12. Linear Independence, Subspaces and Dimension

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_n$ such that (1) not all c_i's are zero and (2) $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$.

Otherwise, they are said to be **linearly independent** : the only scalars $c_1, \ldots, c_n$ that satisfy $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ are $c_1 = \cdots = c_n = 0$.

The collection of non-zero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ determines a **subspace** of $\mathbb{R}^m$, which is the set of all linear combinations $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ over different choices of $c_1, \ldots, c_n \in \mathbb{R}$. The **dimension** of this subspace is the size of the **largest possible, linearly independent** sub-collection of the (non-zero) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Row and Column Rank (Optional)

0 points possible (ungraded)

Suppose $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. The rows of the matrix, $(1,3)$ and $(2,6)$, span a subspace of dimension

1

Answer: 1 . This is the **row rank** of $\mathbf{A}$.

The columns of the matrix, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ span a subspace of dimension

1

Answer: 1 . This is the **column rank** of $\mathbf{A}$.

We will be using these ideas when studying **Linear Regression**, where we will work with larger, possibly rectangular matrices.

Solution:

In both cases, the two vectors are linearly dependent.

$$2 \cdot (1,3) - (2,6) = (0,0)$$

$$3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Rank of a matrix (Optional)

0 points possible (ungraded)

In general, row rank is always equal to the column rank, so we simply refer to this common value as the **rank** of a matrix.

What is the largest possible rank of a 2×2 matrix?

2

✔ **Answer:** 2

What is the largest possible rank of a 5×2 matrix?

2

✔ **Answer:** 2

In general, what is the largest possible rank of an $m \times n$ matrix?

- ○ m
- ○ n
- ◉ $\min(m, n)$
- ○ $\max(m, n)$
- ○ None of the above

✔

Solution:

In general, the rank of any $m \times n$ matrix can be at most $\min(m, n)$, since rank = column rank = row rank. For example, if there are five columns and three rows, the column rank cannot be larger than the largest possible row rank – the largest possible row rank for three rows is, unsurprisingly, 3. The opposite is also true if there are more rows than columns. If a matrix has two columns and six rows, then the row rank cannot exceed the column rank, which is at most 2.
In general, a matrix $\mathbf{A}$ is said to have **full rank** if $\operatorname{rank}(\mathbf{A}) = \min(m, n)$. (note the $=$, instead of $\leq$).

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Examples of Rank (Optional)

0 points possible (ungraded)

What is the rank of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

1 Answer: 1

What is the rank of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$?

2 Answer: 2

What is the rank of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$?

0 Answer: 0

What is the rank of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$?

2 Answer: 2

What is the rank of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$?

3

Answer: 3

Solution:

1. The set of rows describe a subspace of dimension 1, spanned by $(1,1)$.
2. This matrix has rank 2, since $(1,-1)$ and $(1,0)$ are linearly independent.
3. This matrix has rank zero. By definition, the rank is equal to the number of nonzero linearly independent vectors.
4. The second and third rows are independent. However, the sum of the second and third rows are equal to the first: $(1,0,1)+(0,1,0)=(1,1,1)$. So this matrix has rank 2.
5. All three rows are independent. An easy way to check is to notice that this matrix is **upper triangular** , with nonzero entries along the diagonal.

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Invertibility of a matrix (Optional)

0 points possible (ungraded)

An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ has full rank, i.e. $\mathrm{rank}\,(\mathbf{A}) = n$.

Which of the following matrices are invertible? Choose all that apply.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

- [ ] $\mathbf{A}$
- [x] $\mathbf{B}$
- [x] $\mathbf{C}$
- [ ] $\mathbf{D}$

✔

Solution:

We saw in a previous exercise that the rank of $\mathbf{A}$ is 1. The rank of $\mathbf{B}$ is 2, since $(1, 2)$ and $(2, 1)$ are linearly independent, since e.g. by Gaussian Elimination one obtains the reduced upper triangular matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3/2 \end{pmatrix}$. In general, an upper triangular matrix with nonzero entries along the diagonal has full rank.
By the same reasoning, $\mathbf{C}$ also has full rank. Finally, $\mathbf{D}$ does not have full rank, since $(\text{row } 1) + (\text{row } 2) + (\text{row } 3) = \vec{0}$.

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? Optional?
Hello, I noticed that some of the section titles have the word "optional" in them. This one doe... 2

? Definition of linearly dependent
Should it not read " (1) all ci's are non- zero" (rather than "not all ci's are zero")? As currently ... 1

Help with Rank of Matrix in R
require(Matrix) #define C as a vector C<-c(1,1,0,0,1,1,0,0,1) #reshape C as a 3x3 matrix dim(C)... 2

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