① RATHOD HAREKRISSNA UPENDRA ② PARTH
17D070001 ③ PRAJVAL

PAGE NO.:

MIC - Assignment - 3

Q3, (a) Let the 2 shapes be z_1, z_2 representations (in the point set of 3N dimension).

First convert them to pre-shape space by subtracting the respective means i.e.

$$\underset{3\times N}{z_1} \rightarrow \underset{3\times N}{z_1} - \underset{3\times N}{\left[\sum_{n=1}^{N} z_{1n}/N\right]}$$

$$\underset{3\times N}{z_2} \rightarrow \underset{3\times N}{z_2} - \underset{3\times N}{\left[\sum_{n=1}^{N} z_{2n}/N\right]}$$

(can also be done in 3N vector format)

Then rescale both shapes such that

$$z_1 \longrightarrow s_1 z_1 \quad \text{s.t.} \quad s_1 = 1/\sum_n (\|z_{1n}\|_2^2)$$

$$z_2 \longrightarrow s_2 z_2 \quad \text{s.t.} \quad s_2 = 1/\sum_n (\|z_{2n}\|_2^2)$$

Let these scaled and translated shapes be z_1^* & z_2^*. Now they are in pre-shape space

then the procrastes (standard) distance b/w them is defined as:

$$d = \min_{\theta} \sum_{n=1}^{N} \| z_{1n} - M_\theta z_{2n}\|_2^2$$

where M_θ is the rotation matrix (here 3×3) to be optimized

$$= \min_{\theta} \sum_{n=1}^{N} z_{1n}^T M_\theta z_{2n}$$

$$= \min_{\theta} tr(z_1^T M_\theta z_2)$$

$$\therefore d = tr(z_1^T M_\theta z_2)$$

$3 \times N$

where Z_1 is ~~$N \times 3$~~ $3 \times N$, M_θ is 3×3, $Z_2 = 3 \times N$ matrix

Now, $tr(Z_1^T M_\theta Z_2) = tr(M_\theta Z_2 Z_1^T)$

& let ~~...~~ $SVD(Z_2 Z_1^T) = U \Sigma V^T$

the optimal $\boxed{M_\theta = V U^T}$ (if $det(VU^T) = +1$)

↳ will be orthogonal

$\therefore$ $\boxed{d = tr(Z_1^T V U^T Z_2)}$ where $SVD(Z_2 Z_1^T) = U \Sigma V^T$

(if $det(VU^T) = -1$, then

$$\boxed{M_\theta = V \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & & -1 \end{pmatrix} U^T}$$

(b) Let all the shapes be: $Z_1\ Z_2 \ldots Z_M$ (M shapes in total) and they are to be divided into K clusters, then the cost func that can be used for the task of clustering:

$$L = \sum_{l=1}^{K} \sum_{n=1}^{|C_l|} \| Z_n - \bar{Z}_l \|_p$$

where $|C_l|$ denote the class elements (number)

& Z_n denote all shapes in the clusters

$\| \cdot \|_p$ denote the standard procrustes distance as defined in part (a)

& $\bar{Z}_l$ denote the mean shape of all the ~~points~~ shape in cluster

(c) We use k-means++ algorithm for clustering, which is relatively less-sensitive to outliers.

Initialization: Initially, choose a shape at random among $\{z_k\}_{k=1,\dots,M}$ shapes

Now choose k-1 other shapes based on cost related to minimum distance among selected points. To be more specific,

pick $z_i \in \{z_\ell\}_{\ell=1,\dots,M}$ at random, $T \leftarrow z_i$
while $|T| < k$:
  pick $z_i \in \{z_\ell\}_{\ell=1,\dots,M}$ w.p. $\propto$ cost(z_i, T)
  $T \leftarrow T \cup \{z_i\}$
$$= \min_{z \in T} \|z_i - z\|_p$$

So now we have carefully initialized k class center points, now assign all remaining shapes to a cluster based on the criteria of minimizing procrustes distance of the shape from center of that cluster

i.e. $\forall z_i \notin T$ do:
$$j = \underset{\ell=\{1,2,\dots,k\}}{\arg\min} \|z_i - z_\ell\|$$
where $z_\ell \in T$.

$$C_j \cup \{z_i\} \rightarrow C_j$$

where C_j denotes the cluster (one of the k)

Hence we have initialized all the shapes as a part of one of the k clusters optimizations - Now we have $\{C_j\}_{j=1,\dots,k}$ cluster, for each cluster, calculate the mean-shape among all the shape in cluster.

The mean-shape can be calculated by:

$$\bar{Z}_\ell = \underset{Z}{\arg\min} \sum_{m=1}^{|C_\ell|} \sum_{n=1}^{N} \| Z_n - (M_\theta)_m Z_{mn} \|_2^2$$

To solve this optimization problem, consider initial Z_ℓ as the current cluster center, then solve for $(M_\theta)_m$ by solving independently for each data point in the cluster as described in part 1.

$(M_\theta)_m = V_m U_m^T$ where $SVD(Z_{mn} Z_n^T) = U_m \Sigma_m V_m^T$
if $\det(VU^T) = +1$

& $(M_\theta)_m = V \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} U^T$ if $\det(VU^T) = -1$

Now that we have optimal parameters, solve for the optimal mean point set by averaging all aligned pointsets ie

$$Z_\ell^{new} = \sum_{m=1}^{|C_\ell|} Z_m^* / |C_\ell| \quad \text{where } Z_m^* = (M_\theta)_m Z_m$$

make Z_ℓ^{new} to be 0 mean & rescale to make norm ~~zero~~ 1 (actually no need for this as it will already be like this for mean part).

Hence we get $\bar{Z}_\ell$ for each cluster (new center)

Now again assign a cluster to every other point following a similar strategy used during initialization.

Check the loss function at this assignment and repeat the optimization step.

We will notice that the loss decreases after each iteration

<u>Convergence criteria:</u>

We define our algorithm to be satisfactorily converged to an optima when

$|L_i - L_{i+1}| < \varepsilon$ where L_i is loss at end of iteration i

& ε is a user defined constant (could be very small or 0)

At such $(i+1)$ iteration, we tell our algorithm has converged and the current cluster assignments are the most optimal