4. Points and Vectors

A list of n numbers can be thought of as a point or a vector in n-dimensional space. In this course, we will think of n-dimensional vectors $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ flexibly as points and as vectors.

Dot Products and Norm

3/3 points (graded)

Notation: In this course, we will use regular letters as symbols for numbers, vectors, matrices, planes, hyperplanes, etc. You will need to distinguish what a letter represents from the context.

Recall the dot product of a pair of vectors a and b:

$$a \cdot b \;=\; a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \qquad \text{where } a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

When thinking about a and b as vectors in n-dimensional space, we can also express the dot product as

$$a \cdot b \;=\; \|a\|\|b\| \cos \alpha,$$

where α is the angle formed between the vectors a and b in n-dimensional Euclidean space. Here, $\|a\|$ refers to the length, also known as **norm**, of a:

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

What is the length of the vector $\begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$?

0.5

Answer: 0.5

What is the length of the vector $\begin{bmatrix} -0.15 \\ 0.2 \end{bmatrix}$?

1/4

Answer: 0.25

What is the angle (in radians) between $\begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$ and $\begin{bmatrix} -0.15 \\ 0.2 \end{bmatrix}$? Choose the answer that lies between 0 and π.

(Type **pi** for the constant π. Enter an exact answer or a decimal accurate to at least 4 decimal places.)

pi/2

Answer: pi/2

STANDARD NOTATION

Solution:

- Plugging into the equation for norm, we get that the length of $\begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$ is equal to $\sqrt{0.4^2 + 0.3^2} = 0.5$. Notice that the ratio of x:y is 3:4 so we can use 3:4:5 triangle to speed up our calculation to find the length of the vector.
- We do the same for $\begin{bmatrix} -0.15 \\ -0.2 \end{bmatrix}$.
- Using the second expression for dot product and rearranging, we get $\alpha = \cos^{-1} \frac{x \cdot y}{\|x\| \|y\|}$. Using the first expression for dot product and plugging it in we get that $\alpha = \cos^{-1} \frac{(0.4)(-0.15)+(0.3)(0.2)}{\sqrt{(0.4)^2+(0.3)^2}\sqrt{(-0.15)^2+(0.2)^2}}$

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Answers are displayed within the problem

Dot Products and Orthogonality

1/1 point (graded)

Given 3-dimensional vectors $x^{(1)} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} a_1 \\ -a_2 \\ a_3 \end{bmatrix}$, when is $x^{(1)}$ orthogonal to $x^{(2)}$, i.e. the angle between them is $\pi/2$?

- [ ] when $2a_1 + 2a_3 = 0$
- [x] when $a_1^2 - a_2^2 + a_3^2 = 0$
- [ ] when $a_1^2 + a_2^2 + a_3^2 = 0$

STANDARD NOTATION

Solution:

Based on the previous equations for the dot product, we find that the angle between $x^{(1)}$ and $x^{(2)}$ is:

$$\alpha = \cos^{-1} \frac{x^{(1)} \cdot x^{(2)}}{\|x^{(1)}\| \|x^{(2)}\|}$$

$$\alpha = \cos^{-1} \frac{a_1^2 - a_2^2 + a_3^2}{a_1^2 + a_2^2 + a_3^2}$$

$x^{(1)}$ is orthogonal to $x^{(2)}$ when $x^{(1)} \cdot x^{(2)} = 0$ or $a_1^2 - a_2^2 + a_3^2 = 0$.

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Unit Vectors

1.0/1 point (graded)

A unit vector is a vector with length 1. The length of a vector is also called its norm.

Given any vector $x,$ write down the unit vector pointing in the same direction as x?

(Enter **x** for the vector x, and **norm(x)** for the norm $\|x\|$ of the vector x.)

x/norm(x)

Answer: x/norm(x)

STANDARD NOTATION

Solution:

We need to scale the vector x so that it is length 1. Right now it is length $\|x\|$ so we need to divide the vector x by $\|x\|$ in order to get the unit vector which points in the same direction.

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Answers are displayed within the problem

Projections

2/2 points (graded)

Recall from linear algebra the definition of the projection of one vector onto another. As before, we have 3-dimensional vectors $x^{(1)} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} a_1 \\ -a_2 \\ a_3 \end{bmatrix}$.

Which of these vectors is in the same direction as the projection of $x^{(1)}$ onto $x^{(2)}$?

- ( ) $x^{(1)}$
- (•) $x^{(2)}$
- ( ) $x^{(1)} + x^{(2)}$

✔

What is the signed magnitude c of the projection $p_{x^{(1)} \to x^{(2)}}$ of $x^{(1)}$ onto $x^{(2)}$? More precisely, let u be the unit vector in the direction of the correct choice above, find a number c such that $p_{x^{(1)} \to x^{(2)}} = cu$.

(Enter **a_1** for a_1, **a_2** for a_2, and **a_3** for a_3.)

$c =$

(a_1^2-a_2^2+a_3^2)/sqrt(a_1^2+a_2^2+a_3^2)

Answer: (a_1^2-a_2^2+a_3^2)/sqrt(a_1^2+a_2^2+a_3^2)

$$\frac{a_1^2-a_2^2+a_3^2}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

STANDARD NOTATION

Solution:

- The definition of the projection of one vector onto another is the part of the first vector which points in the same direction as the second vector. Thus the projection of $x^{(1)}$ onto $x^{(2)}$ points in the direction of $x^{(2)}$
- The vector has magnitude $||x^{(1)}|| \cos \alpha$. From our previous result $\alpha = \cos^{-1} \frac{x^{(1)} \cdot x^{(2)}}{||x^{(1)}|| \, ||x^{(2)}||}$, the projection thus has magnitude $\frac{x^{(1)} \cdot x^{(2)}}{||x^{(2)}||}$. Plugging in our values for $x^{(1)}$ and $x^{(2)}$ we get $\frac{a_1^2-a_2^2+a_3^2}{\sqrt{a_1^2+a_2^2+a_3^2}}$.

Hence, to find the final vector projection, we scale the unit vector in the direction of the vector projection, which is $\frac{x^{(2)}}{||x^{(2)}||}$ by the length, $||p_{x^{(1)} \to x^{(2)}}||$. So the answer is $||p_{x^{(1)} \to x^{(2)}}|| \frac{x^{(2)}}{||x^{(2)}||}$

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Answers are displayed within the problem

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2

Dot Products and Orthogonality
Good morning. I´m sorry but I don´t understand the questions. May You explain to me, please? ...
5

The standard notation on pi is not working
Why it keeps saying Invalid math syntax when I use pi instead of the symbol.
2

Unit Vector
Please, clarify what are you asking to write in the Unit vector question. Is it a vector or what? Beca...
4

[Staff] Please clarify Question on Projections - It's not Clear what is being asked
Hi @Staff, Could you please clarify what you are asking for in the questions regarding the "Projecti...
10

Shouldn't be considered (something)^(1/2) equal to sqrt(something)?
One of the answers dind't work using ^(1/2) but only with sqrt().
3

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