Optimal Advertising Expenditure Strategies of Two Successive Generations of Consumer Durables

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Abstract

The central role of advertising is to provide information and to serve as a means of product differentiation. The informational advertising is particularly important for new or visibly improved products (i.e. for product innovations). Advertising/promotions of innovative products can help the innovators to be informed about the features and benefits of the new innovations. A well planned advertising/promotions strategy may raise the demand function by adding new customers by altering the tastes and preferences of consumers and in that way change the shape of the demand function. Thus, drawing up an optimal advertising plan over time for a new product that is under the influence of diffusion phenomenon is one of the central importances in the field of marketing. There are some models in marketing that discussed the dynamics of advertising expenditure but in the light of single generation product only. In comparison high technology products have received less attention. This paper deals with the determination of optimal advertising expenditure policies of two-generation consumer durables and also introduces a framework for modeling innovation diffusion for two competing generations that incorporates advertising influence.

1. Introduction

Advertising and promotions is bringing a service/product to the attention of potential and current customers. It is one of the key marketing tools managers have at their disposal to influence their customers into acquire a new product. Research, as reported by Rogers (1962), has shown that advertising works best on innovators and early adopters. Advertising and promotions are best carried out by implementing advertising and promotions plan. The goals of the plan should depend very much on the overall goals and strategies of the organization, and the results of the marketing analysis, including the positioning statement. The plan usually includes identification of the target markets, what message company wants to convey, how it will be conveyed and how much money is budgeted for this effort. Successful advertising depends very much on knowing the preferred methods and styles of communications of the target markets.

It is well-known that advertising expenditures affect the future demand for the product as well as that of the present. An optimal decision rule for this market conduct must take into account both the present and the future net revenue of the firm which advertises, under dynamic conditions. Drawing up an optimal advertising plan over time for a new product that is under the influence of diffusion phenomenon is one of the central importances in the field of marketing. There are several approaches dealing with this problem reviewed by Sethi (1977) and Feichtinger et al. (1994). In comparison, optimization of advertising expenditures for technology generations has received less attention. Generally technological innovations come in generations and often an organization launched the new innovations without withdrawing the earlier one from the market. Thus a consumer can also map the new generational product on the merit of earlier generations. So in such a dynamic situation, for a manager it's indispensable to have clear advertising expenditure plan for the competing generations. But due to the complex nature of the problem research in this area is very limited.

The purpose of the present paper is to suggest optimal advertising expenditure policies for a technology product having two generations in the market by making use of demand functions that explicitly incorporates the advertising component under different dynamic conditions. Our results suggest that optimal advertising is determined by the advertising effectiveness, discount rate, and the shadow prices of both the generation products. Depending upon the interplay among these factors, the optimal advertising takes monotonically increasing or monotonically decreasing shape. The structure of this article is as follows. The following sections of this paper are the theoretical background, advertising policies analyzed for a general diffusion model, model development, advertising policies analyzed for the proposed model. Finally, the article concludes with a discussion on the extension and limitations and future avenues of research.

2. Literature Review

Since the pioneering work of Fourt and Woodlock (1960) many marketing scientists have proposed sales growth models to measure the effectiveness / success of a new idea or new product among target populations. This high level of interest in measuring the diffusion of innovation has resulted in a large body of publications (Mahajan et al. (2000)). In comparison, the diffusion modelling of multiple generation products have received less attention, though the number of publications in this area is steadily growing (Islam and Meade (2006)). The growing interest in this area can be attributed to reducing lifecycles of products and dominance of high technology products in the market. The basic mixed innovation diffusion model was proposed by Bass (1969), since then it has become the standard for further development and modification. This is of the form

$$g = \frac{dN(t)}{dt} = p(\overline{N} - N(t)) + q(\overline{N} - N(t))N(t) \qquad (1)$$

where, $\overline{N}$ is the potential market size, ‘ p ’ is innovation coefficient and ‘ q ’ is the imitation coefficient.

Norton and Bass (1987) proposed the first model to capture the diffusion effects for multiple generation products, which is built upon the Bass model. During the model development they assumed that the coefficients of innovation and imitation remain unchanged from generation to generation of technology. Islam and Meade (1997) have tested the hypothesis of coefficient constancy across generation of Norton-Bass model. Their empirical work relaxed the assumption of constant coefficient of Norton-Bass model. They proposed that the coefficients of later generation technology are constant increment/decrement over the coefficients of the first generation. Mahajan and Muller (1996) proposed a model that is an extension of Bass model to capture simultaneously both the substitution and diffusion patterns for each successive generation of technological products. The specification of the Bass like model is that they don’t contain any deterministic explanatory variable.

Many authors consider the impact of marketing variables on new product diffusion (Robinson and Lakhani (1975), Bass and Bultez (1982), Kalish (1983, 1985), Kamakura and Balasubramanium (1988), Horsky (1990), Sethi and Bass (2003)). These models incorporate the pricing effects on diffusion. Thompson and Teng (1996) derived the optimal price and quality policies and tried to establish a relationship between these two marketing strategies and the diffusion process. According to them under certain conditions higher prices imply higher quality and under the other conditions the optimal price declines over time while the product quality improves. Bayus (1992), Speece and

MacLachlan (1995) and Hardie and Putsis (Jr.) (2001) proposed models for technology generations by incorporating price as an explanatory variable.

Other models incorporate the effects of advertising on diffusion (Horsky and Simon (1983), Simon and Sebastian (1987), Dockner and Jorgensen (1988)).Horsky and Simon (1983) modified the Bass model by incorporating the effects of advertising in Bass' innovation coefficient. Thompson and Teng (1984) in their dynamic oligopoly price-advertising model had incorporated learning curve production cost. Dockner and Jorgensen discussed the optimal advertising policies for diffusion model under monopolistic product situation. Bass et al. (1994) include both price and advertising in their Generalized Bass Model (GBM). The authors found that when the rate of changes in pricing and advertising were kept constant the proposed GBM no longer gives better fit than the Bass model (1969). Many models have already been developed to study the optimal policies of different marketing variables. But most of them based on single generation framework only. This paper deals with the determination of optimal advertising expenditure policies for two generational product under different marketing conditions, in which advertising expenditure is a control variables whose optimal values are to be determined under the finite time-horizon.

3. Model Formulation

In this section we begin our analysis by stating a general model with a very few assumptions. Here we are considering sales model for two generations of a technology product by restricting our analysis to the case of a monopolistic firm that controls its advertising expenditure under finite planning horizon.

Let 't' denote the time such that $0 \le t \le T$. The length of the planning period, T, is fixed.

$x_i \ (i = 1,2) = x_i(t)$ = Cumulative sales volume (demand) of the i^{th} generation product by time 't'.

$A_i \ (i = 1,2) = A_i(t)$ =The firm's rate of advertising expenditure on i^{th} generation product at time 't'.

The diffusion models for two generation case can be given as:

$$\dot{x}_1 = \dot{x}_1(t) = g_1\big(x_1(t), x_2(t), A_1(t), A_2(t)\big);\ \ x_1(0) = x_{10} \ge 0 = \text{a fixed value} \tag{2}$$

$$\dot{x}_2 = \dot{x}_2(t) = g_2(x_1(t), x_2(t), A_1(t), A_2(t));\ x_2(0) = x_{20} \geq 0 = \text{a fixed value} \tag{3}$$

Functions (2) and (3) are twice differentiable, and

$g_{1A_1}, g_{2A_2} > 0$; $g_{1A_1A_1}, g_{2A_2A_2} < 0$ and $g_{1A_1A_2} = g_{1A_2A_1}$; $g_{2A_1A_2} = g_{2A_2A_1}$

Learning curve phenomenon is introduced by assuming that the marginal cost $c_i\ (i = 1,2)$ depend on cumulative sales (production) such that marginal cost decrease with increase in cumulative output. Where,

$$c_1 = c_1(x_1) \text{ and } c_2 = c_2(x_2);\ \frac{dc_1(x_1)}{dx_1} = c_{1x_1} \leq 0 \text{ and } \frac{dc_2(x_2)}{dx_2} = c_{2x_2} \leq 0 \tag{4}$$

We are also assuming that the firm charges a constant price $p_i(i = 1,2)$ over the planning period and also $(p_i - c_i(t)) > 0; \forall t$. Ideally, the optimization problem should include prices and other marketing-mix variable (Kalish (1983, 1985), Thompson and Teng (1996)). But as we are interested on determining the influence of advertising expenditures affect of two successive generations of durables on both the present and the future net revenue of the firm and as well as on the overall diffusion, we deliberately avoided the cost pricing effects.

The instantaneous profit stream to the firm is given by

$$P(x, A) = (p_1 - c_1(x_1))g_1(x_1, x_2, A_1, A_2) + (p_2 - c_2(x_2))g_2(x_1, x_2, A_1, A_2) - A_1 - A_2 \tag{5}$$

The objective here is to determine an optimal policy for Advertisement expenditure over the fixed time interval from $t = 0$ to $t = T$ such that the total discounted profit

$$J(x, A) = \int_0^T e^{-rt}\left[(p_1 - c_1)g_1 + (p_2 - c_2)g_2 - A_1 - A_2\right]dt \tag{6}$$

is maximized, subject to the constraints (2) and (3).

where, r is the discount rate.

The control variable $A_i\ (i = 1,2) = A_i(t)$ are twice differentiable in 't' and satisfy $A_i(t) \geq 0,\ \forall t$.

To solve the problem, we apply Pontryagin Maximum principle. Dropping the time notation, the current value Hamiltonian can be written as:

$$H = H(x_1, x_2, A_1, A_2, \lambda, \mu) = (p_1 - c_1 + \lambda)g_1 + (p_2 - c_2 + \mu)g_2 - A_1 - A_2 \tag{7}$$

where, $\lambda = \lambda(t)$ and $\mu = \mu(t)$ are the current-value costate variable (shadow price of x_1 and x_2 respectively) that satisfies the differential equations:

$$\dot{\lambda} = r\lambda + c_{1x_1} g_1 - g_{1x_1}(p_1 - c_1 + \lambda) - g_{2x_1}(p_2 - c_2 + \mu) \tag{8}$$

$$\dot{\mu} = r\mu - g_{1x_2}(p_1 - c_1 + \lambda) + c_{2x_2} g_2 - g_{2x_2}(p_2 - c_2 + \mu) \tag{9}$$

with the transversality condition at $t = T$

$$\lambda(T) = 0 \text{ and } \mu(T) = 0 \tag{10}$$

The physical interpretation of the current value Hamiltonian H can be given as follows: $\lambda(t)$ and $\mu(t)$ stands for the future benefits from first and second generation (at time 't') of having one more units produced. Thus the current value Hamiltonian is the sum of current profit $[(p_1 - C_1)g_1 + (p_2 - C_2)g_2]$ and the future benefit $[\lambda g_1 + \mu g_2]$. In short, H represents the instantaneous total profit of the firm at time 't'.

For an optimal non-negative $A_i (i = 1,2)$ we have the first order necessary condition that $H_{A_i} = 0$

i.e.,

$$H_{A_1} = (p_1 - c_1 + \lambda)g_{1A_1} + (p_2 - c_2 + \mu)g_{2A_1} - 1 = 0 \tag{11}$$

$$H_{A_2} = (p_1 - c_1 + \lambda)g_{1A_2} + (p_2 - c_2 + \mu)g_{2A_2} - 1 = 0 \tag{12}$$

The second order condition for H-maximization are:

$$H_{A_1A_1} < 0 \Rightarrow (p_1 - c_1 + \lambda)g_{1A_1A_1} + (p_2 - c_2 + \mu)g_{2A_1A_1} < 0 \tag{13}$$

$$H_{A_2A_2} < 0 \Rightarrow (p_1 - c_1 + \lambda)g_{1A_2A_2} + (p_2 - c_2 + \mu)g_{2A_2A_2} < 0 \tag{14}$$

and,

$$\Delta = \begin{vmatrix} H_{A_1A_1} & H_{A_1A_2} \\ H_{A_2A_1} & H_{A_2A_2} \end{vmatrix} > 0 \tag{15}$$

also,

$$H_{A_1A_2} = (p_1 - c_1 + \lambda)g_{1A_1A_2} + (p_2 - c_2 + \mu)g_{2A_1A_2} \tag{16}$$

$$H_{A_2A_1} = (p_1 - c_1 + \lambda)g_{1A_2A_1} + (p_2 - c_2 + \mu)g_{2A_2A_1} \tag{17}$$

Now integrating (8) and (9) with condition (10), we have the future benefit of one more unit produced from each generation

$$\lambda = \int_t^T e^{-rs}\left[g_{1x_1}(p_1 - c_1 + \lambda) + g_{2x_1}(p_2 - c_2 + \mu) - c_{1x_1} g_1\right]ds \tag{18}$$

and, $$\mu = \int_t^T e^{-rs}\left[g_{1x_2}\left(p_1 - c_1 + \lambda\right) + g_{2x_2}\left(p_2 - c_2 + \mu\right) - c_{2x_2} g_2\right] ds \tag{19}$$

Defining $\beta_i \ (i = 1,2)$=elasticity of demand with respect to the advertisement expenditure of the i^{th} generation product $= \left(\frac{A_i}{g_i}\right) g_{iA_i}$ and the cross-elasticities are $\beta_{ij} \ (i, j = 1,2) = -\left(\frac{A_j}{g_i}\right) g_{iA_j}$ and using (11) and (12), we can derive the optimal values of advertisement expenditure as:

$$A_1^* = (p_1 - c_1 + \lambda) g_1 \left[\beta_1 - \left(\frac{(p_1 - c_1 + \lambda) g_1}{(p_2 - c_2 + \mu) g_2}\right) \beta_{21}\right] \tag{20}$$

$$A_2^* = (p_2 - c_2 + \mu) g_2 \left[\beta_2 - \left(\frac{(p_1 - c_1 + \lambda) g_1}{(p_2 - c_2 + \mu) g_2}\right) \beta_{12}\right] \tag{21}$$

(20) and (21), can be interpreted as, at optimum the advertisement expenditure of any generational product is directly proportional to its advertisement elasticity.

3.1 A Subclass of the General Formulation

As we have already discussed the demand of a product to large extent depend on advertising expenditures in addition to the price and consumer incomes. Advertising expenditures may raise the demand function by adding new customers by altering the tastes and preferences of consumers and in that way alter the shape of the demand function. In early period of generational products life-cycle advertising can play pivotal role to stimulate the innovators and the early adopters. To get the clear insight about the role of the advertisement expenditure for generational goods, in this section we are assuming that the sales of each generational product are function of advisement expenditure only i.e. the sales figure of both the generational product depend on advisement function not on word-of-mouth influence. In this case equation (18) and (19) becomes

$$\lambda = -\int_t^T e^{-rs} c_{1x_1} g_1 ds \tag{22}$$

and, $$\mu = -\int_t^T e^{-r} c_{2x_2} g_2 ds \tag{23}$$

The above two equations represent the future benefit of selling one more unit from each generation hence, it can't be negative, unless there is no saturation effect on demand. Now, taking time-derivative of (11) and (12), we have (for details, see appendix A)

$$\dot{A}_1 = -\frac{r\alpha}{\Delta}\left[H_{A_2A_2} - \beta H_{A_1A_2}\right] \tag{24}$$

$$\dot{A}_2 = \frac{r\alpha}{\Delta}\left[H_{A_2A_1} - \beta H_{A_1A_1}\right] \tag{25}$$

where, $\alpha = \lambda g_{1A_1} + \mu g_{2A_1}$; $\beta = \dfrac{\lambda g_{1A_2} + \mu g_{2A_2}}{\lambda g_{1A_1} + \mu g_{2A_1}}$ and $\Delta > 0$

From (24) and (25) it can be observed that for the case $r = 0$ optimal policies reduces to static problem and the advertising expenditure of both the generation remains constant for the entire planning horizon. But by the model assumptions the diffusion process of both the generation depends on the advertising expenditure, and can't be kept it constant. In general conditions are identified in the Appendix in which optimal prices decrease and increase. Now, if the planning horizon is long enough and also the discount rate is positive, then depending on the sign of α and β we have the following general advertising expenditure strategies:

Theorem 1. If $\beta < 0$; $g_1 = g_1(A_1, A_2)$ and $g_2 = g_2(A_1, A_2)$

	Conditions	Results	
		$\alpha > 0$	$\alpha < 0$
Case1	$H_{A_2A_2} - \beta H_{A_1A_2} > 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 < 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 > 0$
Case2	$H_{A_2A_1} - \beta H_{A_1A_1} > 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 > 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 < 0$
Case3	$H_{A_2A_1} = H_{A_1A_2} < 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 < 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 > 0$

Proof. See Appendix A.

Case 1 suggest that when $H_{A_2A_2} - \beta H_{A_1A_2} > 0 \;\Rightarrow\; H_{A_1A_2} > 0$ and from case 2 we find that, when $H_{A_2A_1} - \beta H_{A_1A_1} > 0 \;\Rightarrow\; H_{A_2A_1} > 0$. The above mathematical equations suggest that the total profit H

at any time 't' can be optimized by increasing (decreasing) advertisement expenditure of both the generational product simultaneously. Case 3 can be interpreted as when $H_{A_2A_1} = H_{A_1A_2} < 0$, optimal advertising expenditure of first and second generation go up or down in opposite direction. Now, for the case $\beta > 0$, we have the following theorem.

Theorem 2. If $\beta > 0$; $g_1 = g_1(A_1, A_2)$ and $g_2 = g_2(A_1, A_2)$

	Conditions	**Results**	
		$\alpha > 0$	$\alpha < 0$
Case1	$H_{A_2A_2} - \beta H_{A_1A_2} > 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 > 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 < 0$
Case2	$H_{A_2A_1} - \beta H_{A_1A_1} < 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 < 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 > 0$
Case3	$H_{A_2A_1} = H_{A_1A_2} > 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 > 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 < 0$

Proof. See Appendix A.

Like theorem 1, here Case 1 suggest that when $H_{A_2A_2} - \beta H_{A_1A_2} > 0 \Rightarrow H_{A_1A_2} < 0$ and from case 2 we find that, when $H_{A_2A_1} - \beta H_{A_1A_1} < 0 \Rightarrow H_{A_2A_1} < 0$. Thus for the above two conditions it can be suggested that to optimize the total profit H at any time 't' the firm should adopt a policy that advertising expenditure of first and second generation go up or down in opposite direction. Case 3 suggested that when $H_{A_2A_1} = H_{A_1A_2} > 0$, total profit H at any time 't' can be optimized by increasing (decreasing) advertisement expenditure of both the generational product simultaneously. To illustrate the above mathematical findings, we now apply theorem 1 and 2 in the following section. To do so, we first develop two-generation demand function and then analyze the optimal advertisement expenditure policies for the model.

4. Special Functional Form

In this section we will first propose two-generation sales growth model and then determine the optimal marketing strategies of advertisement expenditure. The model is based on the following assumptions:

- Once an adopter adopts a new technology, he/she doesn't revert to earlier generation later.

- New adopters (first time buyers) will purchase only that particular generation product for which he/she will get the maximum utility. Utility can be expressed as a function of advertising expenditure and the goodwill of all the available generations in the market and goodwill of the product depend on the word-of-mouth influence.
- Sales of a second-generation durable come from two sources:
  I. *New Purchasers (First time Buyers)*: Those who have for the first time adopted the product.
  II. *Repeat Purchasers*: Those adopters who have bought the earlier generation and now upgrade to latest technology.
- Each adopter can purchase exactly one product unit and she/he makes no further purchases of the product generations that they have adopted. And also each adopter after having made the first purchase may make a repeat purchase of exactly one unit in each successive generation or they can skip a generation product and can wait for more advanced one.

The process of incorporating a new technology is a process, which involves the diffusion of knowledge about the characteristic of the technology. Here in this model we will discuss a multiple generation of innovations. Second generation is introduced into the market before its predecessor is withdrawn from the market place. Two component of the model are adoptions due to the new purchase and repeat sales. We can propose our basic equation as:

$$CumulativeAdopters_j(t) = NewPurchasers_j(t) + \mathrm{Re}\,peatPurchasers_j(t)$$

where 'j' is the index representing the generation of a particular technology and x_j is the cumulative number of adopters in the j^{th} generation. Thus,

$$x_j(t) = x_j^/(t) + R_j(t) \tag{26}$$

where, $x_j^/(t)$ is the cumulative number of first time purchasers and $R_j(t)$ is the cumulative number of repeat purchasers of a j^{th} generation technology product . Here, we are assuming a monopoly market situation and each adopter can adopt a one unit of product from each generation. In general we do expect the potential market size to increase monotonically after every τ_j (where, τ_j introduction time of j^{th} generation product) but it may not hold for all generation of technological product. The proposed model shows continuous flow dynamic character. The model for different market situations can be build as follows:

Case 1. When a single generation product is in the market place

When there exist only a single generation product in the market, the cumulative sales pattern of that generation can be described by the following model, which is an extension of basic mixed innovation diffusion model proposed by Bass (1969):

$$\dot{x}_1(t) = \frac{dx_1(t)}{dt} = (\overline{N}_1 - x_1(t))\, f_1(A_1(t), x_1(t)) \qquad (27)$$

where,

$\overline{N}_1$ is the potential of the first generation product.

$f_i(A_i(t), x_i(t))$ = Diffusion effect on i^{th} generation demand at time t ($i = 1,2$).

$f(A, x)$ can be specified as Horsky-Simon (1983) mode. The functional form of diffusion equation is:

$$f_1(A_1, x_1) = a_1 + a_1' \ln A_1 + b_1\left(\frac{x_1}{\overline{N}_1}\right)$$

where,

a_1 = innovation coefficient, a_1' = reaction coefficient for advertising, b_1 = imitation coefficient.

Case 2. When two generation products are in the market place

When there is two generation of products in the market, the potential purchasers of first generation technology product come to the influence of both the advertising/promotion and word-of-mouth influence of the second generation product. As a result a fraction of the adopters (say, $\gamma(t)$) who would have otherwise adopted the first generation product instead adopt the latest technology and the remaining $[1 - \gamma(t)]$ will adopt the first generation product. Let us define the parameter $\gamma(t)$ as the switching parameter. Two generational demand functions are:

$$\dot{x}_1(t) = \frac{dx_1(t)}{dt} = (\overline{N}_1 - x_1(t))\, f_1(A_1(t), x_1(t))(1 - \gamma(t)) \qquad (28)$$

$$\dot{x}_2(t) = \frac{dx_2(t)}{dt} = (\overline{N}_2 - x_2'(t))\, f_2(A_2(t), x_2(t)) + (\overline{R}_2(t) - R_2(t))\, f_2'(A_2(t), x_2(t)) + \gamma(t)(\overline{N}_1 - x_1(t))\, f_1(A_1(t), x_1(t)) \qquad (29)$$

where, the skipping parameter ($\gamma(t)$) can be defined as

$$\gamma(t)=\omega(t)\left[\frac{f_2(A_2(t),x_2(t))}{f_1(A_1(t),x_1(t))+f_2(A_2(t),x_2(t))}\right]; \qquad \omega(t)=\begin{cases}0, & if \quad t<\tau\\ 1, & if \quad t\geq\tau\end{cases} \tag{30}$$

$\overline{N}_2$ is the potential of the second generation product due to first time purchasers.

$x_i(t)$ = Rate of adoption of i^{th} generation product ($i=1,2$).

$f_2^{/}(A_2(t),x_2(t))$ = Diffusion effect on repeat purchasers of 2^{nd} generation product at time t.

$R_2(t)$ is the cumulative number of repeat purchasers of second generation technology, who have already purchased the earlier one.

$t_2=t-\tau$; τ is the introduction time of second generation product.

The buyers of first generation product can become the potential purchasers of the second generation technology and if none of the adopters of new technology drops out of the market in the later generation, the total number of potential repeat purchasers in the second generation is equal to the summation of all prior purchasers of the first generations i.e., we can conclude that the potential repeat purchasers of the second generation product is the function of the already adopters of the first generation product. Thus the possible repeat purchasers for 2^{nd} generation technology product can be expressed as a function of adoption of first generation product and can be written as follows:

$$\text{Potential repeat purchasers} = \overline{R}_2(t)=\left[\sum_{i=1}^{t-1} n_1(i)\right] \qquad \text{i.e. } \overline{R}_2(t)=F(x_1(t)) \tag{31}$$

$n_1(i)$: Sales of first generation product due to first time purchasers at time t. Now, if we assume that $f_2(A_2(t),x_2(t))=f_2^{/}(A_2(t),x_2(t))$, then equation (28) can be written as:

$$\overset{\bullet}{x}_2(t)=\frac{dx_2(t)}{dt}=(\overline{x}_2(t)-x_2(t))\,f_2(A_2(t),x_2(t))+\gamma(t)(\overline{N}_1-x_1(t))\,f_1(A_1(t),x_1(t)) \tag{32}$$

where, $\overline{x}_2(t)=\overline{N}_2+\overline{R}_2(t)$; $x_2(t)=x_2^{/}(t)+R_2(t)$ and

$$f_2(A_2(t),x_2(t))=a_2+a_2^{/}\ln A_2(t)+b_2\left(\frac{x_2(t)}{\overline{x}_2(t)}\right)$$

where,

a_2 = innovation coefficient, $a_2^{/}$ = reaction coefficient for advertising, b_2 = imitation coefficient.

Thus the two generational demand functions can be written as:

$$\dot{x}_1(t) = g_1(x_1(t), x_2(t), A_1(t), A_2(t)) = f_1(A_1(t), x_1(t))(\overline{N}_1 - x_1(t))[1-\gamma(t)] \quad (33)$$

$$\dot{x}_2(t) = g_2(x_1(t), x_2(t), A_1(t), A_2(t)) = f_2(A_2(t), x_2(t))[(\overline{x}_2(t)) - x_2(t)] + \gamma(t) f_1(A_1(t), x_1(t))(\overline{N}_1 - x_1(t)) \quad (34)$$

where,

$$g_{iA_i} > 0;\ g_{iA_j} < 0;\ g_{iA_iA_i} < 0 \text{ and } g_{iA_iA_j} = g_{iA_jA_i}, \text{ where } (i, j = 1,2) \quad (35)$$

$$\text{also, } \gamma_{A_1} < 0;\ \gamma_{A_2} < 0;\ \gamma_{A_1A_1} > 0;\ \gamma_{A_2A_2} > 0 \text{ and } \gamma_{A_1A_2} = \gamma_{A_2A_1} \quad (36)$$

Thus from (13)-(17), we have

$$H_{A_1A_1} = (p_1 - c_1 + \lambda)g_{1A_1A_1} + (p_2 - c_2 + \mu)g_{2A_1A_1} < 0$$

$$H_{A_2A_2} = (p_1 - c_1 + \lambda)g_{1A_2A_2} + (p_2 - c_2 + \mu)g_{2A_2A_2} < 0$$

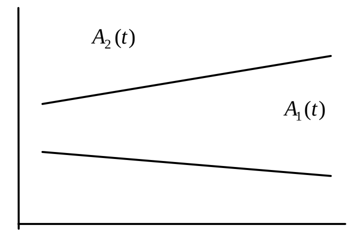
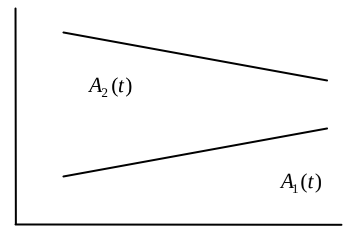
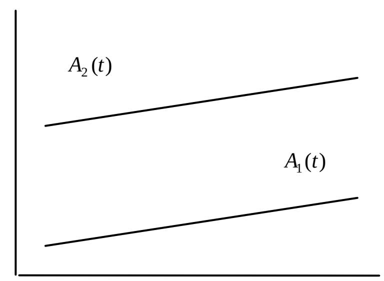
$$H_{A_1A_2} = H_{A_2A_1}$$

4.1. Optimal Advertisement Expenditure Strategies

In this sub-section, we analyze the optimal advertisement expenditure policies for the two product generations by a monopolist subject to the evolution of equation (33) and (34). Recalling theorem 1 and 2, we can propose the following theorem for the demand equation (33) and (34).

Theorem 3. For constant price p_i $(i = 1,2)$ of the i^{th} generation with $g_1 = g_1(A_1, A_2)$ and $g_2 = g_2(A_1, A_2)$, then the following holds:

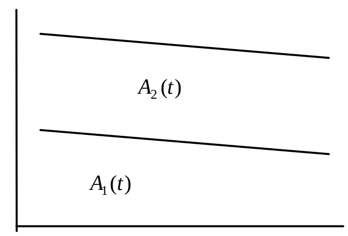
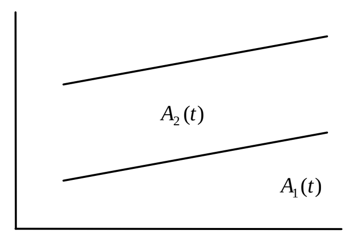
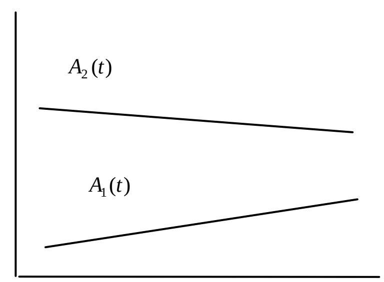
i. If the future benefit on selling first generation product is less than or equal to that of second generation product than the optimal advertisement expenditures path of both the generational product depend on the following conditions.

Conditions	Results	
$H_{A_2A_2} - \beta H_{A_1A_2} > 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 > 0$	
$H_{A_2A_1} - \beta H_{A_1A_1} < 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 < 0$	
$H_{A_2A_1} = H_{A_1A_2} > 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 > 0$	

Proof: See Appendix B.

From the first case if $H_{A_2A_2} - \beta H_{A_1A_2} > 0$, the firm should maintain a policy that advertisement expenditure of first and second generation move in opposite direction. In this situation the optimal strategy is to increase the advertising of second generation product gradually over the planning horizon and decrease the advertising expenditure on first generation product, so that the adoption rate of second generation product can climb up. On the other hand if $H_{A_2A_1} - \beta H_{A_1A_1} < 0$ the suggested optimal strategy is to increase the advertising of first generation product gradually over the planning horizon and decrease the advertising expenditure of second generation product, so that first generation sales attain maturity stage early before the second generation product cannibalize its market. Finally for $H_{A_2A_1} = H_{A_1A_2} > 0$ the suggested optimal policy in such situation is to increase the advertising of both the product generation gradually over the planning horizon to counter balance the decrease in current sales caused by current penetration.

ii. For $\lambda >> \mu$ the optimal advertisement expenditures path of both the generational product depend on the following conditions.

Conditions	Results	
$H_{A_2A_2} - \beta H_{A_1A_2} > 0$	$\dot{A}_1 < 0$ & $\dot{A}_2 < 0$	
$H_{A_2A_1} - \beta H_{A_1A_1} > 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 > 0$	
$H_{A_2A_1} = H_{A_1A_2} < 0$	$\dot{A}_1 > 0$ & $\dot{A}_2 < 0$	

Proof: See Appendix B.

The results of theorem 2 can be interpreted as: as long as $H_{A_2A_2} - \beta H_{A_1A_2} > 0$ varying advertisements expenditure of second generation product is more stimulating then first generation to increase the overall profitability. The optimal policy in such situation is advertise heavily in the beginning of the planning-horizon in order to stimulate innovators or early adopters and then as the market settles down the advertisement expenditure goes down steadily. In this situation the optimal price path of both the generation will follow the same trend as $H_{A_1A_2} > 0$. This type of pattern can be seen when the diffusion effects are positive for both the product generation throughout their planning horizon or the initial heavy advertising/promotion targeting to a particular market segment has done well. On the other hand if $H_{A_2A_1} - \beta H_{A_1A_1} > 0$ investing more on advertising expenditure for first generation product is more stimulating then second generation to raise the overall profitability. The suggested optimal policy in this situation is to increase advertising expenditure gradually over the planning horizon to counter balance the decrease in current sales caused by current penetration and the optimal price path of both the generation will follow the same trend as $H_{A_2 A_1} > 0$. Policies for increasing advertising expenditure occur when the diffusion effects are negative for both the product generation throughout their planning horizon. This can be due to negative word-of-mouth communication generated by the users of the product. Finally $H_{A_2 A_1} = H_{A_1 A_2} < 0$ indicates that the firm should maintain a policy that advertisement expenditure of first and second generation move in opposite direction. In this situation the optimal strategy is to increase the advertising of first generation and reduce the advertising of second generation product, so that the first generation product can attain saturation point as early as possible before the sales of second generation picks up. Otherwise, the first generation product may go out of the market due to the high quality/features of second generation product.

The results above are generally consistent with previous work as suggested by different authors for single generation product. Here, we proposed a model, which can optimize the advertising expenditure for two-generation product. In this research paper, we have shown that the investment pattern on advertising or promotions of earlier generation product depends on the diffusion rate of the latest generation and vice-versa. We have also observed that for both the generational product the optimal advertisement expenditure increases or decreases monotonically over the entire planning horizon. Thus, the optimal investment on advertising/promotion either be initially low and picks up later on or it is very high initially and slows down later on. In theorem 3, the overall results have been summarized diagrammatically to show the structure of the time path of optimal advertising expenditure of the two generation diffusion models (33) and (34).

Conclusions

This paper discusses the optimal advertising expenditure strategy for two generation models under dynamic situation. Also a two generation diffusion model has been proposed in the light of Horsky and Simon (1983) model. The results discussed in this paper are consistent with the work of Dockner and Jorgensen (1988) for single generation product. The proposed model incorporates the replacement behavior of first generation purchasers when an advanced generation is introduced in the market. Conditions are found in which optimal advertising expenditure decrease and increase. In all the cases we observed that optimal policies behave monotonically. The theoretical results obtained here confirm and extend the prior results in the literature. Finally, the model can be extended in several ways, e.g. by extending the monopolistic model for a duopolistic or oligopolistic market. The model can also be extended for n-generation product situation in the dynamic environment.

Appendix

A. Proof of conditions and theorem of General Formulation

Taking time-derivative on (11), we have

$$\left(\dot{c}_1+\dot{\lambda}\right)g_{1A_1}+(p_1-c_1+\lambda)\left[g_{1A_1A_1}\dot{A}_1+g_{1A_1A_2}\dot{A}_2+g_{1A_1x_1}\dot{x}_1+g_{1A_1x_2}\dot{x}_2\right]+\left(\dot{c}_2+\dot{\mu}\right)g_{2A_1}+(p_2-c_2+\mu)$$
$$\left[g_{2A_1A_1}\dot{A}_1+g_{2A_1A_2}\dot{A}_2+g_{2A_1x_1}\dot{x}_1+g_{2A_1x_2}\dot{x}_2\right]=0$$

$$\Rightarrow\left(-c_{1x_1}\dot{x}_1+r\lambda+c_{1x_1}g_1-(p_1-c_1+\lambda)g_{1x_1}-(p_2-c_2+\mu)g_{2x_1}\right)g_{1A_1}+\dot{A}_1\left[(p_1-c_1+\lambda)g_{1A_1A_1}+(p_2-c_2+\mu)\right.$$
$$\left.g_{2A_1A_1}\right]+\dot{A}_2\left[(p_1-c_1+\lambda)g_{2A_1A_2}+(p_2-c_2+\mu)g_{2A_1A_2}\right]+\left(-c_{2x_2}\dot{x}_2+r\mu+c_{2x_2}g_2-(p_1-c_1+\lambda)g_{1x_2}\right.$$
$$\left.-(p_2-c_2+\mu)g_{2x_2}\right)g_{2A_1}=0$$
$$\Rightarrow H_{A_1A_1}\dot{A}_1+H_{A_1A_2}\dot{A}_2=d_1 \tag{A.1}$$

where,

$$d_1=-r\lambda g_{1A_1}-r\mu g_{2A_1}+g_{1A_1}\left[(p_1-c_1+\lambda)g_{1x_1}+(p_2-c_2+\mu)g_{2x_1}\right]+g_{2A_1}\left[(p_1-c_1+\lambda)g_{1x_2}+(p_2-c_2+\mu)\right.$$
$$\left.g_{2x_2}\right]$$

Taking time-derivative on (12), we have

$$\left(\dot{c}_1+\dot{\lambda}\right)g_{1A_2}+(p_1-c_1+\lambda)\left[g_{1A_2A_1}\dot{A}_1+g_{1A_2A_2}\dot{A}_2+g_{1A_2x_1}\dot{x}_1+g_{1A_2x_2}\dot{x}_2\right]+\left(\dot{c}_2+\dot{\mu}\right)g_{2A_2}+(p_2-c_2+\mu)$$
$$\left[g_{2A_2A_1}\dot{A}_1+g_{2A_2A_2}\dot{A}_2+g_{2A_2x_1}\dot{x}_1+g_{2A_2x_2}\dot{x}_2\right]=0$$

$$\Rightarrow\left(-c_{1x_1}\dot{x}_1+r\lambda+c_{1x_1}g_1-(p_1-c_1+\lambda)g_{1x_1}-(p_2-c_2+\mu)g_{2x_1}\right)g_{1A_2}+\dot{A}_1\left[(p_1-c_1+\lambda)g_{1A_2A_1}+(p_2-c_2+\mu)\right.$$

$$\left.g_{2A_2A_1}\right]+\dot{A}_2\left[(p_1-c_1+\lambda)g_{1A_2A_2}+(p_2-c_2+\mu)g_{2A_2A_2}\right]+\left(-c_{2x_2}\dot{x}_2+r\mu+c_{2x_2}g_2-(p_1-c_1+\lambda)g_{1x_2}\right.$$

$$\left.-(p_2-c_2+\mu)g_{2x_2}\right)g_{2A_2}=0$$

$$\Rightarrow H_{A_2A_1}\dot{A}_1+H_{A_2A_2}\dot{A}_2=d_2 \qquad \text{(A.2)}$$

where,

$$d_2=-r\lambda g_{1A_2}-r\mu g_{2A_2}+g_{1A_2}\left[(p_1-c_1+\lambda)g_{1x_1}+(p_2-c_2+\mu)g_{2x_1}\right]+g_{2A_2}\left[(p_1-c_1+\lambda)g_{1x_2}+(p_2-c_2+\mu)g_{2x_2}\right]$$

Solving (A.1) and (A.2), we have

$$\dot{A}_1=\frac{1}{\Delta}\left[d_1H_{A_2A_2}-d_2H_{A_1A_2}\right] \qquad \text{(A.3)}$$

$$\dot{A}_2=-\frac{1}{\Delta}\left[d_1H_{A_2A_1}-d_2H_{A_1A_1}\right] \qquad \text{(A.4)}$$

Assuming that the cumulative sales of both the generations are function of advertising only and word-of-mouth influence are not important, then

$$d_1=-r\lambda g_{1A_1}-r\mu g_{2A_1} \qquad \text{and} \qquad d_2=-r\lambda g_{1A_2}-r\mu g_{2A_2}$$

$$\therefore (A.3)\ \&\ (A.4)\Rightarrow$$

$$\dot{A}_1=\frac{1}{\Delta}\left[\left(-r\lambda g_{1A_1}-r\mu g_{2A_1}\right)H_{A_3A_2}-\left(-r\lambda g_{1A_2}-r\mu g_{2A_2}\right)H_{A_1A_2}\right]$$

$$=-\frac{r\left(\lambda g_{1A_1}+\mu g_{2A_1}\right)}{\Delta}\left[H_{A_3A_2}-\left(\frac{\lambda g_{1A_2}+\mu g_{2A_2}}{\lambda g_{1A_1}+\mu g_{2A_1}}\right)H_{A_1A_2}\right]$$

$$\dot{A}_2=-\frac{1}{\Delta}\left[\left(-r\lambda g_{1A_1}-r\mu g_{2A_1}\right)H_{A_2A_1}-\left(-r\lambda g_{1A_2}-r\mu g_{2A_2}\right)H_{A_1A_1}\right]$$

$$=\frac{r\left(\lambda g_{1A_1}+\mu g_{2A_1}\right)}{\Delta}\left[H_{A_2A_1}-\left(\frac{\lambda g_{1A_2}+\mu g_{2A_2}}{\lambda g_{1A_1}+\mu g_{2A_1}}\right)H_{A_1A_1}\right]$$

Let, $\alpha=\lambda g_{1A_1}+\mu g_{2A_1}$ and $\beta=\dfrac{\lambda g_{1A_2}+\mu g_{2A_2}}{\lambda g_{1A_1}+\mu g_{2A_1}}$; then

$$\dot{A}_1=-\frac{r\alpha}{\Delta}\left[H_{A_2A_2}-\beta H_{A_1A_2}\right] \qquad \text{(A.5)}$$

$$\dot{A}_2 = \frac{r\alpha}{\Delta}\left[H_{A_2A_1} - \beta H_{A_1A_1}\right] \tag{A.6}$$

Lemma 1. When $\beta < 0$

1.1. If $H_{A_2A_2} - \beta H_{A_1A_2} > 0$ *then* $H_{A_1A_2} > 0$ *and* $H_{A_2A_1} - \beta H_{A_1A_1} < 0$

Proof: We have $H_{A_2A_2} < 0$ and $H_{A_2A_2} - \beta H_{A_1A_2} > 0 \Rightarrow H_{A_1A_2} > 0$. Also,

$$\begin{vmatrix} H_{A_1A_1} & H_{A_1A_2} \\ H_{A_2A_1} - \beta H_{A_1A_1} & H_{A_2A_2} - \beta H_{A_1A_2} \end{vmatrix} = \begin{vmatrix} H_{A_1A_1} & H_{A_1A_2} \\ H_{A_2A_1} & H_{A_2A_2} \end{vmatrix} = \Delta > 0$$

$$\Rightarrow H_{A_1A_1}\left(H_{A_2A_2} - \beta H_{A_1A_2}\right) - H_{A_1A_2}\left(H_{A_2A_1} - \beta H_{A_1A_1}\right) > 0 \tag{A.7}$$

Now, as $H_{A_1A_1} > 0$, (A.7) will be true iff $H_{A_2A_1} - \beta H_{A_1A_1} < 0$.

1.2. If $H_{A_2A_1} - \beta H_{A_1A_1} < 0$ *then* $H_{A_2A_1} > 0$ *and* $H_{A_2A_2} - \beta H_{A_1A_2} < 0$

Proof: Similar to lemma 1.1.

1.3. If $H_{A_2A_1} = H_{A_1A_2} < 0$ *then* $H_{A_2A_2} - \beta H_{A_1A_2} < 0$ *and* $H_{A_2A_1} - \beta H_{A_1A_1} < 0$

Proof: It can be proved using lemma 1.1 and 1.2.

Lemma 2. When $\beta > 0$

2.1. If $H_{A_2A_2} - \beta H_{A_1A_2} > 0$ *then* $H_{A_1A_2} > 0$ *and* $H_{A_2A_1} - \beta H_{A_1A_1} < 0$

2.2. If $H_{A_2A_1} - \beta H_{A_1A_1} < 0$ *then* $H_{A_2A_1} > 0$ *and* $H_{A_2A_2} - \beta H_{A_1A_2} < 0$

2.3. If $H_{A_2A_1} = H_{A_1A_2} < 0$ *then* $H_{A_2A_2} - \beta H_{A_1A_2} < 0$ *and* $H_{A_2A_1} - \beta H_{A_1A_1} < 0$

Proof: Similar to lemma 1.

Proof of theorem 1 and 2: From lemma 1 and lemma 2, we can easily prove these theorems.

B. Proof of theorem of Special Functional Form

$$\alpha = \lambda g_{1A_1} + \mu g_{2A_1} = \lambda \frac{a_1^{/}}{A_1}\left(\overline{N}_1 - x_1\right)\left(1-\gamma^2\right) + \mu \frac{a_1^{/}}{A_1}\left(\overline{N}_1 - x_1\right)\gamma^2 = \frac{a_1^{/}}{A_1}\left(\overline{N}_1 - x_1\right)\left[\left(1-\gamma^2\right)\lambda + \gamma^2\mu\right] > 0 \tag{B.1}$$

and, $\beta = \dfrac{\lambda g_{1A_2} + \mu g_{2A_2}}{\lambda g_{1A_1} + \mu g_{2A_1}}$

Now, $\lambda g_{1A_2} + \mu g_{2A_2} = \lambda\left[- f_1\left(\overline{N}_1 - x_1\right)\gamma_{A_2}\right] + \mu\left[f_{2A_2}\left\{\overline{x}_2 - x_2\right\}\right] + \mu\left[f_1\left(\overline{x}_1 - x_1\right)\gamma_{A_2}\right]$

$$= f_1\left(\overline{x}_1 - x_1\right)\frac{a_2^{/}}{A_2 f_2}\gamma(1-\gamma)(\mu-\lambda) + \mu\frac{a_2^{/}}{A_2}\left[\overline{x}_2 - x_2\right] = \mu\frac{a_2^{/}}{A_2}\left[\left\{\overline{x}_2 - x_2\right\} + \frac{f_1 a_2^{/}}{A_2 f_2}\left(\overline{x}_1 - x_1\right)\gamma(1-\gamma)\left(1-\frac{\lambda}{\mu}\right)\right] \tag{B.2}$$

$$\therefore \beta = \begin{cases} positive & when \quad \lambda \le \mu \\ negative & when \quad \lambda >> \mu \end{cases} \tag{B.3}$$

Proof of theorem 3: Theorem 3 can be proved by using condition (B.3) along with lemma 1 and lemma 2.

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