 \cos x_3 \sin x_3} - x_3^2 \cancel{- x_2 \cos x_3 \sin x_3}$

$\dot{V}(x) = -x_1^2 \cos^2 x_3 - x_3^2$

$\dot{V}(x) = 0 \quad \forall \{x_1 = 0, x_3 = 0, x_2 \in \mathbb{R}$

$\dot{V}(x) < 0 \quad \forall \ x_i \in \mathbb{R} - \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$

$\therefore \dot{V}(x) \rightarrow$ N.S.D. $\Rightarrow$ Stable.

Applying LaSalle:

$\Omega_c = \{x \mid \dot{V}(x) \leq 0\} \Rightarrow S = \{x \in \Omega_c \mid x_1 = 0, x_3 = 0\}$

$x_3 = 0 \quad \therefore \dot{x}_3 = 0$

$\therefore \ -x_3 - \frac{x_2}{x_3}\cos x_3 \sin x_3 = 0$

$\frac{-x_3^2 - x_2 \cos x_3 \sin x_3}{x_3} = 0$

We cannot substitute $x_3 = 0$ directly

$\lim_{x_3 \to 0}\left[\frac{-x_3^2 - \frac{2x_2 \cos x_3 \sin x_3}{2}}{x_3}\right] = 0$

$\lim_{x_3 \to 0}\left(\frac{-x_3^2 - \frac{x_2}{2}\sin 2x_3}{x_3}\right) = 0$

differentiate w/ x_3

$\frac{-2x_3 - x_2/\cancel{2}\cos 2x_3 \cdot \cancel{2}}{1} = 0$

$-2x_3 - x_2 \cos 2x_3 = 0$

$-x_2 \cos(0) = 0$

$\boxed{x_2 = 0}$

$M = \{0, 0, 0\}$ $\therefore$ Local Asymptotical Stability proved.

$D = \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \end{cases}$

as $\|x\| \to \infty \quad V(x) \to \infty \quad \therefore$ Radially Unbounded

$\therefore$ GAS.