Problem Set 1

Tuesday, September 13, 2022 5:12 PM

Problem 1 (5 points)

The equations of motion for a quadrotor moving in a 2D plane are given by:

$$m\ddot{x} = -(F_1 + F_2)\sin\theta$$
$$m\ddot{y} = (F_1 + F_2)\cos\theta - mg$$
$$I\ddot{\theta} = L(F_2 - F_1)$$

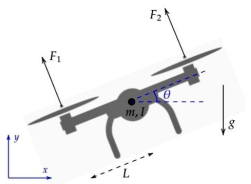


Figure 1: A planar quadrotor

The coordinates (x, y, θ) represent the position and orientation of the quadrotor, m and I are the mass and moment of inertia, L denotes the length of the quadrotor's arm, and g is the gravitational acceleration. F_1 and F_2 denote the thrust forces generated by the propellers.

a) Defining the control inputs $u_1 := (F_1 + F_2)$ and $u_2 := (F_2 - F_1)$ and denoting the horizontal and vertical positions of the quadrotor as the outputs w, construct a state-space model for this system in the form of $f(z, u)$ and $w = h(z, u)$ (we have denoted the state vector by z and the output vector by w to avoid confusion with the horizontal position x and the vertical position y).

b) Find the equilibrium point of the system associated with $x^* = x_e$, $y^* = y_e$, $u_1^* = mg$ and $u_2^* = 0$.

c) Linearize the system around the equilibrium point found in part (b), and explicitly write out the state and output equations corresponding to the linearized dynamics.

A.) ⓐ

$$\left.\begin{matrix} u_1 = F_1 + F_2 \\ u_2 = F_2 - F_1 \\ u_3 = g \end{matrix}\right] \Rightarrow u = \begin{bmatrix} F_1+F_2 \\ F_2 - F_1 \\ g \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$z = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \quad dz = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{F_1+F_2}{m}\sin\theta \\ \dot{y} \\ \frac{F_1+F_2}{m}\cos\theta - g \\ \dot{\theta} \\ \frac{L}{I}(F_2 - F_1) \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{u_1}{m} s z_5 \\ z_4 \\ \frac{u_1}{m} c z_5 - u_3 \\ z_6 \\ \frac{L}{I} u_2 \end{bmatrix}$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}$$

→ State Space & Output Equ.^{n}

ⓑ Finding equ.^{m} pt: (Given: $x^* = x_e$ (z_1^*), $y^* = y_e$ (z_3^*), $u_1^* = mg$, $u_2^* = 0$, $u_3^* = g$)

$$dz = 0 \Rightarrow \begin{cases} z_2^* = 0 \longrightarrow z_2^* = 0 \\ -\frac{u_1^*}{m} s z_5^* = 0 \longrightarrow -\frac{mg}{m} s z_5^* = 0 \Rightarrow s z_5^* = 0 \Rightarrow z_5^* = [0, \pi] \\ z_4^* = 0 \longrightarrow z_4^* = 0 \\ \frac{u_1^*}{m} c z_5^* - u_3^* = 0 \longrightarrow \frac{mg}{m} . c z_5^* - g = 0 \Rightarrow c z_5^* = 1 \Rightarrow \boxed{z_5^* = 0} \\ z_6^* = 0 \longrightarrow z_6^* = 0 \\ \frac{L}{I} u_2^* = 0 \end{cases}$$

$$\boxed{z^* = \begin{bmatrix} x_e \\ 0 \\ y_e \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

→ Equilibirium pts.

ⓒ $\bar{z} = z - z^*$

* State Space Matrix

$$\therefore \bar{z} = \begin{bmatrix} z_1 - x_e \\ z_2 \\ z_3 - y_e \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

$$\frac{\partial f}{\partial z}(z^*, u^*) = \begin{bmatrix} \partial f_1/\partial z_1 & \partial f_1/\partial z_2 & \cdots & \partial f_1/\partial z_k \\ \partial f_2/\partial z_1 & \partial f_2/\partial z_2 & \cdots & \partial f_2/\partial z_k \\ \vdots & & & \vdots \\ \partial f_k/\partial z_1 & \partial f_k/\partial z_2 & \cdots & \partial f_k/\partial z_k \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{u_1}{m} c z_5^* & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{u_1}{m} s z_5^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u_1/m & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{u} = u - u^*$$
$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \bar{u} = \begin{bmatrix} u_1 - mg \\ u_2 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} u_1 - mg \\ u_2 \end{bmatrix} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial z}(z^*, u^*) = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{s z_5^*}{m} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{c z_5^*}{m} & 0 & -1 \\ 0 & 0 & 0 \\ 0 & \frac{L}{I} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/m & 0 & -1 \\ 0 & 0 & 0 \\ 0 & L/I & 0 \end{bmatrix} \xrightarrow[u_3 = 0]{\text{because}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & L/I \end{bmatrix}$$

$$\boxed{\dot{\bar{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u_1/m & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \\ \bar{z}_5 \\ \bar{z}_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & L/I \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}}$$

⇒ Linearized State Space Equ.^{n}.

* Output Matrix

$$w = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} \quad w^* = \begin{bmatrix} x_e \\ y_e \end{bmatrix}$$

$$\bar{w} = \begin{bmatrix} z_1 - x_e \\ z_3 - y_e \end{bmatrix} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_3 \end{bmatrix}$$

$$w = \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} = h(z,u)$$

$$\frac{\partial h}{\partial z} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \ \bar{\omega} = \frac{\partial h}{\partial z}(z^*, u^*)\bar{z} + \frac{\partial h}{\partial u}(z^*, u^*)\bar{u}$$

$$\boxed{\bar{\omega} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \\ \bar{z}_5 \\ \bar{z}_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}} \rightarrow \text{Linearized Output Equ}^{n}.$$

Problem 2 (5 points)

Consider the one-link robot in Figure 2, where θ denotes the angle of the link with the horizontal, τ the torque applied at the joint, (x, y) the position of the tip, l the length of the link, I its moment of inertia, m the mass at tip, g gravity's acceleration, and b the friction coefficient. This system evolves according to the following equation of motion:

$$I\ddot{\theta} = -b\dot{\theta} - mg\cos\theta + \tau$$

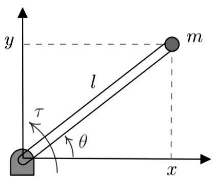


Figure 2: Single-link robot

a) Compute the state-space model for the system with $u = \tau$ as the input and the vertical position of the tip y as the output.
   Please denote the state vector by z to avoid confusion with the horizontal position of the tip x, and write the model in the form:

$$\dot{z} = f(z, u), \quad y = h(z, u)$$

   for appropriate functions f and h. Do not forget the output equation!

b) Find the equilibrium points of the system when $\tau(t) = 0$.

c) Compute the linearization of the system around the equilibrium point with "upward" configuration. From your answer, can you predict if there will be problems when one wants to control the tip position close to this configuration just using feedback from y?

d) Assume $m = 0.1$, $I = 1$, $b = 0$, and $g = 10$. Determine the stability properties of the equilibrium point in part (c). Is the linearized system around that equilibrium point controllable?

A.) (a) $I\ddot{\theta} = -b\dot{\theta} - mg\cos\theta + \tau$

$$z = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{1}{I}(b\dot{\theta} + mg\cos\theta - \tau) \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{1}{I}(b \cdot z_2 + mgcz_1 - \tau) \end{bmatrix}$$

$$u = [\tau]$$

$$y = [l\sin\theta] = [l\sin z_1]$$

$$\boxed{\begin{aligned} \dot{z} &= f(z,u) = \begin{bmatrix} z_2 \\ -\frac{1}{I}(bz_2 + mgcz_1 - u) \end{bmatrix} \\ y &= h(z,u) = [l\sin z_1] \end{aligned}} \Rightarrow \text{State Space \& Output Equ}^{n}.$$

(b) Finding Equ^{n}. pt.: $\dot{z} = f(z^*, u^*) = 0$

$$\rightarrow \begin{cases} z_2^* = 0 \\ -\frac{1}{I}(bz_2^* + mgcz_1^* - u^*) = 0 \end{cases} \Bigg|_{\forall u^* = 0}$$

$$\rightarrow z_2^* = 0$$

$$\rightarrow -\frac{1}{I}(b(0) + mgcz_1^* - (0)) = 0$$

$$mgcz_1^* = 0$$

$$cz_1^* = 0 \longrightarrow \boxed{z_1^* = \pi/2, 3\pi/2} \ \text{Equilibirium pts.}$$

$$\therefore \boxed{(z_1^*, z_2^*, u^*, y^*) = (\pi/2, 0, 0, 0) \ \& \ (3\pi/2, 0, 0, 0)}$$

(c) Linearization around $z_1^* = \pi/2$ pos^{n}.

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{I}(-mgsz_1^*) & -\frac{1}{I}(b) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mg}{I} & -\frac{b}{I} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ -\frac{1}{I}(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/I \end{bmatrix}$$

$$\frac{\partial h}{\partial z} = \begin{bmatrix} l_1cz_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\frac{\partial h}{\partial u} = [0]$$

$$\bar{z} = \begin{bmatrix} z_1 - \pi/2 \\ z_2 \end{bmatrix} \qquad \bar{u} = [u - 0] = [u]$$

$$\boxed{\begin{aligned} \dot{\bar{z}} &= \begin{bmatrix} 0 & 1 \\ \frac{mg}{I} & -b/I \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I \end{bmatrix} [\bar{u}] \\ \bar{y} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + [0][\bar{u}] \end{aligned}} \rightarrow \begin{aligned} &\dot{\bar{z}} = A\bar{z} + B\bar{u} \\ &\text{Linearized State Space \& O/P Equ}^{n}.: \\ &\bar{y} = C\bar{z} + D\bar{u} \end{aligned}$$

If we try to use $\bar{y}$ as feedback, our control strategy would fail due to value of $\bar{y}$ being a constant 'zero'.

(d) Assume $m = 0.1$, $I = 1$, $b = 0$, $g = 10$

$$\therefore \dot{\bar{z}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\bar{u}]$$

$$\bar{y} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + [0][\bar{u}]$$

Stability of a linearized equ^n about an equ^m pt. can be calculated using eigen vectors.

$$|A-\lambda I| = 0$$

$$\left|\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right| = 0$$

$$\lambda^2 - 1 = 0$$

$$\boxed{\lambda_1 = 1,\ \lambda_2 = -1}$$

To find controllibility, Rank $(A-\lambda I \quad B) = n$ → in our case, n should be 2

$\lambda = 1 \Rightarrow \text{Rank}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = 2$

$\lambda = -1 \Rightarrow \text{Rank}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2$

Both ranks are '2', ∴ System is controllable at both λ_i.

Problem 3 (5 points)

A unicycle robot moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system:

$$\dot{p}_x = v\cos\theta, \quad \dot{p}_y = v\sin\theta, \quad \dot{\theta} = \omega,$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}^T \in \mathbb{R}^2$.

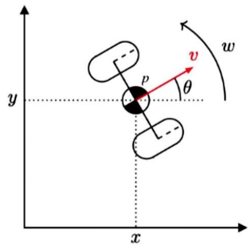


Figure 3: Unicycle model

a) Construct a state-space model for this system in the form of $f(x,u)$ with state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x\cos\theta + (p_y - 1)\sin\theta \\ -p_x\sin\theta + (p_y-1)\cos\theta \\ \theta \end{bmatrix}$$

and output $y := \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2$

b) Compute a local linearization for this system around the equilibrium point $x^* = 0$, $u^* = 0$.

c) It is given that the trajectory $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$, $\forall t \geq 0$ is a solution to the system. Show that a local linearization of the system around this trajectory results in an LTI system.

A) (a)

$\dot{p}_x = v\cos\theta$

$\dot{p}_y = v\sin\theta$

$\dot{\theta} = \omega$

→ Coordinates of Robot

$$u = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} P_x c\theta + (P_y - 1)\sin\theta \\ -P_x\sin\theta + (P_y-1)\cos\theta \\ \theta \end{bmatrix}$$

$$dx = \begin{bmatrix} \dot{P}_x c\theta - P_x s\theta\cdot\dot{\theta} + \dot{P}_y s\theta + (P_y-1)s\theta\cdot\dot{\theta} \\ -\dot{P}_x s\theta - P_x c\theta\cdot\dot{\theta} + \dot{P}_y c\theta - (P_y-1)s\theta\cdot\dot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} vc^2\theta + x_2\cdot\dot{\theta} + vs^2\theta \\ -\cancel{vc\theta s\theta} - x_1\dot{\theta} + \cancel{vs\theta c\theta} \\ \dot{\theta} \end{bmatrix}$$

$$\boxed{\dot{x} = \begin{bmatrix} v + x_2\cdot\omega \\ -x_1\cdot\omega \\ \omega \end{bmatrix}} \Longrightarrow \text{State Space Representation}$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{\therefore y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix}} \Longrightarrow \text{Output Equ}^n\text{ of form: } Y = CX + DU$$

(b) Linearization about $x^* = 0$; $u^* = 0$

$\dot{x} = f(x,u) \qquad y = h(x,u)$

$$\therefore \frac{\partial f}{\partial x}(x^*, u^*) = \begin{bmatrix} 0 & \omega^* & 0 \\ -\omega^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial u}(x^*, u^*) = \begin{bmatrix} 1 & x_2^* \\ 0 & -x_1^* \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial h}{\partial x}(x^*, u^*) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u}(x^*, u^*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{v} \\ \bar{\omega} \end{bmatrix} \rightarrow \dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{v} \\ \bar{\omega} \end{bmatrix} \rightarrow \bar{y} = C\bar{x} + D\bar{u}$$

⇒ Linearized State Space & Output Matrix

(c) $\omega(t) = v(t) = 1$

$P_x(t) = \sin t$

$P_y(t) = 1 - \cos t$

$\theta(t) = t$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \sin t\cdot\cos t - \cos t\cdot\sin t \\ -\sin^2 t - \cos^2 t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}$$

$$u(t) = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x}(t) = x(t) - \cancel{x^*(t)} = x(t)$$

$\bar{u}(t) = u(t) - \cancel{u^*(t)} = u(t)$

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\dot{\bar{x}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\bar{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\bar{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

This $\dot{\bar{x}}$ & $\bar{y}$ matrix are a proof that after linearlization; The system is linear time invarient due of absence of 't' term.

Problem 4 (5 points)

a) The second-order system

$$\ddot{\omega} + g(\omega)\dot{\omega} + \omega = 0 \quad (*)$$

with equilibrium point $\omega = \dot{\omega} = 0$. Construct a state-space model for the system. Determine for which values of $g(0)$ we can guarantee convergence to the origin based on the local linearization.

Note. Equation $(*)$ is called the Liénard equation and can be used to model several different mechanical systems, for distinct choice of the function $g(\cdot)$

b) Consider the nonlinear system:

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

Linearize the system around the trajectory $(x_1, x_2) = (\sin t, \cos t)$.

Is the linearized system time-invariant or time-varying?

A.) ⓐ $\ddot{\omega} + g(\omega)\cdot\dot{\omega} + \omega = 0$

$$x = \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad dx = \begin{bmatrix} \dot{\omega} \\ \ddot{\omega} \end{bmatrix} = \begin{bmatrix} \dot{\omega} \\ -g(\omega)\dot{\omega} - \omega \end{bmatrix} = \begin{bmatrix} x_2 \\ -g(x_1)\cdot x_2 - x_1 \end{bmatrix}$$

$$\therefore \boxed{dx = \begin{bmatrix} x_2 \\ -g(x_1)\cdot x_2 - x_1 \end{bmatrix}} \Rightarrow \text{State Space model}$$

$*$ Linearization at $\omega = \dot{\omega} = 0 \Rightarrow x_1 = x_2 = 0$

$$\frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} 0 & 1 \\ -g'(x_1^*)\cdot x_2^* - 1 & -g(x_1^*) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix}$$

$$\dot{\bar{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix}}_{\text{A matrix}} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

for convergence to origin, the system should be stable, i.e. $Re(\lambda_i) < 0$

$\therefore$ For stability, Find eigen values:

$|A - \lambda I| = 0$ Assume $-g(0) = k$

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & k \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ -1 & k-\lambda \end{bmatrix} \right| = 0$$

$(-\lambda)(k-\lambda) - (-1) = 0$

$-\lambda k + \lambda^2 + 1 = 0$

$\lambda^2 - k\lambda + 1 = 0$

$$\therefore \lambda = \frac{-(-k) \pm \sqrt{k^2 - 4(1)(1)}}{2(1)} = \frac{k \pm \sqrt{k^2-4}}{2}$$

$$\lambda = \frac{k \pm \sqrt{k^2-4}}{2}$$

$$\Rightarrow \frac{k \pm \sqrt{k^2-4}}{2}$$

Situation 1:

$k^2 > 4$, real roots

$\lambda = \frac{k \pm \sqrt{k^2-4}}{2}$

$Re(\lambda_i) < 0$

i.e. $\frac{k \pm \sqrt{k^2-4}}{2} < 0$

$k \pm \sqrt{k^2-4} < 0$

$k + \sqrt{k^2-4} < 0$ $k - \sqrt{k^2-4} < 0$

$k < -\sqrt{k^2-4}$ → This is a negative no. as a whole $k < \sqrt{k^2-4}$

Since k is less than -ve no., i.e. $k<0$ Here k can only be -ve

i.e. $|k| > |\sqrt{k^2-4}|$

$\therefore k^2 > k^2 - 4$

$\cancel{k^4} > \cancel{k^4} - 8k^2 + 16$

$-k^2 + 2 < 0$

$k^2 - 2 > 0$

$k^2 > 2$

$|k| > \sqrt{2}$

Since $k < 0$

i.e $k < -\sqrt{2}$

$k = -g(0)$

$-g(0) < -\sqrt{2}$

$\boxed{g(0) > \sqrt{2}}$ — ②

Ⓐ $k < 0$

$k^2 > k^2 - 4$

⇓

$k < -\sqrt{2}$

$\therefore \boxed{g(0) > \sqrt{2}}$ — ③

Situation 2:

$k^2 < 4$, complex root.

i.e. $\lambda = \frac{k \pm i\sqrt{4-k^2}}{2}$

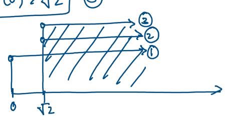
i.e. $Re(\lambda_i) < 0$

i.e. $k < 0$

$k = -g(0)$

i.e. $-g(0) < 0$

$\therefore \boxed{g(0) > 0}$ — ①



$\boxed{\text{i.e. } g(0) > \sqrt{2} \text{ for guarenteo convergence}}$

6

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} x_2 + x_1 - x_1^3 - x_1 x_2^2 \\ -x_1 + x_2 - x_1^2 x_2 - x_2^3 \end{bmatrix}$$

Linearization about trajectory : $x^* = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

$$\frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} 1 - 3x_1^{2*} - x_2^{2*} & 1 - 2x_1^* x_2^* \\ -1 - 2x_1^* x_2^* & 1 - x_1^{2*} - 3x_2^{2*} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 3\sin^2 t - \cos^2 t & 1 - 2\sin t \cos t \\ -1 - 2\sin t \cos t & 1 - \sin^2 t - 3\cos^2 t \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{1} - 2\sin^2 t - \cancel{1} & 1 - \sin 2t \\ -1 - \sin 2t & \cancel{1} - \cancel{1} - 2\cos^2 t \end{bmatrix}$$

$$= \begin{bmatrix} -2\sin^2 t & 1 - \sin 2t \\ -(1 + \sin 2t) & -2\cos^2 t \end{bmatrix}$$

$$\boxed{\dot{\bar{x}} = \begin{bmatrix} -2\sin^2 t & 1 - \sin 2t \\ -(1 + \sin 2t) & -2\cos^2 t \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \quad \forall \; \bar{x} = \begin{bmatrix} x_1 - \sin t \\ x_2 - \cos t \end{bmatrix}}$$

The resulting linearized System is time varient due to presence of 't' dependent terms.