Linear Algebra and Probability Take Home Midterm[1]

[1] draft

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Instructions:

- Open book take–home midterm. Consult any written sources. Visit the library! Internet use allowed, feel free to work with colleagues.
- Solve as many problems as you can and find useful. All questions here for good reasons.
- Midterm due as soon as you are done. We can discuss any remaining questions after Spring break.

1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if for all $x, y \in \mathbb{R}^n$ and all $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

   A symmetric matrix $P = A^T + A$ is called positive–definite if all its eigenvalues are positive.

   Show that a quadratic function $f(x) = x^T P x$ is a convex function if and only P is positive–definite.

2. You are given sets of data $y_i \in \mathbb{R}^{n_y}$ and $x_i \in \mathbb{R}^{n_x}, i \in 1, 2, ..., n$, and would like to study the conjecture that they are linearly related, that is, $y = Ax$. Strict equality will rarely hold, because of experimental error and measurement uncertainty. Devise a criterion to judge whether a given value of A is better or worse than another. It should be based on the error accumulated over the n data samples. Ideally, investigate 2 ideas to define adequate criteria: one probabilistic, another deterministic.

   Using your criterion, obtain if possible a formula for the best value of A. Start with the scalar case, $\mathbb{R}^{n_y} = \mathbb{R}^{n_x} = 1$, then try to move to the multidimensional case.

   (even in the unlikely event that the linear model is actually correct, whatever that means)

3. Consider a random variable $x_0 \in \mathbb{R}^n$ and a sequence of random variables related by the recursion

$$x_{k+1} = A x_k.$$

   First compute the expectation and variance of x_1 as a function of the expectation and variance of x_0. Then find the expectation and variance of x_k for a general k. What can you say about the asymptotic behavior expectation and variance of x_k, for $k \to \infty$?

   Hint: consider — you guessed! — the eigenvalues and eigenvectors of A.

   Also consider the scalar case, $n = 1$, which helps understand the vector one.

4. Using the convolution formula for the sum of random variables, determine and plot the probability density function of the average of 2 independent random variables which have identical uniform distributions.

   Now find the density of the average of 2 independent instances of the random variable you obtained previously. Iterate the procedure a few times, and observe the resulting probability density.

   To obtain the density of the resulting random variable, use any method you prefer: calculus, numerical or symbolic integration software, a convenient discrete approximation, Monte–Carlo style simulation by drawing random samples and recording histograms, or other.