where $K \in E^n$ is a vector of constants to be determined. The maximum principle for fixed-end-point problems can be restated as (2.32) with $x(T) = \alpha$ and without $\lambda(T) = S_x[x^*(T), T]$. The resulting two-point boundary value problem has initial and final values on the state variables, whereas both initial and terminal values for the adjoint variables are unspecified, i.e., $\lambda(0)$ and $\lambda(T)$ are constants to be determined.

In Exercises 2.9 and 2.18, you are asked to solve the fixed-end-point problems given there.

2.5 Solving a TPBVP by Using Spreadsheet Software

A number of examples and exercises found in the rest of this book involve finding a numerical solution to a two-point boundary value problem (TPBVP). In this section we shall show how the GOAL SEEK function in the EXCEL spreadsheet software can be used for this purpose. We will solve the following example.

Example 2.7 Consider the problem:

$$\max\left\{J = \int_0^1 -\frac{1}{2}(x^2 + u^2)dt\right\}$$

subject to

$$\dot{x} = -x^3 + u, \; x(0) = 5. \tag{2.76}$$

Solution. We form the Hamiltonian

$$H = -\frac{1}{2}(x^2 + u^2) + \lambda(-x^3 + u),$$

where the adjoint variable λ satisfies the equation

$$\dot{\lambda} = x + 3x^2\lambda, \; \lambda(1) = 0. \tag{2.77}$$

Since u is unconstrained, we set $H_u = 0$ to obtain $u^* = \lambda$. With this, the state equation (2.76) becomes

$$\dot{x} = -x^3 + \lambda, \; x(0) = 5. \tag{2.78}$$

Thus, the TPBVP is given by the system of equations (2.77) and (2.78).

In order to solve these equations we discretize them by replacing dx/dt and $d\lambda/dt$ by

$$\frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \text{ and } \frac{\Delta \lambda}{\Delta t} = \frac{\lambda(t + \Delta t) - \lambda(t)}{\Delta t},$$

respectively. Substitution of $\Delta x / \Delta t$ for $\dot{x}$ in (2.78) and $\Delta \lambda / \Delta t$ for $\dot{\lambda}$ in (2.77) gives the discrete version of the TPBVP:

$$x(t + \Delta t) = x(t) + [-x(t)^3 + \lambda(t)] \Delta t, \; x(0) = 5, \tag{2.79}$$

$$\lambda(t + \Delta t) = \lambda(t) + [x(t) + 3x(t)^2 \lambda(t)] \Delta t, \; \lambda(1) = 0. \tag{2.80}$$

In order to solve these equations, open an empty spreadsheet, choose the unit of time to be $\Delta t = 0.01$, make a guess for the initial value $\lambda(0)$ to be, say -0.2, and make the entries in the cells of the spreadsheet as specified below:

Enter $-$ 0.2 in cell A1.
Enter 5 in cell B1.
Enter = A1 + (B1 + 3 * (B1^2) * A1) * 0.01 in cell A2.
Enter = B1 + (-B1^3 + A1) * 0.01 in cell B2.

Note that $\lambda(0) = -0.2$ shown as the entry -0.2 in cell A1 is merely a guess. The correct value will be determined by the use of the GOAL SEEK function.

Next blacken cells A2 and B2 and drag the combination down to row 101 of the spreadsheet. Using EDIT in the menu bar, select FILL DOWN. Thus, EXCEL will solve equations (2.79) and (2.80) numerically from $t = 0$ to $t = 1$ in steps of $\Delta t = 0.01$, and that solution will appear as entries in columns A and B of the spreadsheet. In other words, the guessed solution for $\lambda(t)$ will appear in cells A1 to A101 and the guessed solution for $x(t)$ will appear in cells B1 to B101. In order to find the correct value for $\lambda(0)$, use the GOAL SEEK function under TOOLS in the menu bar and make the following entries:

Set cell: A101.
To value: 0.
By changing cell: A1.

It finds the correct initial value for the adjoint variable as $\lambda(0) = -0.10437$, which should appear in cell A1, and the correct ending value of the state variable as $x(1) = 0.62395$, which should appear in cell B101. You will notice that the entry in cell A101 may not be exactly zero as instructed, although it will be very close to it. In our example, it is -0.0007. By using the CHART function, the graphs of $x(t)$ and $\lambda(t)$ can be printed out by EXCEL as shown in Figure 2.6.

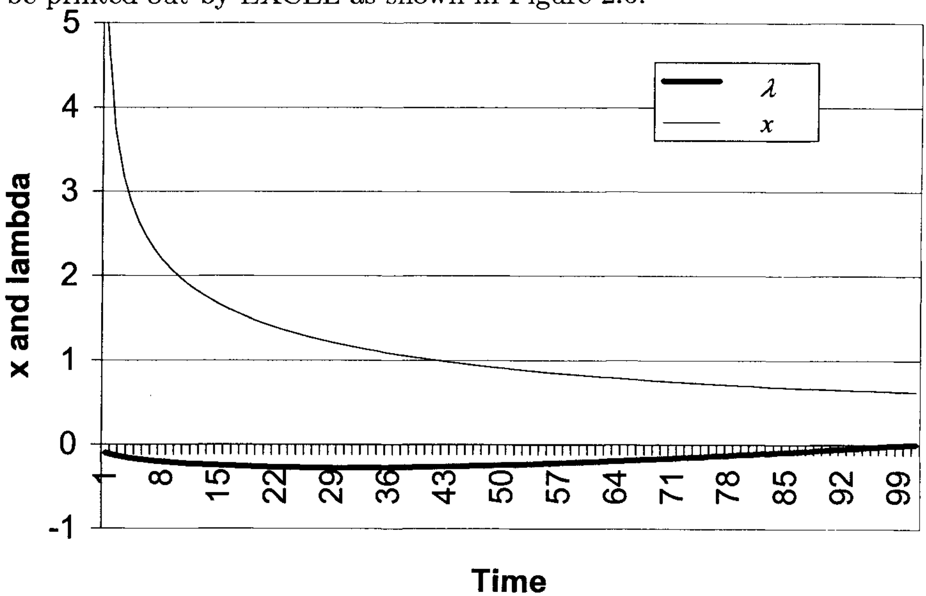

Figure 2.6: Solution of TPBVP by EXCEL

EXERCISES FOR CHAPTER 2

2.1 (a) In Example 2.1, show $J^* = -\frac{1}{2}$.

(b) In Example 2.2, show $J^* = 0$.

(c) In Example 2.3, show $J^* = -\frac{1}{6}$.

(d) In Example 2.4, show $J^* = -\frac{1}{6}$.

2.2 Rework Example 2.5 with $F = 2x - 3u$.

2.3 Show that both the Lagrange and Mayer forms of the optimal control problem can be reduced to the linear Mayer form (2.5).

2.4 Complete Example 2.5 by writing the optimal $x^*(t)$ in the form of integrals over the three intervals $(0, t_1), (t_1, t_2)$, and $(t_2, 2)$ shown in Figure 2.5.
[Hint: It is not necessary to actually carry out the numerical evaluation of these integrals unless you are ambitious.]