SECTION 2

MODULE 2 : Determinants

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the square matrix of order 2. Then the determinant of A is denoted by $det(A)$ or $|A|$ and is evaluated as

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be the square matrix of order 3. Then

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(eh - gf) - b(di - gf) + c(dh - eg)$$

NOTE : If $A = [a]$ then

$$det(A) = det(a) = a.$$

Example

Find $det(A)$ if A is given by

(i) $\begin{bmatrix} 2 & -3 \\ 4 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 \\ 6 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & -1 & 3 \\ 6 & 4 & 16 \\ 8 & 5 & 8 \end{bmatrix}$ (vi) $\begin{bmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{bmatrix}$

Solution :

$$
\begin{aligned}
(i)\ \ det(A) &= \begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix} \\
&= 2 \times 9 - 4 \times (-3) = 18 + 12 = 30
\end{aligned}
$$

$$
\begin{aligned}
(iv)\ \ det(A) &= \begin{vmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{vmatrix} \\
&= 1\begin{vmatrix} 6 & 2 \\ 7 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix} \\
&= 1(6-14) - 5(2-6) + 3(14-18) \\
&= -8 + 20 - 12 = 0
\end{aligned}
$$

(ii) 18 (iii) 1 (v) -182 (vi) 0

Minor and Cofactor

Definition

The minor of an element in a determinant is the determinant obtained by suppressing the row and the column in which the particular element occurs.

In the $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

The minor of $a = d$, minor of $b = d$, minor of $c = b$ and minor of $d = a$.

In $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$,

The minor of $a = \begin{vmatrix} e & f \\ h & i \end{vmatrix}$

The minor of $b = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$

The minor of $c = \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

The minor of $d = \begin{vmatrix} b & c \\ h & i \end{vmatrix}$

The minor of $e = \begin{vmatrix} a & c \\ g & i \end{vmatrix}$

The minor of $f = \begin{vmatrix} a & b \\ g & h \end{vmatrix}$

The minor of $g = \begin{vmatrix} b & c \\ e & f \end{vmatrix}$

The minor of $h = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$

The minor of $i = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$

Example

(1) Find minor of each element of

$$\text{(i)}\begin{bmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{bmatrix} \qquad \text{(ii)}\begin{bmatrix} 2 & -1 & 3 \\ 6 & 4 & 16 \\ 8 & 5 & 8 \end{bmatrix} \qquad \text{(iii)}\begin{bmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{bmatrix}$$

Ans : (i) The minor of $1 = -8,\ \ 5 = -4,\ \ 3 = -4,\ \ 2 = -16,\ \ 6 = -8,\ \ 2 = -8,\ \ 3 = -8,\ \ 7 = -4, 1 = -4.$

(2) Find value of

$$\begin{vmatrix} 1 & z & -y \\ -z & 1 & x \\ y & -z & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Ans : $1 + x^2 + y^2 + z^2$, $(x - y)(y - z)(z - x)$

Definition

The Cofactor of an element in i^{th} row and j^{th} column is $(-1)^{i+j}$ times its minor.

In $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$,

The cofactor of a=$(-1)^{1+1}\times$ The minor of $a = \begin{vmatrix} e & f \\ h & i \end{vmatrix}$

The cofactor of b=$(-1)^{1+2}\times$ The minor of $b = -\begin{vmatrix} d & f \\ g & i \end{vmatrix}$

The cofactor of c=$(-1)^{1+3}\times$ The minor of $c = \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

The cofactor of d=$(-1)^{2+1}\times$ The minor of $d = -\begin{vmatrix} b & c \\ h & i \end{vmatrix}$

The cofactor of e=$(-1)^{2+2}\times$ The minor of $e = \begin{vmatrix} a & c \\ g & i \end{vmatrix}$

The cofactor of f=$(-1)^{2+3}\times$ The minor of f $= -\begin{vmatrix} a & b \\ g & h \end{vmatrix}$

The cofactor of g=$(-1)^{3+1}\times$ The minor of g $= \begin{vmatrix} b & c \\ e & f \end{vmatrix}$

The cofactor of h=$(-1)^{3+2}\times$ The minor of h $= -\begin{vmatrix} a & c \\ d & f \end{vmatrix}$

The cofactor of i=$(-1)^{3+3}\times$ The minor of i $= \begin{vmatrix} a & b \\ d & e \end{vmatrix}$

Example

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

Solution :

The cofactor of $0 = (-1)^{1+1} \begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

Example

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

Solution :

The cofactor of $0 = (-1)^{1+1} \begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

The cofactor of $1 = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ 2 & 6 \end{vmatrix} = -2$

Example

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

Solution :

The cofactor of $0 = (-1)^{1+1} \begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

The cofactor of $1 = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ 2 & 6 \end{vmatrix} = -2$

The cofactor of $-1 = (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$

Example

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

Solution :

The cofactor of $0 = (-1)^{1+1}\begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

The cofactor of $1 = (-1)^{1+2}\begin{vmatrix} 2 & 5 \\ 2 & 6 \end{vmatrix} = -2$

The cofactor of $-1 = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$

The cofactor of $2 = (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = -10$

Example

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

Solution :

The cofactor of $0 = (-1)^{1+1}\begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

The cofactor of $1 = (-1)^{1+2}\begin{vmatrix} 2 & 5 \\ 2 & 6 \end{vmatrix} = -2$

The cofactor of $-1 = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$

The cofactor of $2 = (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = -10$

The cofactor of $0 = (-1)^{2+2}\begin{vmatrix} 0 & -1 \\ 2 & 6 \end{vmatrix} = 2$

Example

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

Solution :

The cofactor of $0 = (-1)^{1+1} \begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

The cofactor of $1 = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ 2 & 6 \end{vmatrix} = -2$

The cofactor of $-1 = (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$

The cofactor of $2 = (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = -10$

The cofactor of $0 = (-1)^{2+2} \begin{vmatrix} 0 & -1 \\ 2 & 6 \end{vmatrix} = 2$

The cofactor of $5 = 2$, The cofactor of $2 = 5$, The cofactor of $4 = -2$,
The cofactor of $6 - 2$,

(2) Find cofactor of each element of the determinant $\begin{vmatrix} 3 & a & b \\ -e & 0 & 4 \\ 7 & y & 1 \end{vmatrix}$

Properties of Determinant

- If the rows and columns of determinant are interchanged, the value of the determinant remain unchanged.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

- The interchange of any rows (or columns) changes the sign of determinant without altering its absolute value

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

- If two rows (or two columns) in determinant are identical, the value is equal to 0.

$$\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} = 0$$

- If the elements of a row (or column) of a determinant are multiplied by a scalar, then the value of the new determinant is equal to same scalar times the value of the original determinant.

$$\begin{vmatrix} a & b & c \\ Kd & Ke & Kf \\ g & h & i \end{vmatrix} = K \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

- If each element of any row (or column) of a determinant is the sum of two or more numbers, then the determinant is expressible as the sum of two or more determinants of the same order.

$$\begin{vmatrix} a+r & b & c \\ d+s & e & f \\ g+t & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} r & b & c \\ s & e & f \\ t & h & i \end{vmatrix}$$

- If each element of a row (or column) of a determinant be added to the equi-multiples of the corresponding elements of one or more rows (or columns), then the value of the determinant is not changed

$$\begin{vmatrix} a+sb+tc & b & c \\ d+se+tf & e & f \\ g+sh+ti & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Example

Evaluate the following determinants using properties.

$$(1)\begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix} \quad (2)\begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \quad (3)\begin{vmatrix} 1 & 374 & 1893 \\ 1 & 372 & 1892 \\ 1 & 371 & 1891 \end{vmatrix}$$

Solution : (1) Let

$$D = \begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix}$$

Example

Evaluate the following determinants using properties.

$$(1)\begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix} \quad (2)\begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \quad (3)\begin{vmatrix} 1 & 374 & 1893 \\ 1 & 372 & 1892 \\ 1 & 371 & 1891 \end{vmatrix}$$

Solution : (1) Let

$$D = \begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & 3 \\ 4 & 3 & 4 \\ 5 & 4 & 5 \end{vmatrix} \qquad \text{applying } C_1 \to C_1 - C_2$$

Example

Evaluate the following determinants using properties.

$$(1)\begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix} \qquad (2)\begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \qquad (3)\begin{vmatrix} 1 & 374 & 1893 \\ 1 & 372 & 1892 \\ 1 & 371 & 1891 \end{vmatrix}$$

Solution : (1) Let

$$\begin{aligned} D &= \begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 2 & 3 \\ 4 & 3 & 4 \\ 5 & 4 & 5 \end{vmatrix} \qquad \text{applying } C_1 \to C_1 - C_2 \\ &= 0 \qquad \text{since 1st and 3rd columns are identical} \end{aligned}$$

(2) Let

$$D = \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix}$$

(2) Let

$$\begin{aligned} D &= \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -333 & -444 \\ 333 & 0 & -666 \\ 444 & 666 & 0 \end{vmatrix} \qquad \text{changing rows into columns} \end{aligned}$$

(2) Let

$$
\begin{aligned}
D &= \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -333 & -444 \\ 333 & 0 & -666 \\ 444 & 666 & 0 \end{vmatrix} \qquad \text{changing rows into columns} \\
&= (-1)^3 \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \qquad \text{taking } (-1) \text{ common from each row}
\end{aligned}
$$

(2) Let

$$
\begin{aligned}
D &= \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -333 & -444 \\ 333 & 0 & -666 \\ 444 & 666 & 0 \end{vmatrix} && \text{changing rows into columns} \\
&= (-1)^3 \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} && \text{taking } (-1) \text{ common from each row} \\
&= -D
\end{aligned}
$$

(2) Let

$$
\begin{aligned}
D &= \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -333 & -444 \\ 333 & 0 & -666 \\ 444 & 666 & 0 \end{vmatrix} \qquad \text{changing rows into columns} \\
&= (-1)^3 \begin{vmatrix} 0 & 333 & 444 \\ -333 & 0 & 666 \\ -444 & -666 & 0 \end{vmatrix} \qquad \text{taking } (-1) \text{ common from each row} \\
&= -D
\end{aligned}
$$

Therefore $D = 0$

(3) Let

$$D = \begin{vmatrix} 1 & 374 & 1893 \\ 1 & 372 & 1892 \\ 1 & 371 & 1891 \end{vmatrix}$$

(3) Let

$$\begin{aligned} D &= \begin{vmatrix} 1 & 374 & 1893 \\ 1 & 372 & 1892 \\ 1 & 371 & 1891 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 371 & 1891 \end{vmatrix} \qquad R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3 \end{aligned}$$

(3) Let

$$\begin{aligned} D &= \begin{vmatrix} 1 & 374 & 1893 \\ 1 & 372 & 1892 \\ 1 & 371 & 1891 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 371 & 1891 \end{vmatrix} \qquad R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3 \\ &= 0 - 2(0-1) + 1(0-1) = 2 - 1 = 1 \end{aligned}$$

Therefore $D = 1$

(4) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} = (p-q)(q-r)(r-p)$

Solution : Let

$$D = \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix}$$

(4) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} = (p-q)(q-r)(r-p)$

Solution : Let

$$\begin{aligned} D &= \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 1 \\ p-q & q-r & r \\ qr-rp & rp-pq & pq \end{vmatrix} \qquad C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3 \end{aligned}$$

(4) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} = (p-q)(q-r)(r-p)$

Solution : Let

$$\begin{aligned} D &= \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 1 \\ p-q & q-r & r \\ qr-rp & rp-pq & pq \end{vmatrix} \qquad C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3 \\ &= (p-q)(q-r)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & r \\ -r & -p & pq \end{vmatrix} \qquad (p-q) \text{ from 1st and } (q-r) \text{ from 2nd column} \end{aligned}$$

(4) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} = (p-q)(q-r)(r-p)$

Solution : Let

$$\begin{aligned}
D &= \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ qr & rp & pq \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 & 1 \\ p-q & q-r & r \\ qr-rp & rp-pq & pq \end{vmatrix} \qquad C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3 \\
&= (p-q)(q-r)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & r \\ -r & -p & pq \end{vmatrix} \qquad (p-q) \text{ from 1st and } (q-r) \text{ from 2nd column} \\
&= (p-q)(q-r)1(-p+r) \\
&= (p-q)(q-r)(r-p)
\end{aligned}$$

Example

Without expanding the determinants show that

(1) $$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

(2) $$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Solution : (1) Let

$$D = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

Solution : (1) Let

$$\begin{aligned} D &= \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \\ &= -\begin{vmatrix} b & a & c \\ y & x & z \\ q & p & r \end{vmatrix} \qquad \text{Interchanging } C_1 \text{ and } C_2 \end{aligned}$$

Solution : (1) Let

$$
\begin{aligned}
D &= \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \\
&= -\begin{vmatrix} b & a & c \\ y & x & z \\ q & p & r \end{vmatrix} && \text{Interchanging } C_1 \text{ and } C_2 \\
&= \begin{vmatrix} y & x & z \\ b & a & c \\ q & p & r \end{vmatrix} && \text{Interchanging } R_1 \text{ and } R_2
\end{aligned}
$$

Solution : (1) Let

$$\begin{aligned}
D &= \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \\
&= -\begin{vmatrix} b & a & c \\ y & x & z \\ q & p & r \end{vmatrix} && \text{Interchanging } C_1 \text{ and } C_2 \\
&= \begin{vmatrix} y & x & z \\ b & a & c \\ q & p & r \end{vmatrix} && \text{Interchanging } R_1 \text{ and } R_2 \\
&= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} && \text{Changing rows into columns}
\end{aligned}$$

Solution : (1) Let

$$\begin{aligned}
D &= \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \\
&= -\begin{vmatrix} b & a & c \\ y & x & z \\ q & p & r \end{vmatrix} && \text{Interchanging } C_1 \text{ and } C_2 \\
&= \begin{vmatrix} y & x & z \\ b & a & c \\ q & p & r \end{vmatrix} && \text{Interchanging } R_1 \text{ and } R_2 \\
&= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} && \text{Changing rows into columns}
\end{aligned}$$

Thus we have shown $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Now

$$\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \qquad \text{Interchanging } R_1 \text{ and } R_2$$

Now

$$\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \qquad \text{Interchanging } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} x & p & a \\ y & q & b \\ z & r & c \end{vmatrix} \qquad \text{Interchanging } C_2 \text{ and } C_3$$

Now

$$\begin{aligned}
\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} &= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} && \text{Interchanging } R_1 \text{ and } R_2 \\
&= \begin{vmatrix} x & p & a \\ y & q & b \\ z & r & c \end{vmatrix} && \text{Interchanging } C_2 \text{ and } C_3 \\
&= \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} && \text{Changing rows into columns}
\end{aligned}$$

Now

$$\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \quad \text{Interchanging } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} x & p & a \\ y & q & b \\ z & r & c \end{vmatrix} \quad \text{Interchanging } C_2 \text{ and } C_3$$

$$= \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \quad \text{Changing rows into columns}$$

Thus we have shown $\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$

Now

$$\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \quad \text{Interchanging } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} x & p & a \\ y & q & b \\ z & r & c \end{vmatrix} \quad \text{Interchanging } C_2 \text{ and } C_3$$

$$= \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \quad \text{Changing rows into columns}$$

Thus we have shown $\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$

Hence $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$

(2) Let

$$D = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

(2) Let

$$
\begin{aligned}
D &= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \\
&= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & bca \\ c & c^2 & cab \end{vmatrix} \qquad \text{Multiply 1st row by } a\text{, 2nd by } b\text{, 3rd by } c
\end{aligned}
$$

(2) Let

$$
\begin{aligned}
D &= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \\
&= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & bca \\ c & c^2 & cab \end{vmatrix} \qquad \text{Multiply 1st row by } a\text{, 2nd by } b\text{, 3rd by } c \\
&= abc\frac{1}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \qquad \text{Taking } abc \text{ common from 3rd column}
\end{aligned}
$$

(2) Let

$$\begin{aligned}
D &= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \\
&= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & bca \\ c & c^2 & cab \end{vmatrix} && \text{Multiply 1st row by } a\text{, 2nd by } b\text{, 3rd by } c \\
&= abc\frac{1}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} && \text{Taking } abc \text{ common from 3rd column} \\
&= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} && \text{Interchanging } C_1 \text{ and } C_3
\end{aligned}$$

(2) Let

$$\begin{aligned}
D &= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \\
&= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & bca \\ c & c^2 & cab \end{vmatrix} && \text{Multiply 1st row by } a\text{, 2nd by } b\text{, 3rd by } c \\
&= abc\frac{1}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} && \text{Taking } abc \text{ common from 3rd column} \\
&= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} && \text{Interchanging } C_1 \text{ and } C_3 \\
&= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} && \text{Interchanging } C_2 \text{ and } C_3
\end{aligned}$$

Cramer's Rule

Consider the system of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

If $d_1 = d_2 = d_3 = 0$ then the system is said to be homogeneous, otherwise it is said to be non-homogeneous system.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \; D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \; D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solution Cases

- If $D \neq 0$, the system of equations has unique solution given by
$$x = \frac{D_1}{D}, \;\; y = \frac{D_2}{D}, \;\; z = \frac{D_3}{D}.$$
- If
$$D = D_1 = D_2 = D_3 = 0,$$
the system of equations has infinitely many solutions.
- If
$$D = 0,$$
and atleast one of D_1 or D_2 or D_3 is nonzero then the system of equations have no solution.

If the given system of equations has a solution then it is called consistent system, otherwise it is called inconsistent system.

Example

Solve the following system of equations
(1)

$$x + 2y + 3z = -5$$
$$3x + y - 3z = 4$$
$$-3x + 4y + 7z = -7$$

Solution : Here $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{vmatrix} = 40$

$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} -5 & 2 & 3 \\ 4 & 1 & -3 \\ -7 & 4 & 7 \end{vmatrix} = -40$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 3 \\ 3 & 4 & -3 \\ -3 & -7 & 7 \end{vmatrix} = 40$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -5 \\ 3 & 1 & 4 \\ -3 & 4 & -7 \end{vmatrix} = -80$$

Here as $D \neq 0$, the system has unique solution given by

$$x = \frac{D_1}{D} = -1, \;\; y = \frac{D_2}{D} = 1, \;\; z = \frac{D_3}{D} = -2.$$

Therefore the solution in

$$x = -1, \;\; y = 1, \;\; z = -2.$$

(2)

$$\begin{aligned} 3x + y + z &= 2 \\ x - 3y + 2z &= 1 \\ 7x - y + 4z &= 5 \end{aligned}$$

Solution : Here $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{vmatrix} = 0$

$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{vmatrix} = 0$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 7 & 5 & 4 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -3 & 1 \\ 7 & -1 & 5 \end{vmatrix} = 0$$

Here as

$$D = D_1 = D_2 = D_3 = 0,$$

the system of equations has infinitely many solutions.

Consider

$$3x + y + z = 2$$
$$x - 3y + 2z = 1$$

and let

$$z = k$$

then we get

$$3x + y = 2 - k$$

and

$$x - 3y = 1 - 2k$$

which on solving gives values of

$$x = \frac{7 - 5k}{10}$$

and

$$y = \frac{5k - 1}{10}$$

Therefore the solution is $x = \frac{7-5k}{10}, \quad y = \frac{5k-1}{10}, \quad z = k$

(3)

$$x + y + 2z = 3$$
$$2x - y + 3z = 4$$
$$5x - y + 8z = 1$$

Solution : Here $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 8 \end{vmatrix} = 0$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 2 \\ 4 & -1 & 3 \\ 1 & -1 & 8 \end{vmatrix} = 8$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 5 & 1 & 8 \end{vmatrix} = -10$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ 5 & -1 & 1 \end{vmatrix} = 30$$

Here as $D = 0$ and atleast one of D_1, D_2, D_3 is nonzero so the system of equations has no solution.

(4)

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y + 2z = 0$$

Solution : Here $D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{vmatrix} = 6, \quad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 0,$

$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 0 & 2 \end{vmatrix} = 0, \quad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0.$

Here as $D \neq 0$, the system has unique solution given by

$$x = \frac{D_1}{D} = 0, \;\; y = \frac{D_2}{D} = 0, \;\; z = \frac{D_3}{D} = 0.$$

Therefore the solution in

$$x = 0, \;\; y = 0, \;\; z = 0.$$

(5)

$$\begin{aligned} 2x - 3y + 7z &= 5 \\ 3x + y - 3z &= 13 \\ 2x + 19y - 47z &= 32 \end{aligned}$$

Ans : No solution

(6)

$$
\begin{aligned}
x - 3y - 8z + 10 &= 0 \\
3x + y - 4z &= 0 \\
2x + 5y + 6z - 13 &= 0
\end{aligned}
$$

Solution : Here $D = \begin{vmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{vmatrix} = 0, \quad D_1 = \begin{vmatrix} -10 & -3 & -8 \\ 0 & 1 & -4 \\ 13 & 5 & 6 \end{vmatrix} = 0,$

$D_2 = \begin{vmatrix} 1 & -10 & -8 \\ 3 & 0 & -4 \\ 2 & 13 & 6 \end{vmatrix} = 0, \quad D_3 = \begin{vmatrix} 1 & -3 & -10 \\ 3 & 1 & 0 \\ 2 & 5 & 13 \end{vmatrix} = 0.$

Here as

$$D = D_1 = D_2 = D_3 = 0,$$

the system of equations has infinitely many solutions.

Consider

$$x - 3y - 8z + 10 = 0$$
$$3x + y - 4z = 0$$

and let

$$z = k$$

then we get

$$x - 3y = -10 + 8k$$

and

$$3x + y = 4k$$

which on solving gives values of

$$x = -1 + 2k$$

and

$$y = 3 - 2k$$

Therefore the solution is $x = -1 + 2k, \quad y = 3 - 2k, \quad z = k$