Discrete Dynamic Programming Models for Pricing Asian Exotic Options

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Abstract

This report presents a discrete dynamic programming (DP) approach to price arithmetic-average Asian exotic options under European, Bermudan, and American exercise styles. The method explicitly models early-exercise flexibility and path dependence through a two-dimensional state space in spot price and running average. The framework integrates Gauss–Hermite quadrature for expectation computation, Monte Carlo validation for convergence studies, and detailed analysis of exercise boundaries, early-exercise premiums, and multi-style sensitivity overlays. Results confirm the theoretical hierarchy $V_{Euro} \leq V_{Berm} \leq V_{Amer}$ and demonstrate the DP model's accuracy and robustness across key parameters such as volatility, strike, interest rate, and averaging frequency.

1 Introduction

Asian options are path-dependent derivatives whose payoffs depend on the average price of the underlying asset over a monitoring period. By averaging over multiple observations, they reduce exposure to short-term volatility and are commonly used in structured financial products and corporate hedging.

Traditional pricing approaches—such as closed-form approximations or Monte Carlo simulation—struggle to efficiently handle early exercise features (as in Bermudan or American variants). Discrete Dynamic Programming (DP) provides a natural solution framework: it recursively computes continuation and exercise values across a discretized (S, A) state grid. This recursive structure captures optimal stopping behavior and facilitates transparent convergence control via spatial and temporal discretization.

The goal of this work is to develop, validate, and interpret a DP-based solver capable of pricing Asian options of multiple styles and to assess its numerical stability, early-exercise behavior, and sensitivity to key parameters.

2 Mathematical Framework

We consider an arithmetic-average Asian option under a standard risk-neutral process. The model discretizes both the underlying spot S and the running average A on finite grids.

2.1 Underlying Dynamics

The risk-neutral dynamics of the underlying are discretized as:

$$S_{t+\Delta t} = S_t e^{(r-q-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\,Z}, \tag{1}$$

where r is the risk-free rate, q is the dividend yield, σ is volatility, and $Z \sim \mathcal{N}(0,1)$.

2.2 Arithmetic Average Update

At each monitoring step,

$$A_{t+\Delta t} = \frac{kA_t + S_{t+\Delta t}}{k+1}, \tag{2}$$

where k denotes the number of past observations included in the running average.

2.3 DP Recursion

Let $V_n(S,A)$ denote the option value at discrete step n. The backward recursion is:

$$V_n(S,A) = \begin{cases} e^{-r\Delta t}\mathbb{E}[V_{n+1}(S',A')], & \text{(continuation)} \\ \max\{\Phi(S,A), e^{-r\Delta t}\mathbb{E}[V_{n+1}(S',A')]\}, & \text{(exercise)} \end{cases} \tag{3}$$

where $\Phi(S,A)$ is the intrinsic payoff (e.g., $\max(K-A,0)$ for a put). The expectation is computed via Gauss–Hermite quadrature using pre-tabulated nodes and weights, offering high accuracy for Gaussian integrals.

2.4 Boundary and Terminal Conditions

At maturity, $V_T(S,A) = \Phi(S,A)$. For sufficiently large or small S or A, boundary conditions are imposed such that $V(S,A)$ tends to intrinsic payoff limits. The backward recursion is performed from $t = T$ to $t = 0$.

3 Numerical Validation: Convergence Study

To establish correctness, the DP model is validated against Monte Carlo (MC) pricing for European Asian options. We examine sensitivity to grid resolution in S and A, and quadrature accuracy via K_{gh} nodes.

3.1 Convergence vs Grid Parameters

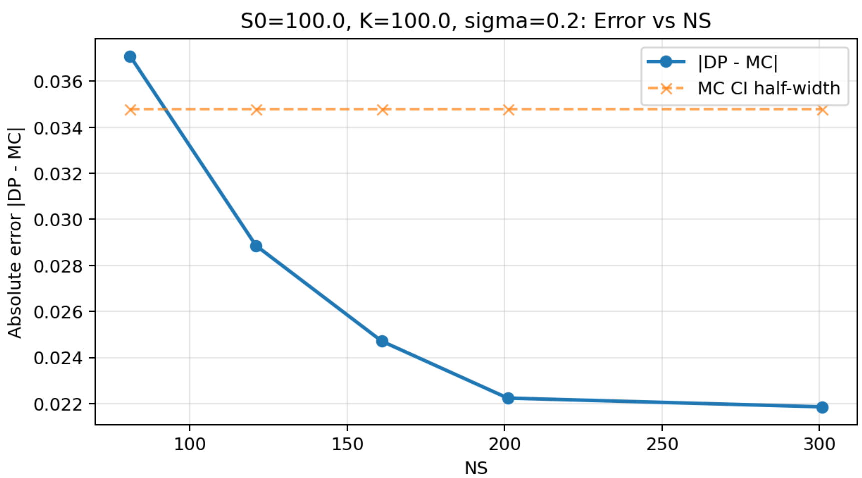

Figure 1: Convergence with increasing number of spot grid points (N_S)

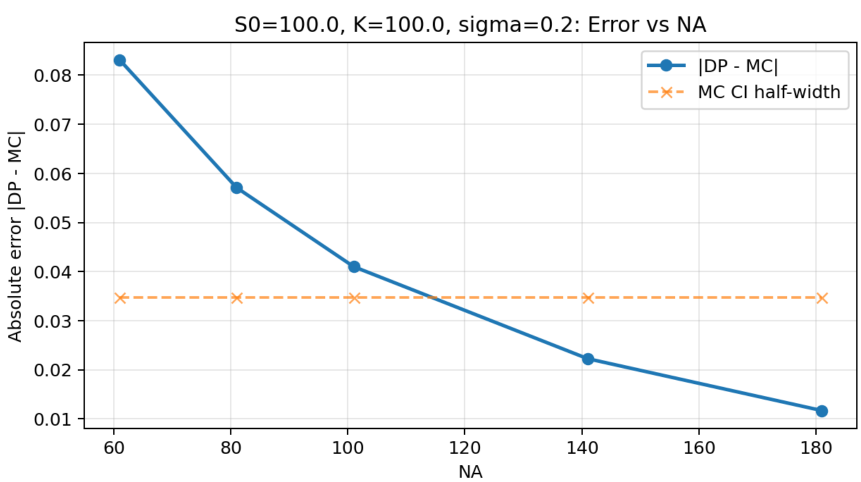

Figure 2: Convergence with increasing number of average grid points (N_A)

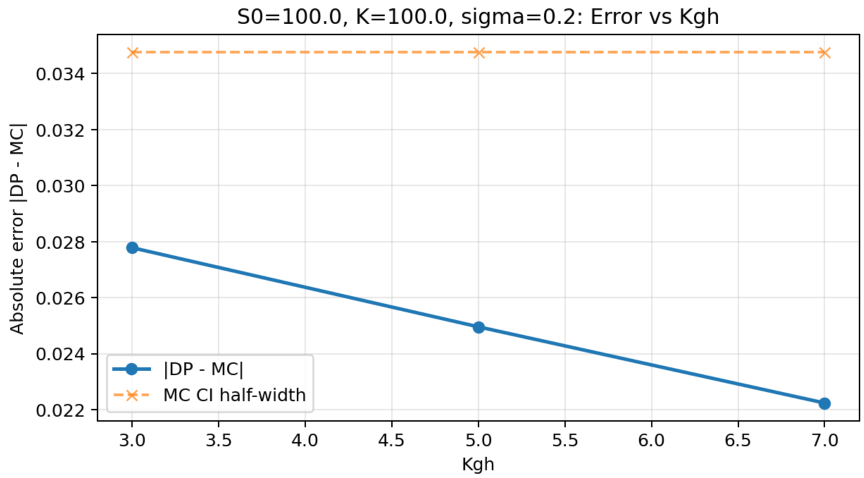

Figure 3: Convergence with increasing number of Gauss–Hermite nodes (K_{gh})

3.2 Interpretation

Error decreases monotonically with finer spatial discretization (N_S, N_A) and higher quadrature order (K_{gh}). The average grid resolution (A dimension) contributes more to accuracy due to its role in representing path-dependence. Beyond $K_{gh} = 5$, quadrature error stabilizes, confirming that primary bias arises from grid discretization rather than numerical integration. The DP results remain within the Monte Carlo 95% confidence interval, validating both numerical and probabilistic consistency.

4 Bermudan Option: Early-Exercise Frontier

For Bermudan options, early exercise is permitted at discrete time steps. At each monitoring point, we compute the optimal exercise region where $\Phi(S, A) > e^{-r\Delta t}\mathbb{E}[V_{n+1}]$.

4.1 Visualizing the Frontier

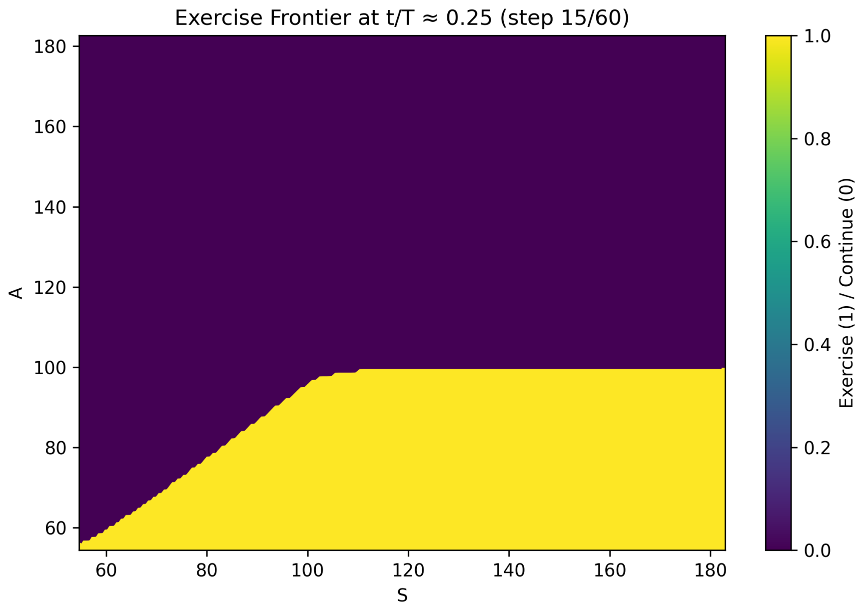

Figure 4: Exercise frontier at 25% maturity

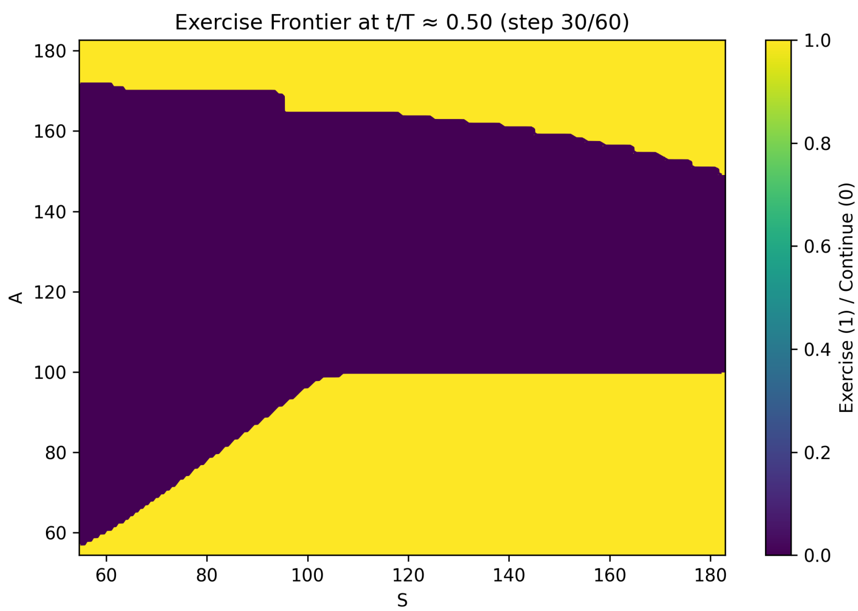

Figure 5: Exercise frontier at 50% maturity

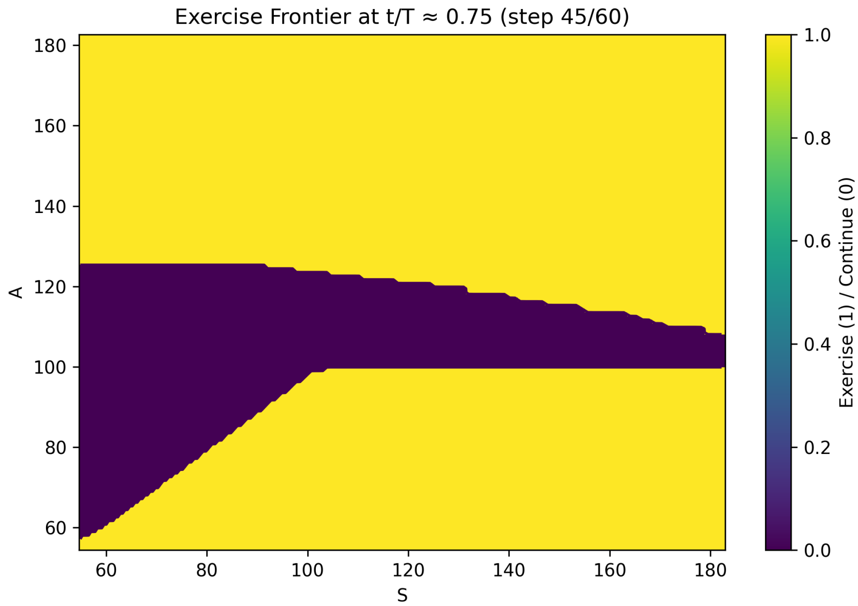

Figure 6: Exercise frontier at 75% maturity

4.2 Interpretation

The yellow region represents the **exercise zone**, and the purple region denotes **continuation**. The frontier shifts inward as time progresses, showing that the optimal exercise region expands closer to maturity. The boundary is upward sloping: higher running averages A require larger S to justify immediate exercise, as the expected future payoff diminishes with increasing A. This pattern captures the natural trade-off between *intrinsic value* and *time value* in early-exercise behavior.

5 Early-Exercise Premium Analysis

We define the early-exercise premium as:

$$\text{Premium} = V_{Berm/Am} - V_{Euro}.$$

This premium quantifies the added value of early exercise flexibility.

5.1 Parameter Variation Studies

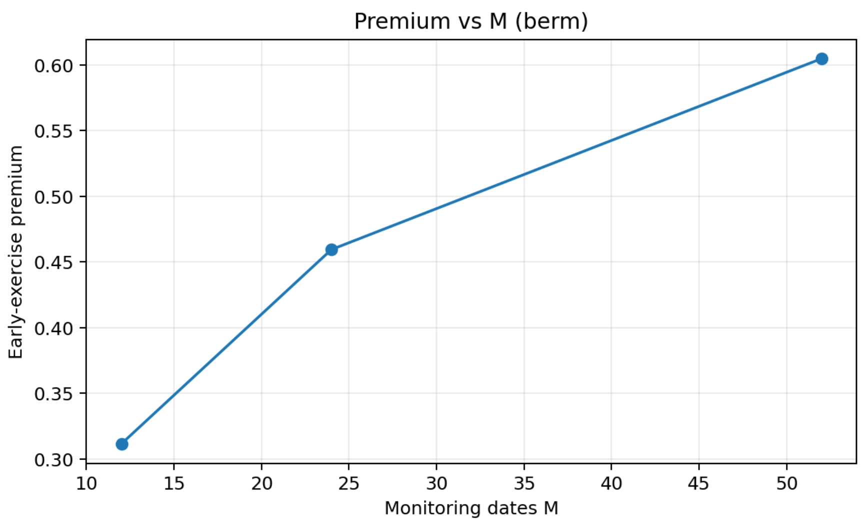

Figure 7: Premium vs number of monitoring steps (M)

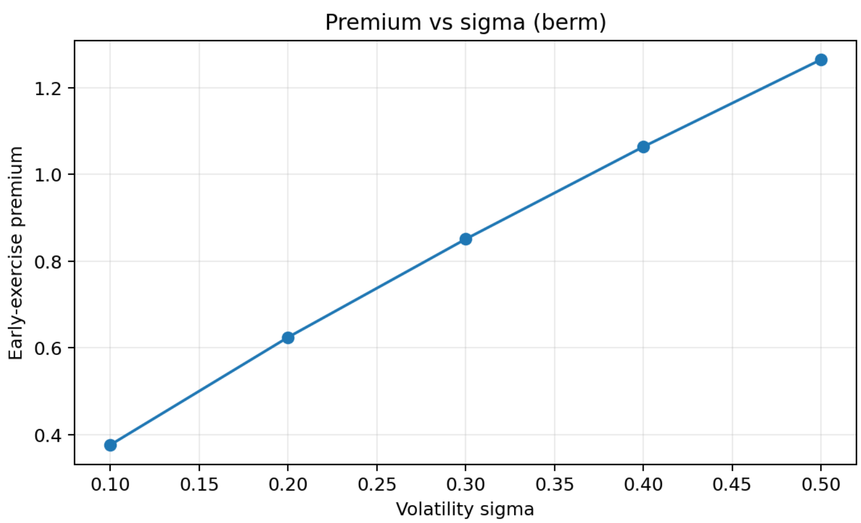

Figure 8: Premium vs volatility (σ)

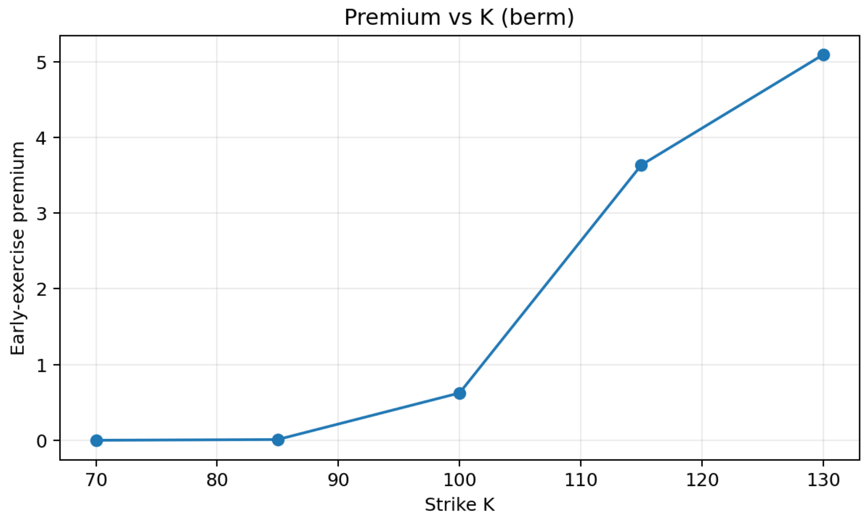

Figure 9: Premium vs strike (K)

5.2 Interpretation

The variable M represents the number of averaging or monitoring dates. Here we set $M = N$, implying that averaging occurs at every DP time step. Increasing M smooths the payoff distribution, reducing volatility in the running average and leading to smaller early-exercise gains. Premiums increase with volatility σ, as uncertainty enhances optionality and thus the benefit of flexible exercise. As strike K increases (deep in-the-money for puts), the premium rises sharply since the immediate payoff dominates. These results align with theoretical intuition and confirm numerical stability.

6 Sensitivity Overlay: European, Bermudan, and American Options

The solver enables simultaneous comparison across exercise styles. For each varying parameter—volatility, strike, rate, and averaging frequency—the three style values satisfy:

$$V_{Euro} \leq V_{Berm} \leq V_{Amer}.$$

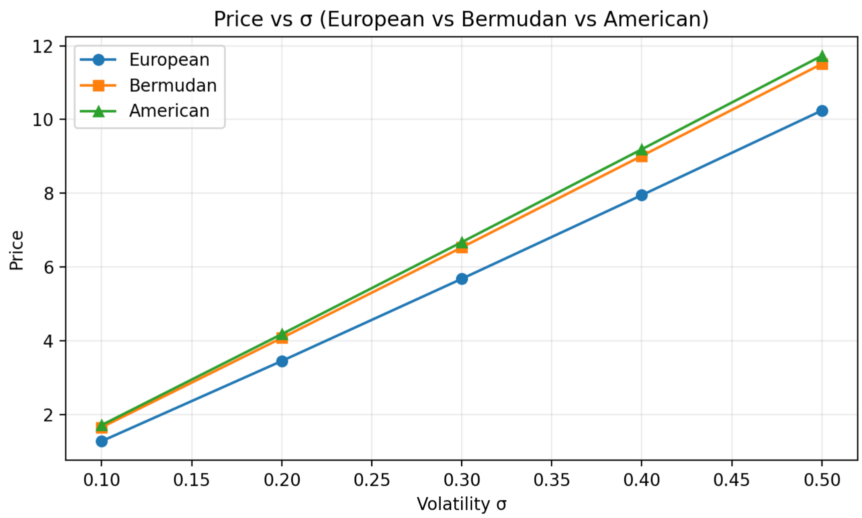

Figure 10: Option price vs volatility

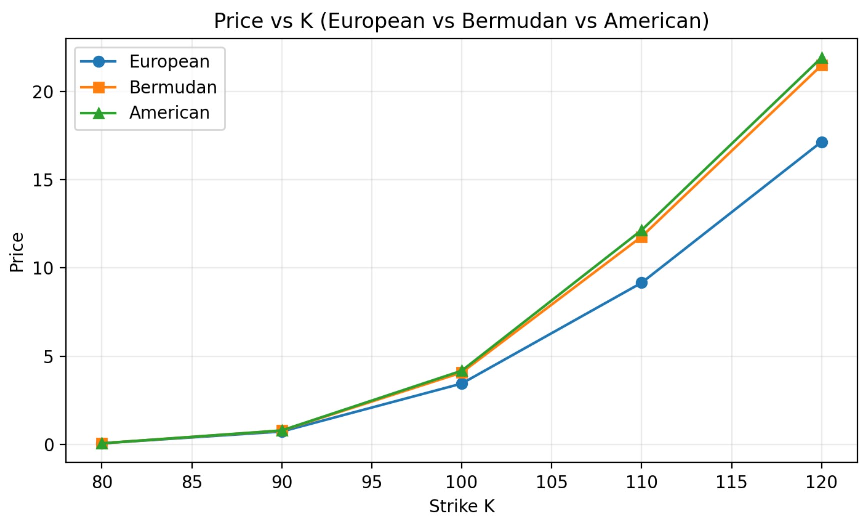

Figure 11: Option price vs strike

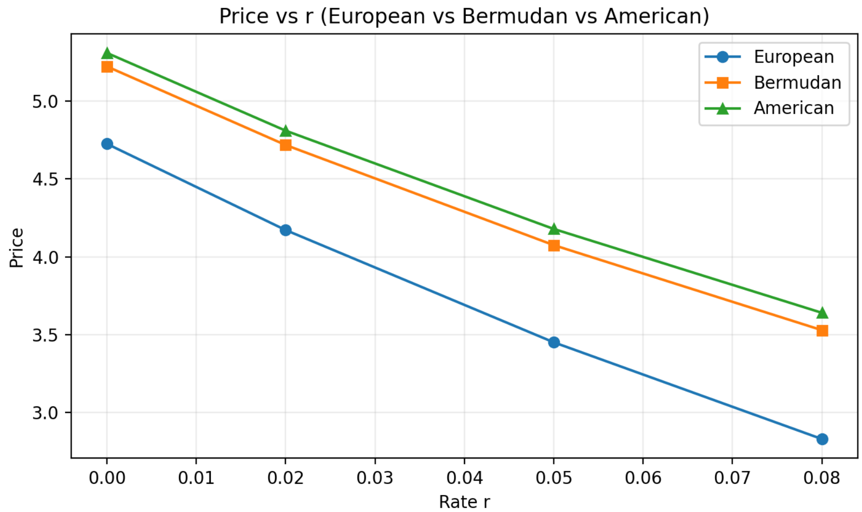

Figure 12: Option price vs interest rate

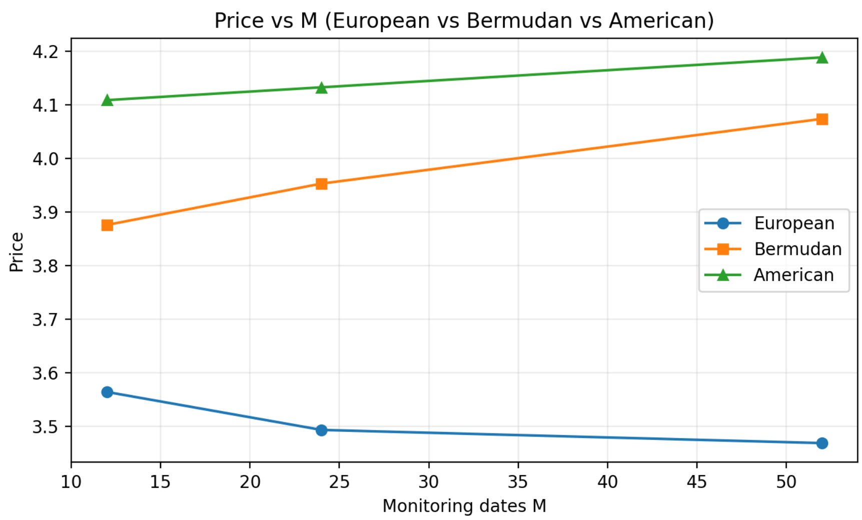

Figure 13: Option price vs monitoring frequency (M)

6.1 Interpretation

Volatility σ increases the value of all option styles, but the incremental gain is largest for American options. Increasing K reduces option value (typical put behavior), and higher interest rate r slightly reduces discounted payoffs. As averaging frequency increases, all prices converge due to reduced path volatility. The consistent ranking of $V_{Euro} < V_{Berm} < V_{Amer}$ across all sensitivities confirms correct implementation of exercise flexibility.

7 Conclusion and Future Work

The discrete DP framework successfully models Asian options with early exercise flexibility. The solver shows strong convergence to MC results, realistic early-exercise boundaries, and theoretically consistent sensitivities. Future improvements include:

- Adaptive non-uniform grids for finer resolution near the exercise frontier.
- Parallelization of DP recursion for high-dimensional state spaces.
- Integration of neural regression to approximate continuation values (Deep DP).
- Extension to multi-asset basket options and stochastic volatility models.

References

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