SECTION 2

MODULE 1 : Matrix Algebra

- Introduction
- Types of Matrices
- Operations of Matrices
- Adjoint Matrices
- Solution of System of Equations by Matrix Inversion Method

Definition

- A matrix is a rectangular array of numbers.
- A matrix with m rows and n columns is called an $m \times n$ matrix.
- A matrix with the same number of rows as columns is called square.
- Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Some Applications:
• Matrices are used in models of communications networks and transportation systems.
• Matrix Algebra can be used in analyzing the relationship between the vertices of a graph and movement of robots.

Notation :

A matrix with m rows and n columns where m, n are positive integers is

$$[a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Types of Matrices

Definition

A matrix is said to be

- Column Matrix if it has only one column.

$$e.g. \quad A = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$

- Row Matrix if it has only one row.

$$e.g. \quad B = \begin{bmatrix} 7 & -2 & 2 \end{bmatrix}$$

- Zero matrix or null matrix if all its elements are zero.

$$e.g. \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

A Square matrix is said to be

- Diagonal matrix if all its non-diagonal elements are zero.

$$e.g. \quad A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- Scalar matrix if its diagonal elements are equal and rest are all zero.

$$e.g. \quad B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

- Identity matrix if diagonal are all 1 and rest are all zero.

$$e.g. \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition

A Square matrix is said to be

- upper triangular matrix if all its elements below main diagonal are zero.

$$e.g.\quad A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

- lower triangular matrix if all its elements above main diagonal are zero.

$$e.g.\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -8 & 2 & -5 \end{bmatrix}$$

Operation of Matrices

Addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The sum of A and B, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its $(i, j)th$ element.

In other words, $A + B = [a_{ij} + b_{ij}]$.

$$e.g. \quad \text{If} \quad A = \begin{bmatrix} 3 & 11 & 1 \\ 1 & 4 & 6 \\ 8 & 0 & -8 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & -1 & 2 \\ 5 & 2 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\text{then} \quad A + B = \begin{bmatrix} 3-5 & 11-1 & 1+2 \\ 1+5 & 4+2 & 6+4 \\ 8+2 & 0+0 & -8+5 \end{bmatrix} = \begin{bmatrix} -2 & 10 & 3 \\ 6 & 6 & 10 \\ 10 & 0 & -3 \end{bmatrix}$$

Properties of Matrix Addition:

If A, B, C are three matrices and 0 is the null matrix then

- Commutativity : $A + B = B + A$
- Associativity : $(A + B) + C = A + (B + C)$
- Existence and Additive identity : $A + 0 = A = 0 + A$
- Existence of Additive inverse : $A + (-A) = 0 = (-A) + A$

Subtraction

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The difference of A and B, denoted by $A - B$, is the $m \times n$ matrix that has $a_{ij} - b_{ij}$ as its $(i, j)th$ element.

In other words, $A - B = [a_{ij} - b_{ij}]$.

$$e.g. \quad \text{If} \quad A = \begin{bmatrix} 3 & 11 & 1 \\ 1 & 4 & 6 \\ 8 & 0 & -8 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & -1 & 2 \\ 5 & 2 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\text{then} \quad A - B = \begin{bmatrix} 3-(-5) & 11-(-1) & 1-2 \\ 1-5 & 4-2 & 6-4 \\ 8-2 & 0-0 & -8-5 \end{bmatrix} = \begin{bmatrix} 8 & 12 & -1 \\ -4 & 2 & 2 \\ 6 & 0 & -13 \end{bmatrix}$$

Multiplication by a Scalar

Let $A = [a_{ij}]$ be $m \times n$ matrix and k be a scalar. Then $k \cdot A$, is the $m \times n$ matrix that has $k \cdot a_{ij}$ as its $(i, j)th$ element.

In other words, $k \cdot A = [k \cdot a_{ij}]$.

$$e.g. \quad \text{If} \quad A = \begin{bmatrix} 3 & 11 & 1 \\ 1 & 4 & 6 \\ 8 & 0 & -8 \end{bmatrix} \quad \text{and} \quad k = 3$$

$$\text{then} \quad k \cdot A = 3 \cdot A = \begin{bmatrix} 3 \cdot 3 & 3 \cdot 11 & 3 \cdot 1 \\ 3 \cdot 1 & 3 \cdot 4 & 3 \cdot 6 \\ 3 \cdot 8 & 3 \cdot 0 & 3 \cdot -8 \end{bmatrix} = \begin{bmatrix} 9 & 33 & 3 \\ 3 & 12 & 18 \\ 24 & 0 & -24 \end{bmatrix}$$

Properties of Scalar Multiplication :

If A, B are two matrices and k, l are scalar then

- $k \cdot (A + B) = k \cdot A + k \cdot B$
- $(k + l) \cdot A = k \cdot A + l \cdot A$
- $(kl) \cdot A = k \cdot (l \cdot A) = l \cdot (k \cdot A)$
- $(-k) \cdot A = -(k \cdot A) = k \cdot (-A)$
- $1A = A$
- $(-1)A = -A$

Multiplication

Let $A = [a_{ij}]$ be $m \times n$ matrix and $B = [b_{ij}]$ be $n \times p$ matrix. The multiplication of A and B is defined if and only if

number of columns of A = number of rows of B.

It is denoted by AB, and it is the $m \times p$ matrix.

$$e.g. \quad \text{If} \quad A = \begin{bmatrix} 3 & 11 & 1 \\ 1 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & -1 & 2 \\ 5 & 2 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\text{then} \quad AB = \begin{bmatrix} 3 & 11 & 1 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} -5 & -1 & 2 \\ 5 & 2 & 4 \\ 2 & 0 & 5 \end{bmatrix} \text{ is a } 2 \times 3 \text{ matrix}$$

$$AB = \begin{bmatrix} (3 \times -5) + (11 \times 5) + (1 \times 2) & (3 \times -1) + (11 \times 2) + (1 \times 0) & (3 \times 2) + (11 \times 4) + (1 \times 5) \\ (1 \times -5) + (4 \times 5) + (6 \times 2) & (1 \times -1) + (4 \times 2) + (6 \times 0) & (1 \times 2) + (4 \times 4) + (6 \times 5) \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 19 & 55 \\ 27 & 7 & 48 \end{bmatrix}$$

Properties of Matrix Multiplication :

If A, B, C are three matrices, 0 is the null matrix and I is identity matrix then

- Non-Commutativity : $AB \neq BA$ in general
- Associativity : $(AB)C = A(BC)$
- Distrubutivity : $A(B+C) = AB + AC$
- Powers : $A^n = AAA \cdots\cdots n$ times
- Existence of Identity : $AI = A = IA$
- $AB = 0$ doesnot necessarily apply $A = 0$ or $B = 0$

Transpose of a Matrix

Let $A = [a_{ij}]$ be $m \times n$ matrix. Then the transpose of $A$is denoted by A^T or A', is an $n \times m$ matrix obtained by interchangingthe rows and columns of A.

$$e.g. \quad \text{If} \quad A = \begin{bmatrix} 3 & 11 & 1 \\ 1 & 4 & 6 \\ 8 & 0 & -8 \end{bmatrix}$$

$$\text{then} \quad A^T = \begin{bmatrix} 3 & 1 & 8 \\ 11 & 4 & 0 \\ 1 & 6 & -8 \end{bmatrix}$$

Examples

(1) Find x, y, z and t if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

Solution :

$$\begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 2z-3 \\ 2y+0 & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$2x+3=9, \quad 2z-3=15, \quad 2y=12, \quad 2t+6=18.$$

$$x=3, \quad z=9, \quad y=6, \quad t=6.$$

(2) If $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 4 \\ -3 & 5 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 9 \\ 9 & 0 & 8 \\ -1 & 1 & 6 \end{bmatrix}$, find the product AB and BA and show that $AB \neq BA$.

Solution :

$$AB = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 4 \\ -3 & 5 & -8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 9 \\ 9 & 0 & 8 \\ -1 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+1(9)+(-3)(-1) & 2(-1)+1(0)+(-3)(1) & 2(9)+1(8)+(-3)(6) \\ 1(1)+0(9)+4(-1) & 1(-1)+0(0)+4(1) & 1(9)+0(8)+4(6) \\ (-3)(1)+5(9)+(-8)(-1) & (-3)(-1)+5(0)+(-8)(1) & (-3)(9)+5(8)+(-8)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -5 & 8 \\ -3 & 3 & 33 \\ 50 & -5 & -35 \end{bmatrix} \tag{1}$$

$$BA = \begin{bmatrix} 1 & -1 & 9 \\ 9 & 0 & 8 \\ -1 & 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 4 \\ -3 & 5 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+(-1)(1)+9(-3) & 1(1)+(-1)(0)+9(5) & 1(-3)+(-1)(4)+9(-8) \\ 9(2)+0(1)+8(-3) & 9(1)+0(0)+8(5) & 9(-3)+0(4)+8(-8) \\ (-1)(2)+1(1)+6(-3) & (-1)(1)+1(0)+6(5) & (-1)(-3)+1(4)+6(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -26 & 46 & -79 \\ -6 & 49 & -91 \\ -19 & 29 & -41 \end{bmatrix} \tag{2}$$

From (1) and (2) we have $AB \neq BA$

(3) If $A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$ then find $A^2 - A - 6I$.

Solution :

$$\begin{aligned} A^2 - A - 6I &= \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 4-(-2)-6 & 1-1+0 \\ 0-0-0 & 9-3-6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

(4) Find A and B if $A+B=\begin{bmatrix}7 & -5\\ 2 & 11\end{bmatrix}$ and $A-B=\begin{bmatrix}3 & 1\\ 6 & 9\end{bmatrix}$

Solution :

$$\begin{aligned}2A=(A+B)+(A-B)&=\begin{bmatrix}7 & -5\\ 2 & 11\end{bmatrix}+\begin{bmatrix}3 & 1\\ 6 & 9\end{bmatrix}\\ &=\begin{bmatrix}10 & -4\\ 8 & 20\end{bmatrix}\end{aligned}$$

Hence, $\quad A=\frac{1}{2}\begin{bmatrix}10 & -4\\ 8 & 20\end{bmatrix}=\begin{bmatrix}5 & -2\\ 4 & 10\end{bmatrix}$

Now $B=\begin{bmatrix}7 & -5\\ 2 & 11\end{bmatrix}-A=\begin{bmatrix}7 & -5\\ 2 & 11\end{bmatrix}-\begin{bmatrix}5 & -2\\ 4 & 10\end{bmatrix}=\begin{bmatrix}2 & -3\\ -2 & 1\end{bmatrix}$

(5) If $f(x) = x^2 - 3x + 3$, find $f(A)$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution :

$$f(A) = A^2 - 3A + 3I$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 6 \\ 4 & 1 & 4 \\ 3 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Singular and Non-singular matrix

A square matrix A is said to be

- Singular if $det(A) = 0$.
- Non-singular if $det(A) \neq 0$.

Adjoint of Matrix

If $A = [a_{ij}]$ is a square matrix of order n and A_{ij} represents the cofactor of the element a_{ij} in the determinant $|A|$, then the transpose of cofactor matrix $[A_{ij}]$ is called the adjoint of A and it is denoted by (adj A).

e.g. If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and cofactor matrix of $A = \begin{bmatrix} -20 & -2 & 8 \\ -10 & 2 & 2 \\ 5 & -2 & -2 \end{bmatrix}$

Then $\text{adj}(A) = \begin{bmatrix} -20 & -2 & 8 \\ -10 & 2 & 2 \\ 5 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} -20 & -10 & 5 \\ -2 & 2 & -2 \\ 8 & -2 & -2 \end{bmatrix}$

Properties of Adjoint :

If A, B are two non-singular matrices of same order, I is the identity matrix then

- (adj AB)=(adj B)(adj A)
- A(adj A)=(adj A)$A = |A|I$
- $|\text{adj } A| = |A|^{n-1}$
- adj(adj A)= $|A|^{n-2}A$

Inverse of matrix in terms of adjoint

If $|A| \neq 0$, then inverse of matrix A exists and is given by

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

Example

(1) Find inverse of (i) $\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 2 \\ -2 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

Solution : (i) Here $|A| = det(A) = \begin{vmatrix} -3 & 2 \\ 6 & -4 \end{vmatrix} = 12 - 12 = 0.$

So we have $det(A) = 0$. Hence A^{-1} does not exists.

(ii) Here $|A| = det(A) = \begin{vmatrix} 4 & 2 \\ -2 & 5 \end{vmatrix} = 20 + 4 = 24.$

So we have $det(A) \neq 0$. Hence A^{-1} exists and

$$A^{-1} = \frac{1}{|A|}(\text{adj } A).$$

Now for $A = \begin{bmatrix} 4 & 2 \\ -2 & 5 \end{bmatrix}$ the cofactor matrix is $\begin{bmatrix} 5 & 2 \\ -2 & 4 \end{bmatrix}$.

Therefore (adj A)$= \begin{bmatrix} 5 & 2 \\ -2 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ 2 & 4 \end{bmatrix}$.

So

$$A^{-1} = \frac{1}{24}\begin{bmatrix} 5 & -2 \\ 2 & 4 \end{bmatrix}.$$

(iii) Here $|A| = det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 1(12-6) - 2(4-3) + 3(2-3) = 1.$

So we have $det(A) \neq 0$. Hence A^{-1} exists and

$$A^{-1} = \frac{1}{|A|}(\text{adj } A).$$

Now for $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ the cofactor matrix is $\begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

Therefore (adj A)$= \begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

So

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(2) If $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$ verify that $A^2 + 3A + 4I = 0$ and hence find A^{-1}.

Solution : Here $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$.

$$\begin{aligned} A^2 + 3A + 4I &= \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} + 3\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & -3 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -1-3+4 & 3-3+0 \\ -6+6+0 & 2-6+4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now $A^2 + 3A + 4I = 0$. Multiply both sides by A^{-1} we get

$$
\begin{aligned}
A^{-1}(A^2 + 3A + 4I) &= A^{-1}(0) \\
A^{-1}(A^2) + A^{-1}(3A) + A^{-1}(4I) &= A^{-1}(0) \\
A + 3I + 4A^{-1} &= 0 \\
A^{-1} &= \frac{1}{4}(-A - 3I)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
A^{-1} &= \frac{1}{4}(-A - 3I) \\
&= \frac{1}{4}\left(-\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \frac{1}{4}\begin{bmatrix} 1-3 & 1-0 \\ -2-0 & 2-3 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} -2 & 1 \\ -2 & -1 \end{bmatrix}
\end{aligned}
$$

Solution of system of equation by matrix inversion method

- Non-homogeneous equations

Three equation with three unknowns

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Matrix form of above system is $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

which can be written as $AX = B$.

If $|A| \neq 0$, then $X = A^{-1}B$.

Solution cases

- If $|A| \neq 0$, then the system is consistent has unique solution.
- If $|A| = 0$, then the system has either no solution or infinite number of solutions
  - If $(\text{adj } A)B \neq 0$, the system has no solution and is therefore inconsistent.
  - If $(\text{adj } A)B = 0$, the system is consistent has infinitely many solutions.

• Homogeneous equations

Three equation with three unknowns

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

Matrix form of above system is $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

which can be written as $AX = B$.

Solution cases

- If $|A| \neq 0$ then the solution of system of equations is trivial i.e., $x = 0$, $y = 0$, $z = 0$.
- If $|A| = 0$ then the system of equations has non-trivial solutions.

Example

Solve the following system of equations by matrix inversion method.

(1) $2x+3y-z=9, \quad x-2y+z=-9, \quad 3x+2y+2z=-1$

Solution : The matrix form of the system is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 1 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -1 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 2(-4-2)-3(2-3)-1(2+6) = -12+3-8 = -17 \neq 0.$

Hence the system of equations has unique solution which is given by

$$X = A^{-1}B.$$

Now

$$A^{-1} = \frac{1}{|A|}(\text{adj } A).$$

The cofactor matrix of $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 1 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 8 \\ -8 & 7 & 5 \\ 1 & -3 & -7 \end{bmatrix}.$

Therefore $\operatorname{adj} A = \begin{bmatrix} -6 & 1 & 8 \\ -8 & 7 & 5 \\ 1 & -3 & -7 \end{bmatrix}^T = \begin{bmatrix} -6 & -8 & 1 \\ 1 & 7 & -3 \\ 8 & 5 & -7 \end{bmatrix}$.

Hence

$$A^{-1} = \frac{1}{-17} \begin{bmatrix} -6 & -8 & 1 \\ 1 & 7 & -3 \\ 8 & 5 & -7 \end{bmatrix}.$$

Therefore $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -6 & -8 & 1 \\ 1 & 7 & -3 \\ 8 & 5 & -7 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -1 \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} 17 \\ -51 \\ 34 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

So

$$x = -1, \;\; y = 3, \;\; z = -2$$

is the solution of given system of equations.

(2) $3x + y + z = 2, \quad x - 3y + 2z = 1, \quad 7x - y + 4z = 5$

Solution : The matrix form of the system is $\begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{vmatrix} = 3(-12+2) - 1(4-14) + 1(-1+21) = -30 + 10 + 20 = 0.$

Now we compute (adj A)B.

The cofactor matrix of $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -10 & 10 & 20 \\ -5 & 5 & 10 \\ 5 & -5 & -10 \end{bmatrix}.$

Therefore adj $A = \begin{bmatrix} -10 & 10 & 20 \\ -5 & 5 & 10 \\ 5 & -5 & -10 \end{bmatrix}^T = \begin{bmatrix} -10 & -5 & 5 \\ 10 & 5 & -5 \\ 20 & 10 & -10 \end{bmatrix}.$

Hence (adj A)$B = \begin{bmatrix} -10 & -5 & 5 \\ 10 & 5 & -5 \\ 20 & 10 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Therefore the system has infinitely many solutions.

Let $z = k$ and consider first two equations $3x + y = 2 - k, \quad x - 3y = 1 - 2k$

The matrix form of this system is $\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-k \\ 1-2k \end{bmatrix}$ i.e., $CX = D$

which gives $X = C^{-1}D$.

Now $|C| = -9 - 1 = -10$ and Cofactor matrix of $C = \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}$

Hence adj $C = \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}$

Therefore $X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-10}\begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2-k \\ 1-2k \end{bmatrix} = \frac{1}{-10}\begin{bmatrix} -7+5k \\ 1-5k \end{bmatrix}$

So

$$x = \frac{7-5k}{10}, \;\; y = \frac{5k-1}{10}, \;\; z = k$$

is the solution of given system of equations.

(3) $x + y + 2z = 3, \quad 2x - y + 3z = 4, \quad 5x - y + 8z = 10$

Solution : The matrix form of the system is $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 8 \end{vmatrix} = 1(-8+3) - 1(16-15) + 2(-2+5) = -6 + 6 = 0.$

Now we compute (adj A)B.

The cofactor matrix of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 8 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 3 \\ -10 & -2 & 6 \\ 5 & 1 & -3 \end{bmatrix}.$

Therefore adj $A = \begin{bmatrix} -5 & -1 & 3 \\ -10 & -2 & 6 \\ 5 & 1 & -3 \end{bmatrix}^T = \begin{bmatrix} -5 & -10 & 5 \\ -1 & -2 & 1 \\ 3 & 6 & -3 \end{bmatrix}.$

Hence $(\text{adj } A)B = \begin{bmatrix} -5 & -10 & 5 \\ -1 & -2 & 1 \\ 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$.

Here we get $|A| = 0$ and $(\text{adj } A)B \neq 0$.

Therefore the system has no solutions.

(4) $x - y + z = 0, \quad x + 2y - z = 0, \quad 2x + y + 2z = 0$

Solution : Given system of equations is homogeneous.

The matrix form of the system is $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{vmatrix} = 1(4+1) + 1(2+2) + 1(1-4) = 6 \neq 0.$

As $|A| \neq 0$, the solution of system of equations is trivial i.e., $x = 0, \; y = 0, \; z = 0.$

(5) $2x + 3y - z = 0, \quad x - y + 2z = 0, \quad x + 2y - z = 0$

Solution : The matrix form of the system is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 2(1-4) - 3(-1-2) - 1(2+1) = 9 - 9 = 0.$

The system of equations has infinitely many solutions.

Let $z = k$ and consider first two equations. $2x+3y = k, \quad x-y = -2k$. The matrix form of this system is $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -2k \end{bmatrix}$ $i.e., CX = D.$

Therefore $|C| = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5$

The cofactor matrix of $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$ and adj $C = \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$

Therefore $X = C^{-1}D = \frac{1}{-5}\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} k \\ -2k \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} 5k \\ -5k \end{bmatrix}$

So

$$x = -k, \;\; y = k, \;\; z = k$$

is the solution of given system of equations.