SECTION 1

MODULE 3 : Elementary Combinatorics

Permutation

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$$P(n,n) = n!$$

Example

(1) If ${}^{n}P_2 = 72$, find the value of n.

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Solution : Here

$$
\begin{aligned}
{}^nP_2 = 72 &\implies \frac{n!}{(n-2)!} = 72 \\
&\implies \frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!} = 72 \\
&\implies n(n-1) = 72 \implies n^2 - n - 72 = 0 \\
&\implies (n-9)(n+8) = 72
\end{aligned}
$$

Hence

$$n = 9.$$

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Solution : Here

$$ {}^{2n+1}P_{n-1} = \frac{(2n+1)!}{(n+2)!} \quad \text{and} \quad {}^{2n-1}P_n = \frac{(2n-1)!}{(n-1)!} $$

$$ \begin{aligned} {}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5 &\implies \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5} \\ &\implies \frac{(2n+1)\times(2n)\times(2n-1)!\times(n-1)!}{(n+2)\times(n+1)\times(n)\times(n-1)!\times(2n-1)!} = \frac{3}{5} \\ &\implies \frac{(2n+1)(2n)}{(n+2)(n+1)(n)} = \frac{3}{5} \\ &\implies \frac{4n+2}{n^2+3n+2} = \frac{3}{5} \\ &\implies 3n^2 - 11n - 4 = 0 \\ &\implies (n-4)(3n+1) = 0 \end{aligned} $$

Hence

$$ n = 4. $$

Combination

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Note :

$$C(n, n) = C(n, 0) = 1$$

Properties of $C(n,r)$ or ${}^nC_r$.

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3. $$ {}^nC_x = {}^nC_y \implies x = y \quad \text{or} \quad x + y = n. $$

Example

(1) If ${}^{18}C_r = {}^{18}C_{r+2}$, then find the value of ${}^{r}C_5$.

Solution : Here ${}^{18}C_r = {}^{18}C_{r+2}$, then by property of ${}^{n}C_r$ we have

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$$r + (r+2) = 18 \implies 2r + 2 = 18 \implies r = 8.$$

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$${}^{r}C_5 = {}^{8}C_5 = \frac{8!}{5! \times (8-5)!} = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} = 56.$$

(2) In how many ways 3 questions can be selected from 9 questions ?

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(2) In how many ways 3 questions can be selected from 9 questions ?

Solution : The required number of ways is

$$ {}^{9}C_3 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

(3) If ${}^{n}C_{x} = 56$ and ${}^{n}P_{x} = 336$, find n and x.

Solution : Given ${}^{n}C_{x} = 56$ and ${}^{n}P_{x} = 336$.
Therefore

$$\frac{n!}{x! \times (n-x)!} = 56 \quad \text{and} \quad \frac{n!}{(n-x)!} = 336.$$

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(3) If ${}^{n}C_x = 56$ and ${}^{n}P_x = 336$, find n and x.

Solution : Given ${}^{n}C_x = 56$ and ${}^{n}P_x = 336$.
Therefore

$$\frac{n!}{x! \times (n-x)!} = 56 \quad \text{and} \quad \frac{n!}{(n-x)!} = 336.$$

So

$$\frac{336}{x!} = 56 \implies \frac{336}{56} = x! \implies x! = 6 \implies x = 3$$

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$${}^nP_x = 336 \implies {}^nP_3 = 336.$$

which gives

$$\frac{n!}{(n-3)!} = 336 \implies \frac{n \times (n-1) \times (n-2) \times (n-3)!}{(n-3)!} = 336$$

$$\implies n \times (n-1) \times (n-2) = 336 = 8 \times 7 \times 6$$

Hence $n = 8$.

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(i) $S(1)$ is true.

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(i) $S(1)$ is true.

(ii) $S(k+1)$ is true whenever $S(k)$ is true.

Then $S(n)$ is true for all positive integer n.

Example

(1) Show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

where $n \geq 1$ by mathematical induction.

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Solution :Let $S(n)$ be the given statement

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$$S(n) : 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

For $n = 1$, LHS $= 1^2 = 1$ and RHS $= \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}\times 1\times(1+1)(2\times 1+1) = 1$.

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Therefore LHS=RHS. Hence $S(1)$ is true.

Assume $S(k)$ is true

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$$

We show that $S(k+1)$ is true

$$S(k+1) : 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3).$$

Now

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right] \\
&= (k+1)\left[\frac{1}{6}(2k^2 + 7k + 6)\right] \\
&= \frac{1}{6}(k+1)(k+2)(2k+3).
\end{aligned}$$

Hence $S(k+1)$ is true. So $S(k+1)$ is true whenever $S(k)$ is true. By Principal of Mathematical Induction $S(n)$ is true for all positive integer n.

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For $n = 3$, LHS is $3^2 = 9$ and RHS is $2n+1 = 7$.

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For $n = 3$, LHS is $3^2 = 9$ and RHS is $2n+1 = 7$. Therefore LHS > RHS. Hence $S(3)$ is true.

Assume $S(k)$ is true

$$k^2 > 2k+1$$

Now we show that $S(k+1)$ is true

$$S(k+1) : (k+1)^2 > 2k+3.$$

Now

$$\begin{aligned}(k+1)^2 = k^2 + 2k + 1 &= (k^2) + 2k + 1 \\ &> 2k + 1 + 2k + 1 \\ &> 2k + 1 + 1 + 1 \\ &= 2k + 3\end{aligned}$$

Hence $S(k+1)$ is true.

So $S(k+1)$ is true whenever $S(k)$ is true. By Principal of Mathematical Induction $S(n)$ is true for all positive integer $n \geq 3$.

(3) For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

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Solution : Let $S(n)$ be the given statement

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For $n = 1$,

(3) For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Solution : Let $S(n)$ be the given statement

$$S(n) : 7^n - 3^n \text{ is divisible by } 4.$$

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For $n = 1$, $7^n - 3^n$ is $7 - 3 = 4$ and it is divisible by 4. Hence $S(1)$ is true.

Assume $S(k)$ is true

$$7^k - 3^k \text{ is divisible by } 4.$$

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Now we show that $S(k+1)$ is true $7^{k+1} - 3^{k+1}$ is divisible by 4.

Now

$$
\begin{aligned}
7^{k+1} - 3^{k+1} &= 7^k 7 - 3^k 3 \\
&= 7^k 7 - 7^k 3 + 7^k 3 - 3^k 3 \\
&= 7^k (7-3) + 3(7^k - 3^k) \\
&= 7^k (4) + 3(7^k - 3^k)
\end{aligned}
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which is divisible by 4.

Hence $S(k+1)$ is true.

So $S(k+1)$ is true whenever $S(k)$ is true. By Principal of Mathematical Induction $S(n)$ is true for all positive integer n.

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Assume $S(k)$ is true

$$(1+x)^k \geq (1+kx)$$

where $x > -1$ Now we show that $S(k+1)$ is true $(1+x)^{k+1} \geq (1+(k+1)x)$ where $x > -1$.

Now

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x)\\ &= 1+kx+x+kx^2\\ &\geq 1+kx+x\\ &= 1+(k+1)x\end{aligned}$$

Hence $S(k+1)$ is true.

So $S(k+1)$ is true whenever $S(k)$ is true. By Principal of Mathematical Induction $S(n)$ is true for all natural number n.