

Graduate Texts in Physics

Reiner M. Dreizler
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Theoretical Mechanics

Theoretical Physics 1

First Edition



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Theoretical Mechanics

Theoretical Physics 1

 Springer

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Additional material to this book can be downloaded from <http://extras.springer.com>
Password: 978-3-642-11137-2

ISSN 1868-4513 e-ISSN 1868-4521
ISBN 978-3-642-11137-2 e-ISBN 978-3-642-11138-9
DOI 10.1007/978-3-642-11138-9
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2010936710

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PREFACE

This series of texts on Theoretical Physics is based on lecture notes of courses given at the Goethe-University in Frankfurt am Main. It will finally contain five volumes covering the subjects

- Theoretical Mechanics
- Electrodynamics and Special Theory of Relativity
- Quantum Mechanics (two volumes)
- Thermodynamics and Statistical Physics.

Mechanics is the basic field of physics. This subject deals with the physical aspects of nature which are more directly accessible to observation. It provides for this reason a solid foundation of most of the concepts used in the description of nature. Theoretical Mechanics is also the area of physics which was the first to be developed and brought to a conclusion on a high mathematical level. The beginning of the development in the 16th and the 17th centuries is characterised by a more systematic approach to the collection of experimental data and the desire to unravel the basic principles active in nature. The experimental side can be documented with the observations of Tycho de Brahe and of Johann Kepler. As examples of the more theoretical efforts the writings of Galilei Galileo and of Isaac Newton may be quoted. These endeavours prepared the ground for the development of more advanced mathematical methods as the infinitesimal and the variational calculus (in particular by the Bernoulli brothers as well as L. Euler and G. Leibniz), which led to a rapid development and formalisation of mechanics. This development reached a final form with the work of J. d'Alembert, J. Comte de Lagrange and Sir W.R. Hamilton at the end of the 18th and the beginning of the 19th centuries.

The first volume (and additional volumes) are organised as follows: the main text is supported by three substantial 'appendices'

- a Mathematical Supplement,
- a Collection of 71 Problems for private study
and
- a set of Auxilliary Remarks on selected points,

which can be accessed/downloaded at the location

<http://extras.springer.com>

via the ISBN of this book.

The 270 page Mathematical Supplement represents our credo that a very close connection of the physical and mathematical material is necessary for a successful understanding of theoretical physics. This was implemented in the original lectures. In a textbook it seemed preferable to separate the mathematical tools from the main text. The mathematical material is presented in a form that is adapted to the needs of physics students. At all relevant points of the main text appropriate references to the chapters and sections of the mathematical supplement are indicated by ● Math.Chap. x.y .

The theory lectures at our university start in the first term. This requires an adiabatic transition to the 'higher' subjects of mechanics. The resulting, slower introduction to theoretical mechanics proper has been retained in the book for didactic reasons and as a means to support an independent start of studies. The first chapter of the book provides a general overview of the areas of physics. The second chapter introduces, in a simpler fashion, the basic kinematic concepts of mechanics. The discussion of the basic conservation laws in chapter 3 also uses initially elementary means but closes with a full application of methods of vector analysis. Starting with the fourth chapter the presentation refrains from compromises with respect to mathematical necessities. On the other hand it should be mentioned that the ever present differential equations are already introduced in the second chapter and are extensively used thereafter.

A Collection of Problems, covering the material of the main text, can also be found on the server. They are marked by the statement ● Probl. x.y at the relevant positions of the main text. This collection covers the material of chapters 2 to 6. It has been prepared in a way that utilises the electronic medium fully so that it can be used for private studies. The core of each problem is a set of consecutive questions and answers which guide the reader step by step towards the solution(s). For those not interested in this guidance, a condensed summary of the solution of each problem and the steps towards it is offered. The electronic medium makes it possible to present more vivid animation or presentation of the material. Also included are JAVA applets, which are to be used as theoretical laboratories. Students are invited to investigate simpler or more advanced problems of motion by variation of the relevant parameters.

The third 'appendix' offers some aid via a limited number of auxiliary calculations (rather than the statement 'after a short calculation one obtains ...') as well as some additional comments. The relevant passages of the text are highlighted by ● D.tail x.y .

We thank our families for patience, understanding and support during this project. We also thank the team of the Springer Verlag in Heidelberg for the friendly cooperation and technical support.

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1 A First Survey

Mechanics deals with 'ordinary objects' moving at 'moderate velocities' (as for example colliding steel balls or planets orbiting around the sun). Elementary particle physics, a topic that will only be discussed very briefly in this introductory course, addresses the properties of minute particles moving in general at much higher velocities. These statements suggest that the areas, which are of interest in physics, can be fitted into a diagram which is characterised by a length and a velocity scale¹ (Fig. 1.1). The concept of length (L) has to be interpreted relatively loosely. As a length one should consider the size of objects (e.g. the diameter of elementary particles, of atoms or of the steel balls and planets mentioned above) as well as the specification of wavelengths (e.g. of water waves, sound waves or electromagnetic waves) or the distances between celestial objects.

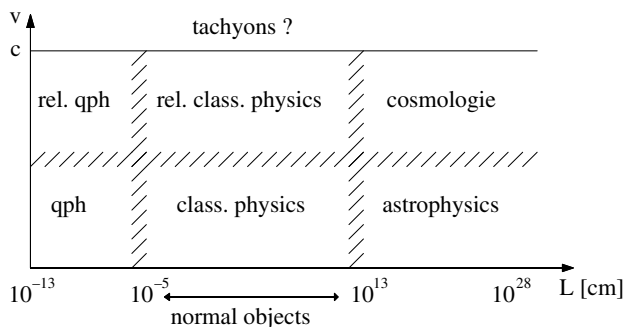


Fig. 1.1. The basic diagram

The length-scale that is needed is quite large. It begins at 10^{-13} cm (the ten billionth - in American usage: the ten trillionth - fraction of a cm). This length corresponds approximately to the diameter of a proton, one of the building blocks of atomic nuclei. It represents the smallest distance that has been measured explicitly so far. The scale ends at 10^{28} cm. This distance represents the present estimate of the diameter of the universe. Within these

¹ following Reference [1]

boundaries all the objects that are addressed in physics can be found. For a subdivision of the scale (in order to cover the enormous range a logarithmic scale should preferably be used) the following examples could be quoted:

- 10^{-12} cm \longrightarrow diameter of atomic nuclei
- 10^{-8} cm \longrightarrow diameter of atoms
- 10^{-6} cm \longrightarrow diameter of large molecules
- 10^{-5} cm \longrightarrow resolution of the best optical microscopes,
corresponding to the wavelength of visible light
- 10^{-1} cm \longrightarrow grain of sand
- 10^2 cm \longrightarrow man (as a measure of all things)
- 10^9 cm \longrightarrow diameter of the earth
- 10^{13} cm \longrightarrow the distance sun to earth
- 10^{17} cm \longrightarrow the distance earth to the next star (α - Centauri).

The range between 10^{-5} to 10^{13} cm (approximately) is the range of the 'ordinary objects' of mechanics.

The velocity (v) scale starts at zero. It ends (by necessity, as is believed on the basis of the special theory of relativity) with the speed of light. This velocity is

$$\begin{aligned} c &= (2.997925 \pm 0.000001) \cdot 10^{10} \text{ cm/s} \\ &\approx 3 \cdot 10^{10} \text{ cm/s} = 300000 \text{ km/s} \\ &= 1.08 \cdot 10^9 \text{ km/h.} \end{aligned}$$

By comparison with the velocity of light the velocity of most objects is rather small

$$\begin{aligned} 300 \text{ km/h} &\approx 3 \cdot 10^{-7} c \\ 10000 \text{ km/h} &\approx 10^{-5} c \\ 10^5 \text{ km/h} &\approx v_{\text{earth around sun}} \approx 10^{-4} c. \end{aligned}$$

If a linear scale is used to represent velocities, it is found that the three examples indicated are in close vicinity to the abscissa of the diagram. Material objects with velocities close to the velocity of light can be found in cosmic radiation. In the upper layers of the atmosphere, particles (e.g. muons characterised by the Greek letter μ) are produced with velocities that come close to the speed of light ($v_{\mu} \approx 0.995 c$). Classical mechanics (with ordinary objects at moderate velocities) is restricted to a small section of the basic diagram (Fig. 1.1) in the vicinity of the abscissa.

Objects in the range below 10^{-5} cm (with a gradual transition at the region of the boundary) have to be discussed in terms of quantum mechanics, a subject that dominated the development of physics in the first half of the 20th century. The fact that atoms are the building blocks of matter was established around 1850 with the recognition of the periodic system of elements. Atoms still constitute the elementary building blocks in chemical reactions. The fact that atoms consist of a nucleus and a cloud of electrons

was recognised in 1913. The relevant experiment was carried out by Geiger, Marsden and Rutherford in Cambridge. In this experiment helium nuclei (α -particles), which were available via the natural radioactivity of polonium, were directed onto a thin gold foil. The significant observation was a strong

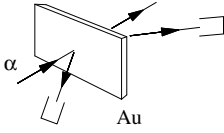
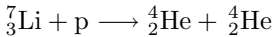


Fig. 1.2. Scattering of α -particles from a gold foil

backscattering of the α -particles. This is only possible if the α -particles hit a 'solid object'. The strength of the backscattering even allowed an estimate of the size of this object, the gold nucleus.

The era of nuclear physics experiments with accelerators started in 1932 with an investigation of the reaction



by Cockroft and Walton. When protons impinge on lithium, two helium nuclei can be produced. The investigation of a large number of reactions involving nuclei revealed that

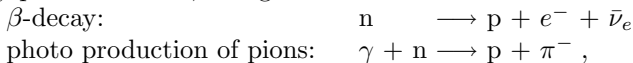
$$\left. \begin{array}{ll} \text{neutrons (n), protons (p)} & \longrightarrow \text{nucleus} \\ \text{electrons (e}^-\text{)} & \longrightarrow \text{electron cloud} \end{array} \right\} \text{atom}$$

could be considered as the basic building blocks of nature. A fourth elementary particle, the photon (γ) - that is the quantum of light - was known at this time and could be included, even though it plays a special role as mediator of the electromagnetic interaction.

With the steady improvement of accelerator techniques and by analysis of cosmic radiation the family of elementary particles has been extended. Up to 1950 the list also contained (in sequence of their discovery)

- the antineutrino ($\bar{\nu}_e$), which is associated with the electron (although it was first conjectured rather than discovered),
- the positron (e^+), the antiparticle of the electron,
- positively and negatively charged muons (μ^\pm), close but more massive relatives of the electron and the positron,
- the family of pions ($\pi^{\pm,0}$),
- the kaon or K meson (K) and the Lambda particle (Λ).

By the year 1960 the number of elementary particles had grown to more than 300. As it was found that a large number of processes involving elementary particles exists, as e.g.



it became obvious that all these elementary particles are, in fact, not elementary after all. They seem to be composed of more fundamental objects.

The decisive hint came from symmetry considerations (a popular game in theoretical physics), in this case the properties of the symmetry group² SU(3). Around 1970 the elementary particles were classified as

exchange particles (gauge particles)	: γ
leptons (light particles)	: $e^-, \nu_e, \mu^-, \nu_\mu$ + antiparticles
hadrons (strongly interacting particles)	: π, n, p, \dots + antiparticles.

The family of hadrons could be divided into mesons (particles of medium mass, as e.g. the pion), and baryons (heavy particles, as e.g. the proton and the neutron). On the basis of symmetry considerations the hadrons could be arranged in so-called SU(3)-multiplets. There exists, for example, an octet (indicated in Fig. 1.3), which contains the neutron and the proton as well as the hyperons Lambda (Λ), Sigma ($\Sigma^{0,\mp}$) and the cascade particles ($\Xi^{0,-}$).

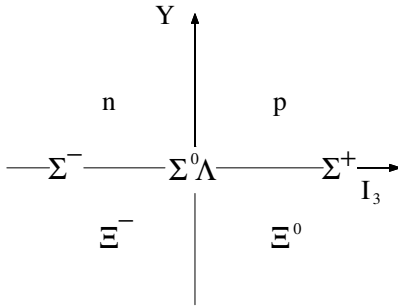


Fig. 1.3. The relatives of the proton and the neutron in the SU(3) classification of elementary particles

This subfamily of elementary particles is characterised by the fact that all members have comparable mass and additional similarities (not specified here). They can be arranged into a specific pattern based on the so-called hypercharge (Y) and isospin (I_3). It was soon realised that the smallest possible family of the SU(3) classification scheme (with three members) did not seem to exist. On the other hand, the larger families (multiplets) could be constructed, using the concepts of group theory, from the basic triplet. For this reason attempts were undertaken to discover these three elementary particles in nature. Their properties could be inferred, to a certain degree, from the multiplet structures. They were named the three quarks (following a quotation from *Finnegans Wake* by James Joyce). Less poetically they were later distinguished by the names of up, down and strange. So far, quarks have

² Three elementary texts on the theory of symmetry groups are indicated in the list of references [2].

not been observed directly. There is, however, sufficient indication of their existence, for example from electron-proton collisions at high energies.

In the meantime (that is since about 1985) the list of elementary particles contains the following entries:

gauge particles	: $\gamma, W^\pm, Z^0, g_1, g_2, \dots, g_8$
leptons	: $e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau$ + antiparticles
quarks	: q_1, q_2, \dots, q_{18} + antiparticles.

- The gauge particles transmit the various basic interactions between leptons and quarks. The photon is responsible for the electromagnetic interaction. The gauge bosons, which have been named W and Z , transmit the weak interaction and the 8 gluons generate the strong interaction between the quarks.
- In addition to electron and muon a heavier lepton, the tauon, was discovered in 1974. Each lepton is associated with a corresponding neutrino and for each of the six particles there exists an antiparticle.
- The total number of quarks is 18. They are characterised by so-called 'inner quantum numbers': the 'flavour' (6 flavours with the names up, down, strange, charm, top and bottom) as well as a kind of charge, the 'colour' (3 colours with arbitrary names, usually red, green and blue, referred to as RGB).

Group theory demands a specific pattern for the composition of hadrons from the basic units (the quarks): mesons, e.g. pions or the K mesons, consist of a quark-antiquark pair ($q\bar{q}$); baryons, for instance neutrons or protons, contain three quarks (qqq). This pattern is realised in nature exactly as demanded by group theory.

Quite a number of aspects of the scenario described have not been explained so far. Nonetheless, the question can be raised: What are quarks composed of? To date any answer to this question is rather speculative. On the other hand, one point is certain. As experiments have to deal with the resolution of very small spatial domains, an experimental answer to such questions will turn out to be expensive. Loosely speaking, the cost is proportional to the amount of energy needed, which is in turn inversely proportional to the size of the domain to be resolved.

This first excursion into the world of quantum physics has introduced:

elementary particle, nuclear and atomic physics.

In addition to these fields the following could be considered:

molecular and solid state physics.

In all these branches of physics effects are encountered which cannot be explained in terms of everyday (classical) experience. The following introductory observations underline this statement:

(a) A classical particle can be at rest at the lowest point of a 'valley'. It is not possible to observe a quantum particle which is at rest at the bottom of a 'valley' (if the term 'valley' is replaced by the term 'potential well'). A quantum particle will oscillate, in a rather uncontrolled manner, around the deepest point (Fig. 1.4). This is a consequence of the (Heisenberg) uncertainty principle, an inherent property of quantum systems.



Fig. 1.4. Uncertainty principle: classical versus quantum particle

(b) If a classical particle moves around in a closed box with a velocity that is not sufficient to penetrate through the walls, it will obviously stay within the box (Fig. 1.5). A quantum particle, by contrast, can penetrate through the 'walls' even if this is not possible on the basis of energy considerations. One example for the occurrence of such tunnel processes in nature is the observation of the natural radioactivity of nuclei, as in the case of the α -decay of polonium mentioned above.



Fig. 1.5. The tunnel effect

Relativistic physics is the proper frame for the discussion of situations in which larger velocities play a role. The basic scenario is quite simple. Two 'observers' (experimenters) move with respect to each other with a constant, relative velocity (Fig. 1.6). Observer 1 could for example be on a train, while Observer 2 is stationed on the embankment. They observe the same experiment, for example, the motion of an object that is thrown up and comes down due to gravity. The question which has to be addressed is the following. Can a transformation law be formulated that allows the calculation of the trajectory as seen by Observer 2, if the relative velocity v_0 and the trajectory registered by Observer 1 are known? It would then be possible to transcribe the results of Observer 1 by analytical means, as indicated in the following example. Observer 1 stands on a moving platform-car, throws an object ver-

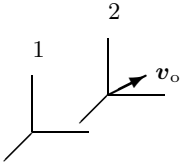


Fig. 1.6. Two coordinate systems in relative motion

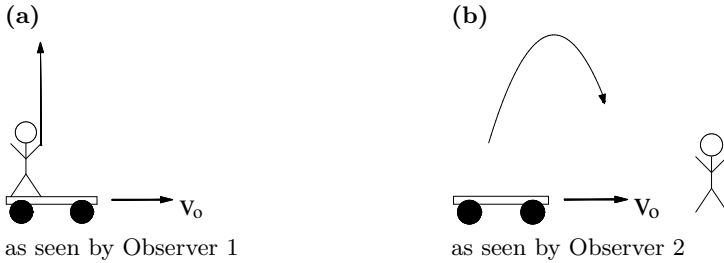


Fig. 1.7. Projectile movement

tically (from his point of view) into the air, finds that it moves vertically up to the highest point and returns along the same trajectory. Observer 2, who follows the experiment from the embankment, registers, as required by the transformation law, that the trajectory of the object is a parabola (Fig. 1.7).

A transformation law that answers the question, the Galilei transformation, had been known long before Einstein. Nonetheless, Einstein raised the question once more (in 1905), but with an additional twist: what is the form of the transformation law if it is assumed that the speed of light has the same value c for *all* observers who move uniformly with respect to each other? From the point of view of everyday experience this assumption sounds absurd. Everybody knows that velocities are added. If someone moves with respect to the ground with velocity v_0 and sets an object in motion (in the same direction) with the velocity v , then the velocity of the object with respect to the ground is $v + v_0$ (Fig. 1.8).

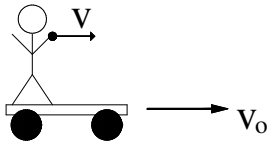


Fig. 1.8. Addition of velocities

This conjecture that Einstein used in the derivation of a new transformation law emerged from a classical experiment which was performed by Michelson and Morley in Cleveland in 1880. The results obtained in 1880

have since been confirmed by a large number of more refined experiments. The experimental statement is: if the velocity v equals the speed of light, then the sum of the two velocities is c and not $c + v_0$.

The derivation of the corresponding transformation laws is simple enough for the case of a constant relative velocity of the two observers. Only basic concepts of mathematics are needed. The resulting 'special theory of relativity' is for this reason readily understood from a mathematical point of view. The consequences of Einstein's transformation laws are, however, far reaching. They have changed the conception of space and time (two basic concepts of physics) in a dramatic fashion. For example, the law leads to the statement, that the mass of an object changes with its speed v according to the relation

$$m(v) = \frac{m(v=0)}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

This formula seems to contradict normal experience. The contradiction arises, however, from the fact that the change of the mass is too small to be detected, even for rather large classical velocities. The mass of a rocket moving at $v = 10^{-5}c \approx 10000$ km/h is increased to (by expansion with the binomial formula³)

$$\begin{aligned} m(10^{-5}c) &\approx m(0)\left(1 + 0.5\left(\frac{v}{c}\right)^2 + \dots\right) \\ &= m(0)(1 + 0.5 \cdot 10^{-10}) \\ &= m(0)(1.000000000005). \end{aligned}$$

This increase in mass cannot be detected even in the most refined experiment. For an object with $v = 0.8c$ one would, on the other hand, find

$$m(0.8c) \approx 1.67 \cdot m(0),$$

that is, nearly a doubling of the mass of the object at rest. The curve $m(v)$ as a function of v is indicated in Fig. 1.9. The region, for which the change of the mass cannot be detected, is called the classical region. The steep rise of the curve for $v \rightarrow c$ explains why the velocity of light is considered to be a natural limit of all velocities. A mass would attain the limiting value $m \rightarrow \infty$ for $v \rightarrow c$. This implies that this limit can not be reached.

Classical mechanics as well as quantum mechanics have to be modified for situations with large velocities. The ramifications of relativistic mechanics are not readily accessible in an earth bound laboratory. In order to observe relativistic effects for macroscopic objects one has to use a laboratory that includes outer space.

³ See ● Math.Chap. 1.3 in the Mathematical Supplement concerning series expansions.

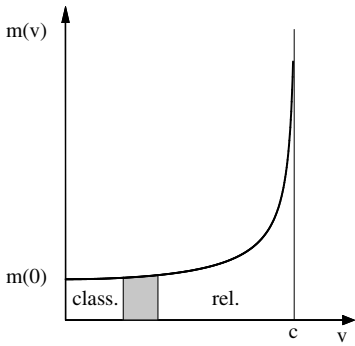


Fig. 1.9. The relativistic variation of the mass of an object with velocity

There is another aspect to be considered. The assumption of a uniform relative velocity leads to the special theory of relativity. For a discussion of the more general situation, frames of reference that are accelerated with respect to each other, the general theory of relativity is the proper tool. It involves, for example, the statement that the acceleration due to gravity and all other accelerations are fully equivalent. This statement is verified by the observation and the interpretation of the advance of the perihelion of the planet Mercury.

The domain of large dimensions is addressed in astrophysics. One of the problems that inhibited astrophysics for many years was the restriction enforced by earth bound observation platforms. In the meantime a variety of satellite missions such as

COBE **CO**smic **B**ackground **E**xplorer for the exploration of the cosmic background radiation

HUBBLE for images of very distant objects

ROSAT **RO**entgen **SAT**ellite for the discovery and survey of X-ray sources

SOHO **SO**lar **H**eliospheric **O**bservatory for the measurement of data relating to the sun

and many others have collected a large amount of data concerning the universe.

As early as 1929 the American astronomer E. Hubble interpreted the red shift of the spectral lines of stars as a Doppler shift. The Doppler shift is well known in acoustics. The sound of the horn of a car or the whistle of a train becomes deeper if the car or train moves away from a stationary observer. The associated increase of the wavelength of sound corresponds to a red shift of the (optical) spectral lines of distant stars. Hubble's observation thus leads to the conjecture of the expansion of the universe. One can imagine that the earth is a point on a balloon that is blown up. All other points on the surface of the balloon recede from this particular point. The expansion of the universe

has, however, only been observed for a very limited period of time. With the extrapolation to larger periods, thousands or millions of years, quite a number of scenarios are possible. It could be that the size of the universe oscillates, so that the present phase of expansion is followed by a phase of contraction. It could also be possible that the universe was originally concentrated in a small region with extremely high matter and energy density and that a 'Big Bang' is the reason for the present expansion.

An answer to the question, how the universe will develop, could possibly be gleaned from the general theory of relativity. According to this theory there exists a critical mass density ρ_c (mass/volume) of the universe. If the actual density is lower than the critical one ($\rho < \rho_c$), then the attractive force of gravity is not sufficient to stop the expansion. On the other hand, if the mass density is larger than the critical value ($\rho > \rho_c$), then gravitation will finally lead to a contraction of the universe. The value of the critical mass density is not known very well. The estimate is

$$\rho_c = (0.3...1.9) \cdot 10^{-29} \text{ g/cm}^3 .$$

The density due to the sum of all visible objects in the universe amounts only to a fraction of the critical density

$$\frac{\rho_c}{200} < \rho_{\text{visible}} < \frac{\rho_c}{100} .$$

It is known, on the other hand (for example, via its gravitational action on visible objects), that invisible matter (known as dark matter) is distributed in the universe. The questions what kind of matter this is or which fraction of the total matter it constitutes can at present not be answered. We do not yet know in which fashion the universe will develop in the (distant) future.

The Big Bang scenario is popular as it combines the present knowledge of the world of small and of large distances. Efforts to prove the correctness of this scenario have been (and are being) undertaken. In the initial state (however it was created) matter is supposed to be so densely packed and hot that composite elementary particles, the hadrons, cannot exist. The original matter consisted of a kind of 'primeval soup', which contains only quarks and gluons, the quark-gluon plasma. A number of experiments involving the collision of heavy nuclei at high impact energies have been undertaken in the hope of finding hints of the quark-gluon plasma in the reaction products. So far the results are not fully conclusive.

When the expanding elementary matter cools down, baryons and mesons will freeze out, leading to the most stable hadrons, the nucleons. These can, via chains of fusion reactions, combine to form the lightest elements and finally the distribution of elements observed at present.

The brief outline of classical physics and the world of quantum phenomena presented here does not cover all fields of physics nor all boundary regions of the basic diagram. For historical reasons electrodynamics (electric and magnetic fields of stationary or moving charge distributions) and thermodynamics (response of matter to changes in temperature and pressure) are

counted as fields of classical physics. As the velocity of light plays a special role in electrodynamics (the emphasis is on 'dynamics' e.g. the generation of electromagnetic waves such as light), it is understandable that the formulation of the theory of relativity was initiated by questions raised in the development of electrodynamics. In a similar fashion, attempts to interpret some of the phenomena of thermodynamics on the basis of the motion of atoms can be regarded as first steps towards quantum mechanics. This statement applies, in particular, to the discovery of the quantum aspects of the black body radiation.

An example for the discussion of areas beyond the boundary set by the velocity of light is the concept of tachyons (a term which translates into 'more than fast particles'), even if they are not as well founded on literature as the quarks. The equations of the theory of relativity permit solutions which would represent a new kind of particle with rather exotic properties. While all the known particles can only have velocities that are smaller than the velocity of light, tachyons can only exist if their velocity is larger than the velocity of light. No specific hint for the existence of tachyons has been discovered so far. This does not contradict the theory of relativity, as not all possible solutions of equations, which can be formulated, need be realised in nature.

2 Kinematics

The basic questions of mechanics are:

1. What causes the motion of objects?
2. How can the motion of objects be described in mathematical terms?

The second question will be answered below, though in a somewhat preliminary fashion. The third chapter will provide an answer to the first question. Examples and more sophisticated methods for the discussion of the motion of mechanical objects follow in the subsequent chapters.

Of all the possible forms of motion, the motion along a straight line is the most simple. This type of motion is an ideal topic for an introduction into the world of mechanics. It will be used to illustrate the transition from a mathematical formulation to an idea of what actually takes place in nature. It is also used to define and to discuss the basic kinematic concepts, that is position, velocity and acceleration. Dynamical aspects, as the determination of trajectories of objects once the acceleration is specified, will be indicated but not discussed fully at this stage.

The three dimensional world is approached step by step via a discussion of trajectories in two space dimensions. Experiments suggest that motion in a plane can be described by a superposition of two independent one dimensional motions. This allows the characterisation of position, velocity and acceleration by vectors with two components. The variety and the complexity that can be obtained in this fashion is illustrated with selected examples.

The discussion of motion in three space dimensions (as e.g. the motion of an object along a helix) relies on the obvious extension. Position, velocity and acceleration are described by vectors with three components. The use of vectors is ideally suited for formal aspects, the discussion of details demands, nonetheless, the use of Cartesian or more general (as e.g. spherical) coordinates.

2.1 One-dimensional motion

A task, which has to be faced in physics, is the transition from experimental observation to a suitable mathematical description and, vice versa, that is the interpretation of a set of formulae in terms of explicit details of the physical

process. This task is illustrated in the next section with three examples for the motion in one space dimension.

2.1.1 Three examples for the motion in one space dimension

The following simple 'experiment' can be performed without elaborate equipment. An object (e.g. a small steel ball) falls under the influence of gravity (Fig. 2.1). The position of the object is recorded as a function of time t (di-

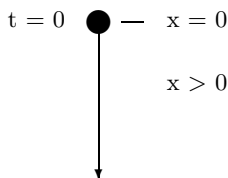


Fig. 2.1. Free fall from rest

mension [T], measured e.g. in seconds [s]), using $t = 0$ to indicate the start of the experiment. The position x (dimension: length [L], e.g. in centimetres [cm]) is read off a scale, that is orientated downwards, with the origin ($x = 0$) at the starting position. The result of the simple free fall experiment could be summarised in a table, recording the distance the object has fallen as a function of the time

t [s]	0	0.1	0.2	0.3	0.4	0.5	0.6
x [cm]	0	5	20	45	80	125	180

The next step is the extraction of more general information from such tables. In the present case, it is useful to construct an x versus t diagram (Fig. 2.2a) for this purpose. If this is done, it could be noticed, that all the points measured lie on a parabola which is described by the function

$$x(t) = a t^2 \quad \text{with} \quad a = 500 \frac{\text{cm}}{\text{s}^2} . \quad (2.1)$$

Notwithstanding the fact, that the outcome of the experiment is well known, the following remarks should be added:

1. The transition from a set of isolated points to a formula of the kind $x(t)$ has to be regarded with some caution. In principle, one would need an infinitely dense set of points, a requirement that would have stopped progress in physics with the first experiment. It is a question of appropriate judgement when the transition from a series of measurements to an explicit formula can be made. One has to keep in mind though, that some interesting phenomena in physics were discovered, when a measurement was repeated with a finer distribution of points.

2. The steel ball that was used in the fictitious experiment is an object with a spatial extension. The expression 'position of the steel ball' therefore needs some comment. In the experiments just described, 'position' could stand for the position of the centre of gravity or the geometric centre of the sphere. The substitution of a point for an extended object is, however, only useful if other forms of motion of the object (as e.g. a rotation of the object during the free fall) do not take place or are not of interest. If the reduction of an extended, massive object to a point is appropriate, the concept of a **mass point**, alternatively referred to as **point particle**, is used. A mass point is an abstraction from reality. It is an object without any spatial extension that carries a mass m (dimension $[M]$, measured e.g. in grams $[g]$).
3. The choice of the origin of a coordinate system to mark the initial position and of the time $t = 0$ for the starting time of the experiment is rather arbitrary. If one moves the initial position to the point x_0 and uses the initial time t_0 , then the outcome of the experiment would be summarised as

$$x(t) = x_0 + 500(t - t_0)^2$$

or

$$x(t) - x_0 = 500(t - t_0)^2 .$$

The last equation and Fig. 2.2b indicate that only length and time intervals matter.

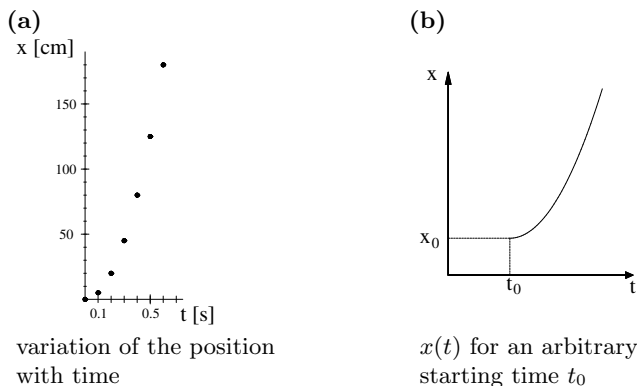


Fig. 2.2. The free fall experiment ($x = x(t)$)

The conclusion to be drawn from these introductory remarks is: the mathematical description of motion in one space dimension relies on the discussion of functions $x(t)$. Two additional examples will be used to illustrate how an

explicit idea of the time development of the motion can be extracted from such equations.

A well used example is the harmonic oscillator which is characterised by the function

$$x(t) = A \sin \omega t \quad A, \omega > 0. \quad (2.2)$$

A and ω are constants. The harmonic oscillator can be found in practically all fields of physics: in mechanics, in electrodynamics or in applications of quantum theory in nuclear and in solid state physics. The story behind this function is best visualised if one uses an x versus ωt diagram. The angle ωt can be measured either in degrees (a full circle corresponds to 360°) or in radians (a full circle equals 2π).

The sine function (2.2), represented in Fig. 2.3, describes the oscillation of a mass point about the equilibrium position $x = 0$, starting at this position at time $t = 0$. The mass point moves first in the direction of positive x -values. At the time $t = \pi/(2\omega)$ it is found, for the first time, at the maximal displacement A from the origin. It returns, at time $t = \pi/\omega$, to the

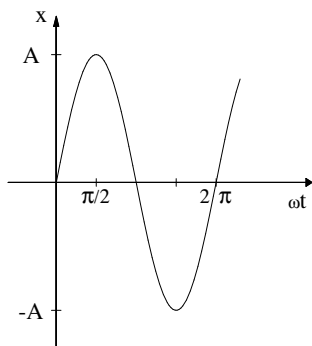


Fig. 2.3. The function $x(\omega t)$ for the harmonic oscillator problem

equilibrium position, moves through it and reaches at time $t = 3\pi/(2\omega)$ its largest negative displacement. At the time $t = 2\pi/\omega$ it has returned to the origin. This pattern is repeated any number of times. For the discussion of the harmonic oscillator the following terms are used:

1. The magnitude of the largest displacement A is the amplitude (dimension $[L]$).
2. The quantity ω is the circular (or angular) frequency with the dimension $[1/T]$.
3. The time for one full oscillation $T = 2\pi/\omega$ is the period.
4. The inverse of the period is the frequency $f = 1/T$ (dimension $[1/T]$).
Frequencies are measured in the units oscillations/second = $s^{-1} \equiv \text{Hertz}$.

The question, what kind of motion of a mass point is characterised by the function

$$x(t) = \frac{g}{k}t - \frac{g}{k^2}(1 - e^{-kt}) , \quad (2.3)$$

is perhaps not answered that quickly (any suggestions?). The constants g and k with the dimensions

$$[k] = [\text{T}^{-1}] \quad [g] = [\text{L}/\text{T}^2]$$

have to be specified, e stands for the well known transcendental number (Euler's number) $e = 2.71828 \dots$.

Knowledge of some properties of the exponential function e^{-kt} is required (representative values can be obtained e.g. with a pocket calculator) for a discussion of the formula (2.3). The exponential function is plotted in Fig. 2.4a. It features (for positive values of kt) a sharp drop from the value 1 at $kt = 0$ to the value zero for large values of kt . The last statement corresponds to

$$\lim_{kt \rightarrow \infty} e^{-kt} = 0$$

in more mathematical terms. A more precise statement on the behaviour of the function for small values of the variable kt is given by the series expansion of the exponential function

$$\begin{aligned} e^{-kt} &= 1 - kt + \frac{1}{2}k^2t^2 - \frac{1}{6}k^3t^3 + \dots (-1)^n \frac{k^n t^n}{n!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{k^n t^n}{n!} . \end{aligned}$$

The series expansion of the exponential function is discussed more fully in section 1.3.1 of the Mathematical Supplement (● Math.Chap. 1.3.1). Chapter 1 of the supplement contains (a rather compact) summary of the analysis of functions of one variable. Basic knowledge of differentiation and integration techniques is assumed.

The two statements on the exponential function are sufficient for a rough discussion of the motion described by the function (2.3). The mass point starts at the time $t = 0$ at the position $x(0) = 0$. For small times (more accurately, small values of kt) the series expansion yields

$$x(t) \xrightarrow{kt \text{ small}} \frac{g}{k}t - \frac{g}{k^2}(1 - 1 + kt - \frac{1}{2}k^2t^2 + \frac{1}{6}k^3t^3 + \dots) .$$

Terms independent of and linear in the time variable cancel, so that the relation

$$x(t) \approx \frac{1}{2}gt^2 - \frac{1}{6}(gk)t^3 + \dots \quad (kt \text{ small}) \quad (2.4)$$

remains. This shows that the motion is initially governed by a quadratic function of time (as in the free fall). For larger times terms proportional to t^3 and higher orders have to be taken into account. For much larger times (large values of kt) the asymptotic behaviour of the exponential function leads to the limiting value

$$x(t) \approx \frac{g}{k}t - \frac{g}{k^2} \quad (kt \text{ large}) . \quad (2.5)$$

On the basis of this information a sketch of the function $x(t)$ in (2.3) can be attempted. The expression (2.4) describes (if one considers only the first term) a parabola. The large time limit (2.5) represents a straight line with the slope g/k and the intercept $-g/k^2$. The actual motion is characterised by the parabola for small values of kt and by the straight line for large values. For intermediate values the motion is described by a curve that interpolates between these limits. The initial free fall motion turns into a uniform motion.

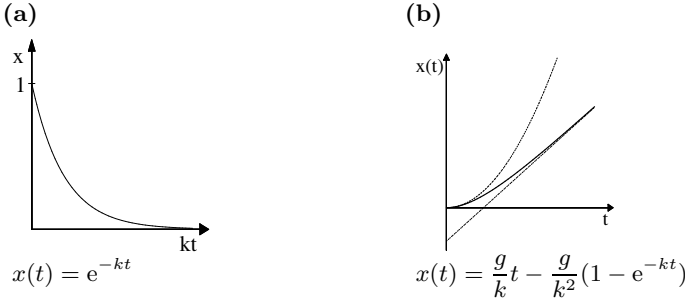


Fig. 2.4. The functions of the third example

The function (2.3) describes the free fall of objects (mass points) if frictional effects (in a simplified approximation associated with the name of Stokes, see Chap. 4.2.2) are taken into account. The resistance due to friction leads to a slowing down of the free fall motion. This statement can be inferred from Fig. 2.4b. For a given interval in the variable x , the interval in the variable t is smaller for the parabola than for the correct curve. This shows that free fall motion, described by the parabola, is faster. The strength of the friction is expressed by the magnitude of the constant k . This constant depends on the medium in which the object falls (e.g. air or a more dense medium as water) as well as on the geometric shape of the object. It should be kept in mind though, that (2.3) only refers to the motion of the centre of mass of the object.

Two limits for the effect of friction can be discussed directly:

- The limit $k \rightarrow 0$. The notation $x(k, t)$ is more appropriate here, as the motion is now discussed as a function of t as well as k . The limiting value

$$\lim_{k \rightarrow 0} x(k, t)$$

follows from (2.4). As each term, with the exception of the first one, contains positive powers of k , which vanish in the limit, one finds

$$\lim_{k \rightarrow 0} x(k, t) = \frac{1}{2}gt^2 .$$

In words: the ideal free fall motion is recovered, if the friction is turned off.

- The limit $k \rightarrow \infty$. Expression (2.5) indicates that the description of the motion deviates from the parabola sooner for larger values of k . For larger k one finds that the slope of the straight line becomes smaller and smaller. The motion becomes slower and slower with increasing k , so that

$$\lim_{k \rightarrow \infty} x(k, t) = 0$$

is obtained in the limit $k \rightarrow \infty$. In words: the object does not move at all. It stays forever at the starting point. The limit $k \rightarrow \infty$ corresponds to an infinitely viscous medium.

The discussion of the three examples for the description of motion in one spatial dimension dealt only with the transcription of the mathematical formulation into a more direct picture of the actual process, demonstrating that a good deal of information can be found in simple formulae. Dynamical aspects, as a discussion of the question what causes the motion of objects, will be outlined in Chap. 3.1. The examples are now used for an introduction of the basic concepts of kinematics, that is velocity and acceleration.

2.1.2 Velocity

The definition of velocity is fashioned after the measuring process (Fig. 2.5a). The position of a mass point at time t_1 is measured as $x(t_1)$ and as $x(t_2)$ at the later time t_2 . The quotient

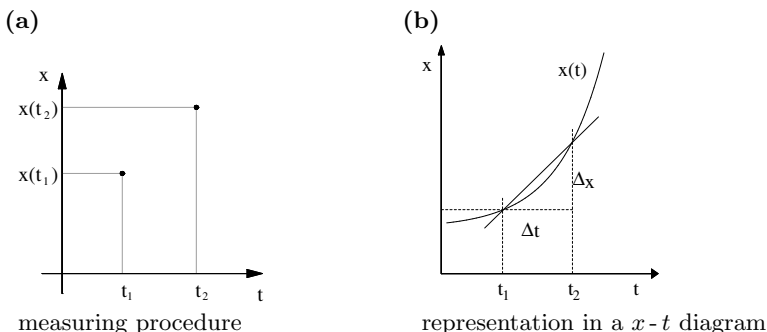


Fig. 2.5. The definition of the average velocity

$$\frac{x(t_2) - x(t_1)}{t_2 - t_1} = \bar{v}(t_1, t_2)$$

represents the **average velocity** (mean velocity) in the time interval $[t_1, t_2]$. The illustration of this equation in the $x-t$ diagram (Fig. 2.5b) shows: the

average velocity represents the slope (more precisely the tangent of the angle of the slope) of the secant through the points (t_1, x_1) and (t_2, x_2)

$$\bar{v} = \frac{\Delta x}{\Delta t} = \tan(\alpha_{\text{sec}}) .$$

The average velocity does not characterise the motion in a sufficient manner. Every curve $x(t)$ through the two points has the same \bar{v} in the interval $[t_1, t_2]$.

For a more detailed picture one needs the concept of the **instantaneous velocity**, which corresponds to the first derivative of the function $x(t)$

$$v(t) = \lim_{\Delta t \rightarrow 0} \left[\frac{x(t + \Delta t) - x(t)}{\Delta t} \right] . \quad (2.6)$$

Alternatively one writes

$$v(t) = \frac{dx(t)}{dt} = \dot{x}(t) = x'(t) .$$

The instantaneous velocity is in general also a function of time, it changes from point to point. This quantity can be interpreted in a simple manner in the $x-t$ diagram: The slope of the secant turns into the slope of the tangent line at the point (t, x) (Fig. 2.6).

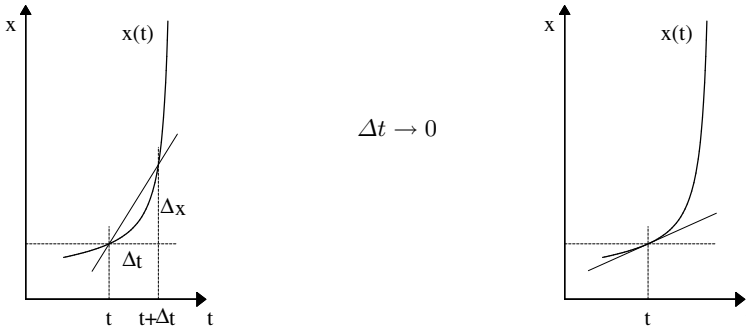


Fig. 2.6. Definition of the instantaneous velocity

The definition indicates, that the instantaneous velocity can not be measured directly. It may be approximated with ever increasing precision, if arbitrarily small time intervals and distances can be measured. The naive limit is, however, always the indefinite expression

$$v_{\text{exp}}(\Delta t \rightarrow 0) \longrightarrow \frac{0}{0} .$$

The instantaneous velocities for the three examples are easily calculated.

- Free fall (in a more general formulation)

$$x(t) = \frac{1}{2}gt^2 \quad v(t) = gt .$$

- Harmonic oscillator

$$x(t) = A \sin \omega t \quad v(t) = A\omega \cos \omega t .$$

- Free fall with friction

$$x(t) = \frac{g}{k}t - \frac{g}{k^2}(1 - e^{-kt})$$

$$v(t) = \frac{g}{k} - \frac{g}{k^2}(-(-k)e^{-kt}) = \frac{g}{k}(1 - e^{-kt}) .$$

In addition to the characterisation of the motion, which can be gained from a discussion of the position $x(t)$, further information can be obtained by a discussion of $v(t)$.

- There is not much to say for the free fall. The velocity increases linearly with time.
- For the case of the harmonic oscillator, it is useful to compare the two functions $x(t)$ and $v(t)$ (Fig. 2.7). The velocity is zero at the turning point

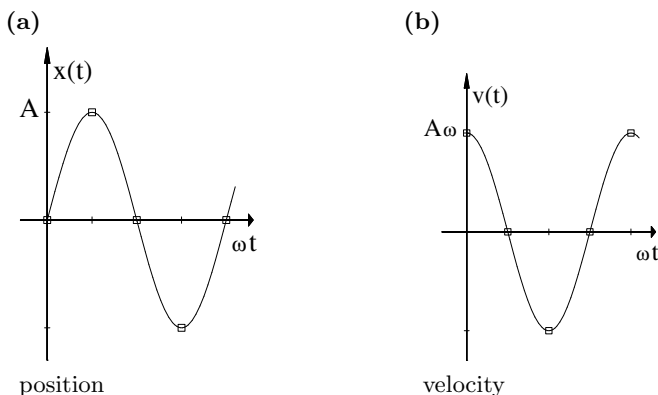


Fig. 2.7. The functions $x(t)$ and $v(t)$ for the harmonic oscillator problem

of the mass. It has a maximum if the object passes through the equilibrium points. The significance of the sign of v can also be recognised. A positive velocity describes (referring to the figure) an upward motion, a negative velocity a downward motion.

- The velocity for the free fall with friction is best discussed in the same fashion as the function $x(t)$ with the aid of the limiting situations. If kt is sufficiently small, so that the series for the exponential function can be restricted to quadratic terms, one finds

$$v(t) = \frac{g}{k} \left(1 - 1 + kt - \frac{1}{2}k^2t^2 + \dots \right) = gt - \frac{1}{2}gkt^2 + \dots .$$

If kt is sufficiently large, the result is

$$v(t) \rightarrow \frac{g}{k}.$$

The velocity increases at first linearly with time, then grows more slowly. For large times a constant final velocity is obtained as a consequence of friction. The final value of the velocity is smaller for a larger frictional constant k (Fig. 2.8).

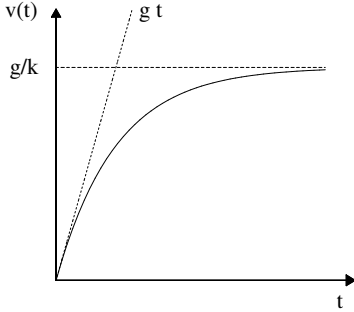


Fig. 2.8. The function $v(t)$ for the free fall with friction

2.1.3 Acceleration

The (instantaneous) velocity is a direct measure of the change of the position with time. The acceleration is a measure of the change of the velocity with time. The corresponding definitions are:

average (mean) acceleration in the interval $[t_1, t_2]$

$$\bar{a}(t_1, t_2) = \frac{v(t_2) - v(t_1)}{t_2 - t_1},$$

instantaneous acceleration at time t

$$a(t) = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \dot{v}(t). \quad (2.7)$$

The instantaneous acceleration is obtained by differentiation of the function $v(t)$, so that there is nothing more to be said from a technical point of view. The derivative of the first derivative is the second derivative with the usual notation

$$a(t) = \frac{d^2x}{dt^2} = \ddot{x}(t).$$

The instantaneous acceleration for the three examples can easily be calculated:

- Free fall: $a(t) = g$. Near the surface of the earth the acceleration due to gravity is constant. The value

$$g = 981 \text{ cm/s}^2 = 9.81 \text{ m/s}^2$$

will be derived in Chap. 3.2.4.1 from the general law of gravitation. The actual value found in a local experiment depends, however, on the geographical latitude, the altitude above sea level and the geological environment.

- The acceleration for the harmonic oscillator $a(t) = -A\omega^2 \sin \omega t$ is once again described by a sine curve, but with a negative sign (Fig. 2.9). This

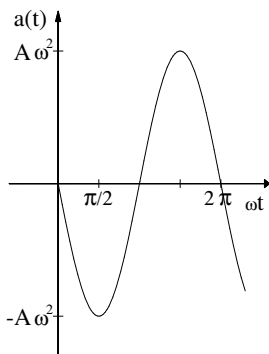


Fig. 2.9. The acceleration $a(t)$ for the harmonic oscillator problem

means that the acceleration is directed against the motion in the first quarter cycle, in the second quarter it produces the return to the equilibrium position, etc. The equation for the acceleration of the harmonic oscillator can also be written in the form

$$a(t) = -\omega^2 x(t) .$$

This indicates that the acceleration is at all times proportional but opposite to the instantaneous displacement. This statement is the characteristic feature of the harmonic oscillator. The last equation can be written in the form

$$\ddot{x}(t) = \frac{d^2 x(t)}{dt^2} = -\omega^2 x(t) . \quad (2.8)$$

This example illustrates for the first time the mathematical problem that has to be addressed in Theoretical Mechanics. The relation (2.8) represents a **differential equation**, that is an equation for the determination of the function $x(t)$ which involves derivatives. The right hand side, the acceleration, has a simple form in this example. According to the standard classification, the differential equation of the harmonic oscillator problem is a homogeneous, linear differential equation of second order with constant coefficients.

Differential equations are an ever present feature of theoretical physics. A first overview and methods of solution for the simplest ordinary differential equations are presented in **Math.Chap. 2**. The discussion of differential equations is continued in **Math.Chap. 6**.

- For the last example one finds

$$a(t) = ge^{-kt} .$$

The acceleration starts at the value g for $t = 0$, but drops (quickly) to the value zero because of friction (Fig. 2.10). In the same fashion as the acceleration approaches the value zero, the velocity becomes constant. Further

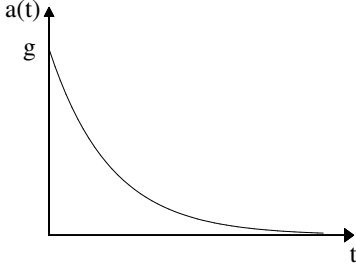


Fig. 2.10. The function $a(t)$ for the free fall with friction

insight is gained by rewriting the result for the acceleration in a simple fashion

$$\begin{aligned} a(t) &= g + ge^{-kt} - g \\ &= g + k\frac{g}{k}(e^{-kt} - 1) \\ &= g - kv(t) . \end{aligned} \tag{2.9}$$

The interpretation of this rearrangement is: the acceleration is composed of two parts. The first part corresponds to a free fall motion. The second term is the contribution due to friction. The law of Stokes, which states that friction is directly proportional to the velocity, can be recognised here. The negative sign signifies that friction acts against the instantaneous direction of motion¹.

There remains the question, whether it is necessary to consider higher order derivatives as e.g. the change of acceleration with time

$$\frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3x}{dt^3} \quad ?$$

The answer to this question is given by Newton's second axiom of mechanics. It implies that a consideration of higher order derivatives is not necessary. This point will be discussed in Chap. 3.

Three (simple) examples for the description of the motion of a mass point have been presented in the present section. Starting with a discussion of the

¹ Alternative forms for the description of frictional effects can be found in the literature.

time change of the position, further characterisation of the motion calls for the calculation and analysis of

the instantaneous velocity $v(t) = dx(t)/dt$

and the instantaneous acceleration $a(t) = dv(t)/dt$.

It should be kept in mind though, that the problem which has to be faced eventually, is exactly the reverse

Start with a specification of the acceleration a . Use this specification to calculate the velocity $v(t)$ and the position $x(t)$ as functions of time.

Given : $a \longrightarrow$ Calculate : $v(t), x(t)$.

2.1.4 First remarks concerning dynamical aspects

The solution of the problem just indicated is usually more difficult than the application of the rules of differentiation, which are sufficient for the calculation of $v(t)$ and $a(t)$ if $x(t)$ is given. For the determination of $x(t)$ and $v(t)$, given the acceleration a , several options, which will be addressed briefly at this stage, can be distinguished.

In the simplest case, a is given as a function of time only: $a = a(t)$. For the calculation of $v(t)$ and $x(t)$ integration (as the inverse operation to differentiation) is required

$$\frac{dv}{dt} = a(t) \longrightarrow v(t) = \int_{t_0}^t a(t') dt' + v(t_0)$$

$$\frac{dx}{dt} = v(t) \longrightarrow x(t) = \int_{t_0}^t v(t') dt' + x(t_0).$$

The second entry in each line is the inverse of the first. The first term on the right hand side is a definite integral from the initial time t_0 to the final time t . In addition to the integrals two integration constants $v(t_0)$ and $x(t_0)$ appear. As the integrals vanish for $t = t_0$, the fact that the starting velocity and the starting position can be specified, has to be taken care of. A free fall experiment could, for instance, begin at any height with an additional velocity in the x - direction (see • Math.Chap. 2.1 for statements on initial value problems).

For the free fall with $a(t) = g$ the velocity is found to be

$$v(t) = \int_{t_0}^t g dt' + v(t_0) = g(t - t_0) + v(t_0).$$

The first term corresponds to the rectangle under the curve $a(t) = g$ between the points t_0 and t (Fig. 2.11a). The result for the position is

$$x(t) = \int_{t_0}^t (g(t' - t_0) + v(t_0)) dt' + x(t_0) \\ = \left(\frac{1}{2} g (t - t_0)^2 + v(t_0)(t - t_0) \right) + x(t_0).$$

The term in the large brackets represents the area under the curve $v(t)$, which is composed (Fig. 2.11b) of a rectangle with the sides $v(t_0)$ and $(t - t_0)$ and a right angle triangle with the legs $g(t - t_0)$ and $(t - t_0)$.

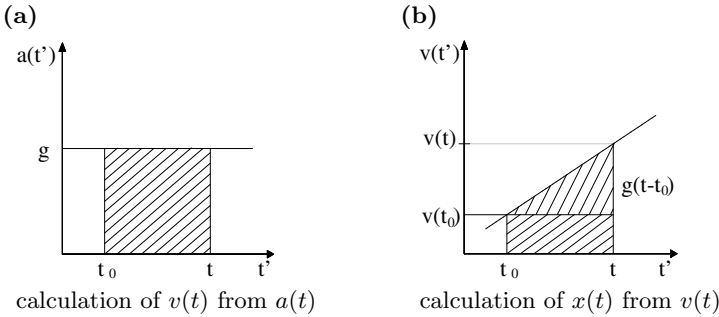


Fig. 2.11. Integration of the equation of motion for the free fall

The second case, the acceleration is given as a function of position, $a = a(x)$, is quite common in applications. The harmonic oscillator problem with

$$a(x) = -\omega^2 x$$

is a relevant, but simple example. Compared with the previous case, determination of $x(t)$ and $v(t)$ is slightly more involved as direct integration is not possible. The differential equation for the unknown function $x(t)$ reads

$$\frac{d^2 x}{dt^2} = -\omega^2 x \quad \text{or} \quad \ddot{x} = -\omega^2 x$$

for the harmonic oscillator and

$$\frac{d^2 x}{dt^2} = a(x) \quad \text{or} \quad \ddot{x} = a(x)$$

in more general situations. The question, that is posed by the differential equation for the harmonic oscillator, can be formulated as: for which function is the second derivative, up to a negative factor, identical with the function itself? The solution (already known) of this differential equation can be obtained with standard techniques (methods for the solution of differential

equations of this type are outlined in ● Math.Chap. 6.1). One should, however, be aware of the fact that the solutions of the related differential equation

$$\ddot{x} = +\omega^2 x$$

(with a positive instead of a negative constant) are exponential and not trigonometric functions.

In the last case, which can again be handled in a simple way, the acceleration is given as a function of velocity alone: $a = a(v)$. The free fall with friction with $a = g - kv$ is a possible example. The corresponding differential equation is best written in the form

$$\frac{dv}{dt} = g - kv$$

or in the general case as

$$\frac{dv}{dt} = a(v) .$$

This type of differential equation can be solved by direct integration

$$\int_{v_0}^v \frac{dv'}{a(v')} = \int_{t_0}^t dt' .$$

Technical details are found in ● Math.Chap. 2.2.1.

The three types of differential equations presented are special cases of differential equations which characterise the motion in one space dimension. In the general case, the acceleration is specified as a function of time, as well as position and velocity $a = a(v, x, t)$. The differential equation

$$\frac{d^2x}{dt^2} = a\left(\frac{dx}{dt}, x, t\right)$$

is an explicit differential equation of second order. The solution of this type of differential equation demands reasonable mathematical skills.

A selection of classes of differential equations, which can be solved analytically, is presented in ● Math.Chap. 2 and in ● Math.Chap. 6. ● Math.Chap. 6.4 contains an outline of numerical methods which have to be used in many instances.

2.2 Problems of motion in two or three dimensions

A free fall experiment can also serve as an introduction to the discussion of motion in more than one space dimension: an object (mass point) does not start from rest, but is given an initial velocity in the horizontal direction. The motion of this object is observed and compared with the motion of a second object, which starts *at the same time and height* with zero initial velocity.

A comparison of the motion of the two objects shows (Fig. 2.12a), that they are always at the same height. This observation is independent of the

value of the initial horizontal velocity of the first object. The conclusion, that can be extracted from this observation, is: the motion in the horizontal direction is independent of the motion in the vertical. Motion in two dimensions can be discussed by a combination of two independent one-dimensional components. The free fall experiment in two dimensions is characterised by a uniformly accelerated motion in the vertical and a uniform motion in the horizontal direction. Denoting the horizontal by y and the vertical (downwards as before) by x , it follows that

$$\begin{aligned} x(t) &= \frac{1}{2}gt^2 & v_x(t) &= gt & a_x(t) &= g, \\ y(t) &= v_{y0}t & v_y(t) &= v_{y0} & a_y(t) &= 0. \end{aligned}$$

Each of the kinematic quantities (position, velocity, acceleration) is characterised by a pair of equations. If an orthogonal coordinate system with an arbitrary orientation of the x -, y - and z - directions is chosen, the position of the object (as a function of time) would have to be described by projection

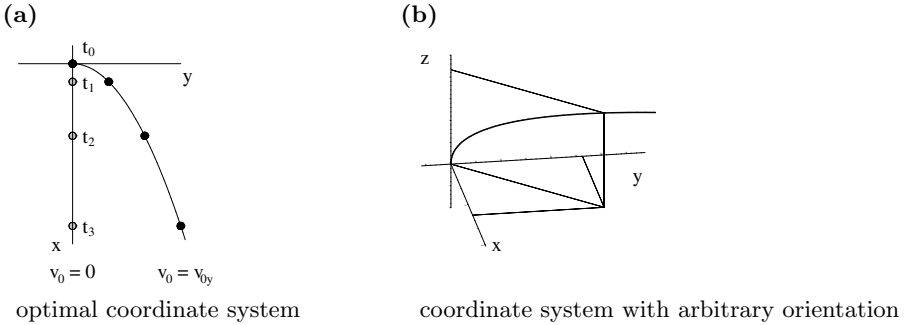


Fig. 2.12. Free projectile motion

onto three coordinate axes. In the three-dimensional world a triple of functions is needed for the characterisation of each of the kinematic quantities

$$\text{position :} \quad (x(t), y(t), z(t)) \quad (2.10)$$

$$\text{velocity :} \quad \left(v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}, v_z = \frac{dz}{dt} \right) \quad (2.11)$$

$$\text{acceleration :} \quad \left(a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt}, a_z = \frac{dv_z}{dt} \right). \quad (2.12)$$

The triples can be handled most efficiently with the concept of **vectors**. Before introducing vector notation it is, however, useful to sample the possibilities that can be generated by a superposition of two one-dimensional motions.

Vector calculus is summarised in ● Math.Chap. 3, which also contains an overview of additional topics of 'Linear Algebra', as matrices, determinants and linear coordinate transformations, as well as some remarks on linear vector spaces and nonorthogonal coordinate systems.

2.2.1 Two-dimensional motion

For the free fall experiment in two dimensions a set of equations of the form $\{x(t), y(t)\}$ is used to characterise the position. This is called a **parametric representation of the trajectory** of the object. The time development of the motion can be followed explicitly (Fig. 2.13), if the points $\{x(t), y(t)\}$ are entered in an x - y diagram.

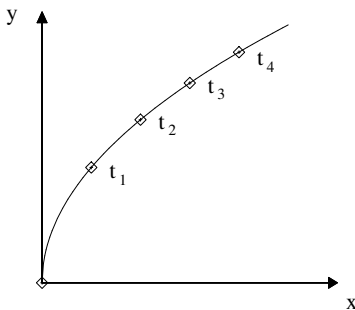


Fig. 2.13. Parametric representation of a trajectory (two-dimensional)

The **equation of the trajectory** itself is obtained by elimination of the time variable from the set of equations

$$x = x(t) \quad y = y(t) .$$

For the simple projectile motion this equation

$$x = \frac{g}{2v_{y0}^2} y^2 \quad \text{or} \quad y = \pm \sqrt{\left(\frac{2v_{y0}^2 x}{g} \right)}$$

is the equation of a parabola, or more precisely, the equation of two branches of a parabola. The equation for the trajectory itself does not contain any information on the time development of the motion.

A large variety of trajectories can be generated by a superposition of harmonic oscillations in two orthogonal directions. These curves are named **Lissajous figures** after the French physicist Jules Lissajous.

2.2.1.1 Superposition of harmonic oscillations. The oscillations in the direction of the x - and the y -coordinates have in general different amplitudes (A), phases (ϕ) and (angular) frequencies (ω)

$$x(t) = A_x \sin(\omega_x t + \phi_x) \quad (2.13)$$

$$y(t) = A_y \sin(\omega_y t + \phi_y) . \quad (2.14)$$

The amplitudes and the phases express the initial positions, e.g. for $t = 0$

$$x(0) = A_x \sin \phi_x \quad v_x(0) = \omega_x A_x \cos \phi_x$$

$$y(0) = A_y \sin \phi_y \quad v_y(0) = \omega_y A_y \cos \phi_y ,$$

the frequencies control the speed of the individual oscillations. It can be demonstrated that only the difference of the phases is relevant, if the coordinate system is chosen appropriately. For this reason one of the phases can, without loss of generality, be set equal to zero. The choice $\phi_y = 0$ yields $y(0) = 0$. The initial position of the oscillating mass point is on the x -axis for this choice of the coordinate system. For simplicity one then writes $\phi_x \equiv \phi$.

The two frequencies are supposed to be equal in the first explicit example: $\omega_x = \omega_y = \omega$. In order to derive the equation of the trajectory one resolves (2.14) in the form $\sin \omega t = y(t)/A_y$ and finds

$$\cos \omega t = \begin{cases} +\frac{1}{A_y} \sqrt{A_y^2 - y(t)^2} & \text{for } -\frac{\pi}{2} \leq \omega t \leq \frac{\pi}{2} \quad \text{mod}(2\pi) \\ -\frac{1}{A_y} \sqrt{A_y^2 - y(t)^2} & \text{for } +\frac{\pi}{2} \leq \omega t \leq \frac{3\pi}{2} \quad \text{mod}(2\pi) . \end{cases} \quad (2.15)$$

The equation for the motion in the x -direction (2.13) can, with the aid of the sum formula of the trigonometric functions, be written as

$$x(t) = A_x (\cos \phi \sin \omega t + \sin \phi \cos \omega t) ,$$

so that the expressions for $\sin \omega t$ and $\cos \omega t$ lead to the relation

$$x = \frac{A_x}{A_y} \left(y \cos \phi \pm \sin \phi \sqrt{A_y^2 - y^2} \right) . \quad (2.16)$$

A general discussion of this equation for the trajectory is possible, but the possibilities are best sampled by looking at some special cases.

Case 1: The two individual oscillations are in phase for $\phi = 0$

$$x(t) = A_x \sin \omega t \quad y(t) = A_y \sin \omega t .$$

The equation for the trajectory, $y = (A_y/A_x)x$, represents a straight line, due to the restricted range of values of the trigonometric functions actually only a section of a straight line (Fig. 2.14a). The superposition of the two linear oscillations yields, in this case, a linear oscillation with the same frequency ω . The mass point oscillates in a direction that is given by $\tan \alpha = (A_y/A_x)$, the maximal displacement from the origin of the coordinate system is $[A_x^2 + A_y^2]^{1/2}$.

Case 2: The parametric representation of the trajectory for the phase difference $\phi = \pm \pi$ is

$$x(t) = -A_x \sin \omega t \quad y(t) = A_y \sin \omega t ,$$

the equation of the trajectory $y = -(A_y/A_x)x$. The result is once more a linear harmonic oscillation, this time along a section of a straight line in the second and fourth quadrant (Fig. 2.14b).

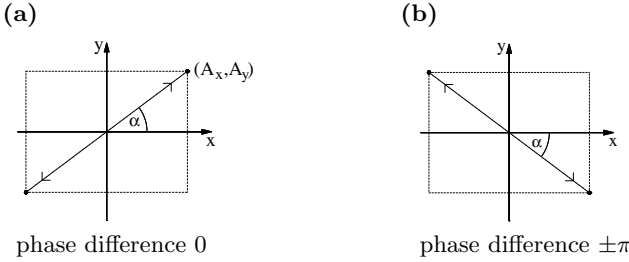


Fig. 2.14. Trajectories of the two-dimensional harmonic oscillator with $\omega_x = \omega_y$, $A_x = A_y$

Case 3: The equation of the trajectory, which follows from

$$x(t) = A \cos \omega t \quad y(t) = A \sin \omega t$$

for the case of equal amplitudes ($A_x = A_y = A$) and a phase difference of $\phi = \pi/2$, is the equation of a circle with radius A

$$x = \pm \sqrt{A^2 - y^2} \quad \text{or} \quad x^2 + y^2 = A^2 .$$

The parametric representation contains the information that the mass point begins at position $(x(0), y(0)) = (A, 0)$ at time $t = 0$ and moves anticlockwise on the circle. The position is $(x(\pi/2\omega), y(\pi/2\omega)) = (0, A)$ at the time $\omega t = \pi/2$. The motion is uniform (Fig. 2.15a), a full revolution is completed in the time $T = 2\pi/\omega$.

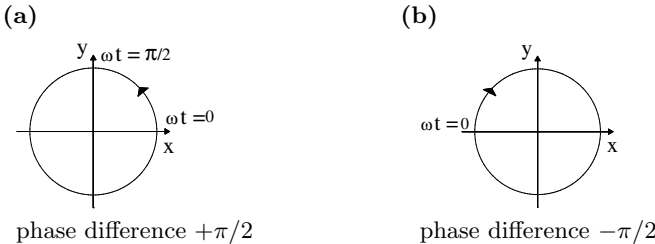


Fig. 2.15. Trajectories of the two-dimensional harmonic oscillator with $\omega_x = \omega_y$, $A_x = A_y$

Case 4: The amplitudes are still equal but the phase difference is $\phi = -(\pi/2)$. The x -coordinate changes its sign to $x(t) = -A \cos \omega t$ in this case. The trajectory is still a circle given by $x^2 + y^2 = A^2$, only the mass point starts this time at the position $(x(0), y(0)) = (-A, 0)$ and traverses the circle uniformly in a clockwise direction (Fig. 2.15b).

Case 5: The parametric representation is

$$x(t) = A_x \cos \omega t \quad y(t) = A_y \sin \omega t$$

for the phase difference $\phi = \pi/2$ and different amplitudes ($A_x \neq A_y$), so that the equation of the trajectory is

$$x = \pm \frac{A_x}{A_y} \sqrt{A_y^2 - y^2} \quad \text{or} \quad \frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1.$$

This trajectory is an ellipse (Fig. 2.16a), which is traversed in the same sense as the corresponding circle starting at the position $(A_x, 0)$. The time for a complete revolution is still $T = (2\pi)/\omega$.

Elimination of the square root in (2.16) yields a more general equation for the trajectory in the case of equal frequencies

$$A_y^2 x^2 - 2A_x A_y x y \cos \phi + A_x^2 y^2 = A_x^2 A_y^2 (\sin \phi)^2. \quad (2.17)$$

It is not difficult (● D.tail 2.1) to establish the fact that the trajectories for $\omega_x = \omega_y$ are in general ellipses, though they are not necessarily oriented along the coordinate axes. Ellipses are also found if the amplitudes are equal $A_x = A_y$, but the phase difference ϕ has no special value (Fig. 2.16b). The circles and the sections of straight lines can be considered as limiting cases of the family of ellipses.

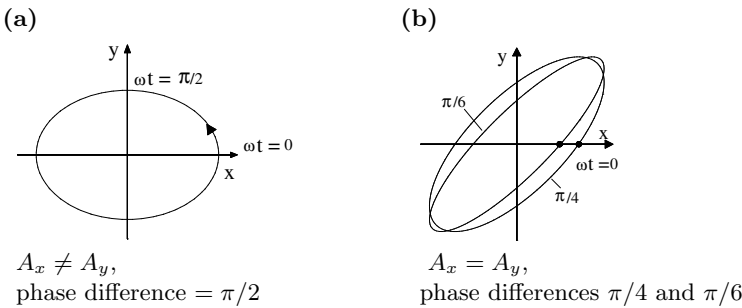


Fig. 2.16. Trajectories of the two-dimensional harmonic oscillator with $\omega_x = \omega_y$

The trajectories become definitely more complex, if the two frequencies are different ($\omega_x \neq \omega_y$). A complete proof of this statement involves a rather lengthy discussion, which is abbreviated by two examples.

In the first example the phase difference is $\phi = \pi/2$, the amplitudes are equal in magnitude ($A_x = A_y = A$), but the frequency of the oscillation in the x -direction is twice as large as the frequency in the y -direction

$$\omega_x = 2\omega, \quad \omega_y = \omega.$$

Elimination of the time from the parametric representation

$$\begin{aligned} x(t) &= A \sin(2\omega t + \pi/2) = A \cos 2\omega t \\ &= A(\cos^2 \omega t - \sin^2 \omega t) \\ y(t) &= A \sin \omega t \end{aligned}$$

relies once more on the relation (2.15)

$$\sin \omega t = \frac{y}{A} \quad \cos \omega t = \pm \sqrt{1 - \frac{y^2}{A^2}}.$$

Insertion into the representation of the x -coordinate yields

$$x = A \left(1 - \frac{y^2}{A^2} - \frac{y^2}{A^2} \right) = \frac{1}{A} (A^2 - 2y^2)$$

or after sorting

$$y = \pm \sqrt{\frac{A}{2}} (A - x).$$

This equation represents a parabola (actually, a section of a parabola, which is located within a square of side $2A$). The curve passes through the points

$$(-A, \pm A), \quad (0, \pm A/\sqrt{2}) \quad \text{and} \quad (A, 0).$$

The mass point starts, at $t = 0$, at the intersection of the parabola and the x -axis (Fig. 2.17a), moves along the upper branch to the point with $x = -A$, reverses its motion and reaches, after traversing the x -axis, the point with $x = -A$ on the lower branch. There it reverses its motion once more and, after crossing the x -axis again, begins the next cycle.

The next example is characterised by the statement

$$(\phi = 0) \quad A_x = A_y = A \quad \omega_x = \omega, \quad \omega_y = 2\omega.$$

The frequency of the oscillation in the y -direction is twice as large as that of the x -direction. The parametric representation

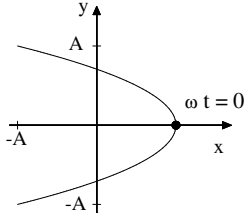
$$\begin{aligned} x(t) &= A \sin \omega t \\ y(t) &= A \sin 2\omega t = 2A \sin \omega t \cos \omega t \end{aligned}$$

leads in this case to the equation of the trajectory

$$y = \pm \frac{2x}{A} \sqrt{A^2 - x^2}.$$

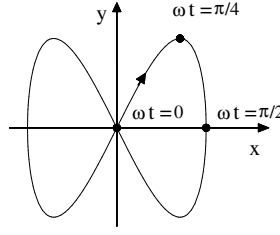
The trajectory is a kind of figure '8' in a square with sides $2A$ around the origin. The mass point traverses, starting at the origin, first the right hand

(a)



$$\omega_x = 2\omega_y = 2\omega, \quad \phi = \pi/2$$

(b)



$$\omega_y = 2\omega_x = 2\omega, \quad \phi = 0$$

Fig. 2.17. Trajectories of the two-dimensional harmonic oscillator for $A_x = A_y = A$ but different frequencies

loop in a clockwise fashion and then the left hand loop counterclockwise. The positive branch of the equation of the trajectory describes the section above the x -axis, the negative branch the lower section (Fig. 2.17b). The variation of the trajectory with the phase difference is shown in Fig. 2.18. It should be noted that the same figure is obtained for $\phi = 0$ and for $\phi = \pi/2$, however the sense of the motion is reversed.

Another selection of possible trajectories is displayed in Figs 2.19 - 2.22. Ellipses with the limiting cases of circles and straight lines are found if the ratio of the frequencies (ω_x/ω_y) is 1 (see more detailed discussion above). The variation of these curves with the phase difference can be observed in Fig. 2.19. Again the reversal of the sense of motion for figures of the same shape and a different phase (e.g. $\phi = 120^\circ$ and 240°) can be noted.

Closed figures are always obtained for a rational ratio of the frequencies

$$\frac{\omega_x}{\omega_y} = \frac{m}{n} \quad (m, n \text{ integer}).$$

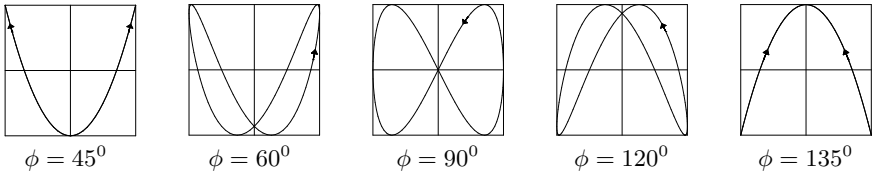


Fig. 2.18. Superposition of oscillations with $2\omega_x = \omega_y$, $A_x = A_y$. Variation of the phase difference ϕ

The number of loops in the figures increases with the magnitude of the integers m and n . Examples are presented in Figs 2.20 - 2.22. The trajectories do not close if the ratio of frequencies is irrational. The mass or oscillator point will, after some time, have covered each point of the rectangle with sides $2A_x$ and $2A_y$.

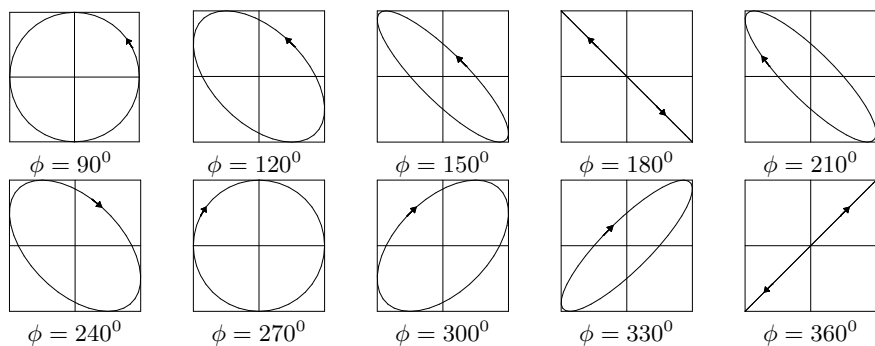


Fig. 2.19. Variation of the trajectories with $A_x = A_y$ and $\omega_x = \omega_y$ with the phase difference ϕ

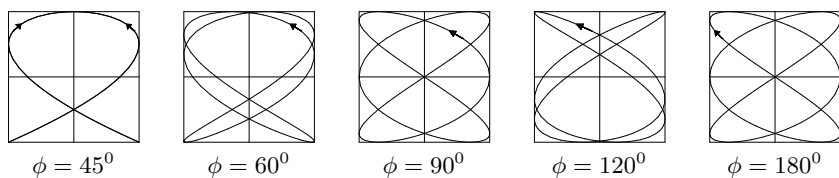


Fig. 2.20. Variation of the trajectories with the phase difference ϕ for the rational ratio $2\omega_x = 3\omega_y$, ($A_x = A_y$)

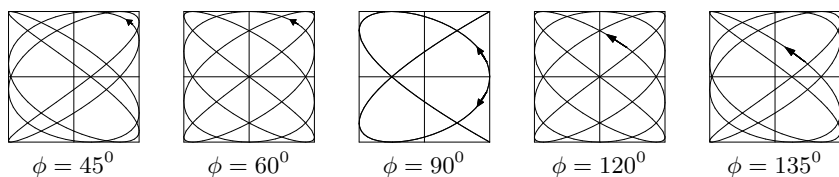


Fig. 2.21. Variation of the trajectories with the phase difference ϕ for the rational ratio $3\omega_x = 4\omega_y$, ($A_x = A_y$)

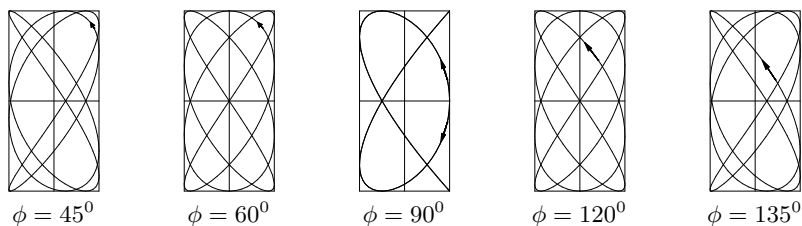


Fig. 2.22. Variation of the trajectories for $3\omega_x = 4\omega_y$, $2A_x = A_y$ with the phase difference ϕ

The first experimental demonstration of the superposition of two linear oscillations by J. Lissajous in 1855 relied on a mechanical device. More modern experimental reproductions of the figures can be obtained with an oscilloscope, which uses alternating voltages at two deflection plates arranged at right angles. As the smallest deviation from a rational ratio of the frequencies yields curves that do not close, an oscilloscope could be used for an extremely accurate, electro-technical comparison of frequencies².

2.2.2 Motion in three spatial dimensions

The set of equations $\{x(t), y(t), z(t)\}$ represents a curve embedded in three-dimensional space. An example is the parametric representation of a helix

$$x(t) = R \cos \omega t \quad y(t) = R \sin \omega t \quad z(t) = bt. \quad (2.18)$$

Here R and b are fixed parameters and t is the variable (e.g. with $0 \leq t \leq \infty$) which is used to trace the space curve. The projection of the helix onto the x - y plane is a circle, which is traversed uniformly. The z -component changes linearly with t by the amount $\Delta z = 2\pi b/\omega$ during the interval of time needed to traverse the circle once. The quantity Δz is called the pitch of the helix (Fig. 2.23).

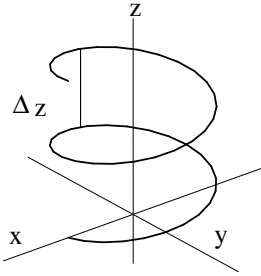


Fig. 2.23. A helix

A second example is characterised by the set of equations

$$x(t) = a \cos \phi \sin \omega t \quad y(t) = b \sin \phi \sin \omega t \quad z(t) = c \cos \omega t \quad (2.19)$$

which depend on four parameters a, b, c and ϕ and the variable t . The equations describe the motion of a mass point on the surface of an ellipsoid which is oriented along the coordinate axes. Figure 2.24a shows the space curve, which runs along a 'longitudinal line' of the ellipsoid through the north and the south poles. For a proof of this statement one needs to eliminate the

² Probl. 2.8 ● offers the possibility to investigate computer-generated Lissajous figures.

parameter t (the time) from the parametric representation. This results in two implicit equations in three variables

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad y = \left(\frac{b}{a} \tan \phi \right) x .$$

The first equation represents the surface of an ellipsoid with the semiaxes a, b, c . The second equation describes, in three-dimensional space, a plane through the z -axis, which is generated by a parallel displacement of the straight line ($y = Ax$) in the x - y plane in the direction of the z -axis. The trajectory corresponds to the intersection of the ellipsoid with the plane (Fig. 2.24b).

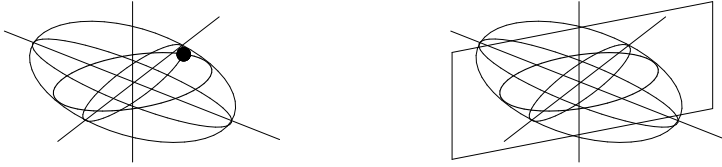


Fig. 2.24. Motion on an ellipsoid of revolution

The representation of curves in space by the intersection of two surfaces is an alternative to the parametric representation via three functions of one variable t . The parametric representation is the preferred tool in theoretical mechanics.

The representation of surfaces in space with the aid of functions of several variables is explained in **Math.Chap. 4**. This chapter also explains concepts as e.g. partial derivatives and integrals of functions of several variables.

The components of the velocity and the acceleration vectors can be calculated by direct differentiation of a given set of coordinates for the position. The motion on the space curve (2.19) can be further characterised by the components of velocity and the acceleration. The results are more compact if the abbreviations

$$A_x = a \cos \phi \quad A_y = b \sin \phi \quad A_z = c$$

and

$$x = A_x \sin \omega t \quad y = A_y \sin \omega t \quad z = A_z \cos \omega t$$

are used. They are

$$v_x = \frac{dx}{dt} = \omega A_x \cos \omega t \quad v_y = \frac{dy}{dt} = \omega A_y \cos \omega t$$

$$v_z = \frac{dz}{dt} = -\omega A_z \sin \omega t$$

for the components of velocity and

$$a_x = \frac{dv_x}{dt} = -\omega^2 A_x \sin \omega t \quad a_y = \frac{dv_y}{dt} = -\omega^2 A_y \sin \omega t$$

$$a_z = \frac{dv_z}{dt} = -\omega^2 A_z \cos \omega t$$

for the components of acceleration. The result shows that this example is another variant of the harmonic oscillator problem (in three dimensions) as

$$a_x = -\omega^2 x \quad a_y = -\omega^2 y \quad a_z = -\omega^2 z .$$

The actual problem that has to be faced, is – exactly as in the case of one-dimensional motion – more difficult. The task is normally not the calculation of the velocity and the acceleration after the specification of the trajectory, but rather:

Calculate the position (vector) $x(t)$, $y(t)$, $z(t)$
 (as well as the velocity (vector) $v_x(t)$, $v_y(t)$, $v_z(t)$),
 given the acceleration (vector) a_x , a_y , a_z and the initial conditions
 $x(t_0)$, $y(t_0)$, $z(t_0)$, $v_x(t_0)$, $v_y(t_0)$, $v_z(t_0)$.

The task is to find the solution of a set of differential equations. This task is relatively simple for the case of three uncoupled differential equations

$$\ddot{x} = a_x(t, x, \dot{x}) \quad \ddot{y} = a_y(t, y, \dot{y}) \quad \ddot{z} = a_z(t, z, \dot{z}) , \quad (2.20)$$

each of the kind that was discussed in Chap. 2.1. In general, a set of three *coupled* differential equations of the form

$$\begin{aligned} \ddot{x} &= a_x(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ \ddot{y} &= a_y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ \ddot{z} &= a_z(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \end{aligned} \quad (2.21)$$

has to be addressed. Examples and appropriate methods for their solution will be discussed in later chapters.

2.2.3 An example for the determination of trajectories in two space dimensions

The differential equations

$$a_x = g - kv_x \quad a_y = -kv_y \quad (2.22)$$

can be used to demonstrate the solution of a set of uncoupled equations of motion in two space dimensions. The acceleration consists of a constant term representing gravity and a frictional contribution in the x -direction (vertical, directed downwards), in the y -direction (horizontal, orientation to the right) there is only friction. The solution describes projectile motion in a homogeneous, viscous medium. The initial conditions

$$t_0 = 0 : x(0) = y(0) = 0 \quad v_x(0) = v_{x0} \quad v_y(0) = v_{y0}$$

state that the mass point starts at the origin with an arbitrary initial velocity.

The method that is used to solve the equations at hand is the method of separation of variables, which has already been used in Chap. 2.2.1 (compare also Chap. 2.1). The differential equation for the velocity component in the x -direction

$$\frac{dv_x}{dt} = g - kv_x$$

is a differential equation of first order. Separation of variables incorporating the initial condition

$$\int_0^t dt = \int_{v_{x0}}^{v_x} \frac{dv'_x}{(g - kv'_x)}$$

yields for the velocity

$$v_x(t) = \dot{x}(t) = \frac{g}{k}(1 - e^{-kt}) + v_{x0}e^{-kt}.$$

This result represents a differential equation for the x -coordinate. The solution can be obtained by direct integration

$$x(t) = \frac{g}{k}t + \frac{g}{k^2}(e^{-kt} - 1) - \frac{v_{x0}}{k}(e^{-kt} - 1). \quad (2.23)$$

The differential equation for the motion in the y -direction is simpler

$$\frac{dv_y}{dt} = -k v_y.$$

As this equation differs from the equation for the x -component only by the fact that $g = 0$, the solution can be written down directly

$$v_y(t) = v_{y0}e^{-kt} \quad y(t) = -\frac{v_{y0}}{k}(e^{-kt} - 1). \quad (2.24)$$

The discussion of the details of the motion relies in this example also on a consideration of the two limiting situations. The series expansion for small values of kt

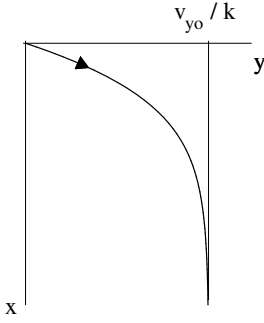
$$(e^{-kt} - 1) = -kt + \frac{1}{2}k^2t^2 - \dots$$

leads to the approximation

$$\begin{aligned} x(t) &\approx v_{x0}t + \frac{1}{2}(g - v_{x0}k)t^2 + \dots \\ y(t) &\approx v_{y0}t - \frac{1}{2}v_{y0}kt^2 + \dots \end{aligned}$$

The trajectory is close to a parabola if frictional effects are small, that is if $v_{x0}k \sim v_{y0}k \ll g$. The time dependence of the coordinates for large values of kt is obtained with the asymptotic behaviour of the exponential function as

$$x(t) \longrightarrow \frac{g}{k}t + \left(\frac{v_{x0}}{k} - \frac{g}{k^2}\right) \quad \text{and} \quad y(t) \longrightarrow \frac{v_{y0}}{k}. \quad (2.25)$$

**Fig. 2.25.** Projectile motion in a viscous medium

The motion is uniform along the vertical axis. The complete trajectory is illustrated in Figure 2.25.

It is also possible to derive an analytical expression for the trajectory. The second equation in (2.24)

$$(e^{-kt} - 1) = -\frac{ky}{v_{y0}}$$

can be resolved in the form

$$t = -\frac{1}{k} \ln \left(1 - \frac{ky}{v_{y0}} \right) .$$

Insertion into (2.23) then leads to the equation for the trajectory

$$x = -\frac{g}{k^2} \ln \left(1 - \frac{ky}{v_{y0}} \right) + \left(\frac{v_{x0}}{v_{y0}} - \frac{g}{kv_{y0}} \right) y .$$

The Lissajous problem of Chap. 2.2.1 with the differential equations

$$\ddot{x} = -\omega_x^2 x \quad \text{and} \quad \ddot{y} = -\omega_y^2 y$$

and the general solution

$$\begin{aligned} x(t) &= A_x \sin(\omega_x t + \phi_x) & v_x(t) &= A_x \omega_x \cos(\omega_x t + \phi_x) \\ y(t) &= A_y \sin(\omega_y t + \phi_y) & v_y(t) &= A_y \omega_y \cos(\omega_y t + \phi_y) \end{aligned}$$

is a second example for a system of simple differential equations of the type (2.20).

2.3 Vectorial formulation of problems of motion

The motion of a mass point in space is characterised by a set of three functions $\{x(t), y(t), z(t)\}$, the parametric representation of the trajectory. The three functions can be interpreted as components of a (time dependent) vector

$$\mathbf{r}(t) = x(t) \mathbf{e}_x + y(t) \mathbf{e}_y + z(t) \mathbf{e}_z . \quad (2.26)$$

The endpoint of the vector $\mathbf{r}(t)$ maps out the trajectory (Fig. 2.26a) in this vectorial description of the time development. It has the advantage that the description of the motion is independent of the choice of a particular coordinate system, even though it may be necessary to resort to a decomposition into components in order to handle details (Fig. 2.26b).

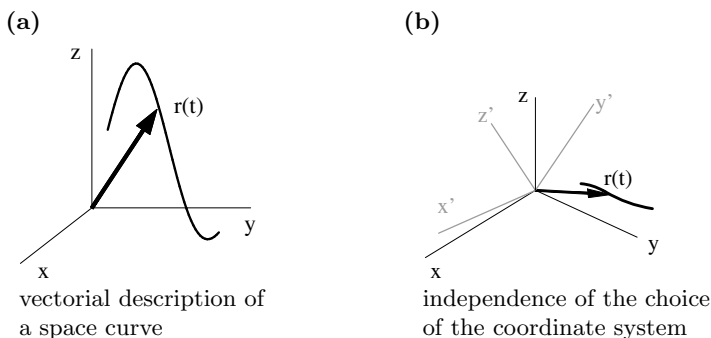


Fig. 2.26. Vectorial description of the motion of a mass point

2.3.1 Basic concepts

The discussion of the **velocity vector** without recourse to a decomposition into components could proceed as follows. The two vectors, which mark the position at times t and $t + \Delta t$, allow the construction of the difference vector $\Delta \mathbf{r} = (\mathbf{r}(t + \Delta t) - \mathbf{r}(t))$ (see Fig. 2.27). This vector, if multiplied with the

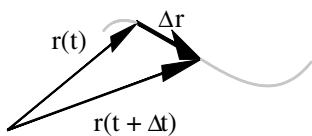


Fig. 2.27. Definition of the velocity vector

scalar $(1/\Delta t)$, is the vector for the average velocity

$$\bar{\mathbf{v}}(t, \Delta t) = \frac{\Delta \mathbf{r}}{\Delta t}.$$

It goes over into the vector for the instantaneous velocity in the limit $\Delta t \rightarrow 0$

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \lim_{\Delta t \rightarrow 0} \frac{(\mathbf{r}(t + \Delta t) - \mathbf{r}(t))}{\Delta t}. \quad (2.27)$$

The first three entries on the left hand side indicate variations in notation, the right hand side is the definition of the actual limiting value. The vector

$\mathbf{v}(t)$ corresponds, geometrically speaking, to a vector in the direction of the tangent line of the space curve in the point $\mathbf{r}(t)$.

The transition from the vectorial formulation to the decomposition into components uses the steps: Refer the two vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$ to a coordinate system that *does not change* in time

$$\bar{\mathbf{v}}(t, \Delta t) = \frac{\Delta x}{\Delta t} \mathbf{e}_x + \frac{\Delta y}{\Delta t} \mathbf{e}_y + \frac{\Delta z}{\Delta t} \mathbf{e}_z$$

and obtain in the limit $\Delta t \rightarrow 0$

$$\mathbf{v}(t) = \dot{x}(t) \mathbf{e}_x + \dot{y}(t) \mathbf{e}_y + \dot{z}(t) \mathbf{e}_z . \quad (2.28)$$


The prerequisite of a coordinate system, which does not change in time, is essential at this point. The discussion is quite different if this is not the case (see e.g. Chap. 6.2 for a treatment of rotating coordinate systems).

The **acceleration vector** is defined correspondingly as the time change of the velocity vector

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = \lim_{\Delta t \rightarrow 0} \frac{(\mathbf{v}(t + \Delta t) - \mathbf{v}(t))}{\Delta t} . \quad (2.29)$$

Again, the actual definition is the limiting value, the remaining entries are variants of notation. The decomposition of this vector with respect to a coordinate system, that does not change with time, is

$$\mathbf{a}(t) = \ddot{x}(t) \mathbf{e}_x + \ddot{y}(t) \mathbf{e}_y + \ddot{z}(t) \mathbf{e}_z . \quad (2.30)$$

The differentiation and the integration of vectors with respect to a parameter (in physics usually the time) will be used from now on. A detailed discussion of relevant mathematical aspects can be found in  Math.Chap. 5.

2.3.1.1 Length of arc, tangent vector and normal vector. A number of quantities that characterise particular aspects of trajectories in space can be deduced from the velocity vector.

- The length of a segment of a trajectory with the endpoints at t_0 and t , the **length of arc** or for short the **arc length**, is given by

$$\begin{aligned} s(t, t_0) &= \int_{t_0}^t [\dot{x}(t')^2 + \dot{y}(t')^2 + \dot{z}(t')^2]^{1/2} dt' \\ &= \int_{t_0}^t [\dot{\mathbf{r}}(t') \cdot \dot{\mathbf{r}}(t')]^{1/2} dt' . \end{aligned}$$

This definition is based on the division of the arc into infinitesimal elements of length

$$ds = [dx^2 + dy^2 + dz^2]^{1/2} ,$$

followed by the introduction of a parameter which traces the curve

$$dx = \dot{x}(t)dt \quad dy = \dot{y}(t)dt \quad dz = \dot{z}(t)dt$$

and finally integration over the complete segment.

- The **tangent vector** \mathbf{e}_T at a point of the trajectory is a unit vector which points in the direction of the tangent line of the curve

$$\mathbf{e}_T(t) = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} .$$

The second entry in this equation constitutes the actual definition. The next entry follows from the chain rule, while the last one uses the definition of the length of arc

$$\frac{ds}{dt} = [\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t)]^{1/2} = |\dot{\mathbf{r}}(t)| .$$

The tangent vector is a unit vector. This can be seen by looking at the scalar product

$$\mathbf{e}_T \cdot \mathbf{e}_T = \frac{\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|^2} = 1 .$$

- The **normal vector**, which is defined by

$$\mathbf{e}_N(t) = \rho(t) \frac{d\mathbf{e}_T(t)}{ds} ,$$

is a unit vector perpendicular to \mathbf{e}_T . As the derivative of the tangent vector with respect to the arc length is not necessarily a vector with the magnitude 1, a normalisation factor has to be introduced

$$\rho(t) = \left| \frac{d\mathbf{e}_T}{ds} \right|^{-1} = \left| \frac{d\mathbf{e}_T}{dt} \frac{dt}{ds} \right|^{-1} = |\dot{\mathbf{r}}| \left| \frac{d\mathbf{e}_T}{dt} \right|^{-1} .$$

The inverse of the magnitude of the derivative of \mathbf{e}_T with respect to the parameter s is referred to as the **radius of curvature** $\rho(t)$. The quantity

$$\kappa(t) = \rho(t)^{-1}$$

is called the **curvature**. Orthogonality of $\mathbf{e}_N(t)$ and $\mathbf{e}_T(t)$ follows from

$$\frac{d(\mathbf{e}_T(t) \cdot \mathbf{e}_T(t))}{dt} = 2 \left(\mathbf{e}_T(t) \cdot \frac{d\mathbf{e}_T(t)}{dt} \right) = 0 .$$

- The complement of the two vectors

$$\mathbf{e}_B(t) = \mathbf{e}_T(t) \times \mathbf{e}_N(t)$$

is the **binormal vector**. The three vectors defined above form a coordinate trihedron.

2.3.2 Vectorial description of motion

The vectorial description of the motion of a point particle is illustrated with the aid of a few examples. The uniform motion on a circle in the x - y plane is the simplest. The position vector is given as

$$\mathbf{r}(t) = (R \cos \omega t) \mathbf{e}_x + (R \sin \omega t) \mathbf{e}_y + 0 \mathbf{e}_z \quad (2.31)$$

or in the standard abbreviation for vectors

$$\mathbf{r}(t) = (R \cos \omega t, R \sin \omega t, 0) . \quad (2.32)$$

The position of the point particle is characterised by the end point of this vector. The particle starts at the position $\mathbf{r}(0) = (R, 0, 0)$ and moves anti-clockwise (Fig. 2.28a) on the circle. The velocity vector (Fig. 2.28b)

$$\mathbf{v}(t) = (-R\omega \sin \omega t, R\omega \cos \omega t, 0) \quad (2.33)$$

has the properties

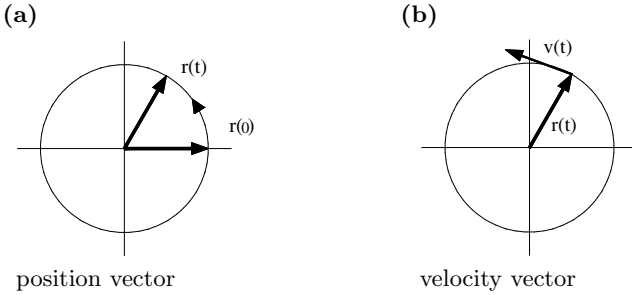


Fig. 2.28. Circular motion

- (1) The magnitude of the vector \mathbf{v} does not depend on time ($v = |\mathbf{v}| = R\omega$).
- (2) The direction of the vector changes with time. The relation $\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$ shows, that the velocity vector and the position vector are orthogonal at all times. The velocity vector rotates in the same sense as the position vector, preceding it by 90° .

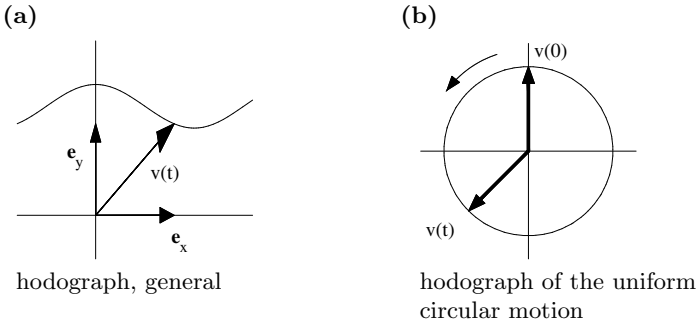


Fig. 2.29. Hodograph

The end points of the velocity vectors, which are arranged so that they emanate from the origin, describe a curve which is called a **hodograph** (Fig. 2.29a, hodos is the Greek word for path). The hodograph for the uniform motion on a circle is also a circle (Fig. 2.29b). In three-dimensional space the parametric representation of a hodograph is composed of the three components of the velocity vector $\{\dot{x}(t), \dot{y}(t), \dot{z}(t)\}$.

The acceleration vector $\mathbf{a}(t)$ for the uniform circular motion is

$$\mathbf{a}(t) = (-\omega^2 R \cos \omega t, -\omega^2 R \sin \omega t, 0) = -\omega^2 \mathbf{r}(t). \quad (2.34)$$

It points at all times towards the centre of the circle, as it is (up to a positive factor) the negative of the position vector. This type of acceleration carries the name **central (or centripetal) acceleration** (Fig. 2.30). The magnitude

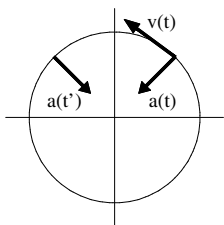


Fig. 2.30. Circular motion: central acceleration

of the acceleration vector is also independent of time

$$a(t) = |\mathbf{a}(t)| = \omega^2 R. \quad (2.35)$$

Relations which are used quite often in the discussion of uniform circular motion are

$$a(t) \equiv a = \omega v = \frac{v^2}{R}. \quad (2.36)$$

In the case of nonuniform motion on a circle the constant circular frequency is replaced by a function $\omega(t)$. The position vector and the resulting velocity vector are

$$\mathbf{r}(t) = (R \cos \omega(t), R \sin \omega(t), 0)$$

$$\mathbf{v}(t) = (-R\dot{\omega}(t) \sin \omega(t), R\dot{\omega}(t) \cos \omega(t), 0).$$

The velocity vector is again tangential to the circle. Its magnitude depends, however, on time in this case ($v = |R\dot{\omega}(t)|$). The vectors \mathbf{r} and \mathbf{v} are still orthogonal

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = 0.$$

Each of the components of the acceleration vector is composed of two functions

$$\mathbf{a}(t) = (-R\ddot{\omega}(t) \sin \omega(t) - R\dot{\omega}(t)^2 \cos \omega(t), \\ R\ddot{\omega}(t) \cos \omega(t) - R\dot{\omega}(t)^2 \sin \omega(t), 0) .$$

The vectorial form

$$\mathbf{a}(t) = \frac{\ddot{\omega}(t)}{\dot{\omega}(t)} \mathbf{v}(t) - \dot{\omega}(t)^2 \mathbf{r}(t) \quad (2.37)$$

shows more clearly that the acceleration is not central for a nonuniform, circular motion.

The motion on a Lissajous ellipse in the x - y plane is characterised by the position vector

$$\mathbf{r}(t) = (a \cos \omega t, b \sin \omega t, 0) . \quad (2.38)$$

The initial position is chosen so that $\mathbf{r}(0) = (a, 0, 0)$ and the sense of circulation is counterclockwise (Fig. 2.31a). The velocity vector (Fig. 2.31b) is

$$\mathbf{v}(t) = (-a\omega \sin \omega t, b\omega \cos \omega t, 0)$$

in this case. The position vector and the velocity vector are only orthogonal at the intersections of the ellipse with the coordinate axes for

$$\omega t = 0, \pi/2, \pi, 3\pi/2, 2\pi$$

as

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = (b^2 - a^2) \omega \cos \omega t \sin \omega t .$$

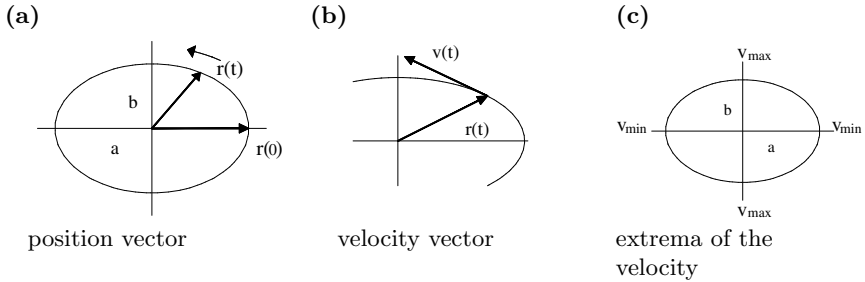


Fig. 2.31. Motion on a Lissajous ellipse

The magnitude of the velocity vector changes with time according to

$$v(t) = \omega [a^2 \sin^2 \omega t + b^2 \cos^2 \omega t]^{1/2} .$$

The motion along the trajectory is not uniform. Consider a situation with $a > b$ (as in Fig. 2.31c). The velocity of the mass is maximal ($v_{\max} = a\omega$) for $\omega t = \pi/2, 3\pi/2, \dots$, that is for points which are closest to the centre

of the ellipse. The velocity is smallest ($v_{\min} = b\omega$) for $\omega t = 0, \pi, \dots$. The time for a complete revolution is, as for uniform motion on a circle, $T = 2\pi/\omega$. The acceleration vector reflects the genesis of the Lissajous ellipse by a superposition of two harmonic oscillations of the same frequency

$$\begin{aligned}\mathbf{a}(t) &= (-\omega^2 a \cos \omega t, -\omega^2 b \sin \omega t, 0) \\ &= -\omega^2 \mathbf{r}(t) .\end{aligned}$$

The uniform motion on a circle and the motion on a Lissajous ellipse are, nonetheless, characterised by the same set of differential equations

$$\ddot{x}(t) = -\omega^2 x(t) \quad \ddot{y}(t) = -\omega^2 y(t) \quad \ddot{z}(t) = -\omega^2 z(t)$$

or in vectorial form

$$\ddot{\mathbf{r}}(t) = -\omega^2 \mathbf{r}(t) . \quad (2.39)$$

The two different trajectories result solely from a difference of the initial conditions

circle	Lissajous ellipse
$\mathbf{r}(0) = (R, 0, 0)$	$(a, 0, 0)$
$\mathbf{v}(0) = (0, \omega R, 0)$	$(0, \omega b, 0)$

For the circle the ratio $v(0)/r(0)$ depends only on the frequency ω which occurs explicitly in the differential equation (2.39). This is not the case for the ellipse.

A more exotic trajectory is described by the vector

$$\mathbf{r}(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}, 0 \right) \quad a > 0 . \quad (2.40)$$

This trajectory is illustrated in Fig. 2.32 for the time interval $0 \leq t \leq \infty$.

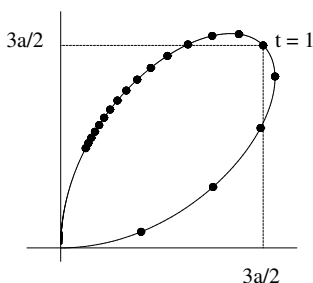


Fig. 2.32. Motion on a Cartesian leaf

At time $t = 0$ the mass point starts at the origin. The initial motion is relatively rapid. At time $t = 1$ (arbitrary units) the mass point has reached the position $(3a/2, 3a/2, 0)$. It then slows down more and more and completes

the (geometrically symmetric) complement of the loop only after an infinite time interval. The trajectory is part of a figure which is named the Cartesian leaf. The discussion of the velocity and the acceleration will be postponed for a while (Chap. 2.3.3.1).

2.3.3 Area theorem

The three examples of the previous section can be used to illustrate a general law, the **law of areas**. This theorem will also be discussed in more general terms in Chap. 3.2.2 under the name of conservation of angular momentum. For an arbitrary curve in space two neighbouring position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$ will span in good approximation a flat triangle, provided the time

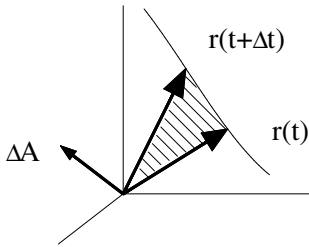


Fig. 2.33. Illustration of the law of areas

interval Δt is sufficiently small. This triangle can be characterised by the cross product

$$\Delta \mathbf{A} \approx \frac{1}{2} (\mathbf{r}(t) \times \mathbf{r}(t + \Delta t)) .$$

The infinitesimal vector $\Delta \mathbf{A}$ is perpendicular to the triangle (Fig. 2.33). Its direction describes the orientation of the triangle in space. The length of the vector corresponds (up to small corrections) to the magnitude of the area which is covered by the position vector in the time interval Δt . As the cross product $\mathbf{r} \times \mathbf{r}$ is a null vector, addition of this quantity does not change anything

$$\Delta \mathbf{A} \approx \frac{1}{2} (\mathbf{r}(t) \times (\mathbf{r}(t + \Delta t) - \mathbf{r}(t)) .$$

Multiplication with $1/\Delta t$ and the limit $\Delta t \rightarrow 0$ leads to the result

$$\begin{aligned} \frac{d}{dt} \mathbf{A}(t) &= \frac{1}{2} \lim_{\Delta t \rightarrow 0} \left\{ \mathbf{r}(t) \times \left(\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right) \right\} \\ &= \frac{1}{2} (\mathbf{r}(t) \times \mathbf{v}(t)) \end{aligned}$$

which can be abbreviated (with a slight change in notation) as

$$\dot{\mathbf{A}}(t) = \frac{1}{2} (\mathbf{r}(t) \times \dot{\mathbf{r}}(t)) . \quad (2.41)$$

The speed, with which a surface is covered by the position vector in time, is, up to a factor $1/2$, equal to the cross product of position vector and velocity vector. The quantity $\dot{\mathbf{A}}(t)$ is called the **areal velocity** (more precisely the vector of the areal velocity). Its dimension is area per time, $[L^2/T]$.

Two related statements follow from the definition (2.41).

- The **law of conservation of areas**, which is often quoted as **area theorem**. The vector of the areal velocity is a constant vector, if a trajectory lies in a plane containing the origin of the coordinate system and if the cross product $\mathbf{r}(t) \times \mathbf{v}(t)$ is independent of time

$$\dot{\mathbf{A}}(t) = \frac{1}{2} \mathbf{C} \quad \dot{\mathbf{C}} = \mathbf{0} . \quad (2.42)$$

Integration over time yields the result

$$\mathbf{A}(t) - \mathbf{A}(t_0) = \frac{1}{2} \mathbf{C} (t - t_0) \quad (2.43)$$

which is abbreviated as

$$\Delta \mathbf{A} = \frac{1}{2} \mathbf{C} \Delta t .$$

The magnitude of this vector

$$|\Delta \mathbf{A}| = \frac{1}{2} |\mathbf{C}| \Delta t \quad (2.44)$$

expresses the area theorem: equal surface areas are covered in equal time intervals under the conditions stated.

- The second statement concerns the question:

For what kind of trajectories is this theorem valid?

An answer is obtained by differentiation of the relation

$$\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{C}$$

with respect to time

$$(\dot{\mathbf{r}}(t) \times \dot{\mathbf{r}}(t)) + (\mathbf{r}(t) \times \ddot{\mathbf{r}}(t)) = \mathbf{0} .$$

As the first term vanishes for all times, the answer is: The area theorem is valid if the condition

$$\mathbf{r}(t) \times \mathbf{a}(t) = \mathbf{0} \quad (2.45)$$

holds. The vector product of the position and acceleration vectors has to vanish for all times. If neither of the two vectors vanishes for all times, this is only possible if

$$\mathbf{a}(t) = \lambda(t) \mathbf{r}(t) . \quad (2.46)$$

The vector $\mathbf{a}(t)$ is for all times directed to or away from the origin of the coordinate system that has been chosen, in other words, if the acceleration is a central acceleration.

The two statements can be summarised in the form:

The relation

$$\dot{\mathbf{A}}(t) = \frac{1}{2} (\mathbf{r}(t) \times \mathbf{v}(t)) = \frac{1}{2} \mathbf{C}$$

is valid for a central acceleration. The motion is planar in this case, the origin of the coordinate system lies in the plane and the area theorem is valid in the form

$$\Delta A \propto \Delta t .$$

The relation

$$(\mathbf{r}(t) \times \mathbf{v}(t)) = \gamma(t) \mathbf{C} \quad (2.47)$$

describes the more general situation that the direction of the vector of the areal velocity is constant but the magnitude changes with time. The area covered by the position vector in a given interval of time is then

$$A(t) - A(t_0) = \Delta A = \frac{C}{2} \int_{t_0}^t \gamma(t') dt' .$$

This relation can be used for the calculation of planar surface areas.

2.3.3.1 Determination of areas with the aid of the areal velocity. It is instructive to discuss the three examples of Chap. 2.3.2 under this heading. For the uniform motion on a circle the cross product of the position and velocity vectors

$$\mathbf{r}(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ R \cos \omega t & R \sin \omega t & 0 \\ -R\omega \sin \omega t & R\omega \cos \omega t & 0 \end{vmatrix} = \omega R^2 \mathbf{e}_z$$

is a constant vector. The area theorem is valid. The increase of the area in a time interval Δt is (Fig. 2.34a)

$$\Delta A = \frac{1}{2} \omega R^2 \Delta t .$$

The well known formula for the area of a circle

$$A(T) = \pi R^2$$

is therefore obtained for a full revolution with

$$\Delta t \rightarrow T = \frac{2\pi}{\omega} .$$

The calculation of arbitrary segments of a circle would proceed in a similar manner. The application of the law of areas allows the calculation of plane areas by sweeping them with the position vector.

For the Lissajous ellipse the law of areas is also valid

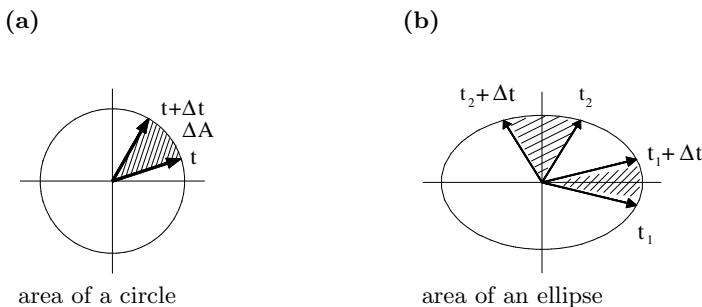


Fig. 2.34. Calculation of areas

$$\mathbf{r}(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a \cos \omega t & b \sin \omega t & 0 \\ -a\omega \sin \omega t & b\omega \cos \omega t & 0 \end{vmatrix} = ab\omega \mathbf{e}_z .$$

The increase and decrease of the position vector and the velocity vector are matched, so that equal areas are covered in equal times. This is the reason why the velocity is larger for points closer to the origin (Fig. 2.34b). The increase of the area in the time interval Δt is

$$\Delta A = \frac{1}{2} ab\omega \Delta t ,$$

for a full revolution one finds

$$A(T) = \frac{1}{2} ab\omega T = ab\pi ,$$

the area of the ellipse.

In order to avoid a possible misunderstanding about the motion on ellipses, it is worthwhile to digress briefly from the current discussion. The Lissajous ellipses, which have been considered above, are obtained as solutions of the differential equations of a two-dimensional harmonic oscillator. They are not identical with the elliptic orbits of planetary motion (Kepler's ellipses, which will be discussed at length in Chap. 4.1). The differential equation for a two-dimensional (or three-dimensional) harmonic oscillator is (compare (2.39), p. 47)

$$\ddot{\mathbf{r}}(t) = -\omega^2 \mathbf{r}(t) .$$

The differential equation for planetary motion is quite different

$$\ddot{\mathbf{r}}(t) = -\frac{k'}{r^3(t)} \mathbf{r}(t) . \quad (2.48)$$

The area theorem holds in both cases. The form of the 'function' in front of the position vector is responsible for the difference in the time development of the motion (see Fig. 2.35a, b).

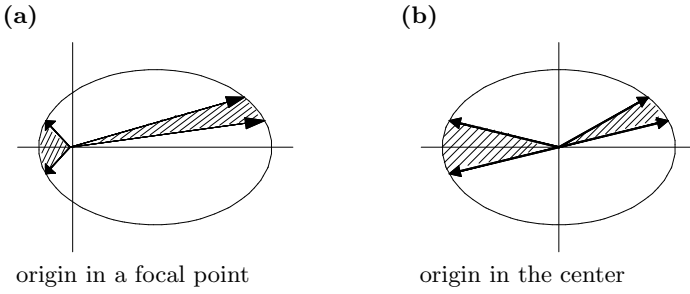


Fig. 2.35. Comparison of Kepler's ellipse (a) with Lissajous' ellipse (b)

The areal velocity for the motion along the border of the Cartesian leaf is determined by

$$\mathbf{r}(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{3at}{(1+t^3)} & \frac{3at^2}{(1+t^3)} & 0 \\ 3a\frac{(1-2t^3)}{(1+t^3)^2} & 3a\frac{(2t-t^4)}{(1+t^3)^2} & 0 \end{vmatrix}$$

(recognise the components of the velocity here), so that

$$\dot{\mathbf{A}}(t) = \frac{9}{2}a^2 \frac{t^2}{(1+t^3)^2} \mathbf{e}_z .$$

The magnitude of the vector of the areal velocity changes with time. The law of areas is not valid. Nonetheless, the area of the leaf (Fig. 2.36) can readily be calculated. Integration of the magnitude of $\dot{\mathbf{A}}$ gives

$$A(t) = \frac{9a^2}{2} \int_0^t \frac{t'^2}{(1+t'^3)^2} dt' .$$

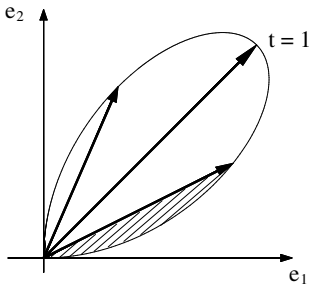


Fig. 2.36. Calculation of areas: Cartesian leaf

Half the area of the leaf is therefore given by

$$A(1) = \frac{9a^2}{2} \int_0^1 \frac{t'^2}{(1+t'^3)^2} dt'.$$

The integral can be calculated with the aid of the substitution

$$\tau = 1 + t'^3, \quad d\tau = 3t'^2 dt', \quad \tau(0) = 1, \quad \tau(1) = 2.$$

The result is

$$A(1) = \frac{3a^2}{2} \int_1^2 \frac{d\tau}{\tau^2} = \frac{3a^2}{2} \left[-\frac{1}{\tau} \right]_1^2 = \frac{3}{4}a^2.$$

For the calculation of the total area the improper integral (improper integrals are discussed in **● Math.Chap. 1.4.1**)

$$A(\infty) = \frac{9a^2}{2} \int_0^\infty \frac{t^2 dt}{(1+t^3)^2}$$

has to be considered. The evaluation is no problem

$$\begin{aligned} A(\infty) &= \frac{3}{2}a^2 \lim_{b \rightarrow \infty} \int_1^b \frac{d\tau}{\tau^2} = \frac{3}{2}a^2 \lim_{b \rightarrow \infty} \left[-\frac{1}{b} + 1 \right] \\ &= \frac{3}{2}a^2. \end{aligned}$$

Obviously the symmetry of the figure demands

$$A(\infty) = 2A(1).$$

The calculation of the area of the Cartesian leaf with standard methods is definitely more involved.

The question of the calculation of areas by tracing the edges of a figure and the connection with standard Riemann integration will be elucidated more fully in **● Math.Chap. 5**. This chapter also contains a discussion of the more general case that the position vector marks an arbitrary surface in space.

2.4 Curvilinear coordinates

The last topic of the chapter entitled 'Kinematics' is an introduction to the subject of curvilinear coordinates.

2.4.1 Coordinates in the plane

The motion in a plane is of particular interest. It is useful to begin the discussion for this reason with the two-dimensional world (and for the reason that details are simpler).

2.4.1.1 Plane polar coordinates. The description of circular motion (whether uniform or not) with a decomposition of the position vector in Cartesian coordinates is possible but not optimal. It is simpler to discuss this type of motion using plane polar coordinates. The basic quantities involved are (Fig. 2.37a)

$$r(t) = [x^2(t) + y^2(t)]^{1/2} \quad \text{length of the vector } \mathbf{r} \quad (2.49)$$

$$\varphi(t) = \arctan \frac{y(t)}{x(t)} \quad \text{angle between } \mathbf{r} \text{ and the } x\text{-axis} . \quad (2.50)$$

These equations are the inverse of the transformation

$$x = r \cos \varphi \quad y = r \sin \varphi . \quad (2.51)$$

The time derivatives of the coordinates $r(t)$ and $\varphi(t)$ are usually denoted as

$$\begin{aligned} \dot{r}(t) &= \frac{dr}{dt} & \dot{\varphi}(t) &= \frac{d\varphi}{dt} = \omega(t) \\ \ddot{r}(t) &= \frac{d^2r}{dt^2} & \ddot{\varphi}(t) &= \frac{d^2\varphi}{dt^2} = \dot{\omega}(t) = \alpha(t) , \end{aligned}$$

in particular, for the uniform circular motion ((2.31), p.44)

$$\begin{aligned} r(t) &= R & \dot{r}(t) &= 0 & \ddot{r}(t) &= 0 \\ \varphi(t) &= \omega t & \dot{\varphi}(t) &= \omega & \ddot{\varphi}(t) &= 0 . \end{aligned}$$

These results might lead to the conclusion that the uniform circular motion is a motion without acceleration. This is, as already discussed, not the case. The incorrect conclusion is due to the fact, that the introduction of polar coordinates is based on a bihedron (a set of two orthogonal basis vectors) that changes with time. The (two-dimensional) Cartesian coordinate system is rigidly fixed in the plane. With polar coordinates a system of basis vectors is introduced which follows the motion of a mass point by changing the orientation of the system. The basis vectors of this time changing coordinate system are (Fig. 2.37b)

$$\begin{aligned} \mathbf{e}_r(t) &\longrightarrow \text{in the direction of the instantaneous position vector} \\ \mathbf{e}_\varphi(t) &\longrightarrow \text{right-handed orthogonal complement.} \end{aligned}$$

The connection between the basis vectors of the Cartesian and the polar coordinate systems is effected by the transformation

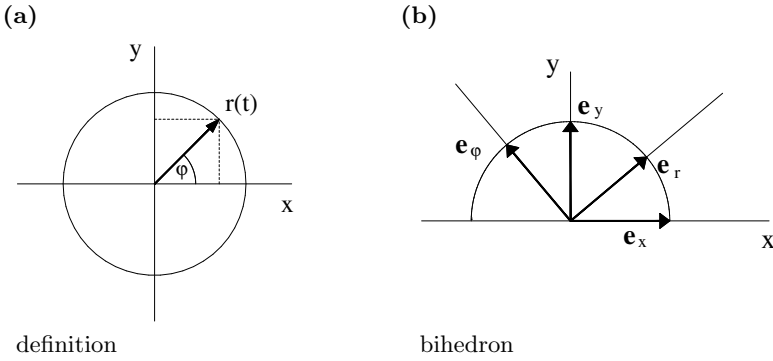
$$\mathbf{e}_r(t) = \cos \varphi(t) \mathbf{e}_x + \sin \varphi(t) \mathbf{e}_y \quad (2.52)$$

$$\mathbf{e}_\varphi(t) = -\sin \varphi(t) \mathbf{e}_x + \cos \varphi(t) \mathbf{e}_y . \quad (2.53)$$

The position vector has by definition the form

$$\mathbf{r}(t) = r(t) \mathbf{e}_r(t) \quad (2.54)$$

in the time dependent polar coordinate system. The time change of the basis vectors has to be taken into account

**Fig. 2.37.** Plane polar coordinates

$$\mathbf{v}(t) = \dot{r}(t) \mathbf{e}_r(t) + r(t) \dot{\varphi}(t) \mathbf{e}_\varphi(t)$$

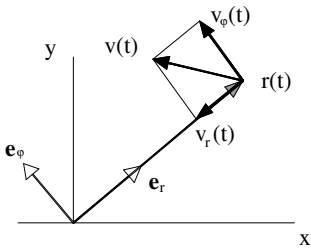
for the calculation of the velocity. The time derivative of the unit vector $\mathbf{e}_r(t)$ can be obtained from the transformation between the two sets of basis vectors as

$$\dot{\mathbf{e}}_r(t) = -\dot{\varphi} \sin \varphi \mathbf{e}_x + \dot{\varphi} \cos \varphi \mathbf{e}_y = \dot{\varphi} \mathbf{e}_\varphi. \quad (2.55)$$

(Prove this relation also by elementary, geometric arguments.) The decomposition of the velocity vector with respect to the basis vectors of the polar coordinate system can therefore be written as (Fig. 2.38)

$$\mathbf{v}(t) = \dot{r} \mathbf{e}_r + r \dot{\varphi} \mathbf{e}_\varphi = v_r \mathbf{e}_r + v_\varphi \mathbf{e}_\varphi. \quad (2.56)$$

The quantity v_r (the projection of \mathbf{v} on the vector \mathbf{e}_r) is the **radial velocity**, the quantity v_φ (in the form radius times angular velocity) is the **azimuthal velocity**. The first term in (2.56) is the velocity that would be registered by

**Fig. 2.38.** Azimuthal vector in plane polar coordinates

an observer, who does not realise that the coordinate system is moving. For such a person the point particle does not leave the direction marked by \mathbf{e}_r . The second term is due to the relative motion of the two coordinate systems. The form $r\dot{\varphi}$ emphasises the fact that the relative motion is more apparent for a larger distance from the origin.

The basis vectors of the moving coordinate system (2.52) and (2.53) satisfy the orthogonality relation

$$\mathbf{e}_r(t) \cdot \mathbf{e}_\varphi(t) = 0$$

for all times. For this reason the magnitude of the velocity vector is

$$v(t) = \sqrt{\dot{r}^2 + r^2 \dot{\varphi}^2}. \quad (2.57)$$

Differentiation of the velocity vector with respect to time yields the components of the acceleration vector

$$\mathbf{a}(t) = \ddot{r} \mathbf{e}_r + \dot{r} \dot{\mathbf{e}}_r + (\dot{r} \dot{\varphi} + r \ddot{\varphi}) \mathbf{e}_\varphi + r \dot{\varphi} \dot{\mathbf{e}}_\varphi. \quad (2.58)$$

The time derivative of the vector \mathbf{e}_φ can also be calculated with the transformation (2.53)

$$\dot{\mathbf{e}}_\varphi = -\dot{\varphi} \cos \varphi \mathbf{e}_x - \dot{\varphi} \sin \varphi \mathbf{e}_y = -\dot{\varphi} \mathbf{e}_r. \quad (2.59)$$

Insertion of the time derivatives of the basis vectors in (2.58) and sorting leads to

$$\mathbf{a}(t) = (\ddot{r} - r \dot{\varphi}^2) \mathbf{e}_r + (2\dot{r} \dot{\varphi} + r \ddot{\varphi}) \mathbf{e}_\varphi = a_r \mathbf{e}_r + a_\varphi \mathbf{e}_\varphi. \quad (2.60)$$

The component a_r is the **radial acceleration**, the component a_φ the **azimuthal acceleration**. From the point of view of the observer moving with the coordinate system the expected term is $\ddot{r} \mathbf{e}_r$, all the remaining terms, which contain derivatives of the angular coordinate, are consequences of the relative motion of the polar coordinate system with respect to the (space fixed) Cartesian system.

Every planar problem can be discussed in terms of polar coordinates. The question is, however: for which class of problems might this approach be most useful? Some examples will prepare the answer:

- (i) The uniform circular motion (once again) with $r(t) = R$ and $\varphi(t) = \omega t$ is characterised by

$$\mathbf{r}(t) = R \mathbf{e}_r \quad \mathbf{v}(t) = R\omega \mathbf{e}_\varphi \quad \mathbf{a}(t) = -R\omega^2 \mathbf{e}_r.$$

- (ii) For nonuniform motion on a circle with $r(t) = R$ and $\dot{R} = 0$, a more general angular function $\varphi(t)$ has to be considered, for example

$$\varphi(t) = \alpha \sin \gamma t$$

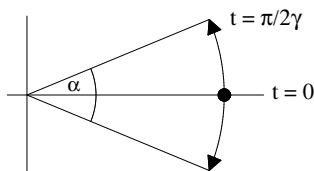
which would describe an oscillation on a section of a circle (Fig. 2.39). The three kinematic vectors are

$$\mathbf{r}(t) = R \mathbf{e}_r \quad \mathbf{v}(t) = R \dot{\varphi} \mathbf{e}_\varphi \quad (2.61)$$

$$\mathbf{a}(t) = -R^2 \dot{\varphi}^2 \mathbf{e}_r + R \ddot{\varphi} \mathbf{e}_\varphi. \quad (2.62)$$

The acceleration is not central as

$$\mathbf{a}(t) \neq f(t) \mathbf{r}(t).$$

**Fig. 2.39.** Oscillation on an arc

- (iii) The motion on a Lissajous ellipse (for details see ● Probl. 2.8). The starting point is in this case

$$r(t) = \sqrt{a^2 \cos^2 \omega t + b^2 \sin^2 \omega t}$$

$$\varphi(t) = \arctan \left(\frac{b}{a} \tan \omega t \right) .$$

The time dependence of the angle is more complicated than $\varphi(t) = \omega t$. The kinematic vectors are therefore

$$\mathbf{r}(t) = r(t) \mathbf{e}_r$$

$$\mathbf{v}(t) = -\frac{1}{r}(a^2 - b^2)\omega \sin \omega t \cos \omega t \mathbf{e}_r + \frac{ab\omega}{r} \mathbf{e}_\varphi$$

$$\mathbf{a}(t) = -\omega^2 r(t) \mathbf{e}_r .$$

The last line follows only after a slightly longer computation, if one starts explicitly with the evaluation of (2.60).

The use of polar coordinates does not seem to be too useful for the last example. Nonetheless the statement can be made:

The choice of polar coordinates is optimal for problems with a central acceleration (central force problems).

This statement can be confirmed by two arguments:

- The first is a discussion of the area theorem in terms of polar coordinates. For this purpose a complete three-dimensional coordinate system, which is spanned by the unit vectors $\mathbf{e}_r(t)$, $\mathbf{e}_\varphi(t)$ and \mathbf{e}_z , is required (Fig. 2.42a, p. 61, with the replacement $\rho \rightarrow r$)³. For a trajectory in the x - y plane the cross product of the position and the velocity vectors is

$$(\mathbf{r} \times \mathbf{v}) = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\varphi & \mathbf{e}_z \\ r & 0 & 0 \\ \dot{r} & r\dot{\varphi} & 0 \end{vmatrix} = r^2 \dot{\varphi} \mathbf{e}_z .$$

This yields for the vector of the areal velocity ((2.41), p. 48)

$$\dot{\mathbf{A}}(t) = \frac{1}{2} r(t)^2 \dot{\varphi}(t) \mathbf{e}_z .$$

³ Such a coordinate system is, as shown below, the basis for the discussion of cylindrical coordinates in three-dimensional space.

The magnitude of this vector is independent of time

$$|\dot{\mathbf{A}}(t)| = \dot{A}(t) = \frac{1}{2}r(t)^2\dot{\varphi}(t) = \text{const.} \quad (2.63)$$

The derivative of this quantity with respect to time vanishes for problems with central acceleration

$$\ddot{A}(t) = r\dot{r}\dot{\varphi} + \frac{1}{2}r^2\ddot{\varphi} = \frac{r}{2}(2\dot{r}\dot{\varphi} + r\ddot{\varphi}) = 0 ,$$

as a comparison of this result with the decomposition of the acceleration vector (2.60) leads to the statement

$$\dot{A} = \text{const.} \quad \longleftrightarrow \quad a_\varphi = 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 . \quad (2.64)$$

If the theorem of area conservation is valid, then the acceleration is central and vice versa. This is the reason why the use of polar coordinates is a more efficient choice in many practical problems, for instance for the determination of the trajectory from a given, radial acceleration.

- The second is the demonstration that the solution of the equations of motion is simpler in terms of plane polar coordinates. The standard form of a central acceleration is often written as

$$\mathbf{a}(t) = -f(r(t)) \mathbf{e}_r , \quad (2.65)$$

where f is an arbitrary function of the separation of the mass point from the origin. Examples are:

$$\begin{aligned} f(r) &= kr : && \text{the two-dimensional harmonic oscillator} \\ f(r) &= \frac{k'}{r^2} : && \text{Kepler's (planetary motion) problem.} \end{aligned}$$

The problem, that is posed, is the determination of $r(t)$ and $\varphi(t)$ for a given function $f(r)$ and the initial conditions $r(0)$, $\dot{r}(0)$, $\varphi(0)$ and $\dot{\varphi}(0)$. The solution can be obtained with the following steps:

- (1) From $a_\varphi = 0$ follows $r^2\dot{\varphi} = C$ or

$$\dot{\varphi} = \frac{C}{r^2} .$$

C is determined by the initial conditions.

- (2) The differential equation for the function $r(t)$ is

$$a_r = -f(r) \quad \text{or} \quad \ddot{r} - r\dot{\varphi}^2 = -f(r) .$$

Insertion of $\dot{\varphi}$ from step (1) gives a differential equation for $r(t)$

$$\ddot{r} = -f(r) + \frac{C^2}{r^3}$$

which has to be solved. How this is done, will be discussed in Chap. 4 (and in ● Math.Chap. 2 and ● Math.Chap. 6). Assuming that the solution has been found, the last step is

- (3) Insert $r(t)$ into the equation for $\dot{\varphi}$ and integrate

$$\varphi(t) - \varphi(0) = \int_0^t \frac{C}{r(t')^2} dt' .$$

The advantage of the use of polar coordinates is apparent, if it is contrasted with the solution of the central acceleration problem in terms of Cartesian coordinates. The system of differential equations

$$\begin{aligned}\ddot{x} &= -\cos \varphi f(r) = -\frac{x}{\sqrt{x^2 + y^2}} f(\sqrt{x^2 + y^2}) \\ \ddot{y} &= -\sin \varphi f(r) = -\frac{y}{\sqrt{x^2 + y^2}} f(\sqrt{x^2 + y^2})\end{aligned}$$

has to be discussed in this case. The solution in terms of Cartesian coordinates is a more difficult task (with the exception of the two-dimensional harmonic oscillator), as the two differential equations are in general coupled.

2.4.1.2 Additional sets of coordinates in plane. There exists a collection of curvilinear coordinates⁴ in two and three space dimensions for the purpose of an optimal adaptation to specific geometrical situations. Instead of presenting this collection in detail, it seems more useful to illustrate the pattern which dictates the choice of a particular set of coordinates.

With the use of Cartesian coordinates, a plane is covered with a grid of orthogonal sets of straight lines ($x = \text{const.}$, $y = \text{const.}$). At each point of the plane there exists a local coordinate system, which is connected with the original choice by a translation. As the orientation of all local coordinate systems is the same as that of the original system and as the kinematic vectors \mathbf{v} and \mathbf{a} are free vectors (they can be moved around in the plane maintaining their length and direction), the decomposition of these vectors is the same in all local systems.

On the other hand, with polar coordinates (Fig. 2.40b) the plane is covered by families of concentric circles and rays ($r = \text{const.}$, $\varphi = \text{const.}$). The

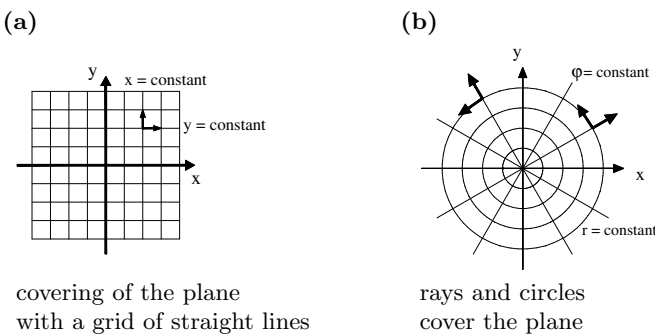


Fig. 2.40. Cartesian and polar coordinates

⁴ The useful reference [3] is unfortunately not in print any more. Check your library.

basis vectors of all local coordinate systems are still orthogonal, but the orientation of the bihedron changes from point to point. The decomposition of the kinematic vectors differs from one local system to the next. For a problem that is adapted to this pattern of coverage, differential equations are, however, decoupled and other quantities can have a simpler form.

An additional example of curvilinear coordinates in a plane are confocal elliptic coordinates (Fig. 2.41), which are set up by a grid of confocal ellipses and hyperbolae. The basis vectors of the local coordinate systems (with the origin at the intersection of the two sets of curves) are orthogonal. The de-

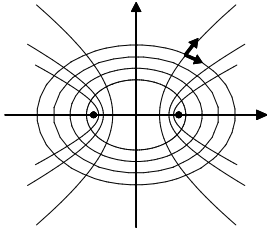


Fig. 2.41. Confocal elliptic coordinates: grid of confocal ellipses and hyperbolae

composition of the kinematical vectors, although it is more complicated in the general case, is simpler if the coordinates are in conformance with the symmetries of the problem (as e.g. for a double star system with one moon).

Alternative sets of orthogonal grids may be useful for the discussion of specific problems. There is no limit to the ingenuity.

2.4.2 Spatial coordinates

There exist two important sets of **curvilinear coordinate systems** in three-dimensional space: cylindrical (or cylinder) coordinates and spherical polar coordinates (for short, spherical coordinates).

2.4.2.1 Cylinder coordinates. Cylinder coordinates are the simplest extension of polar coordinates in the plane. The position of a point P in space is in this case specified by the following quantities (Fig. 2.42a):

$\rho(t) \rightarrow$ shortest distance of P from the z - axis

$\varphi(t) \rightarrow$ angle between the line $\overline{OP'}$ and the x - axis
 $(\overline{OP'}$ is the projection of the line \overline{OP}
 - origin to P - onto the x - y plane)

$z(t) \rightarrow$ shortest distance of P from the x - y plane.

In analogy to the coverage of a plane by grids of curves, points in space are described by the intersection of families of orthogonal surfaces (Fig. 2.42b).

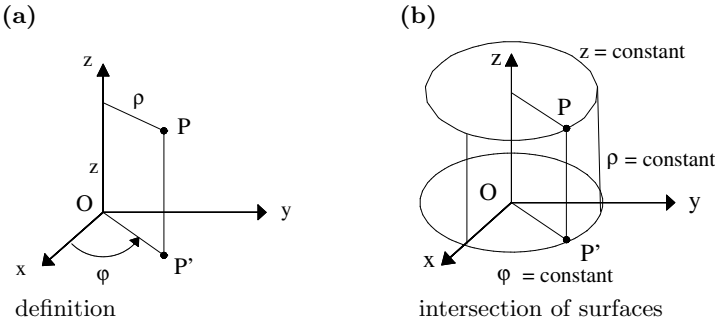


Fig. 2.42. Cylinder coordinates

The intersection of a cylinder ($\rho = \text{const.}$) with a plane ($z = \text{const.}$) and a half plane ($\varphi = \text{const.}$) is used for cylinder coordinates. The transformation between Cartesian and cylinder coordinates is given by

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad z = z, \quad (2.66)$$

the inverse is

$$\rho = \sqrt{x^2 + y^2} \quad \varphi = \arctan \frac{y}{x} \quad z = z. \quad (2.67)$$

These coordinates are also referred to a time changing (local) coordinate system (Fig. 2.43a, 2.43b), which is characterised by the basis vectors

$$\mathbf{e}_\rho(t), \mathbf{e}_\varphi(t), \mathbf{e}_z. \quad (2.68)$$

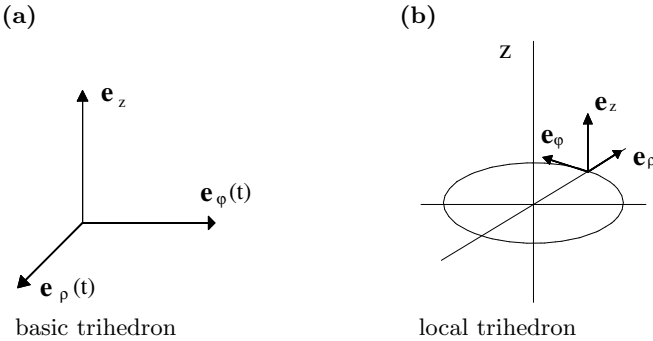


Fig. 2.43. Cylinder coordinates

The decomposition of the kinematic vectors with respect to the system of cylinder coordinates does not differ greatly from the case of polar coordinates (compare (2.54), (2.56) and (2.60))

$$\mathbf{r}(t) = \rho \mathbf{e}_\rho(t) + z \mathbf{e}_z \quad (2.69)$$

$$\mathbf{v}(t) = \dot{\rho} \mathbf{e}_\rho(t) + \rho \dot{\varphi} \mathbf{e}_\varphi(t) + \dot{z} \mathbf{e}_z \quad (2.70)$$

$$\mathbf{a}(t) = (\ddot{\rho} - \rho \dot{\varphi}^2) \mathbf{e}_\rho(t) + (\rho \ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) \mathbf{e}_\varphi(t) + \ddot{z} \mathbf{e}_z . \quad (2.71)$$

The orthogonality of the local trihedron leads to the following magnitudes of these vectors

$$r(t) = \sqrt{\rho^2 + z^2} \quad (2.72)$$

$$v(t) = \sqrt{\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2} \quad (2.73)$$

and a similar expression for $a(t)$.

2.4.2.2 Spherical coordinates. In terms of spherical coordinates a point P in space is characterised by the following quantities (Fig. 2.44a):

$r(t) \rightarrow$ distance of the point from the origin

$\theta(t) \rightarrow$ polar angle (angle between \overline{OP} and the z -axis)

$\varphi(t) \rightarrow$ azimuthal angle (angle between the projection of \overline{OP} onto the x - y plane and the x -axis).

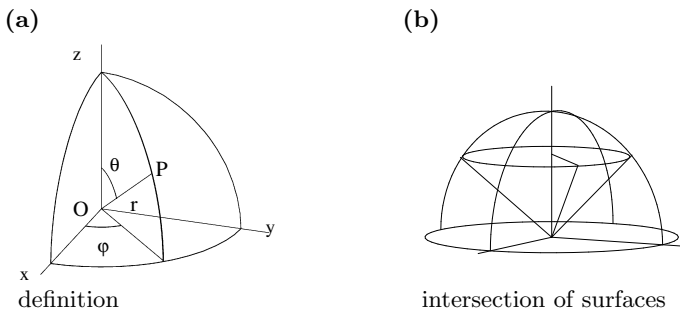


Fig. 2.44. Spherical coordinates

This characterisation corresponds to the specification of a point as an intersection (Fig. 2.44b) of concentric spheres ($r = \text{const.}$), cones ($\theta = \text{const.}$, with the range $0 \leq \theta \leq \pi$) and half planes ($\varphi = \text{const.}$, with the range $0 \leq \varphi \leq 2\pi$).

The set of transformations between Cartesian and spherical coordinates are

$$\begin{aligned} x &= r \cos \varphi \sin \theta \\ y &= r \sin \varphi \sin \theta \\ z &= r \cos \theta . \end{aligned} \quad (2.74)$$

The first two equations describe the projection of the position vector onto the x - y plane and a subsequent projection on each of the coordinate axes.

The last equation corresponds to a direct projection of the position vector on the z -axis. The inverse of these transformations is

$$r = \sqrt{x^2 + y^2 + z^2} \quad \varphi = \arctan \frac{y}{x} \quad \theta = \arctan \frac{\sqrt{x^2 + y^2}}{z}. \quad (2.75)$$

The orthogonal, local trihedron (Fig. 2.45) is specified by

$\mathbf{e}_r(t)$ unit vector in the radial direction

$\mathbf{e}_\theta(t)$ unit vector orthogonal to \mathbf{e}_r ,
tangential to the sphere in the direction of a circle of longitude

$\mathbf{e}_\varphi(t)$ orthogonal complement defined by $\mathbf{e}_\varphi = \mathbf{e}_r \times \mathbf{e}_\theta$,
a unit vector tangential to the sphere in the direction of a
circle of latitude.

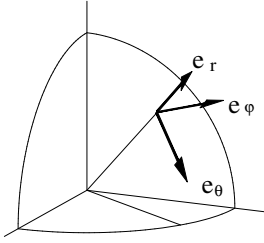


Fig. 2.45. Spherical coordinates: local trihedron

In order to determine the decomposition of the kinematical vectors with respect to the local trihedron the transformation between the Cartesian and the spherical basis vectors is needed. The first of these transformations is (Fig. 2.46a)

$$\mathbf{e}_r = (\sin \theta \cos \varphi) \mathbf{e}_x + (\sin \theta \sin \varphi) \mathbf{e}_y + (\cos \theta) \mathbf{e}_z. \quad (2.76)$$

This corresponds to the Cartesian decomposition of a vector from the origin to a point with $r = 1$, in which the Cartesian coordinates are expressed in terms of spherical coordinates (2.74). The vector \mathbf{e}_θ points along a circle of longitude and is rotated by an angle $\pi/2$ with respect to \mathbf{e}_r (Fig. 2.46b). If the angle θ in (2.76) is replaced by $\theta + \pi/2$, the result

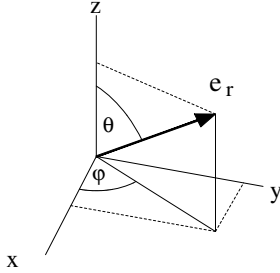
$$\mathbf{e}_\theta = (\cos \theta \cos \varphi) \mathbf{e}_x + (\cos \theta \sin \varphi) \mathbf{e}_y + (-\sin \theta) \mathbf{e}_z \quad (2.77)$$

is obtained. The third unit vector (Fig. 2.47, with a view along the z -axis) is

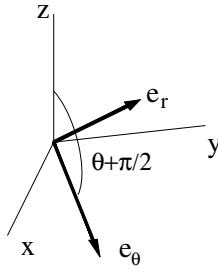
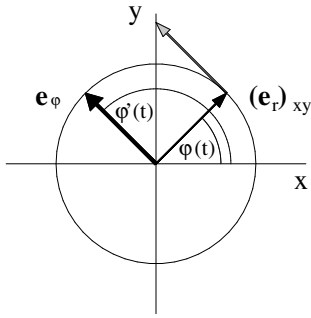
$$\mathbf{e}_\varphi = -\sin \varphi \mathbf{e}_x + \cos \varphi \mathbf{e}_y. \quad (2.78)$$

The statement follows directly from the definition of this vector by a cross product $\mathbf{e}_\varphi = \mathbf{e}_r \times \mathbf{e}_\theta$ or with the argument: the vector is tangential to a circle of latitude. Hence it does not possess a z -component ($\theta = \pi/2$). It is orthogonal to the projection of \mathbf{e}_r onto the x - y plane. The angle with respect

(a)

definition of the vector \mathbf{e}_r

(b)

definition of the vector \mathbf{e}_θ **Fig. 2.46.** Definition of the unit vectors \mathbf{e}_r and \mathbf{e}_θ **Fig. 2.47.** Definition of the vector \mathbf{e}_φ

to the x -axis is thus $\varphi' = \varphi + \pi/2$. Replacing φ by φ' and θ by $\pi/2$ in (2.76) yields e.g. the x -component, namely $-\sin \varphi$ (see also ● Probl. 2.10).

The position vector in spherical coordinates is by definition

$$\mathbf{r}(t) = r(t) \mathbf{e}_r. \quad (2.79)$$

The velocity vector is calculated via

$$\begin{aligned} \mathbf{v}(t) &= \dot{\mathbf{r}}(t) \\ &= \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_r. \end{aligned}$$

The time derivative of the basis vector \mathbf{e}_r can be obtained from the relation (2.76) as

$$\begin{aligned} \dot{\mathbf{e}}_r &= \dot{\theta}(\cos \theta \cos \varphi \mathbf{e}_x + \cos \theta \sin \varphi \mathbf{e}_y - \sin \theta \mathbf{e}_z) \\ &\quad + \dot{\varphi}(-\sin \theta \sin \varphi \mathbf{e}_x + \sin \theta \cos \varphi \mathbf{e}_y). \end{aligned}$$

Direct comparison with the vectors \mathbf{e}_θ and \mathbf{e}_φ in (2.77) and (2.78) leads to

$$\dot{\mathbf{e}}_r = \dot{\theta} \mathbf{e}_\theta + \dot{\varphi} \sin \theta \mathbf{e}_\varphi, \quad (2.80)$$

so that finally the relation

$$\mathbf{v}(t) = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\varphi} \sin \theta \mathbf{e}_\varphi \quad (2.81)$$

results for the velocity vector. The polar angle and its time derivative are

$$\theta = \frac{\pi}{2} \quad \text{and} \quad \dot{\theta} = 0 ,$$

if a mass point moves only in the x - y plane, so that

$$\mathbf{v}(t) = \dot{r} \mathbf{e}_r + r\dot{\varphi} \mathbf{e}_\varphi .$$

This limit agrees with the result (2.56) for plane polar coordinates.

The decomposition of the acceleration vector $\mathbf{a}(t) = \dot{\mathbf{v}}(t)$ can be found in the same manner. The calculation and the final result are, however, quite cumbersome. For this reason it will not be given here. The result can be found in \bullet D.tail 2.2.

This concludes the discussion of the most important sets of curvilinear coordinates. They will be used shortly. The transition from Cartesian to curvilinear coordinates, that has been presented here in an elementary fashion, is not the most elegant. The discussion of the Lagrange formulation of mechanics in Chap. 5.3 provides a more elegant and general method for the introduction of 'generalised' coordinates.

3 Dynamics I: Axioms and Conservation Laws

Three axioms, first formulated by Newton, are the foundation of classical mechanics. The first axiom addresses the question of appropriate systems of reference (inertial systems) for the discussion of mechanical problems, the second introduces the basic equations of motion. The third axiom can be considered as an attempt to comment on the fundamental interactions that are found in nature.

Concepts that are introduced with the axioms are the mass (in particular the inertial mass) and forces. Once these basic concepts are established, additional quantities such as momentum, work, energy, angular momentum, torque etc. can be discussed. If the sole aim of mechanics were the study of the motion of objects (mass points or objects composed of mass points) one could concentrate on the second axiom. The additional quantities just indicated are, however, distinguished by the fact that they satisfy, under suitable conditions, conservation laws. These laws can be used to extract partial statements on or gain insight into mechanical systems with relatively little effort.

The present chapter contains a detailed discussion of the three axioms and a step by step approach towards the basic dynamical concepts of mechanics.

3.1 The axioms of mechanics

The three axioms were published in the main work of Newton 'Philosophiae naturalis principia mathematica', in 1687. It is necessary to address the concepts of

force and **mass**

before a discussion of the actual axioms can be undertaken.

3.1.1 The concept of force

The word 'force' is part of everyday language. If a person pulls at an object or pushes it, one would state: 'the object is subjected to a force'. A series of simple (thought) experiments are, nonetheless, useful in order to gain a more precise notion of this concept.

1. A force is a vectorial quantity \mathbf{F} . The direction of the push or pull makes a difference. Two forces of the same strength, which are applied in (exactly) opposite directions, cancel each other.
2. The strengths of different forces (the magnitudes of the vector) can be compared if the forces are applied to a spring (Fig. 3.1). The extension of a spring is (according to Hooke's law, assuming the extension is not too large) proportional to the force applied ($x \propto F$). This allows a comparison of forces (more precisely, of the magnitude of forces) by comparison of the extensions $F/F' = x/x'$.

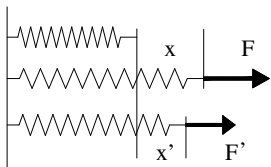


Fig. 3.1. Simple comparison of forces

3. The application of a force to an object, that can move freely, leads to a motion of the object or to a change of its motion. Here mass plays a role. The more 'massive' an object is, the smaller is the change of its motion for a given force.

Such simple considerations can be helpful in illustrating the idea of forces. For a quantitative discussion it is necessary to consider the second of Newton's axioms, which states in its simplest form

$$\mathbf{F} = m\mathbf{a}. \quad (3.1)$$

This equation should be read as follows. There exists a victim, the mass point m . This object reacts with an acceleration $\mathbf{a} = \mathbf{F}/m$ if a force \mathbf{F} is applied.

The equation (3.1) constitutes a simple form of the basic **law of motion** of mechanics. The acceleration of a mass point m is determined by the force, which acts on it. The knowledge of the acceleration and the initial conditions at a time t_0 allows the calculation of the motion of the object for $t > t_0$ (as discussed in terms of simple examples in Chap. 2.2).

This argumentation can be reversed. The acceleration of an object of mass m can be extracted from its trajectory. Knowledge of the acceleration then implies knowledge of the force (the total force) that influences the motion of the object. The second axiom is the tool which can be used to put both the concept of force as well as the concept of mass on a quantitative basis.

$$\begin{array}{ccccc}
 & \cdot 1/m & & \text{differential equation} & \\
 \mathbf{F} & \xrightarrow{\quad} & \mathbf{a} & \xrightarrow{\quad} & \mathbf{r}(t) . \\
 & \xleftarrow{\quad} & & \xleftarrow{\quad} & \\
 & \cdot m & & \text{differentiation} &
 \end{array}$$

3.1.2 Inertial and gravitational mass

In order to obtain a more precise idea of the concept of mass the following thought experiment can be performed: accelerate two masses m_1 and m_2 from rest with the same force \mathbf{F} . It is not necessary at this stage to actually measure the strength of the force. The only requirement is equality of the forces in the two experiments. On the basis of the axiom the resulting linear motion is characterised by the statement

$$F = m_1 a_1 = m_2 a_2 \quad \longrightarrow \quad \frac{m_1}{m_2} = \frac{a_2}{a_1} .$$

The accelerations a_1 and a_2 can be measured (for example by analysis of the time change of the position $x_i(t)$). The measured (e.g. constant) accelerations yield the ratio of the two masses. For an absolute statement one needs a 'standard mass'. This has been defined by international convention. It is the mass of 1 kilogram [kg], which is stored in Sèvres in the vicinity of Paris. With the choice of a standard mass, the second axiom provides a dynamical method to determine masses quantitatively. A more massive object will experience a smaller acceleration for a given force. The mass is a measure of the resistance of an object (mass point) against changes of motion. Masses which are determined with this dynamical method carry the name **inertial mass**.

The introduction of a standard mass and the second axiom lead to a unit of measurement for the **force**. In the SI System (Système International with the basic units meter, kilogram (corresponding to the dimension [M]) and second) a force, which accelerates a mass of 1 kg with 1 m/s^2 is called 1 Newton (N)

$$1 \text{ N} = 1 \frac{\text{kg m}}{\text{s}^2} .$$

The unit of the force in the CGS-system (centimetre, gram, second, still preferred in Theoretical Physics) is dyne with the symbol dyn

$$1 \text{ dyn} = 1 \frac{\text{g cm}}{\text{s}^2} .$$

The factor for the conversion of these units is $1 \text{ N} = 10^5 \text{ dyn}$.

The term 'inertial mass' needs to be discussed more explicitly. The determination of masses in everyday life will in most cases not be based on the dynamical method indicated. Normally one uses a pair of scales (Fig. 3.2a).

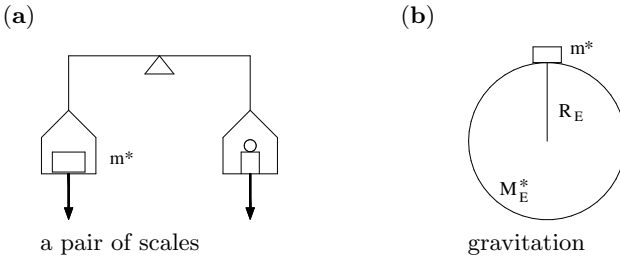


Fig. 3.2. Illustration of the concept of the gravitational mass

The gravitational force of the earth, which acts on the object and on the weights, is compared in this method. For the gravitational force between two mass points with m_1^* and m_2^* at a separation r Newton gave the relation (see Chap. 3.1.6)

$$F = \gamma \frac{m_1^* m_2^*}{r^2}. \quad (3.2)$$

The strength of the interaction is determined by the gravitational constant γ (see Chap. 3.2.4 for a more extensive discussion of the law of gravitation). For the moment it is sufficient to state: even if the objects on the scales and the earth are no mass points, one may use this formula for the specification of the weight G . The weight at the surface of the earth is (Fig. 3.2b)

$$G = m^* \left[\frac{M_E^* \gamma}{R_E^2} \right] = m^* g. \quad (3.3)$$

The gravitational constant γ and the gravitational acceleration g , which has already been introduced in Chap. 2.1, are related via the mass of the earth M_E^* and its radius R_E (assuming a spherical shape). It should be noted that the weight of an object corresponds to the force, which the earth exerts on the object with mass m^* .

There is no reason, why the mass m^* , which enters into the law of gravitation, and the mass m , which measures the resistance against a change of its motion, should be identical. For this reason the mass in the law of gravitation is termed the **gravitational mass** (and denoted by m^*). The equation of motion that describes the free fall in the vicinity of surface of earth has to be specified correctly as

$$m\mathbf{a} = m^*\mathbf{g}$$

or in the form of a differential equation

$$\ddot{\mathbf{r}} = \frac{m^*}{m}\mathbf{g}.$$

Starting in the year 1915 (the date of the formulation of the general theory of relativity by A. Einstein) attempts have been undertaken to find out,

whether there is any difference between the two masses. At the present time the answer is

$$m = m^*$$

with an accuracy of $\Delta m/m = (m - m^*)/m \approx 10^{-10}$. This corresponds to an accuracy of $10^{-8}\%$.

The reader may wonder, why one goes to such lengths with respect to a point which seems like splitting hairs. The reason is the question of the validity of the general theory of relativity, which will not be detailed at all at this point. In order to answer the question implied, it is sufficient to describe another (rather hypothetical) experiment, which however explains the main point (see Fig. 3.3). The laboratory, in which this experiment will be performed, is a large closed container. The laboratory is first located on earth.

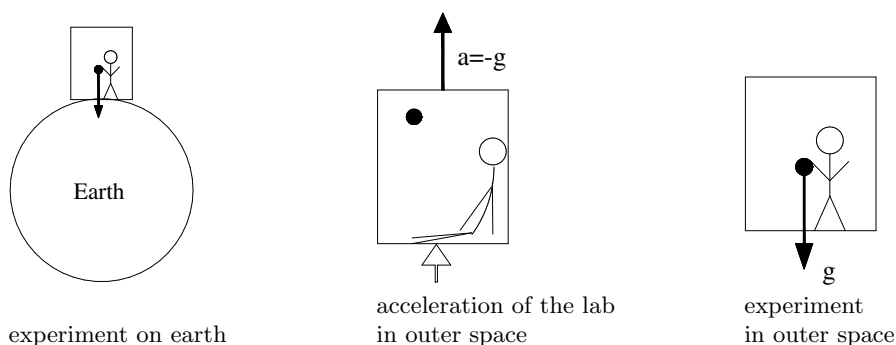


Fig. 3.3. Concerning the difference between the inertial and gravitational masses

A physicist in the laboratory weighs an object and notes

$$G = m^* g .$$

While the physicist takes a rest, the lab is transported into outer space to a position, which is free of gravitational effects. The container (and its contents) is then accelerated with a constant acceleration $\mathbf{a} = -\mathbf{g}$. The physicist (still sleeping) and the object first float in the container, but after the onset of the acceleration they are deposited onto one side and are accelerated with the container. The last two statements describe the situation from the point of view of an outside observer. The physicist (now awake again) interprets the situation as follows. He/she measures an apparent acceleration $\mathbf{a}' = +\mathbf{g}$, which emulates the original acceleration due to gravity. The second measurement of the 'weight' would result in

$$G' = mg$$

in this case. It is the inertial mass which is recorded (even if the experimenter does not know this). It would be possible to distinguish the two situations

(position on the earth at rest with gravitation and position in outer space with a nongravitational acceleration) via $\mathbf{G} \neq \mathbf{G}'$, if the two masses were different ($m \neq m^*$). This should, according to Einstein, not be possible. The general theory of relativity is based on the postulate: there is, as a matter of principle, no difference between the acceleration due to gravity or any other cause. The theory would not be valid, if the relation

$$m^* = m$$

would not be correct. This equality will always be used in the chapters that follow.

3.1.3 The axioms

After these preparatory remarks the discussion of the axioms proper can be undertaken. The axioms are

Axiom 1:

A mass point is at rest or moves with a constant velocity, if it is not subjected to any forces.

Axiom 2:

The equation of motion

$$\frac{d}{dt}(mv) = \mathbf{F} \quad \xrightarrow{\dot{m}=0} \quad m\mathbf{a} = \mathbf{F} \quad (3.4)$$

is valid, if a force \mathbf{F} acts on a mass point m . This statement is an extension of the simpler form $m\mathbf{a} = \mathbf{F}$, which is valid for a mass which does not change with time. The extended version covers situations with masses that change with time.

Axiom 3:

If a mass point m_1 exerts a force \mathbf{F}_{12} on a mass m_2 , then the mass m_2 exerts a force \mathbf{F}_{21} on m_1 . The two forces have the same magnitude but point in opposite directions.

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \quad (3.5)$$

This compact formulation of the axioms definitely calls for a more extensive comment. Concerning the first axiom, the following remarks apply.

3.1.4 The first axiom: inertial systems

This axiom has already been formulated by G. Galilei using similar words. It is also called the principle of inertia. At first glance it seems to be superfluous, because the second axiom leads, for $\mathbf{F} = \mathbf{0}$, to the statement

$$\frac{d}{dt}(mv) = \mathbf{0} \longrightarrow mv = \text{const.} \quad \longrightarrow \quad v = \frac{\text{const}}{m}.$$

The specific inclusion of the axiom makes nonetheless good sense. It expresses the fact, that laws of motion can only have a precise meaning if they are referred to an appropriate frame of reference. The state of rest and the state of uniform motion are equated in the axiom. The description of the time development of the motion of an object from the point of view of two observers with different frames of reference is completely equivalent, provided the frames move with respect to each other with a uniform velocity.

The details of the corresponding argumentation are: two coordinate sys-

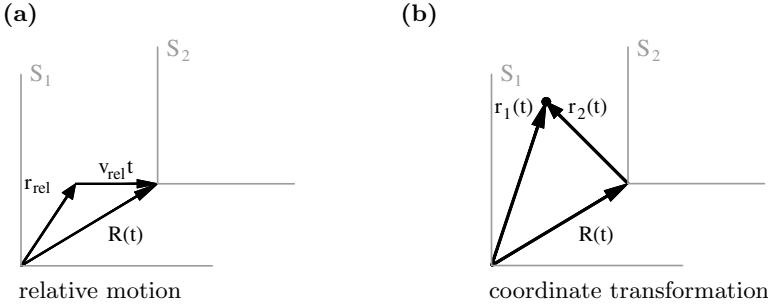


Fig. 3.4. The Galilei transformation

tems S_1 and S_2 move with a uniform velocity with respect to each other. The position of the origin of system S_2 as seen from the perspective of S_1 is given by (Fig. 3.4a)

$$\mathbf{R}(t) = \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}} t. \quad (3.6)$$

The position of a mass point can be measured by each of the two observers. The relation

$$\mathbf{r}_1(t) = \mathbf{R}(t) + \mathbf{r}_2(t) = \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}} t + \mathbf{r}_2(t), \quad (3.7)$$

where $\mathbf{r}_1(t)$ is the position at time t from the point of view of S_1 and $\mathbf{r}_2(t)$ the position at time t from the point of view of S_2 , relates the two observations (see Fig. 3.4b). This equation, which describes a coordinate transformation, is called a **Galilei transformation**¹.

A relation between the velocities of the object as viewed from the two systems is obtained, if the transformation (3.7) is differentiated with respect to time

$$\mathbf{v}_1(t) = \mathbf{v}_{\text{rel}} + \mathbf{v}_2(t). \quad (3.8)$$

¹ An implicit assumption is used in the formulation of this transformation law. It is assumed, that the measurement of time is independent of the frame of reference: $t_1 = t_2 = t$. This apparently reasonable assumption does not hold according to the special theory of relativity. In classical mechanics it can be assumed to be (approximately) correct.

The velocity \mathbf{v}_1 is the velocity of the object from the point of view of S_1 , the velocity \mathbf{v}_2 from the point of view of S_2 . The formula (3.8), which is known as the **theorem of addition of velocities**, allows a direct conversion of the two points of view.

Differentiating once again leads to the statement

$$\mathbf{a}_1(t) = \mathbf{a}_2(t).$$

This result may be interpreted in the following fashion: the accelerations of a mass point as registered by each of the two observers are equal. This implies, that the forces as seen by each of two observers have to be equal as well, if it is assumed that the mass is independent of the point of view

$$m\mathbf{a}_1 = m\mathbf{a}_2 \implies \mathbf{F}_1 = \mathbf{F}_2.$$

For each of the observers the same form of the equation of motion is valid

$$m\mathbf{a}_1 = \mathbf{F}_1 \quad m\mathbf{a}_2 = \mathbf{F}_2.$$

The equations of motion of the two observers are completely equivalent. Any differences of the actual trajectories registered are solely due to the difference of the initial conditions.

The following 'experiment' (see Fig. 3.5 and compare the arguments in Chap. 1) illustrates the consequence of these remarks: Assume that the two frames of reference (two-dimensional) coincide at time $t = 0$ and that system S_2 moves (with respect to S_1) with the velocity v_{rel} in the direction of the positive x -axis. The observer in S_2 throws an object (from his point of view) vertically into the air. He/she records a linear free fall motion, which is characterised by the differential equation $m\mathbf{a}_2 = m\mathbf{g}$ and the initial conditions

$$\mathbf{r}_2(0) = (0, 0), \quad \mathbf{v}_2(0) = (0, v_0).$$

An observer associated with S_1 registers a parabola, which is characterised by the differential equation $m\mathbf{a}_1 = m\mathbf{g}$ and the initial conditions

$$\mathbf{r}_1(0) = (0, 0), \quad \mathbf{v}_1(0) = (v_{\text{rel}}, v_0).$$

The explicit solutions of the problems of motion of the two observers are

$$\begin{aligned} \mathbf{r}_1(t) &= (v_{\text{rel}}t, -\frac{g}{2}t^2 + v_0t) & \mathbf{v}_1(t) &= (v_{\text{rel}}, -gt + v_0) \\ \mathbf{r}_2(t) &= (0, -\frac{g}{2}t^2 + v_0t) & \mathbf{v}_2(t) &= (0, -gt + v_0). \end{aligned}$$

These results lead to the following equations for the trajectories in the two systems

$$\begin{aligned} y_1 &= -\frac{g}{2v_{\text{rel}}^2}x_1^2 + \frac{v_0}{v_{\text{rel}}}x_1 \\ x_2 &= 0 \quad \text{with} \quad y_{2,\text{max}} = \frac{v_0^2}{2g}. \end{aligned}$$

The preceding discussion can be summarised in the form: the question of rest or uniform motion is a question of the point of view. The equation of motion does not change with the point of view. All coordinate systems, that move uniformly with respect to each other, are, in this sense, equivalent. Such systems are called **inertial systems**. The first axiom should be considered as a definition of the concept of an inertial system.

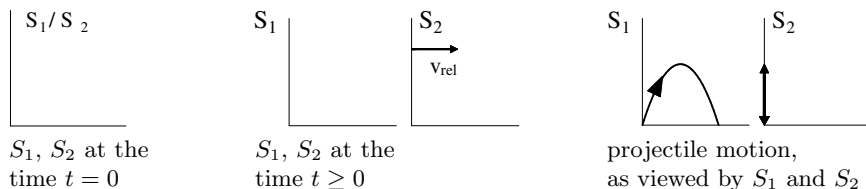


Fig. 3.5. Projectile motion as viewed from two different inertial systems

The situation is quite different if an inertial system S_1 and a system S_2 which is accelerated with respect to S_1 is considered. The two systems are not equivalent in this case. The simplest situation occurs for a uniform relative acceleration of system S_2 . The equation of motion in system S_1 is

$$m\ddot{\mathbf{r}}_1 = \mathbf{F}_1.$$

The Galilei transformation should now be replaced by the transformation

$$\mathbf{r}_1(t) = \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}t + \frac{\mathbf{a}_{\text{rel}}}{2}t^2 + \mathbf{r}_2(t). \quad (3.9)$$

The graphical representation of this equation still corresponds to Fig. 3.4b, but the actual form of $\mathbf{R}(t)$ differs. Differentiation of (3.9) leads to

$$\ddot{\mathbf{r}}_1 = \mathbf{a}_{\text{rel}} + \ddot{\mathbf{r}}_2 \longrightarrow m\ddot{\mathbf{r}}_1 = m\mathbf{a}_{\text{rel}} + m\ddot{\mathbf{r}}_2$$

so that the replacement of $m\ddot{\mathbf{r}}_1$ in the equation of motion yields

$$m\ddot{\mathbf{r}}_2 = (\mathbf{F}_1 - m\mathbf{a}_{\text{rel}}) = \mathbf{F}_2. \quad (3.10)$$

The force that acts in S_2 is different from the force in S_1 . The additional term is solely a consequence of the relative acceleration. For this reason it is called an **apparent force**. Apparent forces are discussed in detail in Chap. 6.2.

The observation of the time development of the motion of an object from the point of view of the earth poses some problems, as a coordinate system associated with the earth is not an inertial system. The earth rotates about the North-South axis as well as around the sun. A rotation is an accelerated form of motion. The effect of the rotation of the earth may be neglected in many instances, so that the earth-bound system can be considered to be (sufficiently) inertial. Effects due to the apparent forces are, on the other hand, definitely observable (e.g. for the Foucault pendulum or, in nature, through the different structure of cyclones on the northern or the southern

hemisphere (see Chap. 6.2)). A coordinate system that is fixed to the sun, is also no inertial system. The sun rotates around the centre of our galaxy. The motion of the sun is, however, a better approximation of a uniform, straight line motion. The solar-bound system is a better approximation of an inertial system than the earth-bound system.

Corrections to the Galilei transformation, which concern the comparison of time scales as well as of masses observed in different inertial reference systems, are necessary. These corrections do not touch on the question of equivalence. They lead, however, to a completely different form of the transformation laws. As they only become apparent, if at least one of the velocities involved in (3.8) is of the order of the velocity of light, they belong to the domain of the special theory of relativity (see Vol. 2). For velocities of the usual kind, the 'relativistic corrections' can safely be neglected.

3.1.5 The second axiom: momentum

The second axiom is, as already discussed, the central law of motion of classical physics. Classical mechanics is a deterministic theory. If the forces, that act on a mass point, and the position and the velocity at a particular time are known, the previous as well as the future history of the motion can be calculated (Fig. 3.6a). This statement is, as indicated in the short excursion on 'chaotic motion' in Chap. 5.4.3, valid for particular classes of problems or at least for more or less short intervals of time.

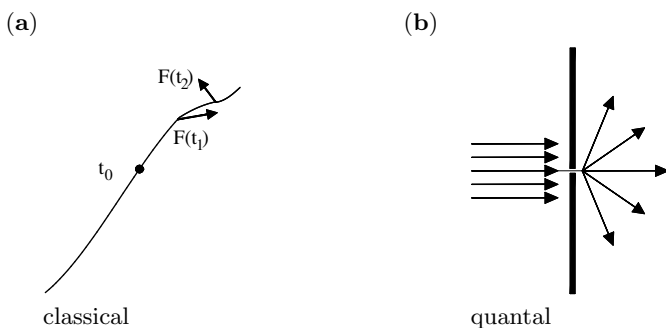


Fig. 3.6. Concerning determinism

The deterministic point of view does not hold for the dynamics of the micro-world (quantum mechanics). For instance, it is, as a matter of principle, not possible to state the (mechanical) initial conditions necessary for the quantum description of the motion of an electron. This is the content of Heisenberg's uncertainty relation: a reasonably accurate determination of the position (for instance with a diaphragm, Fig. 3.6b) forbids an equally accurate determination of the velocity. The uncertainty relation is the reason that the

motion of micro-particles can only be analysed in terms of probabilities (see Vol. 3).

The quantity $m\mathbf{v}$, which enters into the more general formulation of the second axiom

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F} \quad (3.11)$$

is the **momentum vector** (for short **momentum**)

$$\mathbf{p} = m\mathbf{v} . \quad (3.12)$$

The second axiom therefore states that the force which acts on a mass point leads to a time change of its momentum. This general formulation of the second axiom is only needed for special problems of classical mechanics, as e.g. for the calculation of the motion of a rocket taking into account the change of its mass due to the loss of fuel (see ● Probl. 3.3-3.5). The fact that momentum is the dynamical quantity for the characterisation of motion and not the velocity can easily be experienced by trying to stop two objects with the same velocity but different mass (e.g. a fly and a truck).

3.1.6 The third axiom: interactions

The formulation of the third axiom is an attempt to comment in a general fashion on the fundamental forces of nature. At Newton's time only gravitation was known. Today it is assumed that there exist four fundamental forces, which will be briefly presented here, even though they are better discussed in the context of elementary particle physics.

The story begins with the law of gravitation (3.2), which Newton derived from the data for the motion of planets², with the aid of the argument

trajectory	→	acceleration	→	force.
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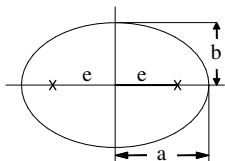
Newton was able to exploit a summary of the details of planetary motion in the form of **Kepler's laws**. These laws, which crowned the observation of planetary motion over more than one thousand years, are:

1. The trajectories of planets are ellipses (Fig. 3.7). The sun is in one of the focal points of the ellipses.
2. The position vector from sun to planet (the radius vector) covers equal areas in equal time intervals (compare Chap. 2.3.3).
3. The squares of the period T of the planets are proportional to the third power of the semi-major axis a

$$T^2 = \kappa a^3 .$$

The constant of proportionality κ is, in good approximation, the same for all the planets.

² A complete discussion of the problem of planetary motion (with all mathematical details) is found in Chap. 4.1.

**Fig. 3.7.** Parameters of the ellipse

A simple derivation of the gravitational law uses the fact that most of the planetary ellipses are quite good circles. In order to characterise the shape of ellipses the concept of **eccentricity** is used. The eccentricity is defined via the distance e between each of the focal points and the centre of an ellipse. For an ellipse with the semiaxes a , b ($a \geq b$) this distance is (Fig. 3.7)

$$e = [a^2 - b^2]^{1/2} .$$

The eccentricity ϵ is obtained by division with the length of the semi-major axis

$$\epsilon = \frac{e}{a} = \left[1 - \frac{b^2}{a^2} \right]^{1/2} \quad 0 \leq \epsilon \leq 1 .$$

It is a measure of the 'flatness' of the ellipse. A circle is characterised by $\epsilon = 0$. The more ϵ approaches the value 1, the flatter is the ellipse. The inverse of the definition above reads

$$\frac{b}{a} = [1 - \epsilon^2]^{1/2} \approx 1 - \frac{1}{2}\epsilon^2 ,$$

where the expansion is valid for $\epsilon \ll 1$. The eccentricities of the nine planets are listed in Table 3.1.

Table 3.1. Planetary data: ratio of the semiaxes

planet	eccentricity ϵ	b/a	
Mercury	0.206	0.978552	(not a bad circle)
Venus	0.007	0.999975	(an excellent circle)
Earth	0.017	0.999855	
Mars	0.093	0.995666	
Jupiter	0.048	0.998847	
Saturn	0.056	0.998431	
Uranus	0.046	0.998941	
Neptune	0.009	0.999959	
Pluto	0.249	0.968503	

The assumption of a circular orbit is quite acceptable, with the possible exception of Mercury and Pluto, even though it is not too incorrect for these two planets either. The situation concerning the orbits can for instance be illustrated by the example of the earth: if the semi-minor axis of the earth were scaled down to 4 cm, the semi-major axis would measure 4.0006 cm.

With the assumption of circular orbits Newton's argumentation can be paraphrased as follows:

- (i) For a uniform motion on a circle (radius R) the formula for the radial acceleration (see (2.36), p. 45)

$$a_R = \frac{v^2}{R}$$

can be used.

- (ii) The speed v for the uniform motion is given by circumference divided by the period (this is Kepler's second law)

$$v = \frac{2\pi R}{T} \implies a_R = (2\pi)^2 \frac{R}{T^2}.$$

- (iii) The magnitude of the central force, that the sun exerts on the planet, can be given as

$$F = m_p a_R = m_p \frac{4\pi^2 R}{T^2}$$

on the basis of the second axiom.

- (iv) Use the third Kepler law ($T^2 = \kappa R^3$) to obtain

$$F = m_p \frac{4\pi^2}{\kappa} \frac{1}{R^2}.$$

The force, that the sun exerts on the planet, depends on the distance between sun and planet as $1/R^2$.

After Newton had found the $1/R^2$ - law for the force, which governs planetary motion — with similar arguments as presented — he noticed that the same law was also responsible for the motion of the moons around the planets (only some of the moons were known in Newton's times). It still took some time until it was realised that the law is universal. The final form of the gravitational force between two mass point m_1 and m_2 is (Fig. 3.8a)

$$\mathbf{F}_{1 \text{ on } 2} = \gamma \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2) \quad (3.13)$$

$$\mathbf{F}_{2 \text{ on } 1} = \gamma \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_2 - \mathbf{r}_1). \quad (3.14)$$

The third axiom $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is obviously satisfied (Fig. 3.8b). The gravitational constant γ has been measured for the first time in 1798 by Cavendish. The (approximate) value is

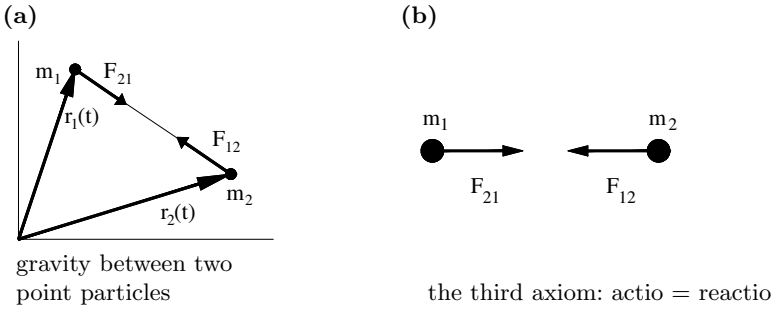


Fig. 3.8. Forces

$$\gamma = 6.674 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg s}^2} = 6.674 \cdot 10^{-8} \frac{\text{cm}^3}{\text{g s}^2}.$$

As a consequence of the 'low' value of this constant, it is found that the gravitational force between two masses (mass points) of a few kilogram, which are separated by a few meters, is very modest.

The determination of the gravitational constant is also of interest for the determination of the mass of the earth. The law of gravitation and the value of the gravitational acceleration at the surface of the earth can be combined, as already demonstrated, to give

$$M_E = \frac{gR_e^2}{\gamma}. \quad (3.15)$$

All the quantities on the right hand side of this equation can be determined experimentally.

The second fundamental force law has been discovered about 100 years (1785) after gravitation. It is **Coulomb's law**, describing the electrostatic force between two point charges (q). The force between two point charges has the same form as the law of gravitation

$$\mathbf{F}_{\text{el},1 \text{ on } 2} = k \frac{q_1 q_2}{r_{12}^3} (\mathbf{r}_2 - \mathbf{r}_1). \quad (3.16)$$

The masses are replaced by charges, the constant k has a different significance, but still the $1/r_{12}^2$ – law for the dependence on the distance between the two charges holds. Electrostatic forces satisfy the third axiom as well. A difference with respect to gravitation is the possibility of attraction and repulsion due to the two possible signs of the charges.

Shortly after the electric force, **magnetic forces** were explored more fully. The magnetic force³ is more complicated. In a (simple) situation, in which two point charges q_1 and q_2 move with constant velocities \mathbf{v}_1 and \mathbf{v}_2 ,

³ A magnetic force can be interpreted as a manifestation of an electric force from the point of view of an observer moving past the source of the electric force according to the special theory of relativity. The fact that the classical Galilei

the mutual magnetic force is given by the formula (electric CGS units, see Vol. 2)

$$\begin{aligned} \mathbf{F}_{\text{magn},12} &= \frac{q_1 q_2}{c^2 r_{12}^3} [\mathbf{v}_2 \times (\mathbf{v}_1 \times \mathbf{r}_{12})] \\ \mathbf{F}_{\text{magn},21} &= \frac{q_1 q_2}{c^2 r_{12}^3} [\mathbf{v}_1 \times (\mathbf{v}_2 \times (-\mathbf{r}_{12}))]. \end{aligned} \quad (3.17)$$

The vector $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ stands again for the separation of the two charges

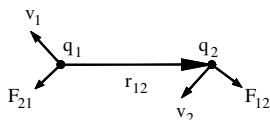


Fig. 3.9. Magnetic forces between two moving charges

and c is the velocity of light. The force vectors indicated in Fig. 3.9 are obtained, if the two charges have the same sign. Magnetic forces do not satisfy the third axiom.

In the last century two additional fundamental forces have been added: Today one distinguishes four fundamental interactions in nature:

- gravitation
 - electromagnetic interaction
- } long range .
- weak interaction
 - strong interaction
- } short

The last two interactions characterise forces between elementary particles. The explicit force can not be written down in the form of simple equations. One assumes (and so far there is no experimental evidence to the contrary), that all four interactions satisfy an extended version of the third axiom.

In order to fill these statements with some life, one has to ask the question: what is the origin of a force? What is the mechanism, with which a mass point m_1 communicates with a mass point m_2 , in order to influence its motion? The fact that the answer is not easily obtained, can be illustrated with a 'theory' of gravitation of the 16th century, that is before the more abstract version of Newton. This theory claimed that planets were pushed on their orbits by angels (Fig. 3.10).

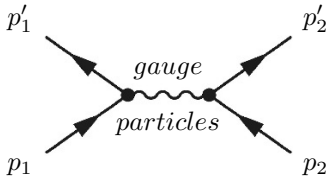
Our present understanding of forces is based on quantum field theory. According to this theory the four fundamental interactions are mediated by the exchange of characteristic field quanta (the gauge particles) between the interacting partners. Interaction corresponds so to speak to the exchange of 'balls' between two objects. It is, however, not clear how one can differentiate

transformation has to be replaced by the more involved Lorentz transformation explains the form and properties of the magnetic forces.



Fig. 3.10. Model of gravitation, 16th century

between repulsion and attraction with such a simplistic picture. In quantum field theory the interactions are represented by **Feynman diagrams**. These pictographs can be translated into corresponding mathematical expressions. The simplest Feynman graphs for the fundamental interactions, which describe the exchange of exactly one gauge particle, have the form indicated in the figure below. Two elementary particles with momenta p_1 and p_2 exchange a gauge particle. This leads to final momenta p'_1 and p'_2 of the two particles. The diagram implies a time axis, which points upwards, and an axis representing space in the horizontal.



Up to the year 1970 the list of gauge particle had the following entries:

- Gravitons, photons, hypothetical intermediate vector bosons and a number of mesons

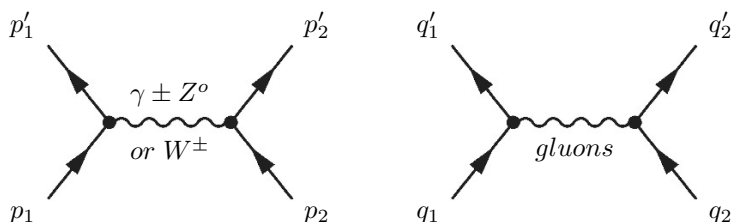
mediated (in the same order)

- gravitation, electromagnetism, the weak and the strong interaction.

Gravitation acts between all particles, the weak interaction between all elementary particles, photons are exchanged between charged particles and the strong interaction is only effective between baryons, as neutron or proton. The third axiom is valid for these interactions but in a different form. For each elementary process (characterised by a 'vertex', the dot in the Feynman diagram, where the gauge particles are emitted or absorbed) momentum conservation is satisfied in the form

$$\mathbf{p}_{\text{in}} = \mathbf{p}_{\text{out}} + \mathbf{p}_{\text{gaugeparticle}}.$$

The momenta of the particles, coming into or leaving the interaction point, and of the gauge particle have to be matched. This statement is, in particular, valid for the electromagnetic interaction. One may therefore infer that the validity of momentum conservation corresponds to the validity of the extended form of the third axiom (see Chap. 3.2.1).



The statements concerning the gauge particles which had been accepted until 1970, have been revised in the meantime. There is no new information on the (still hypothetical) gravitons. The electromagnetic and the weak interactions have been unified. The gauge particles are either a neutral particle (a mixture of photon and the neutral vector boson which has been called Z^0) or a vector boson, with either a positive or a negative charge. The picture concerning the strong interaction has changed completely. The former meson exchange theory has been replaced by quantum chromodynamics. The fundamental particles which interact, are the quarks, the 'balls' are the gluons. The particles, which beforehand were classified as interacting strongly, are composite particles. Mesons are composed of a quark-antiquark pair, baryons contain three quarks. The force that was originally considered to be the strong force is only a weak emanation of the strong interaction inside the hadrons.

It remains to be seen whether our conception about the nature of the fundamental forces will sound as naive in 500 years as the story about the angels pushing planets.

After this comment on the three axioms one should ask the question about their practical benefit. The answer will be given in the form of two comments:

1. The solution of the central equation of motion for given forces. In the case of one mass point the following set of differential equations has to be discussed

$$\begin{aligned} m\ddot{x} &= F_x(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ m\ddot{y} &= F_y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ m\ddot{z} &= F_z(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \end{aligned}$$

in vectorial summary

$$m\ddot{\mathbf{r}} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}). \quad (3.18)$$

The right hand side (three functions, each depending in the general case on 7 variables) is specified. The task is the determination of the function $\mathbf{r}(t)$. Some simple examples have already been discussed in Chap. 2. More sophisticated examples will be found in the chapters starting with Chap. 4.

2. It is on the other hand also useful to pose the question: Can general statements for mechanical systems without reference to specific forces or situations be obtained? The answer to this question are the conservation

laws (for momentum, angular momentum and energy). The conservation laws, as discussed in mechanics, are considered to be a consequence of the axioms (and general statements on the nature of the forces). The conservation laws are, on the other hand, more accessible than the axioms from an experimental point of view. For this reason it is quite sensible to base the system of physics on an axiomatic foundation using the conservation laws. This approach is in so far more adequate as the conservation laws are valid in the world of quantum mechanics. The second axiom, by contrast, is not.

The explicit application of the axioms begins with the discussion of the conservation laws.

3.2 The conservation laws of mechanics

Conservation laws are general statements concerning systems of mass points, which are subjected to external influences (the standard term is 'external forces') and which interact with each other (internal forces). Momentum conservation is the most transparent of the three basic conservation laws of mechanics. It is therefore opportune to start the discussion with this subject.

3.2.1 The momentum principle and momentum conservation

The simplest 'system' that can be considered is a system of two point particles.

3.2.1.1 Systems with two point particles. The centre of gravity (centre of mass) should be used as point of reference for a vectorial description of this system. The position of the centre of mass with respect to an arbitrary coordinate system is defined by (Fig. 3.11)

$$\mathbf{R}(t) = \frac{1}{M}(m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)) \quad M = m_1 + m_2 . \quad (3.19)$$

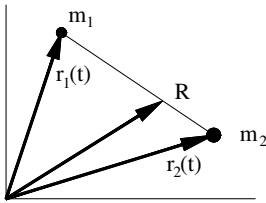


Fig. 3.11. Definition of the centre of mass of two point particles

M is the total mass of the system of mass points. The centre of mass lies on the line connecting the two masses and divides this line in the ratio of the two masses

$$\begin{aligned}\mathbf{R} - \mathbf{r}_1 &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M} - \frac{m_1 + m_2}{M} \mathbf{r}_1 = \frac{m_2}{M} (\mathbf{r}_2 - \mathbf{r}_1) \\ \mathbf{R} - \mathbf{r}_2 &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M} - \frac{m_1 + m_2}{M} \mathbf{r}_2 = \frac{m_1}{M} (\mathbf{r}_1 - \mathbf{r}_2) .\end{aligned}$$

The time derivative of the vector \mathbf{R} is the velocity of the centre of mass

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{M} (m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2) . \quad (3.20)$$

The momentum of the centre of mass is therefore, according to the general definition mass times velocity

$$\mathbf{P} = M\mathbf{V} = m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = \mathbf{p}_1 + \mathbf{p}_2 . \quad (3.21)$$

The momentum of the centre of mass is identical with the total momentum of the system.

The system is referred to as an **(ideal) closed system**, if the two masses are isolated from external influences. No external forces (usually denoted by a capital letter \mathbf{F}) act on the masses

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{0} .$$

The masses move solely under the influence of their mutual interaction, the internal forces (denoted by a lower case letter \mathbf{f}). These are assumed to satisfy the third law

$$\mathbf{f}_{12} + \mathbf{f}_{21} = \mathbf{0} .$$

The vectorial equation of motion for the two mass points

$$\frac{d\mathbf{p}_1}{dt} = \mathbf{f}_{21} \quad \frac{d\mathbf{p}_2}{dt} = \mathbf{f}_{12}$$

can be added to give

$$\frac{d}{dt} \mathbf{P}(t) = \frac{d}{dt} (\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{f}_{21} + \mathbf{f}_{12} = \mathbf{0} . \quad (3.22)$$

On the basis of the assumption, that the third axiom is valid for the internal forces, the second axiom leads to the result: the time derivative of the total momentum equals zero for a closed system of (two) point particles. A direct consequence of this result is the statement

$$\mathbf{P}(t) = \mathbf{P}(t_0) \quad \text{respectively} \quad \mathbf{R}(t) = \mathbf{R}(t_0) + \mathbf{V}(t_0) t . \quad (3.23)$$

The total momentum has the same value for each instant of time. The centre of mass (CM) is, depending on the initial conditions, at rest or in uniform motion. This fact is independent of the motion of the two masses under the influence of the internal forces (see Fig. 3.12a). These statements are equivalent forms of the **principle of momentum** as expressed by (3.22) or the law of **momentum conservation** of the system of two particles expressed explicitly by (3.23).

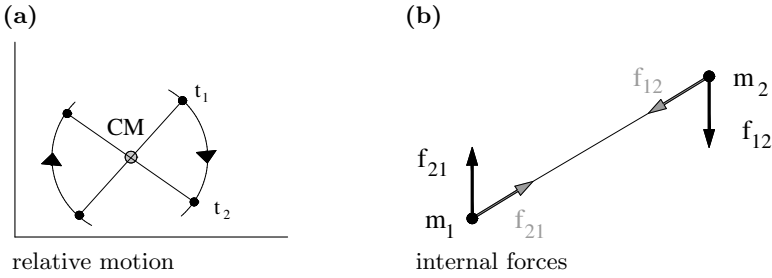


Fig. 3.12. Momentum principle

One point has to be emphasised: for the application of the third axiom, which has been used to establish momentum conservation, only the property that the internal forces are antiparallel and equally strong, is required. It is not necessary that the force vectors are directed along the line connecting the two masses (Fig. 3.12b).

The two mass system is called an **open system** if external forces (\mathbf{F}_1 , \mathbf{F}_2) act in addition to the internal forces on the two masses. The equation of motion for the two masses are then

$$\frac{d}{dt}(\mathbf{p}_1) = \mathbf{F}_1 + \mathbf{f}_{21} \quad \frac{d}{dt}(\mathbf{p}_2) = \mathbf{F}_2 + \mathbf{f}_{12}.$$

If the internal forces satisfy the third axiom, the sum of the two equations of motion yields

$$\frac{d}{dt}\mathbf{P} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}. \quad (3.24)$$

This statement can be interpreted in the following fashion: The system of two masses behaves as if the total mass was concentrated in the centre of gravity. The vector sum of the external forces (the total external force) acts on the centre of gravity and determines its motion. The statement (3.24) is also referred to as the principle of momentum. The momentum principle is identical with momentum conservation for the system, if the total external force vanishes for all times $\mathbf{F}(t) = \mathbf{0}$.

3.2.1.2 Examples for the momentum principle and momentum conservation. It is possible to make partial statements on problems of motion with the aid of the conservation law for the total momentum without a knowledge or the specification of the internal forces. Here are two simple examples to illustrate this point.

- A spring (Fig. 3.13) is compressed between the masses m_1 and m_2 . The spring is supposed to simulate a mutual interaction, which satisfies the third axiom. At time $t = 0$ the velocities \mathbf{v}_1 and \mathbf{v}_2 are equal to zero $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{0}$ and thus $\mathbf{P}(0) = \mathbf{0}$. After the spring has been released, the two masses move (after an initial phase of acceleration) with constant

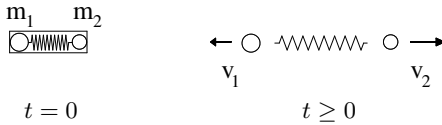


Fig. 3.13. 'Experiment' relating to momentum conservation

velocity in opposite directions. According to the conservation law, the statement

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = \mathbf{0}$$

is valid for all later times. For instance, \mathbf{v}_2 can be calculated if the two masses and \mathbf{v}_1 are known. If one of the masses and the two velocities are measured, the second mass can be determined, etc. The detailed forces exerted by the spring need not be known, as long as it is assured, that the spring acts on each of the masses with the same strength and in opposite direction. Variants of this simple method are used to determine the masses of elementary particles via the analysis of collision processes.

- The second example is the (simplified) ballistic pendulum (Fig. 3.14). A mass m_2 moving with velocity \mathbf{v}_2 hits a larger mass m_1 , originally at rest. After the mass m_2 is embedded in mass m_1 the total system moves with velocity \mathbf{v} . It is certainly difficult to analyse the action of the forces during the impact. As long as it can be assumed that they satisfy the third axiom (they are atomic, that is electrostatic forces), one can state

$$m_2 \mathbf{v}_2 = (m_1 + m_2) \mathbf{v}.$$

If the masses and the final velocity \mathbf{v} is determined, the initial velocity \mathbf{v}_2 of the mass m_2 can be calculated with this simple relation (● Probl. 3.8).

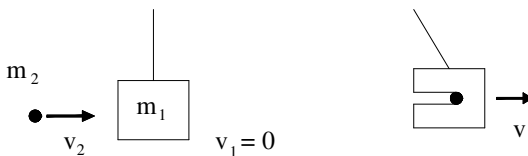


Fig. 3.14. Momentum conservation: ballistic pendulum

- The following example illustrates the application of the momentum principle. A projectile moves, under the influence of gravity, on a parabolic trajectory (Fig. 3.15). At time t_0 it explodes, separating into two fragments. The two fragments represent the system before and after the time t_0 . The forces between them are internal forces, the gravitation due to the earth is an external force. Before the time t_0 ($t \leq t_0$) the two segments move together according to the equation of motion

$$\frac{d}{dt}[(m_1 + m_2)\mathbf{V}] = (m_1 + m_2)\mathbf{g} \quad \longrightarrow \quad \frac{d}{dt}\mathbf{P} = M\mathbf{g}.$$

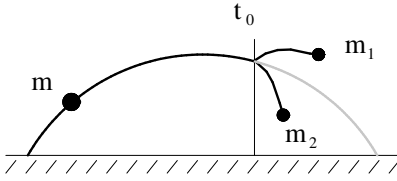


Fig. 3.15. Projectile motion, experiment in an open system

The internal forces must balance so that the two pieces do not separate. The explosion delivers an impulse of force of short duration to each of the parts. After a phase of acceleration (for $t > t_0$) the following equations of motion are valid

$$\frac{d}{dt}(m_1 \mathbf{v}_1) = m_1 \mathbf{g} + \mathbf{f}_{21} \quad \frac{d}{dt}(m_2 \mathbf{v}_2) = m_2 \mathbf{g} + \mathbf{f}_{12}. \quad (3.25)$$

There exist now internal forces between the fragments which are assumed to obey the third law. The initial conditions for the motion of the two fragments are determined by the properties of the impulse of force. For times after the explosion the situation can be characterised by the statements:

1. The centre of gravity satisfies the same equation of motion as before the explosion. With the initial conditions for the centre of gravity at t_0 the motion continues on the parabolic orbit

$$\frac{d}{dt}(\mathbf{P}) = M\mathbf{g} \quad \mathbf{P} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = M\mathbf{V}. \quad (3.26)$$

2. It is useful to employ a **centre of mass system** for the discussion of the motion of the fragments. The origin of this coordinate system is at all times the centre of gravity. As the centre of mass is accelerated in the

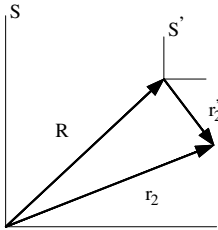


Fig. 3.16. Relation between the laboratory system S and the centre of mass system S'

present example, the centre of mass system is not an inertial system. The position of the two masses with respect to the centre of gravity (\mathbf{r}') are (see Fig. 3.16)

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{r}'_1 \quad \mathbf{r}_2 = \mathbf{R} + \mathbf{r}'_2,$$

the corresponding velocities

$$\mathbf{v}_1 = \mathbf{V} + \mathbf{v}'_1 \quad \mathbf{v}_2 = \mathbf{V} + \mathbf{v}'_2.$$

The left hand side of the first of the equations of motion (3.25) yields

$$\begin{aligned}\frac{d}{dt}(m_1\mathbf{v}_1) &= \frac{d}{dt}m_1(\mathbf{V} + \mathbf{v}'_1) = \frac{m_1}{M} \frac{d}{dt}(\mathbf{P}) + \frac{d}{dt}(m_1\mathbf{v}'_1) \\ &= m_1\mathbf{g} + \frac{d}{dt}(m_1\mathbf{v}'_1),\end{aligned}$$

if the velocity transformation for the first mass is inserted and the momentum principle (3.26) is used.

Comparison with the right hand side of the equation of motion leads to

$$\frac{d}{dt}(m_1\mathbf{v}'_1) = \mathbf{f}_{21}.$$

A corresponding statement

$$\frac{d}{dt}(m_2\mathbf{v}'_2) = \mathbf{f}_{12}$$

is obtained for the second mass. These equations indicate that the motion of the two fragments with respect to the centre of mass is governed solely by internal forces.

3.2.1.3 Systems of N point particles. The extension of the discussion to the general case, a system of N point particles $m_1, m_2, m_3, \dots, m_N$, is not difficult. The mass points can move freely as e.g. the sun, planets, moons and asteroids in our planetary system or the approximately 10^{24} molecules in a volume of gas. The mass points can also have fixed positions with respect to each other. Such a system is referred to as a solid or a rigid body. A body is called (ideally) rigid as long as it does not change its shape under the influence of external forces. The internal forces in a rigid body have to balance completely (Coulomb forces augmented by some refinements act within a molecule or a crystal) as the mass points would otherwise move with respect to each other on a macroscopic scale. The degree of rigidity of a body depends on its reaction towards additional external forces.

For a system of mass points (whether it is rigid, deformable or consists of freely moving point particles) the following definitions are used:

Total mass:

$$M = \sum_{i=1}^N m_i = m_1 + m_2 + \dots + m_N. \quad (3.27)$$

Position of the centre of mass:

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \mathbf{r}_i. \quad (3.28)$$

Velocity of the centre of mass:

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{M} \sum_i m_i \mathbf{v}_i. \quad (3.29)$$

Total momentum (momentum of the centre of mass):

$$\mathbf{P} = M\mathbf{V} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i . \quad (3.30)$$

Position of a mass point with respect to the centre of mass:

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R} . \quad (3.31)$$

The equation of motion of the k -th mass point is

$$\begin{aligned} \frac{d\mathbf{p}_k}{dt} &= \mathbf{F}_k + \mathbf{f}_{1k} + \mathbf{f}_{2k} + \dots + \mathbf{f}_{Nk} \\ &= \mathbf{F}_k + \sum_{\substack{i=1 \\ i \neq k}}^N \mathbf{f}_{ik} . \end{aligned} \quad (3.32)$$

Besides the external force F_k there are $(N - 1)$ internal forces due to the other masses that act on the k -th mass point. With the understanding

$$\mathbf{f}_{kk} = \mathbf{0} \quad (k = 1 \dots N)$$

in conformity with the third axiom

$$\mathbf{f}_{ik} = -\mathbf{f}_{ki}$$

for $i = k$, the sum in (3.32) can be taken to run from 1 to N

$$\sum_{\substack{i=1 \\ i \neq k}}^N \longrightarrow \sum_{i=1}^N .$$

Addition of the vectorial equations of motion (3.32) for all masses gives

$$\sum_{k=1}^N \frac{d\mathbf{p}_k}{dt} = \sum_{k=1}^N \mathbf{F}_k + \sum_{k=1}^N \sum_{i=1}^N \mathbf{f}_{ik} .$$

The time derivative of the total momentum can be recognised on the left hand side

$$\sum_{k=1}^N \frac{d\mathbf{p}_k}{dt} = \frac{d}{dt} \sum_{k=1}^N (\mathbf{p}_k) = \frac{d}{dt} \mathbf{P}(t) .$$

The first term on the right side is the sum of all external forces

$$\mathbf{F}_{ext} = \sum_k \mathbf{F}_k .$$

The double sum on the right hand side vanishes, if the internal forces satisfy the third axiom, as one finds

$$\begin{aligned} \sum_{ik} \mathbf{f}_{ik} &= - \sum_{ik} \mathbf{f}_{ki} \quad (\text{with the third axiom}) \\ &= - \sum_{ik} \mathbf{f}_{ik} \quad (\text{renaming the indices}) , \end{aligned}$$

as only the null vector can be equal to its negative. This statement demonstrates that only external forces contribute on the right hand side. The resulting relation

$$\dot{\mathbf{P}} = \sum_k \mathbf{F}_k = \mathbf{F}_{ext} \quad (3.33)$$

is the momentum principle for an arbitrary system of mass points with internal forces which satisfy the third axiom:

The time change of the total momentum of a system of mass points, for which the internal forces satisfy the third axiom, is determined by the sum of the external forces.

An alternative formulation is:

The centre of gravity of a system of mass points, with internal forces which satisfy the third law, moves as if the total mass was concentrated in the centre of gravity and as if the sum of the external forces acts on it.

Momentum is **conserved** if the sum of the external forces vanishes for all times

$$\sum_k \mathbf{F}_k = \mathbf{0} \quad \longrightarrow \quad \dot{\mathbf{P}} = \mathbf{0} \quad \longrightarrow \quad \mathbf{P}(t) = \mathbf{P}(t_0). \quad (3.34)$$

Momentum conservation applies in particular to a closed system, for which no external forces are present ($\mathbf{F}_k = \mathbf{0} \quad (k = 1, 2, \dots)$). This is usually expressed briefly as

The centre of gravity is at rest or in uniform motion in a closed system.

The momentum principle is a very useful instrument for the discussion of mechanical systems. It is, however, often applied in connection with the energy principle (see Chap. 3.2.3ff). For this reason no additional examples will be presented at this point.

The moment of the momentum is the angular momentum. A conservation law can also be stated for this dynamical quantity, if appropriate conditions apply. The discussion of this quantity is slightly more involved, as the construction of a moment involves a cross product of vectors.

3.2.2 The angular momentum principle and angular momentum conservation

A quantity of the form $\mathbf{r} \times \mathbf{vector}$ is the **moment of a vector**. The angular momentum is the moment of the momentum. Its properties can therefore be extracted from the equation of motion. The simplest case is the angular momentum of a single mass point.

3.2.2.1 Angular momentum of a mass point. Consider a mass point, which moves on a given trajectory. Its (orbital) **angular momentum** is defined as the vector product of the radius vector with the momentum vector (Fig. 3.17a)

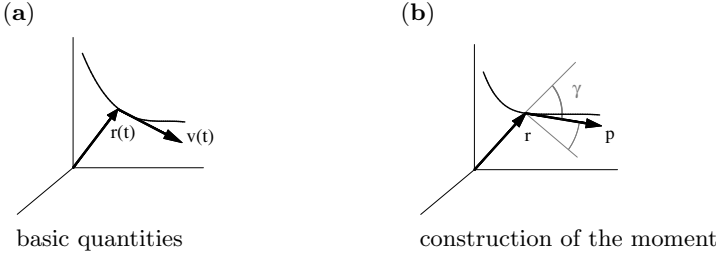


Fig. 3.17. Definition of the angular momentum of a point particle

$$\mathbf{l}(t) = m[\mathbf{r}(t) \times \mathbf{v}(t)] = \mathbf{r}(t) \times \mathbf{p}(t) . \quad (3.35)$$

The angular momentum is, up to a simple factor, identical with the areal velocity, which has been introduced in Chap. 2.3.3

$$\dot{\mathbf{A}}(t) = \frac{1}{2}(\mathbf{r}(t) \times \mathbf{v}(t)) \quad \mathbf{l}(t) = 2m\dot{\mathbf{A}}(t) .$$

The previous discussion of the areal velocity can therefore be complemented by a more direct approach to the dynamical aspects involved.

The magnitude of the angular momentum vector is

$$|\mathbf{l}| = rp|\sin \gamma| = rp|\cos(\pi/2 - \gamma)| ,$$

where the angle γ is the angle between the vectors \mathbf{r} and \mathbf{p} . It corresponds therefore, as indicated in Fig. 3.17b, to the projection of the momentum vector on a direction perpendicular to the radius vector \mathbf{r} . It is a measure of the instantaneous rotation about the origin of a chosen coordinate system. This measure is proportional to the distance of the point particle from the origin. The direction of the angular momentum vector describes the sense of the instantaneous rotation (according to the right hand rule, see ● Math.Chap. 3.1.1). For the situation depicted in Fig. 3.18a the vector points out of the page, for the situation of Fig. 3.18b into the page⁴.

The equation for the change of the angular momentum with time can be obtained by taking the moment of the equation of motion for the mass point

$$\frac{d}{dt} \mathbf{p} = \mathbf{F} \quad \longrightarrow \quad \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F} .$$

The left hand side equals the time derivative of the angular momentum as

⁴ The end point of the vector is indicated by the 'tip' of the arrow (\odot), the starting point by the 'base' of the arrow (\otimes).

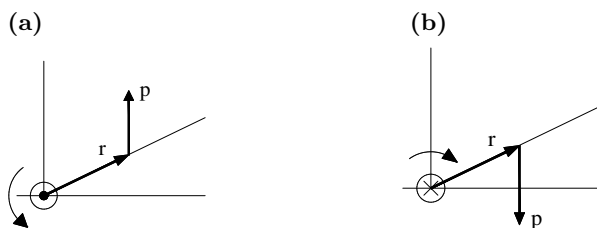


Fig. 3.18. Direction of the angular momentum vector

$$\dot{\mathbf{l}} = (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}}) = (\mathbf{r} \times \dot{\mathbf{p}}) .$$

The first term vanishes because of $(\dot{\mathbf{r}} \times \mathbf{p}) = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}}) = \mathbf{0}$. The right hand side features the **moment of force** or **torque**

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \quad |\mathbf{M}| = rF \sin \theta_{r,F} . \quad (3.36)$$

The torque is responsible for a change of the angular momentum with time. The factor r , the distance between the point of reference and the point where the force is applied, is the arm of the force. The effect of a force on the rotational motion is larger for a larger arm. This statement is verified in everyday life: in order to open a door, one would apply the force as far removed from the hinges and not close to them (Fig. 3.19).



Fig. 3.19. Illustration of the torque

The equation determining the time dependence of the angular momentum is sometimes referred to as the **principle of angular momentum**

$$\boxed{\frac{d}{dt} \mathbf{l}(t) = \mathbf{M}(t) .} \quad (3.37)$$

Angular momentum is **conserved**, if the torque vanishes for all times

$$\boxed{\mathbf{M} = \mathbf{0} \quad \longrightarrow \quad \dot{\mathbf{l}} = \mathbf{0} \quad \longrightarrow \quad \mathbf{l}(t) = \mathbf{l}(t_0) .} \quad (3.38)$$

This implies that the angular momentum is conserved if

1. no force (uniform motion on a straight line),

2. or a central force

is applied (compare the discussion of the law of areas in Chap. 2.3).

Both the torque and the angular momentum have the form $\mathbf{r} \times \mathbf{vector}$. For this reason they depend on the choice of a common point of reference. The conservation law is only valid, if the vector $\mathbf{M}(t)$ vanishes for all times in a chosen coordinate system.

This statement can be illustrated by two examples: the uniform motion along a straight line and a uniform circular motion. The uniform motion along a straight line is characterised by the statement $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t$, so that the relation

$$\mathbf{l} = m(\mathbf{r}(t) \times \mathbf{v}_0) = m(\mathbf{r}_0 \times \mathbf{v}_0) = \mathbf{const.}$$

follows. The constant (in time) angular momentum vector points into the page for the situation indicated in Fig. 3.20a. The motion of a mass point along a straight line might not necessarily be associated with a rotation. A second glance at the figure demonstrates, however, that this motion is indeed a rotation about the origin, be it only by an angle of 180° .

The motion could also be viewed from a coordinate system \mathbf{S}' , for which the origin is a point of the straight line (Fig. 3.20b). The position of the point particle

$$\mathbf{r}'(t) = \mathbf{r}'_0 + \mathbf{v}_0 t$$

with

$$\mathbf{r}'_0 = C\mathbf{v}_0$$

leads to

$$\mathbf{l}' = m(\mathbf{r}'(t) \times \mathbf{v}_0) = m(C + t)(\mathbf{v}_0 \times \mathbf{v}_0) = \mathbf{0}$$

in this case. A radial motion is observed in the system \mathbf{S}' rather than a rotation about the origin.

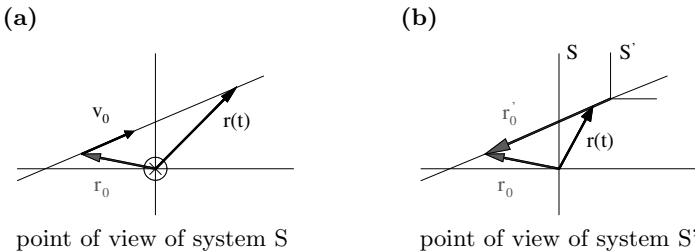


Fig. 3.20. Angular momentum for the motion along a straight line

A similar result can be obtained for the example of uniform circular motion. The equations (Chap. 2.4.1)

$$\mathbf{r} = R\mathbf{e}_r \quad \mathbf{v} = R\omega\mathbf{e}_\varphi$$

lead to

$$\mathbf{l} = mR^2\omega\mathbf{e}_z = \text{const.}$$

if the origin of the coordinate system is the centre of the circle (Fig. 3.21a). On the other hand, the coordinates and the velocity components of the point

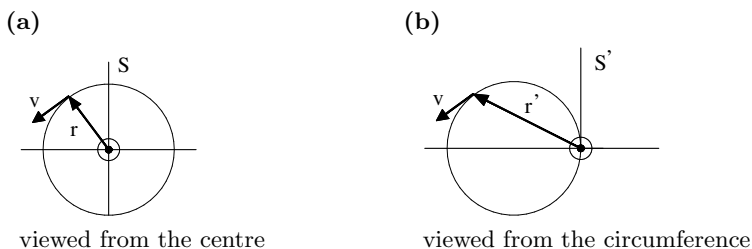


Fig. 3.21. Angular momentum for circular motion

particle are given by

$$\begin{aligned} x' &= x - R = R(\cos \omega t - 1) & y' &= R \sin \omega t \\ \dot{x}' &= -R\omega \sin \omega t & \dot{y}' &= R\omega \cos \omega t, \end{aligned}$$

if the rotation is described from the point of view of a coordinate system with the origin on the circumference of the circle (e.g. on the x -axis, Fig. 3.21b). The magnitude of the angular momentum

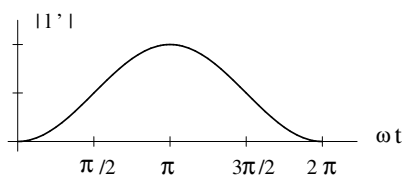


Fig. 3.22. Time variation of the angular momentum if viewed from the circumference

$$\mathbf{l}' = m(x'\dot{y}' - y'\dot{x}')\mathbf{e}_z = mR^2\omega(1 - \cos \omega t)\mathbf{e}_z$$

changes with time (see Fig. 3.22) from the point of view of this coordinate system.

The angular momentum of a mass point (object) is only defined completely, if the position of the starting point of the vector $\mathbf{r}(t)$ is specified (and independent of time).

3.2.2.2 The angular momentum for systems of N mass points. The discussion of the angular momentum of one mass point has to be generalised to the case of a system of mass points. The consideration of moments necessitates a slightly more involved argument. All N mass points of the system have to be characterised from the point of view of *one particular* system of reference. The angular momentum of the k -th particle is (Fig. 3.23a)

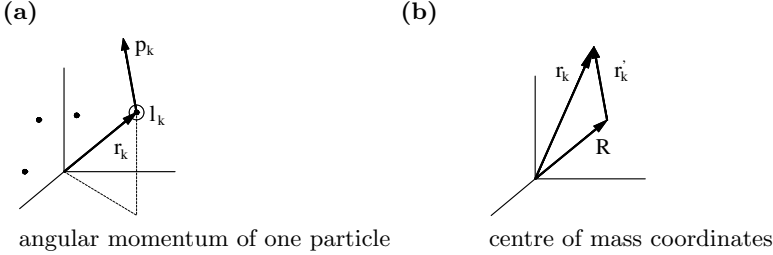


Fig. 3.23. Angular momentum in a system of point particles

$$\mathbf{l}_k(t) = \mathbf{r}_k(t) \times \mathbf{p}_k(t) .$$

The total angular momentum of the system is defined as the vectorial sum of all individual angular momenta

$$\mathbf{L}(t) = \sum_{k=1}^N \mathbf{l}_k(t) = \sum_{k=1}^N \mathbf{r}_k(t) \times \mathbf{p}_k(t) . \quad (3.39)$$

The fact that angular momentum differs from the linear momentum is demonstrated by the following argument. The centre of gravity can be brought into play by (Fig. 3.23b)

$$\mathbf{r}_k = \mathbf{r}'_k + \mathbf{R} \quad \mathbf{v}_k = \mathbf{v}'_k + \mathbf{V} .$$

Insertion into (3.39) yields

$$\begin{aligned} \mathbf{L} &= \sum_k m_k [(\mathbf{R} + \mathbf{r}'_k) \times (\mathbf{V} + \mathbf{v}'_k)] \\ &= \sum_k m_k [(\mathbf{R} \times \mathbf{V}) + (\mathbf{R} \times \mathbf{v}'_k) + (\mathbf{r}'_k \times \mathbf{V}) + (\mathbf{r}'_k \times \mathbf{v}'_k)] . \end{aligned}$$

The replacement $\sum_k m_k = M$ can be used in the first term. This term corresponds to the angular momentum of the centre of mass with respect to a chosen coordinate system

$$\mathbf{L}_{\text{CM}} = \mathbf{R} \times \mathbf{P} .$$

The third term vanishes for the transformation to the centre of mass system

$$\sum_k m_k \mathbf{r}'_k = \sum_k (m_k \mathbf{r}_k - m_k \mathbf{R}) = M\mathbf{R} - M\mathbf{R} = \mathbf{0} .$$

This argument can also be applied to the second term

$$\sum_k m_k \mathbf{v}'_k = \frac{d}{dt} \sum_k m_k \mathbf{r}'_k = \mathbf{0} .$$

The last term is the sum of the individual angular momenta referred to the centre of mass

$$\sum_k \mathbf{l}'_k = \sum_k m_k (\mathbf{r}'_k \times \mathbf{v}'_k) .$$

The total angular momentum of a system of N mass points can therefore be decomposed in the following fashion

$$\mathbf{L} = (\mathbf{R} \times \mathbf{P}) + \sum_k (\mathbf{r}'_k \times \mathbf{p}'_k)$$

or (3.40)

$$\mathbf{L}(t) = \mathbf{L}_{\text{CM}}(t) + \sum_k \mathbf{l}'_k(t) .$$

The total angular momentum of the system is equal to the angular momentum of the centre of mass (referred to a chosen coordinate system) *plus* the sum of the angular momenta of the individual mass points referred to the centre of mass.

The time derivative of the total angular momentum

$$\dot{\mathbf{L}}(t) = \sum_k \dot{\mathbf{l}}_k(t)$$

has to be analysed next. The time derivative of the individual angular momenta is

$$\dot{\mathbf{l}}_k = \mathbf{r}_k \times \dot{\mathbf{p}}_k .$$

The equation of motion (3.32) for the time derivative of the linear momenta gives for $\dot{\mathbf{L}}$

$$\dot{\mathbf{L}}(t) = \sum_k (\mathbf{r}_k \times \mathbf{F}_k) + \sum_{ik} (\mathbf{r}_k \times \mathbf{f}_{ik}) . \quad (3.41)$$

The second term on the right hand side vanishes, if the internal forces \mathbf{f}_{ik} satisfy the third axiom *and* if they are oriented along the lines connecting the relevant pairs of mass points.

This statement can be verified by first renaming the summation indices

$$\sum_{ik} (\mathbf{r}_k \times \mathbf{f}_{ik}) = \sum_{ik} (\mathbf{r}_i \times \mathbf{f}_{ki}) ,$$

so that the relation

$$\sum_{ik} (\mathbf{r}_k \times \mathbf{f}_{ik}) = \frac{1}{2} \sum_{ik} \{(\mathbf{r}_k \times \mathbf{f}_{ik}) + (\mathbf{r}_i \times \mathbf{f}_{ki})\}$$

follows. For forces satisfying the third axiom, this contribution to the time derivative of the total angular momentum can be written as

$$\sum_{ik} (\mathbf{r}_k \times \mathbf{f}_{ik}) = \frac{1}{2} \sum_{ik} [(\mathbf{r}_k - \mathbf{r}_i) \times \mathbf{f}_{ik}] .$$

This expression does not vanish in general. It vanishes, however, if the internal forces have the form of scalar functions of the position vectors involved times the vectorial distance of the two masses

$$\mathbf{f}_{ik} = f_{ik}(\mathbf{r}_i, \mathbf{r}_k) (\mathbf{r}_k - \mathbf{r}_i) .$$

This requirement concerning the internal forces *goes beyond* the requirement that led to the momentum principle. It is satisfied for gravitation and electrostatic forces.

The first term on the right hand side of (3.41) is the sum of torques of the external forces. This quantity is the **total torque**

$$\mathbf{M} = \sum_k (\mathbf{r}_k \times \mathbf{F}_k) = \sum_k \mathbf{M}_k . \quad (3.42)$$

The total torque can be decomposed in the same way as the total angular momentum. The transformation to the centre of mass

$$\mathbf{r}_k = \mathbf{R} + \mathbf{r}'_k$$

leads to the decomposition

$$\begin{aligned} \mathbf{M} &= \sum_k \left\{ (\mathbf{R} \times \mathbf{F}_k) + (\mathbf{r}'_k \times \mathbf{F}_k) \right\} \\ &= (\mathbf{R} \times \mathbf{F}) + \sum_k (\mathbf{r}'_k \times \mathbf{F}_k) \\ &= \mathbf{M}_{\text{CM}} + \mathbf{M}'_{\text{ext}} . \end{aligned} \quad (3.43)$$

The total torque is the sum of the torque of the centre of mass and the torque of the external forces with respect to the centre of mass system.

The discussion can be summarized in the angular momentum principle for a system of N mass points

If the internal forces of the system satisfy the third axiom and if they are directed along the line joining relevant pairs of mass points, then the time derivative of the total angular momentum equals the sum of the moments of the external forces

$$\dot{\mathbf{L}}(t) = \mathbf{M}(t) \quad \text{or} \quad \sum_k \dot{\mathbf{l}}_k = \sum_k (\mathbf{r}_k \times \mathbf{F}_k) . \quad (3.44)$$

The decomposition with respect to the centre of mass

$$\mathbf{L} = (\mathbf{R} \times \mathbf{P}) + \sum_k (\mathbf{r}'_k \times \mathbf{p}'_k) \quad \text{and} \quad \mathbf{M} = (\mathbf{R} \times \mathbf{F}) + \sum_k (\mathbf{r}'_k \times \mathbf{F}_k)$$

is often useful.

Angular momentum **conservation** of the system is found if the total torque vanishes (e.g. for a closed system with $\mathbf{F}_k = \mathbf{0}$ for all k)

$$\boxed{\mathbf{M}(t) = \mathbf{0} \longrightarrow \dot{\mathbf{L}}(t) = \mathbf{0} \longrightarrow \mathbf{L}(t) = \mathbf{L}(t_0) .} \quad (3.45)$$

One additional point should, finally, be emphasised: The angular momentum principle is (as the momentum principle) a vectorial statement. Each vector equation is a summary of three scalar equations.

3.2.2.3 Examples for the application of the angular momentum principle. Angular momentum plays a central role in the discussion of the motion of rigid bodies (Chap. 6.3). For this reason only a few additional examples will be discussed here.

The first remark addresses the mechanism of the lever, which has been used since antiquity (it is said to have been discovered by Archimedes). A lever is a primitive machine, which consists of a bar, that can revolve about a perpendicular axis. An equilibrium condition can be realised, if two constant

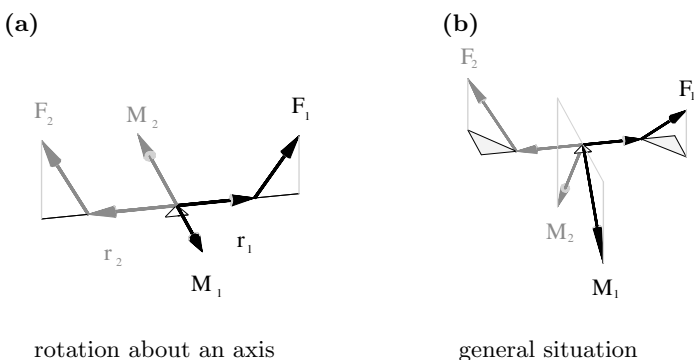


Fig. 3.24. Lever principle

forces (\mathbf{F}_1 and \mathbf{F}_2) with the arms \mathbf{r}_1 and \mathbf{r}_2 (Fig. 3.24a) are applied in a plane perpendicular to the axis of rotation so that

$$\mathbf{M}_1 + \mathbf{M}_2 = (\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) = \mathbf{0} . \quad (3.46)$$

The sum of the torques with respect to the axis of rotation vanishes for an equilibrium situation. If the two force vectors are not in a plane perpendicular to the axis of rotation (Fig. 3.24b, ● D.tail 3.1) part of the action of the moment has to be compensated by the bearing. Otherwise the whole machine will move. A direct application of the lever is illustrated in Fig. 3.25. In order to lift an object of mass M without any additional aid, a force $-M\mathbf{g}$ is

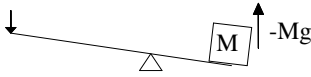


Fig. 3.25. Lever application

required. With a lever it is possible to use a much smaller force (in principle an arbitrary small force for a very long arm).

The dumb-bell is an example of a simple rigid body. It consists of two masses m_1 and m_2 , which are connected by a rigid, weightless rod. The dumb-bell is supposed to rotate about an axis, which is perpendicular to

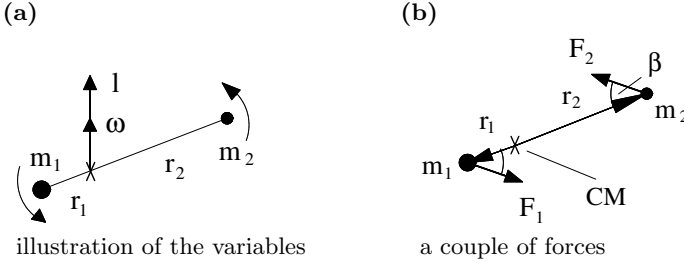


illustration of the variables

a couple of forces

Fig. 3.26. Motion of a dumb-bell

the connecting rod and which passes through the centre of mass (CM) of the system (Fig. 3.26a). The total angular momentum of the dumb-bell with respect to the centre of mass is obtained by considering the equations for the motion of the two masses. The relevant statements for the first mass in suitable cylindrical coordinates, referred to the centre of mass, are (Fig. 3.26b)

$$\begin{aligned} \mathbf{r}_1(t) &= r_1 \mathbf{e}_{r_1} \\ \mathbf{v}_1(t) &= r_1 \omega(t) \mathbf{e}_{\varphi_1} \quad (\dot{r}_1 = 0) . \end{aligned}$$

The distance of the mass m_1 from the centre of mass is r_1 (use centre of mass coordinates without a prime, in contrast to Chap. 3.2.2.2) and $\omega(t)$ is the angular velocity. The angular momentum of m_1 is then

$$\mathbf{l}_1(t) = \mathbf{r}_1(t) \times \mathbf{p}_1(t) = m_1 r_1^2 \omega(t) \mathbf{e}_z .$$

The angular velocity can be characterised by a vector, which marks the instantaneous axis of rotation. In the present situation this vector has the form $\boldsymbol{\omega}(t) = \omega(t) \mathbf{e}_z$. A similar result can be obtained for the second mass m_2

$$\mathbf{l}_2(t) = \mathbf{r}_2(t) \times \mathbf{p}_2(t) = m_2 r_2^2 \omega(t) .$$

The total angular momentum of the system is therefore

$$\mathbf{L}(t) = \mathbf{l}_1(t) + \mathbf{l}_2(t) = (m_1 r_1^2 + m_2 r_2^2) \omega(t) . \quad (3.47)$$

The factor in front of the angular velocity is the **moment of inertia** of the dumb-bell

$$I = (m_1 r_1^2 + m_2 r_2^2) . \quad (3.48)$$

The moment of inertia is (as the angular momentum) defined with respect to a chosen coordinate system, in the present case with respect to the centre of mass system. The relation between angular momentum and angular velocity is

$$\mathbf{L}(t) = I\boldsymbol{\omega}(t)$$

in the present example. The general form of this relation will be discussed in Chap. 6.3.3 in connection with the motion of rigid bodies.

The similarity of this relation with the definition of linear momentum $\mathbf{p} = m\mathbf{v}$ invites the remark: the moment of inertia characterises the resistance of a rigid body against changes of the rotational motion. The importance of the contribution of the arms r_i towards the size of the moment of inertia is emphasised by the quadratic dependence on the arms.

The rotation of the dumb-bell is determined by the equation

$$\dot{\mathbf{L}}(t) = I\dot{\boldsymbol{\omega}}(t) = \mathbf{M}(t) .$$

The time derivative of the angular velocity is the angular acceleration (often denoted by $\dot{\boldsymbol{\omega}} = \boldsymbol{\alpha}$).

The centre of mass of the dumb-bell will not move (or move uniformly) if a **couple of forces** (see Fig. 3.26b) is applied to the masses. The two forces $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ of the couple must be equally strong and antiparallel while including the same angle with respect to the connecting rod of the dumb-bell at all times, so that $\dot{\mathbf{P}} = \mathbf{F}_1(t) + \mathbf{F}_2(t) = \mathbf{F}(t) = \mathbf{0}$. The torque of the couple of forces is

$$\mathbf{M}(t) = \sum_k (\mathbf{r}_k \times \mathbf{F}_k(t)) .$$

The motion is uniform if no torque is applied. Otherwise the angular motion is accelerated.

The last remark on the subject of angular momentum is a simple illustration of the use of angular momentum conservation. A dumb-bell with equal masses $m_1 = m_2 = m$ is assumed to rotate uniformly (with constant angular velocity) about an axis through the centre of mass, which is perpendicular to the axis of the dumb-bell (Fig. 3.27). The masses can be moved on the

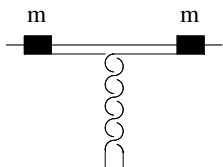


Fig. 3.27. Mechanism of the pirouette

fictitious rod by a mechanism which imitates internal forces (that is in the

direction of the centre of mass). As no external forces are applied, angular momentum is conserved

$$\mathbf{L}(t) = \mathbf{L}(0) .$$

With (3.47) this implies in detail

$$[2mr(t)^2]\omega(t) = [2mr(0)^2]\omega(0) .$$


The angular velocity changes according to

$$\omega(t) = \frac{I(0)}{I(t)} \omega(0) ,$$

if the moment of inertia changes with time. If the masses are pulled inwards, the moment of inertia is decreased. The dumb-bell rotates faster. This illustrates the mechanism of a pirouette.

The third dynamical quantity, the energy, is a central concept of physics. It plays a role in all fields of physics and not only in mechanics. The discussion of this concept and of the related concept of work is one of the main tasks of theoretical mechanics.

3.2.3 Energy and energy conservation for a mass point

The discussion of energy and work relies on the concept of a field (see also  **Math.Chap. 5.1**), in particular a field of force. It is, for this reason, useful to introduce this concept first.

3.2.3.1 Vector fields. An example of a vector field, the gravitational force field, can be visualised in the following fashion: place a mass point M at the origin of a coordinate system. Examine the whole space with a 'test mass' which experiences the gravitational force (see (3.13), p. 79)

$$\mathbf{F}(\mathbf{r}) = -\gamma \frac{mM}{r^3} \mathbf{r} .$$

Attach (at least in thought) at each point of space (Fig. 3.28a) a vector $\mathbf{F}(\mathbf{r})$ representing the gravitational force. All vectors on the surface of a sphere have the same length and are directed towards the origin (where the mass M remains). The larger the radius of the sphere, the shorter are the vectors. This association of vectors with the points of space is called a vector field (Fig. 3.28b), in the present case the gravitational force field

$$\mathbf{F}(\mathbf{r}) = \left(F_x(x, y, z), F_y(x, y, z), F_z(x, y, z) \right) . \quad (3.49)$$

A stationary (that is independent of time) vector field is described by three functions of three variables, for the example under discussion explicitly

$$\mathbf{F}(\mathbf{r}) = -\gamma \frac{mM}{r^3} (x, y, z) \quad r = [x^2 + y^2 + z^2]^{1/2} . \quad (3.50)$$

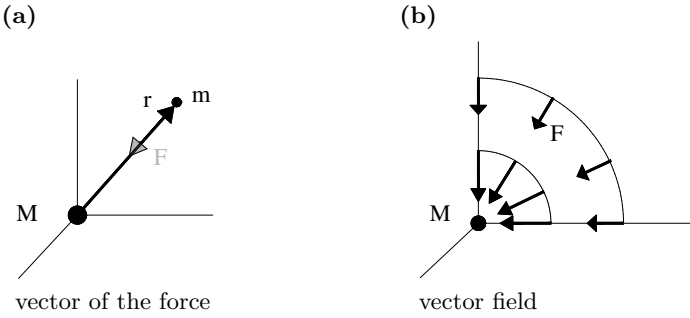


Fig. 3.28. Illustration of the gravitational field of force

The situation discussed can be characterised in a different fashion. The quantity

$$\mathbf{G}(\mathbf{r}) = \frac{1}{m} \mathbf{F}(\mathbf{r}) = -\gamma \frac{M}{r^3} \mathbf{r} \quad (3.51)$$

is also a vector field, which is called the **gravitational field** of the mass M . With the introduction of this field, a model for the action of the forces between the two masses m and M is suggested (Fig. 3.29): start with empty space. By placing the mass M at the position $(0, 0, 0)$ the space is 'modified'. The presence of the mass M establishes the gravitational field \mathbf{G} . In the last step a (test) mass m is positioned at a point \mathbf{r} . The mass m is then subjected to the force $\mathbf{F} = m\mathbf{G}$. The action of the force involves therefore

- the postulate of the existence of a gravitational field \mathbf{G} ,
- the exploration of this field with a test mass in order to observe the force \mathbf{F} .

The obvious question 'Does the field \mathbf{G} exist?' cannot be answered. The field can only be experienced in experiments via the force \mathbf{F} . However, the model of a gravitational field (and a good number of other fields) has turned out to be extremely useful.

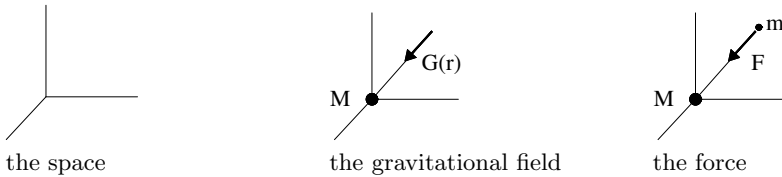


Fig. 3.29. Concerning the gravitational field

A representation of a vector field in terms of field lines can be used instead of the characterisation by arrows attached to space points. The field lines correspond to lines, which are tangent to the field vectors. The representation

of fields via field lines will be taken up in connection with electrostatic fields in Vol. 2.

An example different from a central field is the uniform gravitational force field in the vicinity of the (flat) surface of the earth

$$\mathbf{F} = (0, 0, -mg) .$$

The corresponding gravitational field is

$$\mathbf{G} = (0, 0, -g) .$$

3.2.3.2 The concepts of work and kinetic energy of a mass point.

It is expedient to introduce the concepts of work and kinetic energy for one mass point in four stages. The mass point moves

- along a straight line in a constant force field $\mathbf{F} = (F, 0, 0)$ in the direction of the field,
- along a straight line in the x -direction in a constant force field

$$\mathbf{F} = (F \cos \alpha, 0, F \sin \alpha)$$

which includes an angle α with respect to the direction of motion,

- along a straight line in a variable force field $\mathbf{F}(x) = (F(x), 0, 0)$ in the direction of the field,
- in a general field of force $\mathbf{F}(\mathbf{r}) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$.

The motion of a mass point m is, in the case $\mathbf{F} = (F, 0, 0)$, a linear and uniformly accelerated motion from a point x_0 to a point $x(t)$. The solution of the equation of motion $\mathbf{F} = m\mathbf{a}$ with the initial conditions $\mathbf{r}(0) = (x_0, 0, 0)$ and $\mathbf{v}(0) = (v_0, 0, 0)$ at time $t = 0$ is

$$v - v_0 = \frac{F}{m} t \quad x - x_0 = v_0 t + \frac{F}{2m} t^2 .$$

Elimination of the time variable from these equations leads to

$$\frac{F}{m} (x - x_0) = v_0 \left(\frac{F}{m} t \right) + \frac{1}{2} \left(\frac{F}{m} t \right)^2 = v_0 (v - v_0) + \frac{1}{2} (v - v_0)^2$$

or

$$F (x - x_0) = \frac{m}{2} v^2 - \frac{m}{2} v_0^2 . \quad (3.52)$$

This result can be interpreted in the following fashion. The expression on the left hand side of this equation is the acting force multiplied by the distance the mass has moved. This quantity is called **work**. More precisely one should say: It is the work which the force F supplies to the mass m by moving it by the distance $(x - x_0)$. The right hand side describes the change of the quantity $mv^2/2$, which depends only on the properties of the object (mass and velocity). This quantity is the **kinetic energy** of the mass point

$$E_{\text{kin}} = T = \frac{m}{2} v^2 = \frac{p^2}{2m} \geq 0 . \quad (3.53)$$

The concept implies that a massive object, which moves with the speed v , has a kinetic energy $T \geq 0$. The result (3.52) can be written in a more suggestive manner as

$$T(t) = T(0) + A(0 \rightarrow t) . \quad (3.54)$$

A symbolic interpretation of this equation is indicated by the flow diagram in Fig. 3.30a. At the time $t = 0$ the kinetic energy of the object is $T(0)$. By applying the force \mathbf{F} (in this example a constant force in the direction of the displacement) work is done. The work supplied raises the kinetic energy of the object by the amount A to $T(t)$.

The mass point would be decelerated in a force field in the opposite direction $\mathbf{F} = (-F, 0, 0)$. The relation (3.52) would then be written as

$$\frac{m}{2} v^2 = \frac{m}{2} v_0^2 - F (x - x_0) .$$

As the object slows down, the work done (on the object) is negative. The situation is illustrated in Fig. 3.30b. The system, the mass point, loses kinetic energy.

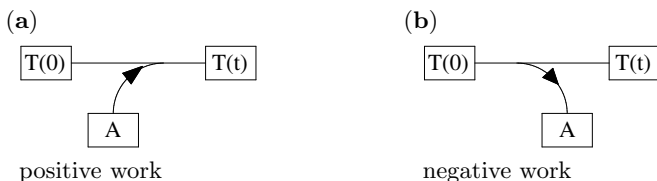


Fig. 3.30. Work: flow diagrams

The second variant of energy supply is: the mass point is moving parallel to the x -axis in a force field with a direction that includes an angle α with respect to the x -axis

$$\mathbf{F} = (F \cos \alpha, 0, -F \sin \alpha) .$$

From a practical point of view, such a situation is only possible if the object moves (without friction) on a suitable support. The component of the force perpendicular to the direction of motion is compensated by the pressure due to the support (Fig. 3.31). Only the x -component of the force influences the motion in the x -direction, so that the result is

$$\frac{m}{2} v^2 = \frac{m}{2} v_0^2 + F (x - x_0) \cos \alpha .$$

The definition of work, according to two cases considered, can be stated as:

Work equals the scalar product of the force vector \mathbf{F} and displacement vector \mathbf{r}
 $A = \mathbf{F} \cdot \mathbf{r} .$

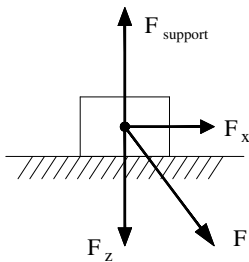


Fig. 3.31. Work in a homogeneous force field, arbitrary direction

This definition is *only valid* if the force field is constant and if the displacement is along a straight line. Obviously, this definition has to be generalised. This will be done after a few remarks concerning the simple definition.

- (1) In order that work (in the sense of the definition above) is supplied to an object, the following conditions have to be satisfied:
 - a) the object has to move,
 - b) the force applied needs to have a nonvanishing component in the direction of the motion.

This point can be emphasised by the following example: a person is holding a stone of 20 kg with arms stretched out. Even if this person is of the opinion that he/she works hard, *no* work is done on the object (the stone) in the sense of the definition as long as the stone does not move

$$\mathbf{r} = \mathbf{0} \longrightarrow A = 0.$$

- (2) The expression for the kinetic energy (a scalar quantity) can be written in the form of a scalar product

$$T = \frac{m}{2} (\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2m} (\mathbf{p} \cdot \mathbf{p}). \quad (3.55)$$

This notation turns out to be correct in general and is e.g. useful if curvilinear coordinates are used.

- (3) The units of energy and work are

$$\text{SI system: } 1 \text{ Joule} = 1 \text{ N m} = 1 \frac{\text{kg m}^2}{\text{s}^2}$$

$$\text{CGS system: } 1 \text{ erg} = 1 \text{ dyn cm} = 1 \frac{\text{g cm}^2}{\text{s}^2}.$$

The conversion factor is $1 \text{ Joule} = 10^7 \text{ erg}$. Other energy units, which are often used, are watt second (Ws corresponding to 1 Joule) and hence the customary kilowatt hour (kWh corresponding to $3.6 \cdot 10^6$ Joule) or electron volt ($1 \text{ eV} = 1.602 \cdot 10^{-19}$ Joule).

- (4) The preliminary definition of work does not apply to the uniform circular motion of a point particle in a central field (Fig. 3.32). Neither is the force field constant nor is the displacement along a straight line. The situation can, however, be discussed directly. The central force, which is

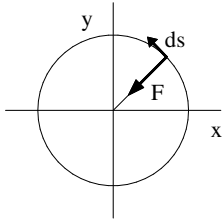


Fig. 3.32. Work during the uniform circular motion

necessary to keep the mass on the circle, is always perpendicular to the **instantaneous** displacement

$$\mathbf{F} \perp d\mathbf{r} \quad d\mathbf{r} = \mathbf{v} dt .$$

It can therefore be expected that no work is done on the mass. The kinetic energy can be calculated with the formula (2.33) for the velocity

$$\mathbf{v}(t) = (-R\omega \sin \omega t, R\omega \cos \omega t, 0) ,$$

which corresponds to the initial condition $\mathbf{v}(0) = (0, R\omega, 0)$, as

$$\frac{m}{2} v^2(t) = \frac{m}{2} v^2(0) = \frac{m}{2} R^2 \omega^2 .$$

No energy is supplied to or subtracted from the uniformly rotating mass as the force is always perpendicular to the displacement.

The displacement along a straight line in the x - direction in a variable force field (Fig. 3.33a)

$$\mathbf{F}(\mathbf{r}) = (F(x), 0, 0)$$

has to be divided into differential displacements dx' , followed by addition of

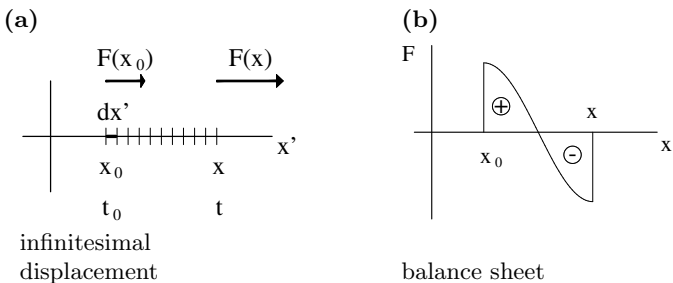


Fig. 3.33. Work in force field, position dependent

the infinitesimal contributions

$$dA = F(x') dx'$$

to the work. The total work for the displacement from a point x_0 to a position x in the limit of an arbitrarily fine division is given by the integral

$$A = \int_{x_0}^x F(x') \, dx' .$$

If the force along the straight line has e.g. the behaviour indicated in Fig. 3.33b, it will first accelerate the mass (assuming motion in the positive x -direction), then slow it down again. The contribution to the total work is first positive, then negative. All possible variants (including the sequence of the limits) are taken care of by the standard definition of the integral.

The relation between the work done and the change of the kinetic energy can be found with the equation of motion in one space dimension

$$m \frac{dv}{dt} = F(x) .$$

Multiplication with dx and integration from the initial to the final position yields

$$m \int_{x_0}^x \dot{v} \, dx' = \int_{x_0}^x F(x') \, dx' .$$

The right hand side represents the work done. The left hand side is reformulated in the following fashion

$$\begin{aligned} \int_{x_0}^x \dot{v} \, dx' &= \\ &\quad [\text{substitution : } x' = x(t'), \quad dx' = v(t') \, dt'] \\ &= \int_{t_0}^t \left[\frac{d}{dt'} v(t') \right] v(t') \, dt' \\ &\quad [\text{reformulate integrand}] \\ &= \frac{1}{2} \int_{t_0}^t \frac{d}{dt'} [v^2(t')] \, dt' \\ &\quad [\text{direct integration}] \\ &= \frac{1}{2} (v^2(t) - v^2(t_0)) . \end{aligned}$$

The final result is, as before, a work-energy relation of the form

$$\frac{m}{2} v^2(t) - \frac{m}{2} v^2(t_0) = \frac{m}{2} v^2 - \frac{m}{2} v_0^2 = A = \int_{x_0}^x F(x') \, dx' . \quad (3.56)$$

The calculation of the work-energy relation for an arbitrary motion in an arbitrary field of force (Fig. 3.34) follows the same pattern. The trajectory of the point particle can be described by the parametric representation

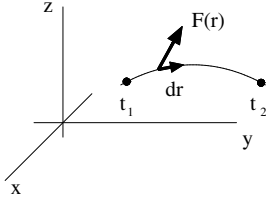


Fig. 3.34. Work in a general force field

$$\mathbf{r}(t) = (x(t), y(t), z(t)) \quad \text{with} \quad t_1 \leq t \leq t_2 ,$$

the (stationary) force field has the form (3.49)

$$\mathbf{F}(\mathbf{r}) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)) .$$

First, the parametric representation is used for a definition of an infinitesimal displacement along the curve for a given instant of time

$$d\mathbf{r} = (dx, dy, dz) = (\dot{x}(t)dt, \dot{y}(t)dt, \dot{z}(t)dt) . \quad (3.57)$$

Every line element $d\mathbf{r}$ contributes an infinitesimal amount of work of the form

$$dA = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} . \quad (3.58)$$

The total work along a section of the trajectory is again obtained by integration, in explicit notation

$$A = \int_{t_1}^{t_2} dt \left\{ F_x(x(t), y(t), z(t)) \dot{x}(t) + F_y(x(t), y(t), z(t)) \dot{y}(t) + F_z(x(t), y(t), z(t)) \dot{z}(t) \right\} . \quad (3.59)$$

The corresponding short hand notation is

$$A = \int_1^2 (F_x dx + F_y dy + F_z dz) \quad (3.60)$$

$$= \int_{K_{12}} \mathbf{F} \cdot d\mathbf{r} . \quad (3.61)$$

The general definition of work involves a **line integral** over the force field. The actual evaluation of the line integral is based on the parametric representation (3.59). The notation implies integration along the given curve starting at the position for the time t_1 to the position for the time t_2 . In (3.60) this rule is abbreviated and in the variant (3.61) the integral is condensed with the aid of the scalar product.

Additional information on line integrals is found in \bullet Math.Chap. 5.3.1.

The starting point for the derivation of the **work-energy** relation is, in this case, the equation of motion for one mass point in three dimensions

$$m\dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}) .$$

Consider the scalar product with \mathbf{dr} and use of the substitution $\mathbf{dr} = \mathbf{v} dt$ on the left hand side to obtain

$$m\dot{\mathbf{v}} \cdot \mathbf{v} dt = \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} .$$

Integration from the starting time t_1 to the final time t_2 then yields on the left hand side

$$\int_{t_1}^{t_2} \dot{\mathbf{v}} \cdot \mathbf{v} dt = \frac{1}{2} \int_{t_1}^{t_2} \left[\frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) \right] dt = \frac{1}{2} (\mathbf{v}(t_2)^2 - \mathbf{v}(t_1)^2) .$$

The right hand side of this equation corresponds to the work which the force field supplies to the mass. The relation in question has in general the form

$$A = \int_1^2 \mathbf{F} \cdot \mathbf{dr} = \frac{m}{2} v_2^2 - \frac{m}{2} v_1^2 . \quad (3.62)$$

The work supplied to the mass leads to a change of the kinetic energy.

The formulation of the energy principle for a mass point in a force field is based on the concept of potential energy. This quantity can only be defined provided certain conditions are met. Two examples are used to illustrate the work-energy relation in more detail and to introduce the concept of potential energy.

3.2.3.3 Energy conservation, potential energy and conservative systems. The situation can be analysed directly for the one dimensional harmonic oscillator. A mass m on a spring (with spring constant k) is at rest at the position $x = 0$. The force, which is necessary to displace the mass by the distance x , is (Fig. 3.35a)

$$F = -kx .$$

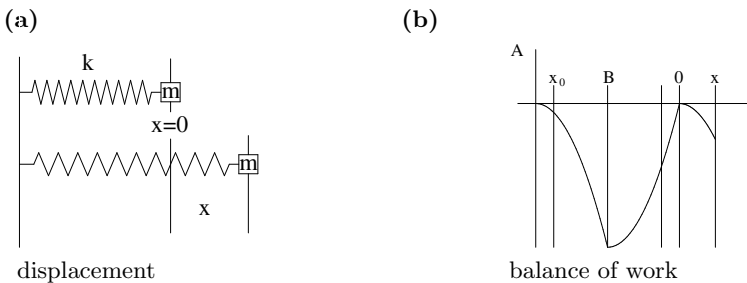


Fig. 3.35. Work and the harmonic oscillator

If the mass moves under the influence of the spring from the position x_0 to the position x , the work supplied by the spring to the mass is

$$A = \int_{x_0}^x (-kx') dx' = \frac{k}{2}x_0^2 - \frac{k}{2}x^2.$$

The total work equals the change of the kinetic energy

$$\frac{k}{2}x_0^2 - \frac{k}{2}x^2 = \frac{m}{2}v^2 - \frac{m}{2}v_0^2.$$

The quantities with the index zero refer to the initial situation, the quantities without an index to the final situation. This relation can e.g. be used to calculate the speed of the mass $|v|$ for each displacement if the initial conditions are specified.

The relative orientation of force and displacement is taken care of in an automatically. Consider, for instance (see Fig. 3.35b), a displacement from x_0 via a maximal displacement B to the position x on the opposite side of the equilibrium position. The following amounts of work are supplied to the mass:

$x_0 \longrightarrow B$ negative work (the spring is stretched)

$B \longrightarrow 0$ positive work (the spring contracts)

$0 \longrightarrow x$ negative work (the spring is compressed).

A different interpretation of the result can be given after the rearrangement

$$\frac{m}{2}v^2 + \frac{k}{2}x^2 = \frac{m}{2}v_0^2 + \frac{k}{2}x_0^2. \quad (3.63)$$

The term $kx_0^2/2$ on the right hand side of this equation can be interpreted as the energy content of the initially stretched (or compressed) spring at time t_0 . A more abstract formulation is: the mass point is endowed with the **potential energy**

$$E_{\text{pot}}(x_0) = U_0 = \frac{k}{2}x_0^2 \quad (3.64)$$

because of its position at the point x_0 in the (one dimensional) force field. The potential energy is the energy, which a mass point possesses because of its position in a field of force. The left hand side of (3.63) represents the sum of the kinetic and potential energy at time t .

The reformulation of the work-energy relation is the energy principle, respectively the **law of energy conservation** for the situation discussed. The total energy of the mass point, the sum of kinetic and potential energy is the same for all times

$$E = E_{\text{kin}} + E_{\text{pot}} \longrightarrow E(t) = E(t_0) \quad \text{or} \quad \frac{dE}{dt} = 0. \quad (3.65)$$

The two points of view (work-energy relation and energy conservation) are illustrated explicitly for the harmonic oscillator problem. The solution of this problem with the initial conditions

$$t_0 = 0 \quad x_0 = B \quad v_0 = 0$$

is

$$x(t) = B \cos \omega t \quad v(t) = -B\omega \sin \omega t \quad \omega = \sqrt{\frac{k}{m}}. \quad (3.66)$$

The relation $T - T_0 = A$ expresses the fact that the kinetic energy of the mass changes due to the action of an external agent (the spring or, more general, a force field). As the force vector acts in part in the direction of the motion in part against the motion, the work supplied to the mass (point particle) oscillates in time

$$A = \frac{k}{2}(x_0^2 - x^2) = \frac{k}{2}B^2(1 - \cos^2 \omega t) = \frac{k}{2}B^2 \sin^2 \omega t.$$

The variation of the kinetic energy with time is the same

$$T - T_0 = \frac{m}{2}(v^2 - v_0^2) = \frac{m}{2}B^2\omega^2 \sin^2 \omega t = \frac{k}{2}B^2 \sin^2 \omega t = A.$$

The relation

$$T + U = T_0 + U_0, \quad (3.67)$$

which is obtained by rearrangement, expresses the fact that the mass point and the spring constitute a closed system. The total energy is conserved in this system. There is, however, an exchange between these two forms of energy (Fig. 3.36a)

$$T(t) = \frac{m}{2}v^2 = \frac{k}{2}B^2 \sin^2 \omega t \quad U(t) = \frac{k}{2}x^2 = \frac{k}{2}B^2 \cos^2 \omega t$$

with time. The total energy

$$E = T(t) + U(t) = \frac{k}{2}B^2$$

does not depend on time and corresponds to the energy initially stored in the spring.

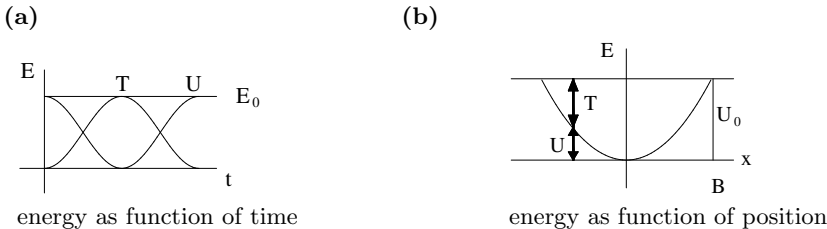


Fig. 3.36. Example: energy situation for the harmonic oscillator

The individual expressions for the kinetic and the potential energies can also be discussed in terms of their dependence on the position in the force

field. In the example, the mass has initially the potential energy $U_0 = U(B)$. The kinetic energy at the starting time is zero. With the change of the position in the force field an exchange between the two forms of energy takes place. Potential energy is changed into kinetic energy and vice versa (Fig. (3.36b)).

The work supplied by the spring depends only on the difference of the potential energies at the initial and the final point and not on the path connecting these points

$$A = U_0 - U = U(x_0) - U(x) . \quad (3.68)$$

A relation like (3.68) characterises a conservative force field. The relation $T - T_0 = A$ is always valid. It is derived from the equation of motion (the second axiom) via mathematical manipulations (and the definition of the relevant quantities).

The path independence can, for instance, be demonstrated explicitly for the example of projectile motion (a two dimensional problem) in the gravitational force field

$$\mathbf{F} = (0, -mg) .$$

The solution of the equations of motion for the initial conditions

$$t_1 = 0 \quad \mathbf{r}(0) = (0, y_1) \quad \mathbf{v}(0) = (v_1, 0)$$

is found to be

$$v_x(t) = v_1 \quad x(t) = v_1 t \quad v_y(t) = -gt \quad y(t) = y_1 - \frac{1}{2}gt^2 .$$

The work due to gravity during the free fall from the position $\mathbf{r}(0)$ to a position $\mathbf{r}(t_2)$ can be obtained by evaluation of the line integral (3.59)

$$A = \int_0^{t_2} (F_x \dot{x} + F_y \dot{y}) dt = \int_0^{t_2} (-mg)(-gt) dt = \frac{m}{2} g^2 t_2^2 .$$

With the solution of the equation of motion this can be written as

$$A = mg \left(\frac{1}{2} g t_2^2 \right) = mg (y_1 - y(t_2)) = mg (y_1 - y_2) .$$

This result can also be obtained directly

$$A = \int_1^2 (F_x dx + F_y dy) = \int_{y_1}^{y_2} (-mg) dy = mg (y_1 - y_2) .$$

The first calculation of the work follows the actual trajectory from the point $(0, y_1)$ to the point (x_2, y_2) (Fig. 3.37a). The integral in the second case is evaluated along the straight line from the point $(0, y_1)$ to the point $(0, y_2)$ (Fig. 3.37b). The segment of the path from $(0, y_2)$ to (x_2, y_2) does not contribute to the work in the present example as $\mathbf{F} \cdot d\mathbf{r} = 0$. Gravity does not do any work along this section of the path. An arbitrary path from the point $(0, y_1)$ to the point (x_2, y_2) (Fig. 3.37c) could also have been chosen. If this

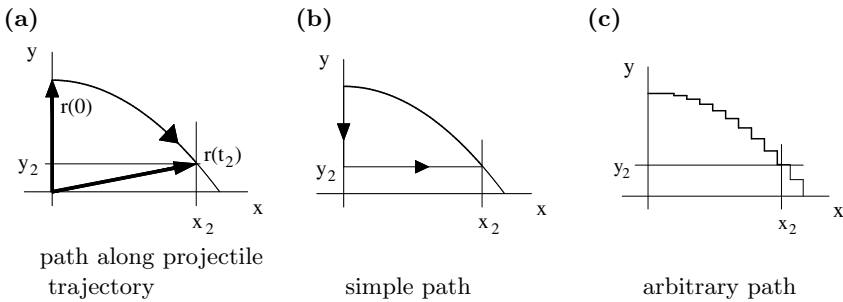


Fig. 3.37. Work in the gravitational field

path is divided into infinitesimal sections it is found that only those sections parallel to the y -axis contribute to the line integral. The argument shows that the same result

$$A = \int_{\text{all } K_{12}} \mathbf{F} \cdot d\mathbf{r} = mg(y_1 - y_2) = U(1) - U(2)$$

is obtained for all paths K_{12} between the starting and the end point.

The gravitational force field of this example is also conservative. It should be noted that the difference of the potential energy between starting point and final point, that is the difference $U(1) - U(2)$ and not $U(2) - U(1)$, appears. This is a question of convention. The standard form of the total energy

$$T_1 + U_1 = T_2 + U_2 = E \quad (3.69)$$

is obtained from

$$A = U_1 - U_2 = T_2 - T_1$$

only by using *this* convention. The law of energy conservation for one point particle

$$\boxed{T + U = E = \text{const.}}$$

is in this example

$$\frac{m}{2}v^2 + mgy = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgy = \text{const.} \quad (3.70)$$

for projectile motion.

Two additional remarks are necessary to complement the discussion of these examples.

- (1) The work integral with conservative forces yields only the difference of potential energies. It would have been possible to extract

$$U = mgy + \text{const.}$$

in the last example. Potential energy is only defined up to a constant of integration. The constant has no physical meaning. An arbitrary value

zero is assigned to the potential energy at the surface of the earth if the value $\text{const.} = 0$ is used.

- (2) The fact that work in the gravitational field $\mathbf{G} = (0, g)$ is independent of the path, can be put to good use. If a mass moves under the influence of gravitation on a slide, the calculation of the motion on the trajectory specified by the slide should include gravitational \mathbf{F}_g as well as **constraining forces** \mathbf{F}_Z due to the slide. The constraining forces are in general quite complicated, but they have one useful property. The force vector is at all times perpendicular to the instantaneous displacement (Fig. 3.38a). Constraining forces do not, for this reason, contribute to the work. For the motion along a slide of arbitrary shape (neglecting frictional effects) energy conservation is valid in the form

$$\frac{m}{2}v^2 + mgy = \frac{m}{2}v_0^2 + mgy_0 .$$

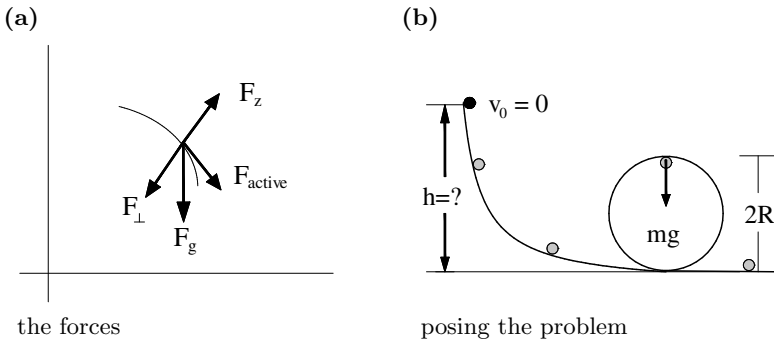


Fig. 3.38. Frictionless slide with a loop

An example for the application of energy conservation in this form is the following problem: consider a frictionless slide with a circular loop (Fig. 3.38b) of radius R . The question is: how far above the ground does one have to start a mass m from rest, so that it just passes through the loop without crashing?

For an answer two ingredients are needed: the velocity at the top of the loop can be related to the starting height h via energy conservation

$$\frac{m}{2}v^2 + mg(2R) = mgh .$$

In order to avoid a crash during the motion through the loop, the centrifugal force due to the motion has to be larger than the gravitational force. The minimal condition is equality of the two forces

$$m \frac{v^2}{R} = mg .$$

If this condition is written in the form

$$\frac{m}{2}v^2 = \frac{1}{2}mgR$$

and inserted into the energy law, the result

$$mgh = \frac{1}{2}mgR + 2mgR$$

or

$$h = \frac{5}{2}R$$

is obtained. If the mass sliding without friction is replaced by a rolling ball, the rotational kinetic energy of the ball has to be included in the discussion (● Probl. 3.10).

Partial answers to relatively complicated problems of motion can be obtained if energy conservation holds. A description of the actual motion (that is tracing the position as a function of time) can only be obtained by solution of the equations of motion including (if applicable) the constraining forces. This is possible but in a general definitely more tedious (see Chap. 5).

The discussion of the energy situation for the motion of one point particle has so far relied on examples. It is therefore appropriate to summarise the discussion in more formal terms at this point.

In order to discuss energy and energy conservation in general terms, some concepts of vector analysis are required. These are introduced and discussed in ● Math.Chap. 5.

3.2.3.4 Formal statement of the law of energy conservation. A stationary conservative force field $\mathbf{F}(x, y, z)$ is characterised by the following equivalent statements

- The **curl** or **rotation** of the field vanishes

$$\begin{aligned} \mathbf{rot} \mathbf{F}(x, y, z) &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{e}_y \\ &+ \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{e}_z = \mathbf{0} . \end{aligned}$$

This expresses the fact that the field \mathbf{F} is **vortex-free**.

- The line integral of a conservative force field between the points \mathbf{r}_1 and \mathbf{r}_2

$$I(1, 2) = \int_1^2 \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

is independent of the path between the two points. This allows the definition of the potential energy of the force field as

$$U(x, y, z) = - \int^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' . \quad (3.71)$$

The lower limit of the line integral is arbitrary.

- It can be represented as the **gradient** (the directional derivative) of a scalar function, the potential energy

$$\mathbf{F}(x, y, z) = -\nabla U(x, y, z). \quad (3.72)$$

The choice of the sign is a question of convention. The inverse of this relation is the line integral (3.71).

The relation between the vanishing of the curl (or rotation) of a vector field and the path independence of the line integral is discussed at length in **• Math.Chap. 5.**

Line integration of the equation for the motion of a mass point

$$\frac{d}{dt} \mathbf{p} = \mathbf{F}(x, y, z)$$

in a conservative field of force between the points \mathbf{r}_1 and \mathbf{r}_2

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{d}{dt} \mathbf{p} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

yields the statement

$$\frac{m}{2} v_1^2 + U(\mathbf{r}_1) = \frac{m}{2} v_2^2 + U(\mathbf{r}_2) = \text{const.}$$

or introducing the initial and final times

$$\frac{m}{2} v(t_1)^2 + U(\mathbf{r}(t_1)) = \frac{m}{2} v(t_2)^2 + U(\mathbf{r}(t_2)). \quad (3.73)$$

The total energy (the sum of the kinetic and the potential energy) for the motion of a mass point in a conservative field of force is a conserved quantity. The manipulation of the equation of motion just described suggests that this energy principle corresponds to the **first integral of the equation of motion.**

By contrast, a general force field can depend on all variables relevant for the motion

$$\mathbf{F} = (F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), F_y(\dots), F_z(\dots)).$$

The evaluation of the work integral

$$A = \int_{t_1}^{t_2} dt \{ F_x(x(t), \dots) \dot{x}(t) + F_y(\dots) \dot{y}(t) + F_z(\dots) \dot{z}(t) \}$$

is only possible if a parametric representation of the path of the mass point is specified. The result can depend on the choice of the path of integration between the initial and final space points. If this is the case, the force field is nonconservative. **Nonconservative fields** are characterised by $\text{rot } \mathbf{F} \neq \mathbf{0}$. Energy conservation is not valid for the motion in force fields of this type. An example is the motion of the mass point in a nonconservative field leading to projectile motion with friction. The x -component of the curl of the force

$$\mathbf{F} = (-k\dot{x}, -k\dot{y}, -k\dot{z} - mg) = -k\mathbf{v} - m\mathbf{g}$$

is for example

$$(\text{rot } \mathbf{F})_x = \frac{\partial}{\partial y}(-k\dot{z} - mg) - \frac{\partial}{\partial z}(-k\dot{y}) .$$

This expression can be evaluated with the aid of the chain rule

$$(\text{rot } \mathbf{F})_x = -k \left\{ \frac{d\dot{z}}{dt} \frac{dt}{dy} - \frac{d\dot{y}}{dt} \frac{dt}{dz} \right\} = -k \left\{ \frac{\ddot{z}}{\dot{y}} - \frac{\ddot{y}}{\dot{z}} \right\} \neq 0 .$$

The work integral depends therefore on the path chosen and a potential function can (in general) not be defined⁵.

3.2.3.5 Conservative force fields and potentials. This section contains a list of conservative forces and the associated potentials which are often used in texts or problems.

- The potential energy of the homogeneous gravitational field near the surface of the earth

$$\mathbf{F} = (0, 0, -mg)$$

is

$$U = mgz + U_0 \quad \text{usually with } U_0 = 0 .$$

- It is easily checked that this force field of the three dimensional anisotropic harmonic oscillator

$$\mathbf{F} = (-k_x x, -k_y y, -k_z z)$$

is conservative

$$\text{rot } \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ k_x x & k_y y & k_z z \end{vmatrix} = \mathbf{0} .$$

The corresponding potential energy is, as indicated before,

$$U = \frac{1}{2}(k_x x^2 + k_y y^2 + k_z z^2) . \quad (3.74)$$

This force field is not a central field. This is only the case for $k_x = k_y = k_z = k$. Energy conservation is valid for the anisotropic oscillator (with different constants k_i) but not angular momentum conservation.

- Arbitrary central fields of force with

$$\mathbf{F} = f(r) \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = f(r) \mathbf{e}_r .$$

⁵ Refer, however, to Chap. 5.3.1 for the definition of generalised potentials.

The first factor describes the strength of the force, the second is a unit vector in the radial direction. The strength of the force depends only on the distance from the centre of the force (e.g. the origin)

$$r(x, y, z) = [x^2 + y^2 + z^2]^{1/2}.$$

The irrotational character of the field can be checked by calculating

$$\text{rot } \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r}f & \frac{y}{r}f & \frac{z}{r}f \end{vmatrix} = \mathbf{e}_x \left(\frac{\partial}{\partial y} \left(\frac{z}{r}f \right) - \frac{\partial}{\partial z} \left(\frac{y}{r}f \right) \right) + \dots$$

The two partial derivatives are equal

$$\frac{\partial}{\partial y} \left(\frac{z}{r}f \right) = z \left\{ \frac{d}{dr} \left(\frac{f(r)}{r} \right) \frac{\partial r}{\partial y} \right\} = \frac{zy}{r} \frac{d}{dr} \left(\frac{f(r)}{r} \right)$$

$$\frac{\partial}{\partial z} \left(\frac{y}{r}f \right) = y \left\{ \frac{d}{dr} \left(\frac{f(r)}{r} \right) \frac{\partial r}{\partial z} \right\} = \frac{yz}{r} \frac{d}{dr} \left(\frac{f(r)}{r} \right)$$

so that $(\text{rot } \mathbf{F})_x = 0$. A corresponding result is found for the other components.

The work integral with the central force

$$A = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 f(r) \left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz \right)$$

is easily evaluated as the expression in the brackets is the total differential of the function $r(x, y, z)$

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz.$$

The evaluation of the line integral reduces to a standard integration in the radial direction (see Fig. 3.39)

$$A = \int_1^2 f(r)dr = \int_1^{2'} f(r)dr.$$

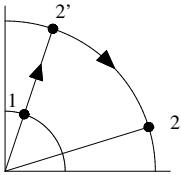


Fig. 3.39. Calculation of the potential energy in a central force field

This result indicates that the direct path of the work integral between the points 1 and 2 is replaced by a path in radial direction between the points

1 and 2' (which provides the result given above) followed by a path along the surface of a sphere from 2' to 2. For this section of the path the force \mathbf{F} is perpendicular to all infinitesimal displacements $d\mathbf{r}$, so that it does not contribute to the integral. The potential energy in a central force field can therefore be written as

$$U(r) = - \int^r f(r') dr' . \quad (3.75)$$

It is a function of the distance from the centre of force (in the present case the origin). The lower limit, which is not specified, corresponds to the constant that can be chosen freely.

Special cases are the isotropic harmonic oscillator with

$$\mathbf{F} = -(kr)\mathbf{e}_r \quad U(r) = \frac{1}{2}kr^2 + U_0$$

and the gravitational force of a mass point M (fixed at the origin) on a mass point m

$$\mathbf{F} = -\gamma \frac{mM}{r^2} \mathbf{e}_r$$

with the potential energy

$$U(r) = +\gamma mM \int^r \frac{dr'}{r'^2} = -\gamma \frac{mM}{r} + U_0 .$$

The constant is normally taken to be zero, so that

$$U(r) = -\gamma \frac{mM}{r} \quad \text{with } U(r \rightarrow \infty) = 0 . \quad (3.76)$$

If the force is represented by the gravitational field of the mass M

$$\mathbf{F}(\mathbf{r}) = m \mathbf{G}(\mathbf{r}) ,$$

a factorisation of the potential energy in the form

$$U(r) = m \Phi(r) = m \left[-\gamma \frac{M}{r} \right] \quad (3.77)$$

offers itself. The quantity Φ defined in this fashion is the **potential** of the gravitational field (for short the gravitational potential). The relation between the gravitational potential and the gravitational field is in general

$$\mathbf{G}(\mathbf{r}) = -\nabla \Phi(\mathbf{r}) , \quad (3.78)$$

in particular for a central field

$$\mathbf{G}(\mathbf{r}) = -\mathbf{e}_r \left\{ \frac{\partial}{\partial r} \Phi(r) \right\} .$$

As the treatment of scalar functions is simpler than that of vector functions, it is (whenever possible) preferable to use a formulation of the laws of mechanics

in terms of scalar functions. This is one the aims of the Lagrange formulation of mechanics (see Chap. 5.3).

A related example is the gravitational force which a homogeneous mass distribution in the form of a sphere (radius R , total mass M) exerts on a mass point m . The corresponding gravitational field is⁶

$$\mathbf{G}(\mathbf{r}) = \begin{cases} -\gamma \frac{M}{r^2} \mathbf{e}_r & r \geq R \\ -\gamma \frac{M}{R^3} r \mathbf{e}_r & r \leq R. \end{cases}$$

The gravitational potential is therefore

$$\Phi(r) = \begin{cases} \int^r \frac{\gamma M}{r'^2} dr' = -\frac{\gamma M}{r} + C_1 & r \geq R \\ \int^r \frac{\gamma M}{R^3} r' dr' = \frac{1}{2} \frac{\gamma M}{R^3} r^2 + C_2 & r \leq R. \end{cases}$$

The standard choice of the constant C_1 is, as indicated above, $C_1 = 0$. The second constant has to be determined by the requirement that the two functions should be continuous on the surface of the sphere

$$\Phi_{\text{inside}}(R) = \Phi_{\text{outside}}(R).$$

This gives

$$C_2 + \frac{1}{2} \frac{\gamma M}{R} = -\frac{\gamma M}{R} \quad \longrightarrow \quad C_2 = -\frac{3}{2} \frac{\gamma M}{R}$$

so that the final result reads

$$\Phi(r) = \begin{cases} -\frac{\gamma M}{r} & r \geq R \\ -\gamma M \left\{ \frac{3}{2R} - \frac{1}{2} \frac{r^2}{R^3} \right\} & r \leq R. \end{cases} \quad (3.79)$$

This function is illustrated in Fig. 3.40. It consists of a section of a parabola

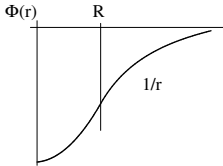


Fig. 3.40. The gravitational potential of a spherical homogeneous mass distribution

inside the sphere which is joined to a section of a hyperbola on the outside. The derivative of the potential is continuous at the junction of the two curves.

⁶ The field is calculated in Chap. 3.2.4.1, p. 127 after the gravitational potential between two point particles is introduced.

The potential energy of a small mass m in the force field of the spherical mass distribution is

$$U(r) = m\Phi(r) .$$

The discussion of the energy principle for one mass point allows a precise definition of the basic concepts involved. The real interest pertains, however, to systems of point particles which are considered in the next section. It is necessary in this case to address the nature of the internal as well as the external forces explicitly.

3.2.4 Energy conservation for a system of mass points

It is again useful to begin the discussion with a system of two point particles which experience only internal forces. The equations of motion are

$$m_1 \dot{\mathbf{v}}_1 = \mathbf{f}_{21} \quad m_2 \dot{\mathbf{v}}_2 = \mathbf{f}_{12} .$$

Each of the forces could in general be a vector function of 13 variables

$$\mathbf{f}_{12} = \mathbf{f}(\mathbf{r}_1, \mathbf{v}_1; \mathbf{r}_2, \mathbf{v}_2; t) \quad \mathbf{f}_{21} = \mathbf{g}(\mathbf{r}_2, \mathbf{v}_2; \mathbf{r}_1, \mathbf{v}_1; t) .$$

The following restrictions can, however, be applied:

- (1) The third axiom demands

$$\mathbf{f}_{12} = -\mathbf{f}_{21} \implies \mathbf{f}(\mathbf{r}_1, \mathbf{v}_1; \mathbf{r}_2, \mathbf{v}_2; t) = -\mathbf{g}(\mathbf{r}_2, \mathbf{v}_2; \mathbf{r}_1, \mathbf{v}_1; t)$$

or in abbreviation

$$\mathbf{f}_{12} = -\mathbf{f}_{21} \implies \mathbf{f}(1, 2, t) = -\mathbf{g}(2, 1, t) .$$

Only one vector function, that changes sign on exchange of the masses, is required

$$\mathbf{f}_{12} = \mathbf{f}(1, 2, t)$$

$$\mathbf{f}_{21} = \mathbf{f}(2, 1, t) = -\mathbf{f}(1, 2, t) .$$

- (2) The first axiom demands the equivalence of all inertial systems. The equivalence is guaranteed if the forces are form-invariant under Galilei transformations (3.7). This condition states that a transformation of the coordinates and of time according to

$$\mathbf{r}'_i = \mathbf{r}_i + \mathbf{v}_{\text{rel}} t + \mathbf{r}_{\text{rel}} \quad (i = 1, 2) \quad \text{and} \quad t' = t$$

should give

$$\mathbf{f}(1, 2, t) = \mathbf{f}(1', 2', t') .$$

This is only possible if the vector function \mathbf{f} depends on the difference of the coordinates and the velocities

$$\mathbf{f}_{12} = \mathbf{f}(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{v}_1 - \mathbf{v}_2, t)$$

$$\mathbf{f}_{21} = \mathbf{f}(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{v}_2 - \mathbf{v}_1, t) .$$

The general case is often not of interest. It can be assumed here that the functions depend only on the coordinates

$$\mathbf{f}_{12} = \mathbf{f}(\mathbf{r}_1 - \mathbf{r}_2) = \{f_x(\mathbf{r}_1 - \mathbf{r}_2), f_y(\mathbf{r}_1 - \mathbf{r}_2), f_z(\mathbf{r}_1 - \mathbf{r}_2)\}$$

$$\mathbf{f}_{21} = \mathbf{f}(\mathbf{r}_2 - \mathbf{r}_1) = -\mathbf{f}(\mathbf{r}_1 - \mathbf{r}_2) .$$

This (internal) interaction between the mass points is called conservative if the following condition is satisfied

$$\mathbf{rot}_1 \mathbf{f}_{21} = \mathbf{rot}_2 \mathbf{f}_{12} = \mathbf{0} . \quad (3.80)$$

The operators act on the coordinates indicated by the index. It is then possible to represent the mutual interaction by the gradient of *one* scalar function V , namely

$$\mathbf{f}_{12} = -\nabla_2 V(\mathbf{r}_1 - \mathbf{r}_2) \quad \mathbf{f}_{21} = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) . \quad (3.81)$$

The third axiom is satisfied automatically. The two gradient operators can be expressed in terms of the gradient operator for the difference of the position vectors $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The relation between the operators ∇_1 and ∇_2 and the operator $\nabla = (\partial_x, \partial_y, \partial_z)$ are

$$\begin{aligned} \nabla_1 &= \sum_{i=1}^3 \mathbf{e}_i \frac{\partial}{\partial x_{i1}} = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \frac{\partial x_i}{\partial x_{i1}} = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} = \nabla \\ \nabla_2 &= \sum_{i=1}^3 \mathbf{e}_i \frac{\partial}{\partial x_{i2}} = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \frac{\partial x_i}{\partial x_{i2}} = - \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} = -\nabla . \end{aligned}$$

The validity of energy conservation for such a conservative interaction can be verified with the following argument. Consider a displacement of m_1 by $d\mathbf{r}_1$

$$m_1 \dot{\mathbf{v}}_1 \cdot d\mathbf{r}_1 = \mathbf{f}_{21} \cdot d\mathbf{r}_1$$

and a displacement of m_2 by $d\mathbf{r}_2$

$$m_2 \dot{\mathbf{v}}_2 \cdot d\mathbf{r}_2 = \mathbf{f}_{12} \cdot d\mathbf{r}_2 .$$

Addition of these equations and line integration from an initial situation

$$t_i \quad \text{with} \quad \mathbf{r}_1(t_i), \mathbf{r}_2(t_i)$$

to a final situation

$$t_f \quad \text{with} \quad \mathbf{r}_1(t_f), \mathbf{r}_2(t_f)$$

gives

$$\int_i^f (m_1 \dot{\mathbf{v}}_1 \cdot d\mathbf{r}_1 + m_2 \dot{\mathbf{v}}_2 \cdot d\mathbf{r}_2) = \int_i^f (\mathbf{f}_{21} \cdot d\mathbf{r}_1 + \mathbf{f}_{12} \cdot d\mathbf{r}_2) .$$

The left hand side (LS) represents (as for a single mass point) the change of the kinetic energy of the two particles

$$\text{LS} = \left[\frac{m_1}{2} v_1(t_f)^2 + \frac{m_2}{2} v_2(t_f)^2 \right] - \left[\frac{m_1}{2} v_1(t_i)^2 + \frac{m_2}{2} v_2(t_i)^2 \right] .$$

The right hand side (RS) can be reformulated via the representation of the forces by the scalar function V

$$\text{RS} = - \int_i^f (\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_1 + \nabla_2 V(\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_2) .$$

The expression within the brackets is the total differential of the potential function V

$$= - \int_i^f dV = V(i) - V(f) .$$

The law of energy conservation for a system of two masses with a conservative internal interaction can therefore be written in the form

$$\left[\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2) \right]_{\text{for each } t} = E_0 . \quad (3.82)$$

The sum of the kinetic energies of the two masses plus the potential energy *between* the two masses is constant in time.

An explicit example is gravitation with the force law

$$\mathbf{f}_{21} = -\mathbf{f}_{12} = -\gamma m_1 m_2 \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

or with $r = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2}$ in detail

$$= -\gamma m_1 m_2 \left\{ \frac{x_1 - x_2}{r^3}, \frac{y_1 - y_2}{r^3}, \frac{z_1 - z_2}{r^3} \right\} .$$

The condition

$$\mathbf{rot}_1 \mathbf{f}_{21} = \mathbf{rot}_2 \mathbf{f}_{12} = \mathbf{0}$$

can easily be checked. The potential energy between the two masses is

$$V(\mathbf{r}_1 - \mathbf{r}_2) = -\gamma \frac{m_1 m_2}{r} = -\gamma \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} , \quad (3.83)$$

where the asymptotic limit $V(r \rightarrow \infty) = 0$ has been chosen. The potential energy is a function of the separation of the two masses.

In the next step external conservative forces for the two particle system are taken into account

$$\frac{d}{dt} \mathbf{p}_1 = \mathbf{F}_1(\mathbf{r}_1) + \mathbf{f}_{21} \quad \frac{d}{dt} \mathbf{p}_2 = \mathbf{F}_2(\mathbf{r}_2) + \mathbf{f}_{12} . \quad (3.84)$$

The external forces can be represented as the gradient of scalar functions

$$\mathbf{F}_1 = -\nabla_1 U_1(x_1, y_1, z_1) \quad \mathbf{F}_2 = -\nabla_2 U_2(x_2, y_2, z_2)$$

if they are conservative, so that the relations

$$\mathbf{rot}_1 \mathbf{F}_1(\mathbf{r}_1) = \mathbf{rot}_2 \mathbf{F}_2(\mathbf{r}_2) = \mathbf{0}$$

are valid. Integration of the equations of motion (3.84) with the same steps as before yields

$$\begin{aligned} (T_1 + T_2)_f - (T_1 + T_2)_i \\ = V(1, 2)_i - V(1, 2)_f + \int_i^f (\mathbf{F}_1 \cdot d\mathbf{r}_1 + \mathbf{F}_2 \cdot d\mathbf{r}_2) . \end{aligned}$$

The two remaining integrals are

$$- \int_i^f (dU_1 + dU_2)$$

so that the law of energy conservation can be stated as

$$[T_1 + T_2 + U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + V(\mathbf{r}_1 - \mathbf{r}_2)]_t = E_0 . \quad (3.85)$$

The sum of the kinetic energies plus the sum of the external potential energies plus the internal potential energy between the masses is a conserved quantity.

An example for such a system is the system earth and moon moving in the gravitational field of the sun (the source of the external forces). The total energy of this two particle system is

$$\frac{m_E}{2} v_E^2 + \frac{m_M}{2} v_M^2 - \gamma \frac{m_E m_S}{r_E} - \gamma \frac{m_M m_S}{r_M} - \gamma \frac{m_E m_M}{|\mathbf{r}_E - \mathbf{r}_M|} = E_0 .$$

The vectors \mathbf{r}_E and \mathbf{r}_M connect the centre of mass of the sun with the centres of mass of the corresponding celestial bodies.

The starting point for the discussion of an arbitrary number of point particles is the set of equations of motion

$$\frac{d}{dt} \mathbf{p}_i = \mathbf{F}_i(\mathbf{r}_i) + \sum_{k=1}^N \mathbf{f}_{ki}(\mathbf{r}_k - \mathbf{r}_i) \quad (i = 1, 2, 3 \dots N) .$$

The indices of the forces indicate the following properties:

- The external forces depend only on the coordinates (\mathbf{r}_i). The functional form could differ for the different masses. This is indicated by the index of the vector function \mathbf{F}_i . In the case of gravitation the difference is only due to different mass factors.
- The internal forces carry indices as well, which could for instance cover the case, that some of the masses are also charged. For instance \mathbf{f}_{12} and \mathbf{f}_{21} could be gravitational forces, while \mathbf{f}_{13} and \mathbf{f}_{31} represent gravitational and electric forces. The internal forces between each pair of masses are characterised by *one* vector function if these forces satisfy the third axiom, e.g. by

$$\begin{aligned} \mathbf{f}_{ik}(\mathbf{r}_i - \mathbf{r}_k) &= g^{(ik)}(|\mathbf{r}_i - \mathbf{r}_k|) (\mathbf{r}_i - \mathbf{r}_k) \\ \mathbf{f}_{ki}(\mathbf{r}_k - \mathbf{r}_i) &= g^{(ik)}(|\mathbf{r}_i - \mathbf{r}_k|) (\mathbf{r}_k - \mathbf{r}_i) . \end{aligned}$$

The argumentation given for the case of two mass points can be repeated, if all the forces involved are assumed to be conservative.

$$\boxed{\nabla_i \times \mathbf{F}_i = \mathbf{0} \quad \nabla_i \times \mathbf{f}_{ki} = \nabla_k \times \mathbf{f}_{ik} = \mathbf{0} .}$$

The argument proceeds as follows: use the equations of motion for the mass with the index i and form the scalar product with $d\mathbf{r}_i$, add the equations of all the masses and integrate from the initial to the final situation. The result of this calculation (which will not be outlined in detail) is the law of energy conservation for a system of point particles with conservative internal and external forces

$$\boxed{\begin{aligned} E = & T_1 + T_2 + T_3 + \dots + T_N \\ & + U_1 + U_2 + U_3 + \dots + U_N \\ & + V_{12} + V_{13} + \dots + V_{1N} \\ & + V_{23} + \dots + V_{2N} \\ & \vdots \\ & + V_{N-1,N} = E_0 . \end{aligned}} \quad (3.86)$$

The total energy of the system is composed of

1. the sum of the kinetic energies of the individual masses

$$T = \sum_{i=1}^N T_i = \sum_{i=1}^N \frac{m_i}{2} v_i^2 = \sum_i \frac{p_i^2}{2m_i} , \quad (3.87)$$

2. the sum of the potential energies of the individual masses due to external force fields

$$U = \sum_{i=1}^N U_i = \sum_{i=1}^N U_i(\mathbf{r}_i) , \quad (3.88)$$

3. the sum of the internal potential energies between all pairs of particles

$$\begin{aligned} V &= \sum_{i < k} V_{ik} = \sum_{i < k} V_{ik}(\mathbf{r}_i - \mathbf{r}_k) \\ &= \frac{1}{2} \sum_{i \neq k} V_{ik}(\mathbf{r}_i - \mathbf{r}_k) . \end{aligned} \quad (3.89)$$

The second line is a consequence of the third axiom, which demands

$$V_{ik}(\mathbf{r}_i - \mathbf{r}_k) = V_{ki}(\mathbf{r}_k - \mathbf{r}_i) .$$

An example for such a system of 'mass points' is our planetary system with sun, planets, moons, asteroids etc., provided the bodies can really be regarded as points. If the sun is chosen as the origin of the coordinate system and the

action of the sun on the other bodies is regarded as an external force, then external and internal forces

$$U = \sum_i \left(-\gamma \frac{m_i m_S}{r_i} \right) \quad V = \sum_{i < k} \left(-\gamma \frac{m_i m_k}{|\mathbf{r}_i - \mathbf{r}_k|} \right)$$

have to be distinguished. All forces should be considered as internal if the sun is not chosen as the origin.

A second example is a rigid body, composed of point particles, moving in a conservative force field. As the internal energies (normally) depend only on the separation of the mass points and as this separation is by definition constant, the internal potential energy does not change in time. The internal potential energy is therefore a constant which is not of interest for the motion of the rigid body as a whole. It can be set equal to zero, so that the energy of a rigid body is

$$E = \sum_i T_i + \sum_i U_i = \text{const.}$$

Before the discussion of the application of the conservation laws to the collision of two mass points (or corresponding point charges), the calculation of the potential or the potential energy of a continuous mass distribution, which has been quoted on page 121, will be outlined.

3.2.4.1 The potential energy of a continuous mass distribution. The discussion of the potential energy of a mass m in the gravitational field of a distribution of N discrete masses (m_i)

$$U(\mathbf{r}) = \sum_{i=1}^N \left(-\gamma \frac{m m_i}{|\mathbf{r} - \mathbf{r}_i|} \right)$$

can be extended to the question: how can the potential energy be calculated if the discrete distribution is replaced by a continuous distribution? The answer is: the N masses have to be replaced by infinitesimal elements (dm_i), which are described by density (ρ) times an infinitesimal volume dV in the continuous limit. The summation has to be replaced by integration (Fig. 3.41)

$$\sum_i dm_i f(\mathbf{r}_i) \longrightarrow \iiint_V \rho(\mathbf{r}') f(\mathbf{r}') dV'$$

The total mass of the distribution can be calculated from the density distribution by

$$M = \iiint_V \rho(\mathbf{r}') dV'.$$

The potential energy of the mass m in the field of the mass distribution is

$$U(\mathbf{r}) = \iiint_V (-\gamma m) \frac{\rho(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|}.$$

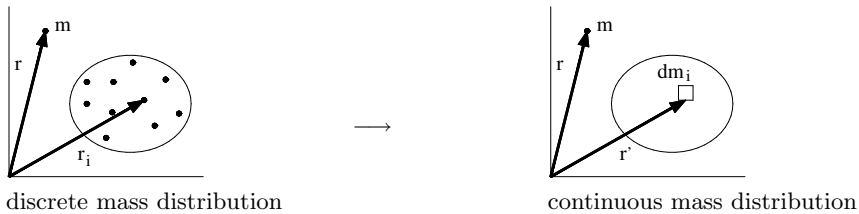


Fig. 3.41. Calculation of the potential energy of mass distributions

The evaluation of the triple integral is not necessarily simple for a general density distribution and an arbitrary volume. Of special interest is the potential energy in the gravitational field of the earth for which an ideal spherical shape is assumed

$$R_E \approx 6.37 \cdot 10^8 \text{ cm} = 6370 \text{ km}$$

$$M_E \approx 5.98 \cdot 10^{27} \text{ g} = 5.98 \cdot 10^{24} \text{ kg} .$$

It will also be assumed that the distribution is isotropic

$$\rho(r) = \begin{cases} \rho(r) & r \leq R_E \\ 0 & r \geq R_E \end{cases} ,$$

so that the density varies only with the radius. The mass is then calculated by the integral

$$M_E = 4\pi \int_0^{R_E} \rho(r') r'^2 dr' .$$

A good choice of the coordinate system for the evaluation of the triple integral for the potential energy is the following: place the origin in the centre of the sphere and let the z -axis run through the mass m . This choice does not amount to a restriction because of the symmetry of the present problem. Choose spherical coordinates so that the integral

$$U(r) = -\gamma m \int_0^{2\pi} d\varphi' \int_0^{R_E} r'^2 \rho(r') dr' \int_0^\pi \frac{\sin \theta' d\theta'}{[r^2 + r'^2 - 2rr' \cos \theta']^{1/2}}$$

has to be considered. The integral over the angle φ' can be written down directly. The substitution

$$x = \cos \theta' \quad dx = -\sin \theta' d\theta'$$

is used for the integration over the angle θ' . Integration over the variable x in the double integral

$$U(r) = -2\pi\gamma m \int_0^{R_E} r'^2 \rho(r') dr' \int_{-1}^1 \frac{dx}{[r^2 + r'^2 - 2rr'x]^{1/2}}$$

yields the standard result

$$I = \int_{-1}^1 \frac{dx}{[r^2 + r'^2 - 2rr'x]^{1/2}} = -\frac{1}{rr'}[r^2 + r'^2 - 2rr'x]^{1/2} \Big|_{-1}^1,$$

but insertion of the limits has to be handled with care. The result for $r > r'$ is

$$I = -\frac{1}{rr'}[(r - r') - (r + r')] = \frac{2}{r},$$

for $r' > r$

$$I = -\frac{1}{rr'}[(r' - r) - (r + r')] = \frac{2}{r'}.$$

The potential energy is

$$U(r) = (-\gamma m)(4\pi) \frac{1}{r} \int_0^R \rho(r') r'^2 dr'$$

or

$$U(r) = -\gamma \frac{mM_E}{r} \quad (r \geq R) \quad (3.90)$$

if the mass m is outside the sphere (the case $r > R$). It is independent of the radial variation of the mass distribution. If the mass m is inside the sphere, U is given by

$$U(r) = (-\gamma m)(4\pi) \left[\frac{1}{r} \int_0^r \rho(r') r'^2 dr' + \int_r^R \rho(r') r' dr' \right].$$

Further evaluation is only possible if $\rho(r')$ is known. The simplest case is a homogeneous mass distribution, $\rho(r') = \rho_0$, for which the potential energy can be calculated as

$$\begin{aligned} U(r) &= (-\gamma m)(4\pi\rho_0) \left[\frac{1}{r} \int_0^r r'^2 dr' + \int_r^R r' dr' \right] \\ &= (-\gamma m)(4\pi\rho_0) \left[\frac{1}{3}r^2 + \frac{1}{2}R^2 - \frac{1}{2}r^2 \right] \\ &= (-\gamma m)(4\pi\rho_0) \left[\frac{1}{2}R^2 - \frac{1}{6}r^2 \right]. \end{aligned}$$

The constant density can be replaced by the total mass

$$\rho_0 = \frac{3}{4\pi} \frac{M_E}{R^3}$$

so that

$$U(r) = (-\gamma mM_E) \left[\frac{3}{2} \frac{1}{R} - \frac{1}{2} \frac{r^2}{R^3} \right] \quad r \leq R. \quad (3.91)$$

For points outside the sphere, the mass distribution looks (independent of the radial form if the distribution) as if the total mass was concentrated in

the centre of mass. The potential energy of a homogeneous distribution has a parabolic form⁷ in the interior of the sphere.

The gravitational field of the (spherical, homogeneous) earth (quoted on p. 121) can now be calculated as

$$r \geq R \quad \mathbf{G} = -\frac{1}{m} \nabla U = -\gamma \frac{M_E}{r^2} \mathbf{e}_r \quad (3.92)$$

$$r \leq R \quad \mathbf{G} = -\frac{1}{m} \nabla U = -\gamma \frac{M_E}{R^3} r \mathbf{e}_r . \quad (3.93)$$

The result for the potential energy can be used to relate the gravitational acceleration near the surface of the earth and the universal gravitational constant. The gravitational potential (3.90) for points outside of the earth (with $r = R_E + h$) is

$$U = -\gamma \frac{mM_E}{R_E} \left(\frac{1}{1 + h/R_E} \right) .$$

It can be expanded in terms of the binomial (or geometrical) series for points *close* to the surface of the earth

$$\begin{aligned} U &= -\gamma \frac{mM_E}{R_E} \left(1 - \frac{h}{R_E} + \frac{h^2}{R_E^2} - \dots \right) \\ &= \text{const} + m \left[\gamma \frac{M_E}{R_E^2} \right] h - m \left[\gamma \frac{M_E}{R_E^3} \right] h^2 + \dots = U_0 + mgh + \dots \end{aligned}$$

for

$$h \ll R_E \approx 6370 \text{ km} .$$

The expression in the first bracket is equal to mgh , so that the relation between the two gravitational constants g and γ is found to be

$$g = \frac{\gamma M_E}{R_E^2} . \quad (3.94)$$

3.2.5 Application: collision problems

The conservation laws (in particular momentum conservation and a simple form of energy conservation) play a special role in the discussion of collision problems. The situation that is usually considered is the following: two masses m_1 and m_2 (point particles) move uniformly along straight lines so that they collide at a certain time. External forces are not present. Either they can be neglected or they are eliminated as in the demonstration experiment using an air cushion rail. As long as the separation of the masses is sufficiently

⁷ Compare Fig. 3.40, which shows the gravitational potential of a spherical, homogeneous mass distribution.

large, the mutual gravitation (or other interaction) can also be neglected. The system before the collision is therefore characterised by

$$\text{Total momentum: } \mathbf{P}_{\text{in}} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$$

$$\text{Total energy: } E_{\text{in}} = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 .$$

The details of the collision process are actually rather complicated. The masses (objects) can be deformed, regain their shape, etc. Energy is released from the collision system in the form of sound and/or heat. It can, however, be assumed – notwithstanding all details – that these processes are controlled by internal forces (interatomic or intermolecular forces, that is electric forces). As these forces satisfy the third axiom, momentum is conserved, so that the relation

$$\mathbf{P}_{\text{in}} = \mathbf{P}_{\text{out}} \quad (3.95)$$

is valid for the sum of the momenta of the two objects independent of these details. Further discussion of the problem depends on the situation concerning the energy. Several cases can be distinguished.

3.2.5.1 The completely elastic collision. The masses retain their original form during the collision and move apart. It is assumed that no energy loss has occurred. This assumption is quite well realised for the collision of billiard or steel balls, if they are not set into rotational motion. The sum of the final momenta is

$$\mathbf{P}_{\text{out}} = \frac{m_1}{2} \mathbf{v}'_1 + \frac{m_2}{2} \mathbf{v}'_2 = \mathbf{P}_{\text{in}} . \quad (3.96)$$

The total energy is again kinetic if the masses are sufficiently far apart

$$E_{\text{out}} = \frac{m_1}{2} v_1'^2 + \frac{m_2}{2} v_2'^2 = E_{\text{in}} . \quad (3.97)$$

It can be asked in how far the velocities after the collision can be determined on the basis of the conservation laws if the velocity components before the collision are known. In three-dimensional space six velocity components would have to be calculated. There are, however, only four equations (three equations for momentum conservation plus one for energy conservation) available. The discussion can be restricted to two space dimensions if the internal forces acting during the collision allow angular momentum conservation. The collision partners will then move in the same plane before and after the collision. In the two-dimensional world three equations are available for the determination of four unknown velocity components. The final situation can still not be determined fully by the conservation laws alone, although some useful partial answers are possible. Only for central elastic collisions (a one-dimensional problem) do the number of relations and the number of unknowns agree.

3.2.5.2 The central elastic collision. The conservation laws for the one-dimensional case are

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2.$$

The direction of the motion is expressed by the sign of the velocity components. The two velocity components after the collision can be determined, if the velocities (and the masses) are known before the collision. A direct combination of the two equations leads to (see ● Probl. 3.19)

$$v'_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_1 + \left(\frac{2m_2}{m_1 + m_2} \right) v_2 \quad (3.98)$$

$$v'_2 = \left(\frac{2m_1}{m_1 + m_2} \right) v_1 - \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_2. \quad (3.99)$$

Any number of special cases can be discussed with the aid of these relations. For instance: the two masses exchange their velocities for $m_1 = m_2 = m$

$$v'_1 = v_2, \quad v'_2 = v_1.$$

The final situation is: the mass m_1 will be at rest after the collision and m_2 will move on with the velocity v_1 , if $m_2 = m$ is initially at rest and is hit by an equally large mass $m_1 = m$ moving with the velocity v_1 .

The relations reduce to

$$v'_1 \approx v_1, \quad v'_2 \approx 2v_1 - v_2$$

for $m_1 \gg m_2$. There is a total reflection of the small mass, if it hits a large mass at rest ($v_1 = 0$). The small mass will move with the velocity $3v$ after the collision, if the two masses move towards each other with the same speed ($v_1 = -v_2 = v$). The large mass continues without a substantial change of its velocity.

3.2.5.3 The noncentral elastic collision. This scenario in two dimensions is sketched in Figure 3.42a. The conservation laws are in this case

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = m_1 \mathbf{v}'_1 + m_2 \mathbf{v}'_2 \quad (2 \text{ equations}) \quad (3.100)$$

$$\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 = \frac{m_1}{2} v'^2_1 + \frac{m_2}{2} v'^2_2 \quad (1 \text{ equation}). \quad (3.101)$$

A complete determination of the four components of the final velocities

$$(v'_{1x}, v'_{1y}, v'_{2x}, v'_{2y})$$

is not possible. It is, however, possible to make useful partial statements, which will be discussed here only for the special case $v_2 = 0$ (the second mass is at rest before the collision). Momentum conservation leads to the

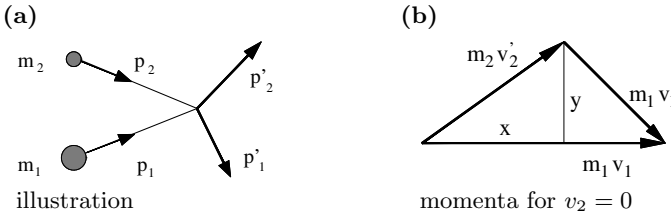


Fig. 3.42. Noncentral elastic collision

statement: the coordinates of the endpoint of the vector \mathbf{p}'_2 satisfy (Fig. 3.42b) the relation

$$x^2 + y^2 = m_2^2 v_2'^2, \quad (m_1 v_1 - x)^2 + y^2 = m_1^2 v_1'^2,$$

where (x, y) stand for the coordinates of the endpoint of \mathbf{p}'_2 . Insertion of this relation into the law of energy conservation yields

$$m_1 v_1^2 = \frac{1}{m_1} [(m_1 v_1 - x)^2 + y^2] + \frac{1}{m_2} [x^2 + y^2].$$

The rearrangement of this equation

$$\left[x - \frac{m_1 m_2}{m_1 + m_2} v_1 \right]^2 + y^2 = \left[\frac{m_1 m_2}{m_1 + m_2} v_1 \right]^2$$

is the equation of a circle. The coordinates of the centre are

$$\mathbf{P}_M = \left(\frac{m_1 m_2}{m_1 + m_2} v_1, 0 \right),$$

the radius is

$$R = \frac{m_1 m_2}{m_1 + m_2} v_1.$$

The tip of the momentum vector \mathbf{p}'_2 lies on a circle, which passes through the starting point of the momentum vector \mathbf{p}_1 . The centre of the circle lies on the vector \mathbf{p}_1 and divides it in the ratio m_2/m_1 (Fig. 3.43a). The situation after the collision permits, as a consequence of the conservation laws, statements as for instance: If the direction *or* the magnitude of one of the momentum vectors is measured after the collision, all other quantities can be calculated.

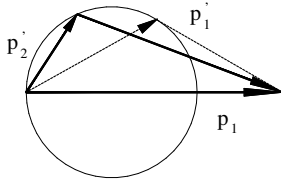
The centre of the circle bisects the momentum vector \mathbf{p}_1 for the collision of two equal masses (Fig. 3.43b). There is always a right angle between the momenta of the masses after the collision.

3.2.5.4 Inelastic collisions. The masses retain (possibly in part) their deformation or energy is lost in a different way (e.g. sound) during the collision process. Momentum conservation is still valid

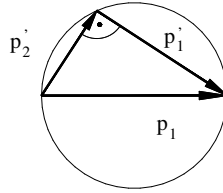
$$\mathbf{P}_{\text{out}} = m_1 \mathbf{v}'_1 + m_2 \mathbf{v}'_2 = \mathbf{P}_{\text{in}}$$

but the balance of energy has to be changed to

(a)

location of the endpoint of p_2'

(b)

the same for the case $m_1 = m_2$ **Fig. 3.43.** Noncentral elastic collision

$$\mathbf{E}_{\text{out}} = \frac{m_1}{2} v_1'^2 + \frac{m_2}{2} v_2'^2 + Q = \mathbf{E}_{\text{in}}.$$

The quantity Q is the energy loss, that is the amount, which has been transformed into other forms of energy. Even for a central collision (one dimensional) not all three quantities (v_1' , v_2' , Q) can be determined with the two equations available in this case.

A number of partial results can, however, also be obtained in this case, as e.g. for

$$m_1 = m_2 = m \quad v_2 = 0$$

with the relations

$$\text{momentum : } v_1 - v_1' = v_2'$$

$$\text{energy : } (v_1 - v_1')(v_1 + v_1') = v_2'^2 + \frac{2Q}{m}. \quad (3.102)$$

Division of the two equations leads to the result

$$(v_1 + v_1') = v_2' + \frac{2Q}{v_2' m}. \quad (3.103)$$

Addition of the first equation in (3.102) and of (3.103) yields for v_2'

$$v_2' = \frac{v_1}{2} \pm \left[\frac{v_1^2}{4} - \frac{Q}{m} \right]^{1/2}. \quad (3.104)$$

This statement only makes sense, if the radicand is positive, that is if

$$\frac{v_1^2}{4} - \frac{Q}{m} \geq 0 \quad \text{or} \quad \frac{1}{2} T_1 \geq Q.$$

The energy loss can at most amount to half of the energy brought in by the incident mass. This restriction is a direct consequence of the conservation laws.

3.2.5.5 Completely inelastic collisions. The deformed masses coalesce during the collision and move on together (e.g. the collision of a steel ball with a lump of putty or the collision of two lumps of putty). The situation after the collision is characterised by

$$\mathbf{P}_{\text{out}} = (m_1 + m_2)\mathbf{v}' = \mathbf{P}_{\text{in}}$$

$$\mathbf{E}_{\text{out}} = \frac{1}{2}(m_1 + m_2)v'^2 + Q = \mathbf{E}_{\text{in}}.$$

It is possible to determine the velocity and the energy loss after the collision both for central collisions as well as for the general case. In each case the number of equations are matched (two quantities and equations in one dimension, three in two dimensions).

The relations for a central collision

$$v' = \frac{1}{m_1 + m_2}(m_1 v_1 + m_2 v_2)$$

$$Q = \frac{m_1 m_2}{2(m_1 + m_2)}(v_1 - v_2)^2$$

allow e.g. the statements: the combined mass $M = (m_1 + m_2)$ will be at rest ($v' = 0$) after the collision, if two equal masses ($m_1 = m_2$) collide with opposite initial velocities of the same magnitude ($v_2 = -v_1$). All initial kinetic energy is transformed into deformation energy ($Q = m_1 v_1^2$).

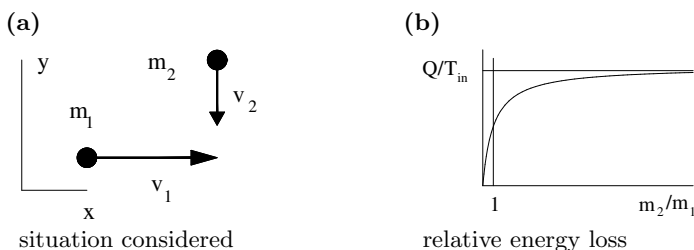


Fig. 3.44. Inelastic Collisions

An example of a two dimensional collision of this kind is the following: two masses, initially on straight line trajectories which are perpendicular to each other, collide (Fig. 3.44a). Momentum conservation states

$$m_1|v_1| + 0 = (m_1 + m_2)v_x \quad \longrightarrow \quad v_x = \frac{m_1}{M}|v_1|$$

$$0 - m_2|v_2| = (m_1 + m_2)v_y \quad \longrightarrow \quad v_y = -\frac{m_2}{M}|v_2|.$$

After the collision the combined masses have a negative velocity component in the y -direction. The final velocity would be $\mathbf{v} = (0.5|v|, -0.5|v|)$ for equal

masses and equal speeds before the collision. The corresponding energy loss is

$$\begin{aligned} Q &= T_{\text{in}} - T_{\text{out}} \\ &= \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 - \frac{M}{2} \left(\frac{m_1^2}{M^2} v_1^2 + \frac{m_2^2}{M^2} v_2^2 \right) \\ &= \frac{1}{2} \frac{m_1 m_2}{M} (v_1^2 + v_2^2) . \end{aligned}$$

It is in particular

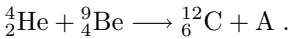
$$\frac{Q}{T_{\text{in}}} = \frac{m_2/m_1}{1 + m_2/m_1}$$

if m_2 is initially at rest ($v_2 = 0$ and is hit by m_1 and taken along). The energy loss grows with the ratio of the masses and is nearly complete if a small mass m_1 hits a big mass m_2 at rest (Fig. 3.44b).

A *historical example* for the application of simple collision theory is the discovery of the neutron by Chadwick in 1932. In this year only two elementary particles were known, the electron e and the proton p . Both particles are charged so that they can be identified by deflection in electric and magnetic fields. In addition, a neutral nuclear radiation, the γ - radiation was known. γ - rays are, as light or radio waves, a special form of electromagnetic radiation. The different forms of radiation are distinguished by their wave length

$$\lambda_\gamma \approx 10^{-10} \text{ cm} \quad \lambda_{\text{light}} \approx 10^{-5} \text{ cm} \quad \lambda_{\text{radio}} \approx 10^4 \text{ cm} .$$

In the year 1932 several experimental groups investigated the nuclear reaction



A Helium nucleus collides with a Beryllium nucleus. Two reaction products can be identified, a carbon nucleus and a 'particle' A , which is electrically neutral. It was relatively easy to find out, that this particle did not correspond to γ - radiation (photons). The energy balance did not match. Chadwick suggested that the 'unknown radiation' could be a massive, neutral particle. In order to verify this hypothesis, two additional experiments were performed, which, in a simplified form, can be described as

Experiment 1: Collision of the neutral 'particle' with an unknown velocity v and mass m with a proton at rest

$$A_v + p_{v_p=0} \longrightarrow A_{v'} + p_{v'_p} .$$

The experiment is analysed with the aid of momentum and energy conservation

$$mv = mv' + m_p v'_p \quad \frac{m}{2} v^2 = \frac{m}{2} v'^2 + \frac{m_p}{2} v_p'^2$$

which leads to the relation

$$v'_p = \frac{2m}{(m_p + m)} v .$$

As neither the mass m nor the velocity v of the new particle were known, a second experiment was necessary.

Experiment 2: Collision of A with a nitrogen nucleus (^{14}N , with mass $M_N \approx 14 m_p$) at rest

$$A_v + {}^{14}\text{N}_{v_N=0} \longrightarrow A_{v'} + {}^{14}\text{N}_{v'_N} .$$

Application of the conservation laws gives in this case

$$v'_N = \frac{2m}{(M_N + m)} v .$$

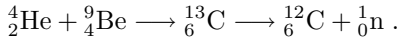
Combination of the two results leads to the ratio

$$\frac{v'_p}{v'_N} = \frac{M_N + m}{(m_p + m)} \approx \frac{14m_p + m}{(m_p + m)} .$$

The ratio of the velocities of the particles colliding with A allow a determination of the unknown mass. This ratio could be determined from tracks in cloud chambers as the emerging reaction partners are charged. The result was

$$\frac{v'_p}{v'_N} \approx 7,5$$

which implies $m \approx m_p$. The nuclear reaction chain, which led to the identification of the neutron, is



The first step is a fusion process in which the carbon isotope ${}^{13}_6\text{C}$ is formed. This nucleus decays into a neutron and the stable carbon isotope ${}^{12}_6\text{C}$.

4 Dynamics II: Problems of Motion

Newton's equations of motion allow, at least in principle, the calculation of the time development of a system of mass points. The prerequisite is that all forces of the system (e.g. as functions of the position of the particles) and the initial conditions for all masses are known. Independent of the technical realisation of such calculations, a distinction is necessary between integrable or chaotic systems. This point will be addressed in Chap. 5.4.3. A system is called integrable if initial conditions that are infinitesimally close will lead to solutions that are infinitesimal close. The solution diverges (exponentially) even for infinitesimally close initial conditions in the case of chaotic systems.

A selection of examples for the motion of *one* mass point by solution of the equations

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

for integrable systems will be presented in this chapter. Additional problems will be discussed in Chap. 5 and 6 after an introduction of more advanced methods. The first problem, that will be considered in this chapter, is a simplified treatment of planetary motion, known as Kepler's problem. This is followed by a presentation of some variants of the problem of oscillations, namely the mathematical pendulum and damped as well as driven oscillators.

4.1 Kepler's problem

A problem, that has occupied mankind since early times, is the understanding of the regular motion of the celestial bodies in our solar system. A sufficiently accurate calculation of this motion is only possible through the nontrivial integration of a set of coupled differential equations for the motion of a large number of interacting objects. The goal that will be pursued in this chapter is more modest. Kepler's laws for the motion of the planets (which are quoted on p. 150 ff) are to be investigated and explained in a simple fashion. The motion of comets and meteorites can be discussed on the same basis.

4.1.1 Preliminaries

It is a well known fact that the solar mass is the dominant mass in our planetary system. A comparison using the mass of the earth (m_E) as a measure emphasises this statement

$$\text{solar mass} \qquad M \approx 333000 m_E$$

$$\text{mass of the lightest planet (Mercury)} \quad m_{Me} \approx \frac{1}{20} m_E$$

$$\text{mass of the heaviest planet (Jupiter)} \quad m_{Ju} \approx 320 m_E .$$

The dominant solar mass is the reason why the orbit of each of the planets is mainly determined by the gravitational action of the sun. The force that a celestial body X exerts on the earth (compare (3.2)) is

$$F_{XE} = \gamma \frac{m_X m_E}{R_{XE}^2} .$$

The ratio of gravitational action of the sun on the earth in comparison with the gravitational action of another body is therefore

$$\frac{F_{SE}}{F_{XE}} = \frac{M}{m_X} \frac{R_{XE}^2}{R_{SE}^2} .$$

The parameters indicated in the first two lines of Table 4.1 are found for the objects in the vicinity of the earth. The ratio of the forces (rounded) given in the last line are calculated with a more accurate value of the solar mass $M = 332\,942 m_E$.

Table 4.1. Gravitational action of the sun and other celestial objects on the earth

	Mars	Venus	Jupiter	Moon	[unit]
m_X	0.107	0.815	318	0.0123	m_E
R_{XE} (min)	0.524	0.277	4.20	0.00257	R_{SE}
F_{SE}/F_{XE}	852 000	31 300	18 500	178	

The only object that has a comparable influence on the earth as the sun (though not really close) is the moon. The motion of the earth, as well as the motion of every other planet, can for this reason, be treated in a first approximation as a two body problem involving only the sun and the celestial body under consideration. The dominant solar mass allows an additional simplification. It is possible to ignore the motion of the sun, so that the two body problem is reduced to a one body problem. This approximation can, as shown on page 152 ff, be corrected easily by the introduction of relative and centre of mass coordinates.

4.1.2 Planetary motion

With the assumptions stated, the motion of each of the planets is determined by the (vectorial) equation of motion

$$m_P \ddot{\mathbf{r}} = -\gamma \frac{M m_P}{r^3} \mathbf{r} . \quad (4.1)$$

As before, M is the solar mass, m_P the mass of the planet and \mathbf{r} the vector from the sun to the planet. This vectorial differential equation characterises the simple Kepler problem.

4.1.2.1 Solution of the equation of motion. The first remark concerns the fact that the mass of the planet can be eliminated. The orbits calculated are not specific for a particular planet. The earth would, for instance, circulate around the sun like Venus, if the earth would be placed on the orbit of Venus and provided with the appropriate initial conditions. The second remark is concerned with the choice of optimal coordinates. Conservation laws are helpful in this respect (and for further discussion). The simple Kepler problem (4.1) is a central force problem. Angular momentum is a conserved quantity. The motion takes place in a plane, which is defined by the initial values $\mathbf{r}(0)$ and $\mathbf{v}(0)$. In view of the central force, plane polar coordinates are therefore the best choice. The corresponding decomposition of the vectorial equation of motion is (see (2.60))

$$a_r = \ddot{r} - r\dot{\varphi}^2 = -\gamma \frac{M}{r^2} \quad (4.2)$$

$$a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0 . \quad (4.3)$$

The initial conditions at $t_0 = 0$ with values of $r_0, \dot{r}_0, \varphi_0, \dot{\varphi}_0$ have to be specified. The second equation expresses (see (2.63)) angular momentum conservation (the law of areas)

$$r^2(t)\dot{\varphi}(t) = A . \quad (4.4)$$

The constant A is determined by the initial conditions as

$$A = r_0^2 \dot{\varphi}_0 = \frac{l_0}{m_P} . \quad (4.5)$$

The angular coordinate $\varphi(t)$ can be determined via separation of variables from

$$\varphi(t) - \varphi_0 = \int_0^t \frac{A}{r(t')^2} dt' \quad (4.6)$$

once $r(t)$ has been calculated. The radial coordinate $r(t)$ is obtained by solution of the differential equation (4.2). As the force is conservative, energy conservation can be utilised for a 'first integration'

$$\frac{m_P}{2} v^2 - \gamma \frac{m_P M}{r} = E_0 , \quad (4.7)$$

where the quantity v^2 is to be expressed in terms of polar coordinates (2.57) as

$$v^2 = \dot{r}^2 + r^2 \dot{\varphi}^2.$$

The radial differential equation to be solved is therefore

$$\frac{1}{2} \dot{r}^2 + \frac{1}{2} \frac{A^2}{r^2} - \gamma \frac{M}{r} = B = \frac{E_0}{m_p} \quad (4.8)$$

if $\dot{\varphi}$ is replaced by A/r^2 and the mass of the planet is eliminated. The constant B is determined by the initial energy (divided by the planetary mass). Separation of variables yields the basic integral of the simple Kepler problem

$$t = \pm \int_{r_0}^r \frac{dr'}{\left[2B + 2\gamma \frac{M}{r'} - \frac{A^2}{r'^2} \right]^{1/2}}. \quad (4.9)$$

The sign has to be chosen, so that the resulting function $t = t(r)$ is a monotonically increasing function. The inverse function $r = r(t)$ can be inserted into the angular momentum conservation law (4.6). The problem would be solved fully after evaluation of this integral.

A similar statement can be made for any central force problem (that is the motion of a mass point m in a potential $\Phi(r) = U(r)/m$)

$$t = \pm \sqrt{\frac{m}{2}} \int_{r_0}^r \frac{dr'}{\left[E_0 - U(r') - \frac{mA^2}{2r'^2} \right]^{1/2}}. \quad (4.10)$$

The method of solution indicated can, unfortunately, not be handled in an analytic fashion for the Kepler problem. The reason is the fact that the inversion of the function $t = t(r)$ can only be carried through numerically. This is illustrated explicitly by the discussion of the meteorite problem in Chap. 4.1.3 or by the discussion of the parametric representation of the planetary motion on p. 153 and ff.

It is necessary to look for a different approach to the problem if an analytical discussion is the aim. Instead of calculating the explicit time development of the motion (the functions $r(t)$ and $\varphi(t)$), the calculation can be restricted to the determination of the possible trajectories of the Kepler problem. The trajectories are characterised by a function $r = r(\varphi)$ – or $\varphi = \varphi(r)$ – which could be obtained from the functions $r(t)$ and $\varphi(t)$ by elimination of the parameter t . The following reformulation of the problem leads to a direct determination of the trajectories. Differentiate the function $r = r(\varphi)$ with respect to time using the chain rule

$$\dot{r} = \frac{d}{dt} r(\varphi) = \frac{dr}{d\varphi} \dot{\varphi} = \frac{A}{r^2} \frac{dr}{d\varphi}.$$

Insert this into the energy expression (4.8)

$$\frac{1}{2} \frac{A^2}{r^4} \left(\frac{dr}{d\varphi} \right)^2 + \frac{1}{2} \frac{A^2}{r^2} - \gamma \frac{M}{r} = B \quad (4.11)$$

and sort with respect to the derivative $dr/d\varphi$

$$\frac{dr}{d\varphi} = \pm \frac{r^2}{A} \left[2B + 2\gamma M \frac{1}{r} - \frac{A^2}{r^2} \right]^{1/2}. \quad (4.12)$$

The sign in this equation determines the direction of the change of φ with the variable r . Separation of variables yields the integral

$$\varphi(r) - \varphi_0 = \pm \int_{r_0}^r \frac{A dr'}{r'^2} \left[2B + \frac{2\gamma M}{r'} - \frac{A^2}{r'^2} \right]^{-1/2}. \quad (4.13)$$

This integral can be evaluated by elementary means as the substitution

$$s' = \frac{1}{r'} \quad \longrightarrow \quad ds' = -\frac{dr'}{r'^2}$$

leads to

$$\varphi(s) - \varphi(s_0) = \mp \int_{s_0}^s \frac{A ds'}{[2B + 2\gamma M s' - A^2 s'^2]^{1/2}}.$$

The primitive of this integral can be found in standard Tables¹

$$\varphi(s) - \varphi(s_0) = \pm \left[\arcsin \left\{ \frac{-A^2 s' + \gamma M}{[\gamma^2 M^2 + 2A^2 B]^{1/2}} \right\} \right]_{s_0}^s. \quad (4.14)$$

In order to simplify the discussion as much as possible, the lower limit of the integral, s_0 , is chosen so that the radius vector and the velocity vector are perpendicular at the intersection of the x -axis and the trajectory (Fig. 4.1). Such points are characterised by $dr/d\varphi = 0$. The differential equation (4.12)

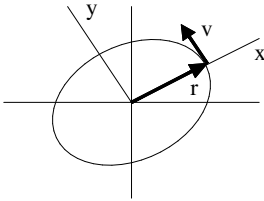


Fig. 4.1. Kepler problem: choice of the coordinate system

leads therefore to

$$\frac{dr}{d\varphi} = 0 \quad \longrightarrow \quad \frac{1}{2} A^2 s_0^2 - \gamma M s_0 = B \quad \left(s_0 = \frac{1}{r_0} \right).$$

Solution of this quadratic equation gives

¹ see references [4].

$$A^2 s_0 = \gamma M \pm [\gamma^2 M^2 + 2A^2 B]^{1/2} .$$

As the quantity B can be negative, it has to be ensured that the radicand is larger than or equal to zero in order to obtain real values for s_0 . It can be demonstrated, after the process of solution is completed (see ● D.tail 4.1), that this is the case.

The points of intersection of the trajectory with the x -axis correspond to the angles $\varphi_0 = 0$ and/or $\varphi_0 = \pi$. The result (4.14) goes over into

$$\begin{aligned} \varphi(s) - \begin{bmatrix} \pi \\ 0 \end{bmatrix} &= \pm \left\{ \arcsin \left[\frac{-A^2 s + \gamma M}{[\gamma^2 M^2 + 2A^2 B]^{1/2}} \right] - \arcsin \begin{bmatrix} -1 \\ +1 \end{bmatrix} \right\} \\ &= \pm \left\{ \arcsin \left[\frac{-A^2 s + \gamma M}{[\gamma^2 M^2 + 2A^2 B]^{1/2}} \right] \pm \frac{\pi}{2} \right\} \end{aligned}$$

if these values are used as lower limits. Resolution with respect to s , that is $1/r$, gives

$$\frac{A^2}{\gamma M} \frac{1}{r} = 1 \pm \left[1 + \frac{2A^2 B}{\gamma^2 M^2} \right]^{1/2} \cos \varphi .$$

The equation of the orbits of the simple Kepler problem, expressed in polar coordinates, is

$$\boxed{\frac{p}{r} = 1 \pm \epsilon \cos \varphi .} \quad (4.15)$$

The parameters p and ϵ depend on the parameters A and B , defined in (4.5) and (4.8), that is on the initial values

$$p = \frac{A^2}{\gamma M} = \frac{l_0^2}{\gamma m_P^2 M} \quad (4.16)$$

$$\epsilon = \left[1 + \frac{2A^2 B}{\gamma^2 M^2} \right]^{1/2} = \left[1 + \frac{2l_0^2 E_0}{\gamma^2 m_P^3 M^2} \right]^{1/2} . \quad (4.17)$$

The initial energy could be replaced by

$$E_0 = \frac{m_P}{2} \dot{r}_0^2 + \frac{1}{2} \frac{l_0^2}{m_P r_0^2} - \gamma \frac{m_P M}{r_0} ,$$

the initial angular momentum by

$$l_0 = m_P r_0^2 \dot{\varphi}_0 .$$

4.1.2.2 Conic sections. Equation (4.15) characterises the conic sections: circles, ellipses, parabolae and hyperbolae. They are referred in the present case to a coordinate system for which the origin is one of the focal points. The Cartesian representation of the conic sections has to be reformulated in terms of appropriate polar coordinates in order to prove this assertion.

A **parabola** is the locus of all points, for which the distance from a given line (the guide line or directrix) and a given point (the focal point) is the

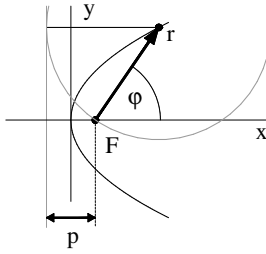


Fig. 4.2. Definition: parabola

same (Fig. 4.2). The standard Cartesian form of the two branches is given by the equations

$$y^2 = +2px \quad \text{for } x \geq 0$$

$$y^2 = -2px \quad \text{for } x \leq 0.$$

The parameter p marks the shortest distance of the focal point from the guide line. The sign specifies whether the parabola opens to the right (plus sign) or to the left. Polar coordinates, which are referred to the two possible focal points, are defined by the transformation

$$x = \pm \frac{p}{2} + r \cos \varphi \quad y = r \sin \varphi.$$

The sign has to correspond to the sign of the normal form. Insertion into the equation of the parabola leads to

$$r^2 \sin^2 \varphi = p^2 \pm 2pr \cos \varphi \quad r^2 = r^2 \cos^2 \varphi \pm 2pr \cos \varphi + p^2$$

$$r = \pm(p \pm r \cos \varphi).$$

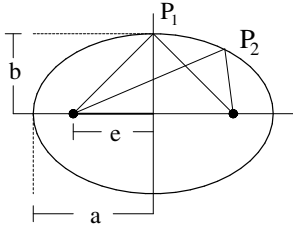
Resolution in the form²

$$\frac{p}{r} = 1 \pm \cos \varphi$$

shows, that equation (4.15) describes a parabola if the parameter ϵ has the value 1.

An **ellipse** is defined as the locus of all points, for which the sum of the distances from two given points is always the same (Fig. 4.3). The Cartesian

² The positive sign in front of the brackets has to be used in order to obtain a positive radial coordinate.

**Fig. 4.3.** Definition: ellipse

principal axis form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

The position of the focal points³ along the semi-major axis (with $a > b$) is

$$x_B = \pm e \quad \text{with} \quad e = [a^2 - b^2]^{1/2} .$$

The polar coordinates referred to the focal points are

$$x = \pm e + r \cos \varphi \quad y = r \sin \varphi$$

so that

$$\begin{aligned} b^2(e^2 \pm 2er \cos \varphi + r^2 \cos^2 \varphi) + a^2 r^2 \sin^2 \varphi &= a^2 b^2 \\ b^4 \mp 2eb^2 r \cos \varphi + e^2 r^2 \cos^2 \varphi &= a^2 r^2 \\ \pm(b^2 \mp er \cos \varphi) &= ar . \end{aligned}$$

As the global sign is again not relevant, the relation

$$\frac{b^2/a}{r} = 1 \pm \epsilon \cos \varphi$$

follows. The numerical eccentricity

$$\epsilon = \frac{e}{a} = \left[1 - \frac{b^2}{a^2} \right]^{1/2}$$

characterises a circle ($\epsilon = 0$) or an ellipse ($0 < \epsilon < 1$).

A **hyperbola** is the locus of all points for which the difference of the distance from two points is constant (Fig. 4.4). The principal axis form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 . \quad (4.18)$$

The position of the focal points on the x -axis is given by

$$x_B = \pm e \quad \text{with} \quad e = [a^2 + b^2]^{1/2}$$

³ The relation $2\sqrt{e^2 + b^2} = (a + e) + (a - e)$ is used (see Fig. 4.3) for the derivation of this equation. It compares the sum of the distances of a locus P_1 on the y -axis from the focal points with the sum of the distances of the locus P_2 on the x -axis.

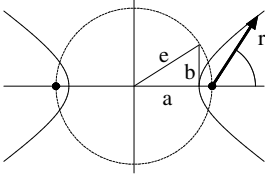


Fig. 4.4. Definition: hyperbola

in this case. Insertion of the transformation to polar coordinates

$$x = \pm e + r \cos \varphi \quad y = r \sin \varphi$$

into the normal form yields

$$a^2 b^2 = b^2 e^2 \pm 2b^2 e r \cos \varphi + b^2 r^2 \cos^2 \varphi - a^2 r^2 \sin^2 \varphi$$

$$a^2 r^2 = b^4 \pm 2eb^2 r \cos \varphi + e^2 r^2 \cos^2 \varphi$$

$$ar = \pm(b^2 \pm er \cos \varphi) .$$

The formula for a hyperbola in polar coordinates referred to the focal points is therefore

$$\frac{b^2/a}{r} = 1 \pm \epsilon \cos \varphi \quad \epsilon = \left[1 + \frac{b^2}{a^2} \right]^{1/2} .$$

The parameter ϵ is larger than 1 .

4.1.2.3 Types of orbits. The parameter p in (4.15) is given by the ratio $p = b^2/a$. It characterises the opening of the parabola or hyperbola, or the 'size' of the ellipse. The numerical eccentricity ϵ fixes the type of trajectory. The sign in the expression $1 \pm \epsilon \cos \varphi$ indicates whether the conic sections are oriented towards the left (plus sign) or to the right (Fig. 4.5). The angular



Fig. 4.5. Role of the sign in Eq. (4.15) of the conic sections

range for the different orbits is:

- The standard range $0 \leq \varphi \leq 2\pi$ applies for an ellipse.
- The range is $-\pi \leq \varphi \leq \pi$ for a parabola and the positive sign, in the case of the negative sign it is $0 \leq \varphi \leq 2\pi$. The different ranges are chosen so that the zeros of $1 \pm \cos \varphi$ occur at the edge of the interval.

- The range is

$$-\arccos(-1/\epsilon) \leq \varphi \leq \arccos(-1/\epsilon)$$

for the hyperbola and $+\epsilon$, in the case of $-\epsilon$

$$\arccos(1/\epsilon) \leq \varphi \leq 2\pi - \arccos(1/\epsilon) .$$

The limits are chosen so that the curves are bounded by the asymptotes.

The parameters p and ϵ of the Kepler orbits (see (4.17))

$$p = \frac{b^2}{a} = \frac{l_0^2}{m_P} \left(\frac{1}{m_P M \gamma} \right) \quad (4.19)$$

$$\epsilon = \left[1 + \frac{b^2}{a^2} \right]^{1/2} = \left[1 + 2E_0 \frac{l_0^2}{m_P} \frac{1}{(m_P M \gamma)^2} \right]^{1/2} \quad (4.20)$$

indicate which initial values characterise the type of trajectory. The type is determined by the initial values of both the angular momentum and the energy, details of the orbits (opening, ratio of the axes, radius) by angular momentum alone. A circular orbit is obtained for $\epsilon = 0$. This corresponds to an energy value

$$E_0 = -\frac{m_P}{2l_0^2} (m_P M \gamma)^2 .$$

Angular momentum and energy (negative) have to be matched for the case of a circle. The parameter ϵ would be imaginary if an energy smaller than the energy characterising the circular orbit is chosen for a given angular momentum. This means: for a given value of the angular momentum the circular trajectory is the one with the lowest energy value possible. Elliptic orbits are characterised by $0 < \epsilon < 1$. These values of ϵ correspond to an energy range

$$E_0(\text{circle}) < E_0 < 0 .$$

Circle and ellipse are possible trajectories of planets. They correspond to trajectories, for which the planet is 'bound' to the sun ($E_0 < 0$). A parabola is obtained for $\epsilon = 1$ or $E_0 = 0$, a hyperbola for $\epsilon > 1$ or $E_0 > 0$. These are possible trajectories of comets. An object enters (at a large distance) the gravitational field of the central body, passes more or less closely and vanishes again into space.

It should be noted that there exist objects in our planetary system with elliptic orbits which are close to a parabolic orbit. These are called *returning comets*. A well known example is Halley's comet (with a period of approximately 76 years). For this 'comet' the maximal distance from the sun is about 60 times larger than the minimal distance.

The orbits of the Kepler problem are mainly determined by the energy. For this reason a simpler (but less detailed) discussion of the Kepler problem is possible. The total energy can be written as

$$E = \frac{m_P \dot{r}^2}{2} + \frac{1}{2} \frac{l_0^2}{m_P r^2} - \gamma \frac{m_P M}{r} . \quad (4.21)$$

The individual terms are kinetic energy of the radial motion, kinetic energy of the rotational motion and potential energy of gravitation. The second term can be interpreted as a potential energy of the radial motion because angular momentum is conserved (a constant of motion). The sum of this **centrifugal contribution** and the gravitational potential energy is referred to as the effective potential energy

$$E = T_{\text{rad}} + U_{\text{cent}} + U_{\text{grav}} = T_{\text{rad}} + U_{\text{eff}} .$$

A plot of $U_{\text{eff}}(r)$ exhibits (provided $l_0 \neq 0$) the characteristics indicated in Fig. 4.6a. The centrifugal term dominates (as $1/r^2$) for $r \rightarrow 0$. The effective potential energy goes over into the gravitational contribution, which decreases more slowly ($1/r$) then the centrifugal term for $r \rightarrow \infty$. The total energy $E_0 = \text{const.}$ is composed of a (mainly negative) contribution due to the potential energy and a (positive) contribution due to the kinetic energy for each value of r .

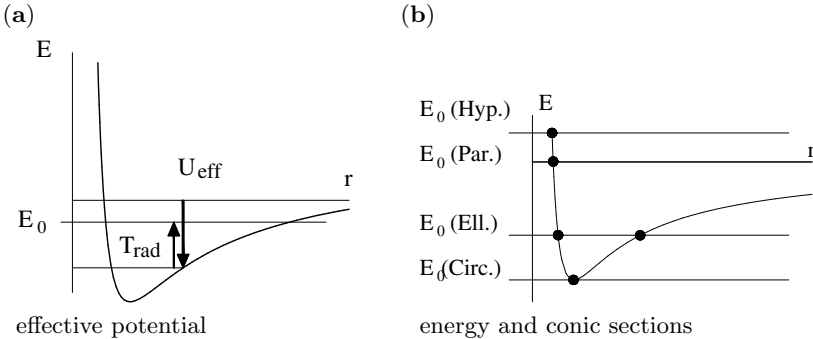


Fig. 4.6. Kepler problem: energy as a function of the radial coordinate

The correlation between the type of orbit and the energy is indicated in Fig. 4.6b. The lowest possible energy value with

$$E_0(\min) = U_{\text{eff}}(R) \quad T_{\text{rad}} = 0$$

is found for the circular orbit, which is distinguished by a constant distance from the origin ($\dot{r} = 0$). There exists two turning points with $\dot{r} = 0$ for initial energies E_0 between the minimal value and zero. The total energy (negative) is composed of the sum of U_{eff} (negative) and T_{rad} (positive) for each value of r between the points closest and furthest from the sun (perihelion and aphelion). A trajectory with two turning points and a finite range of r -values corresponds to an ellipse (the explicit form of the orbit can, however, not be extracted from energy considerations). There is no binding to the central

body for $E_0 \geq 0$. A parabola with $E_0 = 0$ still has two (quasi) turning points: one near the sun, the other at infinity. The parabola is the limit of an infinitely stretched ellipse. A hyperbola with $E_0 > 0$ is distinguished by a single turning point in the vicinity of the sun.

With respect to angular momentum the following statement can be gleaned from Fig. 4.7:

- (1) The inner turning points of the ellipses, parabolae and hyperbolae are further from the sun and the binding energy of a circular orbit (measured by the magnitude of the negative energy) is weaker for larger value of l_0 (Fig. 4.7a).
- (2) The effective potential is identical with the gravitational potential for $l_0 = 0$. Possible trajectories are straight lines in a radial direction towards or away from the sun (Fig. 4.7b). Energy values with $E_0 < 0$ (the objects rises and falls) or with $E_0 \geq 0$ (the object moves away from or moves towards the sun) are possible.

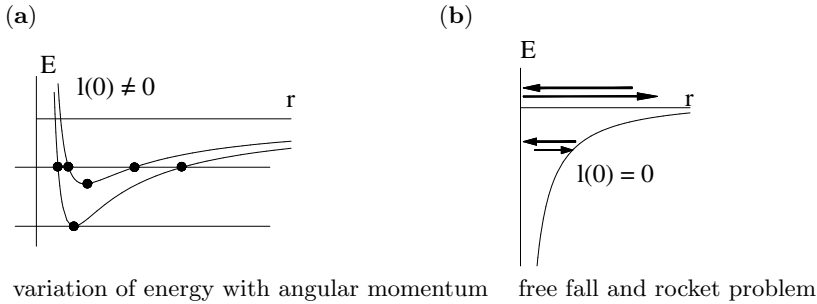


Fig. 4.7. Kepler problem: angular momentum

With the solution of the planetary problem two Kepler laws have been verified:

Planetary orbits are ellipses.
The law of areas is valid.

It remains to discuss the third.

4.1.2.4 The third Kepler law. For the discussion of the third law

$$T^2 \propto a^3 \quad \text{for planets with } a > b$$

a relation between the data of the ellipse and the initial conditions is needed. The definitions (4.19) and (4.20)

$$\epsilon^2 = 1 - \frac{b^2}{a^2} = 1 + \frac{2A^2B}{\gamma^2 M^2} \longrightarrow \frac{b^2}{a^2} = -\frac{2A^2B}{\gamma^2 M^2} \quad (4.22)$$

$$p = \frac{b^2}{a} = \frac{A^2}{\gamma M} \longrightarrow \frac{b^2}{a} = \frac{A^2}{\gamma M} \quad (4.23)$$

yield

$$\frac{1}{a} = -\frac{2B}{\gamma M} = -\frac{2E_0}{\gamma M m_P}.$$

The semi-major axis is only determined by the energy (and the masses). A statement on the period T of an elliptic orbit follows from the law of areas

$$\frac{dF}{dt} = \frac{1}{2}A.$$

F stands for the area, while A is given by (4.5). Integration leads to a relation

$$\int_{\text{ellipse}} dF = \pi ab = \frac{1}{2}A \int_0^T dt = \frac{1}{2}AT$$

that can be used to derive the third Kepler law

$$T^2 = \frac{4\pi^2 a^2}{A^2} b^2 = \frac{4\pi^2 a^2}{A^2} \left(\frac{A^2}{\gamma M} a \right) = \frac{4\pi^2}{\gamma M} a^3. \quad (4.24)$$

The following remarks can be offered:

1. The period of the planets is determined by the energy. It is independent of the angular momentum.
2. The constant depends only on the solar mass (and the gravitational constant). This result is quite well confirmed in nature. One finds

$$\left. \frac{T^2}{a^3} \right|_{\text{planet}} \longrightarrow (0.985 - 1.005) \left. \frac{T^2}{a^3} \right|_{\text{earth}}.$$

The solar mass can be determined from the data (a, T) for the planets within the limits given. The deviation indicated is due to two reasons: Neglect of the motion of the sun and the 'perturbation' due to the motion of neighbouring planets, moons, etc.

4.1.2.5 Motion of the sun. The neglect of the motion of the sun can be corrected directly. If the system sun-planet (Fig. 4.8) is treated as a two body

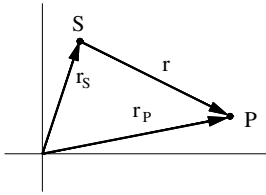


Fig. 4.8. Kepler problem: two body aspects

problem (and not with the additional assumption of a fixed sun as a one body problem) the equations of motion

$$M\ddot{\mathbf{r}}_S = \gamma \frac{M m_P}{r^3} \mathbf{r} \quad \mathbf{r} = \mathbf{r}_P - \mathbf{r}_S \quad (4.25)$$

$$m_P\ddot{\mathbf{r}}_P = -\gamma \frac{M m_P}{r^3} \mathbf{r} \quad (4.26)$$

have to be addressed. The motion of the centre of gravity

$$\mathbf{R} = \frac{1}{m_P + M} (m_P \mathbf{r}_P + M \mathbf{r}_S)$$

is characterised by

$$\ddot{\mathbf{R}} = \mathbf{0} \longrightarrow \dot{\mathbf{R}} = \text{const.}$$

The motion of the centre of gravity is, as expected, uniform and therefore not of interest. The equation responsible for the relative motion is

$$\ddot{\mathbf{r}} = -\gamma \frac{(m_P + M)}{r^3} \mathbf{r} . \quad (4.27)$$

This equation is not essentially different from the differential equation (4.1) of the one particle problem. The mass of the sun M in the results of this problem can simply be replaced by the total mass $M + m_P$ in order to incorporate the motion of the sun. The third Kepler law corrected for the motion of the sun reads therefore

$$T^2 = \frac{4\pi^2}{\gamma(M + m_P)} a^3 . \quad (4.28)$$

As the mass of the sun is so dominant there is only a small difference with respect to (4.24). The corrected formula explains, in a reasonable measure, the variation of the factor of proportionality that has been indicated above.

The motion of the sun itself is best discussed in the centre of mass system, which is characterised by

$$\mathbf{R} = \mathbf{0} \longrightarrow \mathbf{r}_S = -\frac{m_P}{M} \mathbf{r}_P .$$

The sun moves opposite the planet on a miniature ellipse (due to the factor $-m_P/M$) according to the two body formulation (Fig. 4.9).

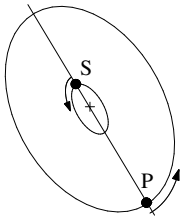


Fig. 4.9. Kepler problem: motion of the sun

4.1.2.6 Additional remarks. The following remarks conclude the discussion of the Kepler problem:

1. The discussion of the two body problem sun-planet can also be applied to the system planet-moon. It is possible in this case to extract the mass of the planet (e.g. of the earth) from the data of the moon (semi-major axis and period) with the aid of the third Kepler law (see ● Problem 4.1).
2. The calculation of the motion of all the planets in our solar system including the mutual interactions is a much larger task. As already the three body problem involving the sun and two planets can not be solved analytically, it is necessary to proceed in a different fashion: in a first step the partial systems consisting of the sun and one planet are considered, as indicated above. In a second step the perturbation of the orbit of each of the planets due to the other planets is calculated in successive approximations⁴.
3. The question, whether the $1/r^2$ - law is correct, can be posed. In order to answer the question a general force law of the form

$$\mathbf{F} \propto -r^\alpha \mathbf{r} \quad (\alpha \text{ arbitrary})$$

has to be considered. The law of areas is valid for each value of the parameter α . The orbits of the corresponding two body problem can be calculated. Elliptic orbits are, however, only possible for $\alpha = -3$ and 0 . The ellipses are only centred on the focal point for $\alpha = -3$. The oscillator problem with $\alpha = 0$ does not allow any unbound (comet) orbits. The observation of elliptic orbits centred on the focal point can be taken as a proof of the validity of the $1/r^2$ law for gravitation (see also ● Probl. 4.2).

4. A *direct integration* of the basic equation (4.9) of the Kepler problem

$$t = \pm \int_{r_0}^r \frac{dr'}{\left[2B + 2\gamma \frac{M}{r'} - \frac{A^2}{r'^2}\right]^{1/2}}$$

is possible, though only in the form of a parametric representation $r = r(\psi)$, $t = t(\psi)$. The integral

$$t = \pm \frac{1}{\sqrt{2|B|}} \int \frac{r' dr'}{\left[-r'^2 + \gamma \frac{Mr'}{|B|} - \frac{A^2}{2|B|}\right]^{1/2}} \quad (4.29)$$

has to be evaluated e.g. for the case of the orbits of planets (with a negative total energy). The notation is the same as used in the derivation of (4.9)

$$B = \frac{E_0}{m_P} \quad A = \frac{l_0}{m_P} .$$

⁴ An overview of the more accurate calculation of planetary motion is found in [5] of the references.

If the definition

$$\epsilon = \left[1 - \frac{2A^2|B|}{\gamma^2 M^2} \right]^{1/2}$$

for the eccentricity and

$$a = \frac{\gamma M}{2|B|} \quad b = a\sqrt{1 - \epsilon^2} = \frac{A}{\sqrt{2|B|}} \quad (a > b)$$

for the semi-major and the semi-minor axes are used, the radicand of the integral (4.29) can be cast into the form

$$-r'^2 + 2ar' - b^2 = a^2\epsilon^2 - (r' - a)^2.$$

The integral (4.29) takes the form

$$t = \pm \sqrt{\frac{a}{\gamma M}} \int \frac{r' dr'}{[a^2\epsilon^2 - (r' - a)^2]^{1/2}}.$$

It can be treated with the substitution

$$(r' - a) = -a\epsilon \cos \psi \quad dr' = a\epsilon \sin \psi d\psi.$$

The resulting elementary integral can be evaluated directly

$$\begin{aligned} t &= \pm \sqrt{\frac{a^3}{\gamma M}} \int d\psi (1 - \epsilon \cos \psi) \\ &= \pm \left(\sqrt{\frac{a^3}{\gamma M}} (\psi - \epsilon \sin \psi) + \text{const.} \right). \end{aligned}$$

It corresponds, together with the substitution used for the variable r , to a parametric representation of the function $r = r(t)$ (only the positive sign is relevant). The initial conditions can, for instance, be chosen so that $\text{const.} = 0$

$$r = a(1 - \epsilon \cos \psi) \quad t = \sqrt{\frac{a^3}{\gamma M}} (\psi - \epsilon \sin \psi). \quad (4.30)$$

This choice of the initial condition corresponds to $\psi = 0$ for $t = 0$ and hence to

$$r(0) = a(1 - \epsilon) = a - e \quad \text{as well as} \quad \dot{r}(0) = 0.$$

The second statement follows by use of the chain rule.

An expression for $\cos \varphi$ can be derived by comparing the parametric representation of $1/r$ (in (4.30)) with the solution (4.15) of the Kepler problem for an elliptical orbit. The result is

$$\cos \varphi = \pm \frac{(\cos \psi - \epsilon)}{(1 - \epsilon \cos \psi)}$$

where the polar coordinates r and φ are referred to the focal point. It can be used to obtain a parametric representation of the Cartesian coordinates of the Kepler ellipses

$$\begin{aligned}x &= \pm a\epsilon + r \cos \varphi = \pm a \cos \psi \\y &= r \sin \varphi = a \sqrt{1 - \epsilon^2} \sin \psi = b \sin \psi .\end{aligned}$$

These equations represent the usual parametric representation of an ellipse in a coordinate system with an origin, which divides the separation of the focal points into equal halves. The sign of the x -coordinate describes the sense of rotation.

The parameter ψ increases by 2π during a full rotation of the planet on the ellipse. The equation for the parametric representation of the time in (4.30) contains therefore explicitly the third Kepler law (4.24). A similar representation is possible for parabolic or hyperbolic orbits (see ● Probl. 4.5).

5. Of interest (in electrodynamics) is the question of the solution of the Kepler problem with a repulsive force (the Coulomb force between charges of the same sign)

$$\mathbf{F} = \frac{\alpha}{r^3} \mathbf{r}, \quad \alpha > 0 .$$

It is easy to show that only orbits with a positive energy are possible. The details of these hyperbolae can be obtained from those of the normal Kepler problem with the replacement of $|\gamma m_{\text{P}} M|$ by $-\alpha$.

4.1.3 Comets and meteorites

The maximal separation of a planet from the central body is finite. By contrast, comets and meteorites can enter the gravitational field of a celestial body from very far away.

4.1.3.1 Meteorites. The problem of meteorites, that is the free fall of a body starting at a large distance from the centre of the earth, can be discussed in the following manner. The potential energy of an object of mass m at points outside the earth (see (3.90)) is

$$U_{\text{grav}}(r) = -\gamma \frac{mM_{\text{E}}}{r} \quad r \geq R_{\text{E}} ,$$

if the earth is treated as a sphere with a (homogeneous) mass distribution. As initial conditions for the free fall the specification

$$\mathbf{r}(0) = r_0 \mathbf{e}_r \quad \mathbf{v}(0) = \mathbf{0} ,$$

which in polar coordinates reads

$$r_0 = r_0 \quad \varphi_0 = \text{arbitrary} \quad \dot{r}_0 = 0 \quad \dot{\varphi}_0 = 0 ,$$

may be used. Angular momentum conservation (4.4) leads to the statement

$$r(t)^2 \dot{\varphi}(t) = r_0^2 \dot{\varphi}_0 = l_0/m = 0 .$$

For $r(t) > R_E \neq 0$ it follows immediately that $\dot{\varphi}(t) = 0$. The object moves in the radial direction towards the earth for the initial conditions given.

The energy conservation law takes the form (4.21)

$$\frac{1}{2} \dot{r}^2 - \gamma \frac{M_E}{r} = \frac{E_0}{m_p} = -\gamma \frac{M_E}{r_0}$$

if the angular momentum is zero. Some simple questions can be answered directly with the aid of this equation, so for instance: what is the speed, with which an object hits the earth, if it starts at a very large distance ($r_0 \rightarrow \infty$) from rest. The answer, neglecting possible frictional effects, is

$$v_E = [2gR_E]^{1/2} \approx 40250 \text{ km/h} .$$

This follows from

$$\frac{1}{2} v_E^2 - \gamma \frac{M_E}{R_E} = 0$$

and the definition (3.94)

$$g = \gamma \frac{M_E}{R_E^2} .$$

It is, on the other hand, necessary to integrate the differential equation (4.8) in order to calculate the actual time development of the motion. A separation of variables (the constant A has the value zero)

$$\frac{dr}{dt} = \pm \left[(2\gamma M_E) \left(\frac{1}{r} - \frac{1}{r_0} \right) \right]^{1/2} ,$$

as well as a rearrangement and integration yields

$$t = \pm \left[\frac{r_0}{2\gamma M_E} \right]^{1/2} \int_{r_0}^r \left[\frac{r'/r_0}{1 - r'/r_0} \right]^{1/2} dr' .$$

The integral on the right hand side can be taken from a suitable Table of integrals or evaluated explicitly. With the substitution

$$\left(\frac{r'}{r_0} \right)^{1/2} = \cos \alpha \quad \longrightarrow \quad r' = r_0 \cos^2 \alpha$$

(reasonable as $0 < r'/r_0 \leq 1$) and

$$dr' = -2r_0 \cos \alpha \sin \alpha d\alpha$$

$$\alpha_{\max} = \arccos \sqrt{r/r_0} \quad \alpha_{\min} = \arccos 1 = 0$$

it is transformed into

$$t = \mp \left[\frac{2r_0^3}{\gamma M_E} \right]^{1/2} \int_0^{\alpha_{\max}} \left[\frac{\cos^2 \alpha \sin \alpha}{\sin \alpha} \right] d\alpha .$$

The remaining integral is

$$\int \cos^2 \alpha d\alpha = \frac{1}{2}(\alpha + \cos \alpha \sin \alpha)$$

so that the result after insertion of the integration limits reads

$$t = \mp \left[\frac{r_0^3}{2\gamma M_E} \right]^{1/2} \left\{ \arccos \sqrt{\frac{r}{r_0}} + \sqrt{\frac{r}{r_0}} \sqrt{\left(1 - \frac{r}{r_0}\right)} \right\}. \quad (4.31)$$

Only the positive sign is relevant for $t > 0$. This result in the form $t = t(r)$ can not be resolved analytically in the form $r = r(t)$. The actual time development of the motion of the meteorite can only be extracted by a numerical inversion. The numerical treatment is illustrated in Fig. 4.10, which shows $t = t(r)$ for the motion of meteorites. The time, that the object has fallen, can

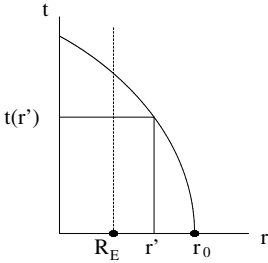


Fig. 4.10. Meteorite problem: Variation of the time of fall with distance

be read off the graph for each distance $R_E \leq r' \leq r_0$. A table of values of the functional relation $r' = r'(t)$ (as accurate as required) or a suitable parametric representation as suggested in Chap. 4.1.2.6 can be prepared instead of the graphical representation indicated in the figure. Such tables are used to a great extent in astronomy.

The result of the meteorite problem (4.31) corresponds to the results of the simple free fall, if one sets

$$r_0 = R_E + h_0 \quad r = R_E + h$$

and expands in the parameter h ($\leq h_0 \ll R_E$).

4.1.3.2 The classical collision problem: orbits of comets. An orbit of a comet results if a mass (with $E_0 > 0$) moves at a given distance past a central body (rather than hitting it directly). This collision problem can be discussed more precisely in the following manner: a mass point m starts at a great distance $r_0 \rightarrow \infty$ from a 'heavy' mass point M with the initial velocity v_0 (Fig. 4.11a). The mass m will, due to the attractive gravitational interaction, move on a hyperbola. It moves on a straight line, the asymptote of the hyperbola, if it is sufficiently far from the heavy mass. The (constant) distance between the asymptote and a parallel line through the centre of M

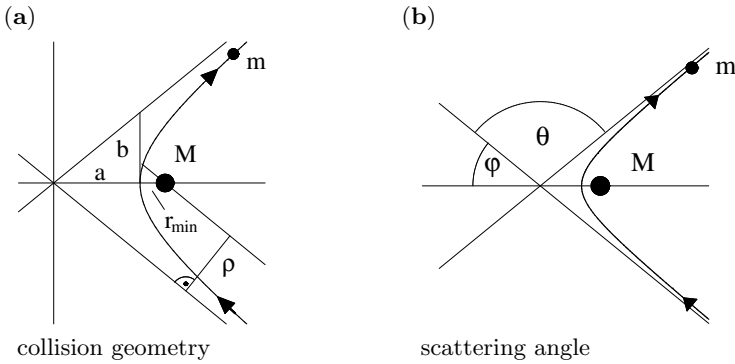


Fig. 4.11. Comets: collision problem

is called the **impact parameter** ρ . The mass m passes the mass M at a minimal distance r_{\min} and completes its orbit moving in the direction of the second asymptote. The angle between the first asymptote and the x -axis, on which the mass M is located, is denoted by φ . Because of the symmetry of the hyperbola, a corresponding angle is found between the second asymptote and the x -axis. The **scattering angle** θ , the angle e.g. between the second asymptote and the extension of the first asymptote (Fig. 4.11b) is therefore $\theta = \pi - 2\varphi$. The initial conditions stated lead to

$$E_0 = \frac{m}{2}v_0^2 \quad l_0 = m\rho v_0$$

or alternatively in terms of the parameters A and B of the Kepler problem to

$$A = \rho v_0 \quad B = \frac{v_0^2}{2}.$$

The angle φ is determined by the parameters a and b of the hyperbola (see (4.18)) in the form $\tan \varphi = b/a$. The parameter of the hyperbola are related to the initial values of the Kepler problem by (4.22) and (4.23), so that the relation

$$\tan \varphi = \frac{b}{a} = \frac{A\sqrt{2B}}{\gamma M} = \frac{\rho v_0^2}{\gamma M} \quad (4.32)$$

can be established. This equation allows the determination of the angle φ , or alternatively the scattering angle θ , via the relation

$$\tan \varphi = \tan \left(\frac{\pi - \theta}{2} \right) = \cot \left(\frac{\theta}{2} \right)$$

for a given impact parameter ρ .

The relation (4.32) is also used, though with some modifications, in quantum mechanics. The calculations in terms of classical mechanics and quantum mechanics lead to the same result for a $1/r$ potential. A difference arises from

the fact that the scattering of a single comet is of interest in the classical problem. In quantum mechanical collision experiments it is not the scattering of one particle but rather the scattering of a whole beam of particles, which move towards the collision centre with equal velocity but different impact parameters (Fig. 4.12). The particles in the beam are scattered under different angles θ , depending on the respective impact parameter. In order to characterise this situation the concept of the **differential cross section** $d\sigma$ is used.

This quantity is defined in the following way: it is proportional to the number of particles dN , which are scattered per unit time into a direction between the angles θ and $\theta + d\theta$. The quantity has to be normalized with respect to the number of particles N , which move in unit time through a unit area of the beam cross section

$$d\sigma = \frac{dN}{N}.$$

The number dN corresponds to the product of N with the area of a circular ring with the radii ρ and $\rho + d\rho$. This area is $2\pi\rho d\rho$. The differential cross section has therefore the dimension of an area (Fig. 4.12) and is given by

$$d\sigma = \frac{dN}{N} = 2\pi\rho d\rho.$$

The dependence of the differential cross section on the scattering angle θ is obtained with the aid of the chain rule as

$$d\sigma = 2\pi\rho(\theta) \left| \frac{d\rho}{d\theta} \right| d\theta.$$

The absolute value of the derivative has to be used to obtain a positive quantity. Together with the relation (4.32) in the form

$$\rho = \frac{\gamma M}{v_0^2} \cot \frac{\theta}{2},$$

the differential cross section is given by the formula

$$d\sigma = \pi \frac{(\gamma M)^2}{v_0^4} \frac{\cos(\theta/2)}{\sin^3(\theta/2)} d\theta.$$

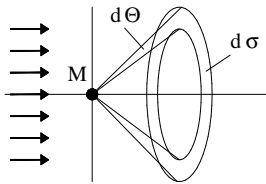


Fig. 4.12. Differential cross section: geometry

Usually, the differential cross section is, because of the cylinder symmetry of the collision system, referred to the solid angle⁵ $d\Omega$

$$d\Omega = 2\pi \sin \theta \, d\theta = 4\pi \sin(\theta/2) \cos(\theta/2) \, d\theta ,$$

so that the final form of the differential cross section is

$$d\sigma = \left(\frac{\gamma M}{2v_0^2} \right)^2 \frac{d\Omega}{\sin^4(\theta/2)} . \quad (4.33)$$

This is the **Rutherford formula**, which has played an important role in the development of quantum mechanics. It has been used, to analyse the scattering of α - particles from Gold nuclei, resulting in a first model of the atom (see Vol. 3). The motion of the α - particles is governed by Coulomb (3.16) rather than gravitational forces. The two forces obey a comparable force law, so that the gravitational factor γM can be replaced by the electric factor $\pm kq^2/m$. Rutherford's formula is valid for attractive (equal charges) as well as repulsive forces (opposite charges).

It should be kept in mind that Rutherford's formula has been derived with respect to the centre of mass system (compare Chap. 4.1.2.5). For the interpretation of experiments in a laboratory system (in which the mass M is originally at rest) a kinematical transformation has to be applied (● Probl. 4.10).

4.2 Oscillator problems

The harmonic oscillator features in many areas of physics (from mechanics to quantum field theory). The reason for this popularity can be pointed out directly. Consider (Fig. 4.13a) a potential energy function $U(x)$ of a mass point (for simplicity in one space dimension), which exhibits a definite minimum. It is always possible to choose a coordinate system, so that the origin and the point with the minimum coincide and that $U(0) = 0$. If the kinetic energy of this mass point in the potential well is smaller than the potential energy of neighbouring maxima of $U(x)$, the mass will only move within a limited range around the origin. The Taylor expansion of the potential energy about the point $x = 0$ is

$$U(x) = U(0) + \left. \frac{dU}{dx} \right|_0 x + \frac{1}{2!} \left. \frac{d^2U}{dx^2} \right|_0 x^2 + \frac{1}{3!} \left. \frac{d^3U}{dx^3} \right|_0 x^3 \dots$$

The first and the second term vanish by construction. The third term is positive, if $x = 0$ is a minimal point. Therefore one may write

$$U(x) = a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \quad a_2 > 0 .$$

For a potential well, which is, as assumed in Fig. 4.13b, symmetric with respect to $x = 0$, all odd powers of the expansion vanish because of $U(-x) = U(x)$. The potential function in the vicinity of the origin is

⁵ This is the solid angle between two cones with the opening angles θ and $\theta + d\theta$.

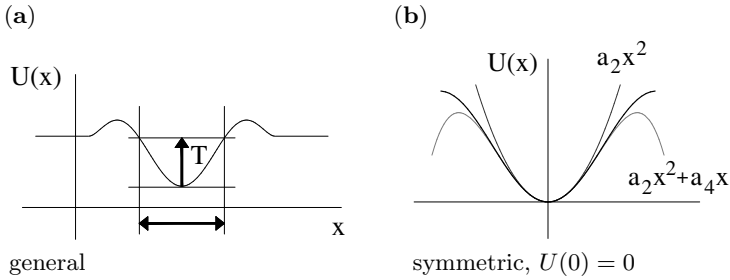


Fig. 4.13. One dimensional potential wells

$$U(x) = a_2 x^2 + a_4 x^4 + \dots$$

The Taylor expansion indicates, that every potential well can in first approximation (that is for small displacements from the equilibrium position) be treated as a harmonic oscillator. This remark can be expressed in a different fashion. The corresponding force acting on the mass point is

$$F(x) = -\frac{dU}{dx} = -2a_2 x - 4a_4 x^3 \dots$$

Every system with a potential well is, in lowest approximation, characterised by a restoring force according to Hooke's law. The stability that is found in nature points towards the omnipresence of such restoring forces. Different oscillating systems as pendulums, oscillating liquid drops, systems of springs, tuning forks, or musical instruments may be discussed in mechanics. In electrodynamics corresponding systems are oscillating electric circuits or transmitting antennae. Oscillating atoms in molecules can, also in first approximation, be considered as harmonic quantum oscillators (although they differ considerably from classical oscillators).

Higher order contributions in the Taylor expansion have to be considered for larger displacements. The oscillations become anharmonic. If the kinetic energy of a mass point is too large, the mass will leave the potential well. A discussion in terms of a power series is in this case not adequate at all. An explicit first example for a classical oscillating system is the mathematical pendulum.

4.2.1 The mathematical pendulum

A mathematical pendulum consists of a fictitious rod of length l , which can rotate about a centre of suspension. At the end of the rod is a mass point m . 'Fictitious' means that the rod is massless and absolutely rigid. The mass point is subjected to the following forces (Fig. 4.14a)

- the gravity mg .
- the (unknown) constraining force due to the rod S .

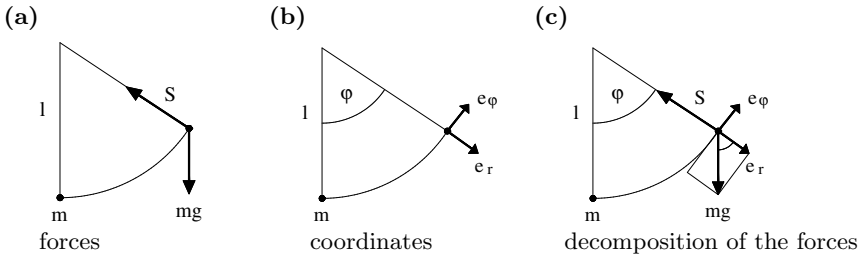


Fig. 4.14. Mathematical pendulum

If the initial conditions are chosen accordingly, the pendulum will move in a plane (for arbitrary initial conditions the pendulum is spherical, see Chap. 5.3.2). Polar coordinates (Fig. 4.14b) are an appropriate choice for the discussion of the motion on the circle that results from the constraint. The forces which act on the mass point are decomposed into a radial and an angular component (Fig. 4.14c)

$$F_r = -S + mg \cos \varphi \quad F_\varphi = -mg \sin \varphi, \quad (4.34)$$

the corresponding decomposition of the acceleration vector (see (2.60)) is

$$a_r = \ddot{r} - r\dot{\varphi}^2 \quad a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi}.$$

A rigid rod is characterised by $r = l$ and $\dot{r} = 0$, $\ddot{r} = 0$. The decomposition of the vectorial equation of motion $m\ddot{\mathbf{r}} = \mathbf{F}$ is therefore

$$\begin{aligned} \text{azimuthal component :} \quad & ml\ddot{\varphi} = -mg \sin \varphi \\ \text{radial component :} \quad & -ml\dot{\varphi}^2 = -S + mg \cos \varphi. \end{aligned} \quad (4.35)$$

The time development of the angle φ (with the initial condition $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \omega_0$) is fully determined by the first equation. Once the solution of this differential equation has been obtained, the constraining force S can be determined from the second equation

$$S(t) = ml\dot{\varphi}(t)^2 + mg \cos \varphi(t).$$

Obviously the relation $S = mg \cos \varphi$, which might be expected from a static argument, is *not* valid.

4.2.1.1 Solution of the differential equation of the pendulum. The differential equation (4.35)

$$\ddot{\varphi} + \frac{g}{l} \sin \varphi = 0 \quad (4.36)$$

does not correspond to the differential equation of the harmonic oscillator. Harmonic oscillations with the solution

$$\varphi(t) = \alpha \sin(\omega t + \beta)$$

are only found for small deflections with the approximation

$$\sin \varphi \approx \varphi$$

and the differential equation

$$\ddot{\varphi} + \omega^2 \varphi = 0 .$$

The angular frequency of the mathematical pendulum is $\omega = \sqrt{g/l}$ in this limit. The period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}} . \quad (4.37)$$

The period depends, in this case, only on the length of the pendulum and the constant g and not on the magnitude of the (small) initial displacement. Pendulums are (nearly) isochronous for small deflections. The manufacturers of pendulum clocks rely on this fact.

The solution of the full differential equation (4.36) with the form $\ddot{\varphi} = F(\varphi)$ requires (see ● Math.Chap. 2.2.1) the substitution

$$\dot{\varphi} = \gamma \quad \ddot{\varphi} = \frac{d\gamma}{dt} = \frac{d\gamma}{d\varphi} \frac{d\varphi}{dt} = \gamma \frac{d\gamma}{d\varphi} .$$

This leads to

$$\gamma \frac{d\gamma}{d\varphi} = -\omega^2 \sin \varphi$$

and after integration via a separation of variables to

$$\frac{1}{2} \dot{\varphi}^2(t) - \frac{1}{2} \dot{\varphi}^2(0) = \omega^2 (\cos \varphi(t) - \cos \varphi(0)) . \quad (4.38)$$

The original variable has been reintroduced in the result quoted. It is (up to a factor ml^2) equivalent to the law of energy conservation. The constraining force does not contribute to the energy balance, as the vector \mathbf{S} is perpendicular to the instantaneous displacement $d\mathbf{r}$ for each instant of time. It would have been possible to begin the discussion directly with the statement

$$\frac{m}{2} v^2 + mgh = E_0 ,$$

followed by an introduction of the geometry of the problem at this point.

The solution of (4.38) shall be discussed here only for the special initial conditions

$$\varphi(0) = 0 \quad \dot{\varphi}(0) = \omega_0 .$$

The mass is at the lowest point at the time $t = 0$ and starts with the angular velocity ω_0 . With respect to the initial velocity two possibilities have to be distinguished:

- (1) ω_0 is large enough, so that the pendulum rotates about the centre of suspension. In this case the differential equation (second integration)

$$\dot{\varphi} = \pm [2\omega^2(\cos \varphi - 1) + \omega_0^2]^{1/2}$$

has to be solved. The signs correspond to a rotation in or against the sense of the clock.

- (2) ω_0 is sufficiently small, so that the pendulum only reaches a maximum deflection φ_m ($0 \leq \varphi_m \leq \pi$). The turning point is characterised by $\dot{\varphi} = 0$, so that the energy principle (4.38) yields

$$-\omega_0^2 = 2\omega^2(\cos \varphi_m - 1) .$$

This relation can be used to eliminate ω_0 from the differential equation (4.38)

$$\dot{\varphi} = \pm \sqrt{2} \omega [\cos \varphi - \cos \varphi_m]^{1/2} . \quad (4.39)$$

The change of the direction of the motion during the oscillation of the pendulum must be reproduced by an appropriate handling of the sign in (4.39). The positive sign has to be used if the angular velocity is $\dot{\varphi} \geq 0$ for the first oscillatory phase ($0 \rightarrow \varphi_m$). In the next phase ($\varphi_m \rightarrow -\varphi_m$) the negative sign characterises the back-swinging mass. The positive sign has to be employed again in the last phase before the completion of the first cycle.

Only the oscillatory motion (for a discussion of the rotating pendulum, see ● Probl. 4.12) with a small value of ω_0 will be discussed in some detail⁶. The motion of the pendulum is periodic in this case, but not harmonic. The solution of (4.39) for the four phases of the pendulum can be obtained with a separation of variables as

$$\begin{aligned} \int_0^{\tau_1} dt &= \frac{1}{\sqrt{2} \omega} \int_0^{\varphi_m} \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \\ \int_{\tau_1}^{\tau_2} dt &= -\frac{1}{\sqrt{2} \omega} \int_{\varphi_m}^0 \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \\ &= \frac{1}{\sqrt{2} \omega} \int_0^{\varphi_m} \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \\ \int_{\tau_2}^{\tau_3} dt &= -\frac{1}{\sqrt{2} \omega} \int_0^{-\varphi_m} \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \\ &= \frac{1}{\sqrt{2} \omega} \int_0^{\varphi_m} \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \end{aligned}$$

⁶ An additional aspect of the motion of the mathematical pendulum is discussed in Chap. 5.4.3 under the heading 'Looking into phase space'.

$$\begin{aligned}\int_{\tau_3}^{\tau_4} dt &= \frac{1}{\sqrt{2} \omega} \int_{-\varphi_m}^0 \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \\ &= \frac{1}{\sqrt{2} \omega} \int_0^{\varphi_m} \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} .\end{aligned}$$

The period of oscillation is the same for all four phases

$$\tau_1 = \tau_2 - \tau_1 = \tau_3 - \tau_2 = \tau_4 - \tau_3 .$$

Integration of (4.39) up to the time $t \leq \tau_1$ gives

$$\omega t = \frac{1}{\sqrt{2}} \int_0^\varphi \frac{d\varphi'}{[\cos \varphi' - \cos \varphi_m]^{1/2}} \quad 0 \leq \varphi \leq \varphi_m \leq \pi . \quad (4.40)$$

The integral on the right hand side cannot be evaluated in an elementary fashion. It is an **incomplete elliptic integral of the first kind** (● Math.Chap. 4.3.4). A normal representation of this special function is obtained with the following substitution

1. Use $\cos \varphi = 1 - 2 \sin^2(\varphi/2)$ and set $k = \sin(\varphi_m/2)$ to obtain

$$\cos \varphi - \cos \varphi_m = 2(k^2 - \sin^2(\varphi/2)) .$$

2. Substitute $\sin(\varphi/2) = k \sin s$. The limits are then

$$\varphi = 0 \rightarrow s = 0 , \quad \varphi = \varphi_m \rightarrow s = \pi/2 .$$

The details of the substitution involve

$$\begin{aligned}\frac{1}{2} \cos(\varphi/2) d\varphi &= k \cos s \, ds \\ d\varphi &= \frac{2k \cos s}{[1 - k^2 \sin^2 s]^{1/2}} ds .\end{aligned}$$

The integrand is

$$\frac{1}{[\cos \varphi - \cos \varphi_m]^{1/2}} = \frac{1}{\sqrt{2}} \frac{1}{[k^2 - k^2 \sin^2 s]^{1/2}} = \frac{1}{\sqrt{2} k \cos s} .$$

If everything is put together, one arrives at

$$\omega t = \int_0^s \frac{ds'}{[1 - k^2 \sin^2 s']^{1/2}} \equiv F(s, k) . \quad (4.41)$$

4.2.1.2 Details of the motion of the pendulum. The elliptic integral $F(s, k)$, or the form $F(s, \varphi_m)$, is listed in Tables. A graphical representation of the function $F(s)$ for the range $0 \leq s \leq \pi/2$ and the dependence on the parameter $\varphi_m = 2 \arcsin(k)$ is shown in Fig. 4.15. The different curves correspond to the values $\varphi_m = 0^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ$ (in the order given, with the lowest curve for $\varphi_m = 0^\circ$). Inversion of these numerical results yields, according to (4.41), the function $\varphi(t)$ for a given value of φ_m .

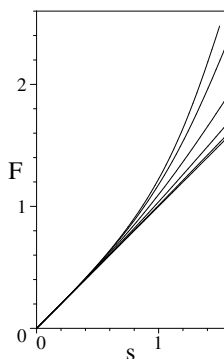


Fig. 4.15. The elliptic integral $F(s, \varphi_m)$ for different maximal deflections φ_m

The time for the duration of a full cycle T can, as a consequence of the periodicity, be calculated via the period of a quarter oscillation

$$T = \frac{4}{\omega} \int_0^{\pi/2} \frac{ds'}{[1 - k^2 \sin^2 s']^{1/2}} = \frac{4}{\omega} F(\pi/2, k). \quad (4.42)$$

The integral with the upper limit $\pi/2$ is a **complete elliptic integral**. Its values could be taken from Fig. 4.15 for $s = \pi/2$. This integral is

$$F(\pi/2, 0) = \int_0^{\pi/2} ds' = \frac{\pi}{2}$$

in the extreme limit $k = 0$ which corresponds to $\varphi_m = 0$. This leads to $T = 2\pi/\omega$, the period of the pendulum in the harmonic limit (4.37).

The function $F(\pi/2, k)$ depends only weakly on k for small values of k , so that this approximation is (as seen in Table 4.2) acceptable for a rather wide range of maximal deflections. As a consequence of the weak dependence

Table 4.2. Variation of the complete elliptic integrals $F(\pi/2, k)$ with the maximal deflection $k = \sin \varphi_m/2$

φ_m	0°	20°	40°	60°	90°
$F(\pi/2, k)$	1.571	1.583	1.620	1.686	1.854

of the elliptic integral on the variable k a series expansion offers itself. The binomial series

$$[1 - x]^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots$$

results in the present situation in

$$[1 - k^2 \sin^2 s']^{-1/2} = 1 + \frac{1}{2}k^2 \sin^2 s' + \frac{3}{8}k^4 \sin^4 s' + \dots$$

A term by term integration using

$$\int_0^{\pi/2} \sin^{2n} s' ds' = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2} \quad n \geq 1, \quad (4.43)$$

which can be obtained via a recursion relation (see ● D.tail 4.2), leads to an expansion for the period T . The corrections to fourth order in k are

$$T = T_{O(k^4)} + \cdots \quad T_{O(k^4)} = \frac{2\pi}{\omega} \left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 \right]. \quad (4.44)$$

The period depends on the maximum deflection. The mathematical pendulum is not isochronous. An idea of the order of magnitude of the correction and of the quality of the approximation up to fourth order is shown in Table 4.3. A

Table 4.3. Comparison of the exact period of the mathematical pendulum with a fourth order approximation

φ_m	0°	20°	40°	60°	90°
$k = \sin \varphi_m/2$	0	0.174	0.342	0.500	0.707
$\frac{\omega}{2\pi} T_{O(k^4)}$	1	1.0077	1.0312	1.0713	1.1602
$\frac{\omega}{2\pi} T_{\text{exact}}$	1	1.0077	1.0313	1.0732	1.1803

deviation of 18% from the harmonic limit is found for a maximum deflection of $\varphi_m = 90^\circ$. The approximation (4.44) is accurate to 1.7 %. The correction of the harmonic limit is used for the construction of mechanical clocks with a high precession (the old astronomical clocks) and for a precise measurement of the gravitational acceleration g using the mathematical pendulum.

The limit $k = 1 \rightarrow \varphi_m = \pi$ with

$$F(\pi/2, 1) = \int_0^{\pi/2} \frac{ds'}{\cos s'} \rightarrow \infty$$

is also of interest. The process takes an infinite amount of time. The initial angular velocity that is required, so that the pendulum just moves to the upright position without turning over, is $\omega_0 = \pm 2\omega = \pm 2\sqrt{g/l}$.

The constraining force S is given by the radial equation (4.35)

$$S(t) = mg \cos \varphi(t) + ml\dot{\varphi}^2(t).$$

It is equal to the sum of the radial component of the gravitational force and the centrifugal force. A rather complicated constraining force is required

in order to keep the mass on the circle. The dependence of the constraining force on the deflection φ can be obtained by eliminating the centrifugal contribution with the energy law (4.38)

$$\dot{\varphi}^2 = \frac{2g}{l}(\cos \varphi - \cos \varphi_m) .$$

This results in

$$S = mg(3 \cos \varphi - 2 \cos \varphi_m) . \quad (4.45)$$

A positive value of S indicates that the (point) mass of the pendulum is pulled in the radial direction by the other two forces. This is always the case

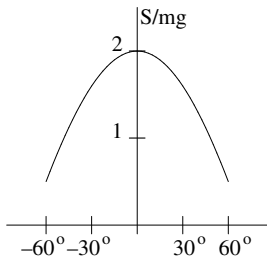


Fig. 4.16. Variation of the guiding force with the deflection for $\varphi_m = 60^\circ$

for $\varphi_m \leq 90^\circ$, as e.g. for the maximal deflection $\varphi_m = 60^\circ$ which is illustrated in Fig. 4.16. The function $S(\varphi)$ can be negative for $\varphi_m > 90^\circ$. Figure 4.17a shows the function $S(\varphi)$ for $\varphi_m = 180^\circ$. The constraining force is negative for deflections larger than 132° ($\cos \varphi = -2/3$). The gravitational component becomes negative and dominates over the positive centrifugal component, so that the mass point would be pulled inwards if the resulting force were not compensated by the rigid rod. The mass will not stay on the circle if it is attached to a string or a flexible wire rather than a rod. A string can only counteract a force in the outward radial direction. The string will start to collapse as soon as the vector sum of the gravitational and centrifugal forces points in the direction of the point of suspension (Fig. 4.17b).

Additional types of pendulums (or forms of oscillations) could be discussed under the present heading, so e.g.:

1. The spherical pendulum, which results for general initial conditions, so that the motion is not restricted to a plane (see Chap. 5.3.2).
2. Pendulums with special guiding systems, as e.g. Huygen's cycloid pendulum, which swings in an isochronous manner independent of the maximum deflection (● Probl. 4.13).
3. The physical pendulum, the rotation of a rigid body about an arbitrary axis (see Chap. 6.3.7).

Some of these examples will be analysed after the introduction of the Lagrangian formulation in Chap. 5. A more direct variant of the (one dimen-

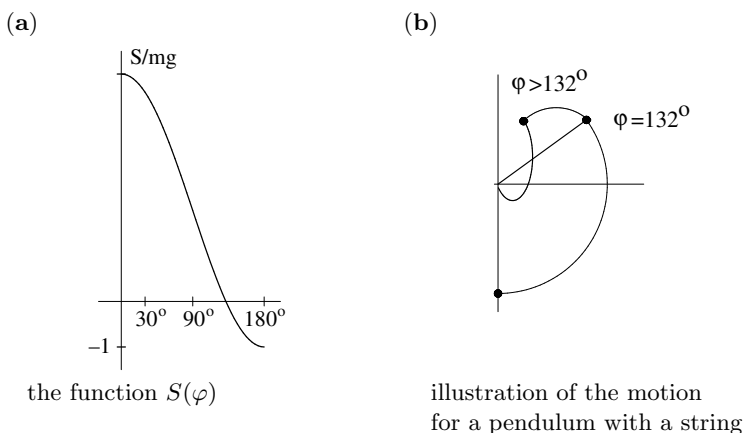


Fig. 4.17. Variation of the guiding force with the deflection for $\varphi_m = 180^\circ$

sional) oscillator problem, which is relevant for the correct operation of some measuring devices, is the damped harmonic oscillator.

4.2.2 The damped harmonic oscillator

The one dimensional differential equation to be discussed in this section is

$$m\ddot{x} + b\dot{x} + kx = 0. \quad (4.46)$$

The second term is a frictional force (Stokes' law: $F_S = -b\dot{x}$), the third represents the linear restoring force (Hooke's law: $F_H = -kx$). The total force is not conservative so that energy is not conserved (see the discussion on page 172). The solution of the homogeneous linear differential equation of second order does not present any difficulties. Introducing the usual abbreviations

$$\beta = \frac{1}{2} \frac{b}{m} \quad \omega_0 = \sqrt{\frac{k}{m}},$$

the differential equations reads

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \quad (4.47)$$

The ansatz with an exponential function $x = e^{\alpha t}$ leads to the characteristic equation

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0$$

with the solution

$$\alpha_{1,2} = -\beta \pm [\beta^2 - \omega_0^2]^{1/2}. \quad (4.48)$$

This result shows that three physically different solutions have to be distinguished: weak and strong damping as well as the aperiodic limit.

4.2.2.1 Discussion of the types of motion. The first is the limit of *weak damping* which is characterised by $\omega_0^2 > \beta^2$. The radicand in (4.48) is negative, so that the roots of the characteristic equation are complex

$$\alpha_{1,2} = -\beta \pm i\omega_1 \quad (\omega_1^2 = \omega_0^2 - \beta^2) .$$

The basic features of complex numbers and functions of complex variables are introduced in Math.Chap. 7.

The general solution can be written as

$$x(t) = e^{-\beta t} \{C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\}$$

or alternatively in real form

$$x(t) = A e^{-\beta t} \cos(\omega_1 t + \delta) . \quad (4.49)$$

The two integration constants are determined by the initial conditions. For instance, the initial conditions

$$x(0) = 0 \quad \dot{x}(0) = v_0 \quad (4.50)$$

give the explicit solution (Fig. 4.18)

$$x(t) = \frac{v_0}{\omega_1} e^{-\beta t} \sin \omega_1 t ,$$

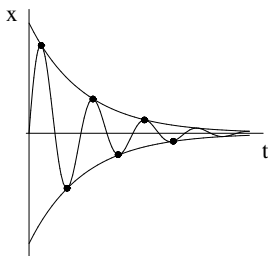


Fig. 4.18. The damped harmonic oscillator: weak damping

which represents a sinusoidal oscillation with the frequency ω_1 , which is damped by a decaying exponential factor. The passage through the equilibrium position $x = 0$ takes place for times with $\omega_1 t = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). The oscillation is periodic in spite of the damping. The local maxima and minima correspond to points for which the the solution and the exponential envelopes coincide (for $t > 0$).

The case of the double root $\beta^2 = \omega_0^2$ is characterised by a specific ratio of the coefficients of the frictional and restoring forces. This case is called the *aperiodic limit*. The general solution is here (see ● Math.Chap. 2.2.2)

$$x(t) = (C_1 + C_2 t) e^{-\beta t} . \quad (4.51)$$

The initial conditions (4.50) leads to

$$x(t) = v_0 t e^{-\beta t}.$$

This function describes a displacement, that increases (initially linearly) up to the time $t = 1/\beta$ and returns, after changing the direction of motion, in an exponential fashion to the equilibrium position (Fig. 4.19).

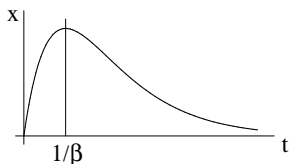


Fig. 4.19. The damped harmonic oscillator: aperiodic limit

Strong damping is characterised by the inequality $\omega_0^2 < \beta^2$. The roots of the characteristic equation are real in this case. As $\beta \geq \sqrt{\beta^2 - \omega_0^2}$ both α_1 and α_2 are negative numbers. The general solution

$$x(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} \quad \alpha_1, \alpha_2 < 0$$

is a superposition of two terms that decrease exponentially with time. The motion is therefore no oscillation (it is often termed a creeping motion). The special solution for the initial condition (4.50), expressed in terms of hyperbolic rather than exponential functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x}),$$

is

$$x(t) = \frac{v_0}{[\beta^2 - \omega_0^2]^{1/2}} e^{-\beta t} \sinh([\beta^2 - \omega_0^2]^{1/2} t). \quad (4.52)$$

The corresponding $x(t)$ -curve (Fig. 4.20) is similar to the curve obtained in the aperiodic limit: a displacement up to a maximum, which is followed

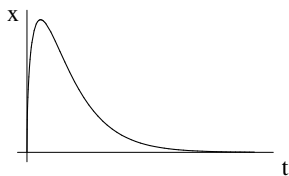


Fig. 4.20. The damped harmonic oscillator: strong damping

by an exponential return to the equilibrium position. The return to the equilibrium position is faster in the aperiodic limit.

The energy situation for the damped oscillator can be analysed as follows: multiply the equation of motion (4.46) by \dot{x} and find

$$m\ddot{x}\dot{x} + kx\dot{x} = -b\dot{x}^2.$$

The left hand side can be reformulated

$$\frac{d}{dt} \left(\frac{m}{2} \dot{x}^2 + \frac{k}{2} x^2 \right) = -b\dot{x}^2 .$$

An interpretation of this result could use the picture that the damped oscillator corresponds to a mass-spring system which is embedded in a viscous fluid (Fig. 4.21). The expressions in the brackets are the kinetic energy of

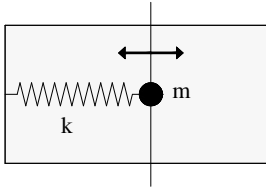


Fig. 4.21. The damped harmonic oscillator: total system

the mass and the potential energy stored in the spring. The change of the total mechanical energy of the mass-spring system with time is negative. The mechanical energy decreases with time according to (● Probl. 4.14)

$$E_{\text{mech}}(t) = E_{\text{mech}}(0) - b \int_0^t \dot{x}(t')^2 dt' .$$

The mechanical energy, which is lost by the mass-spring system, is converted to another form of energy, mainly heat, of the whole mass-spring-fluid system. Energy conservation of the total system (provided it is isolated from its surroundings) can nonetheless be formulated as

$$\frac{d}{dt} \left(E_{\text{mech}}(t) + Q(t) \right) = 0$$

where $Q(t)$ is the (time changing) heat content of the total system.

Forced oscillations constitute another variant of the oscillator problem. Such oscillations can be characterised by the differential equation

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

or

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \quad f = F/m . \quad (4.53)$$

The quantity ω_0 is the **eigenfrequency** of the system. The mass-spring system is subjected to an additional, time dependent external force. Both, the case of free oscillations ($\beta = 0$) and damped oscillations ($\beta \neq 0$) can be considered. A simple example for an external force is a harmonic force.

4.2.3 Forced oscillations: harmonic restoring forces

The external force might e.g. be specified in the form

$$f(t) = \gamma \cos \omega t . \quad (4.54)$$

The oscillating mass is driven in a periodic manner with the external frequency $f = \omega/2\pi$.

The solution of the inhomogeneous differential equation (4.53) can be written in the form (● Math.Chap. 2.2.2)

$$x(t) = x_{\text{hom}}(C_1, C_2, t) + x_{\text{part}}(t) .$$

The general solution of the homogeneous equation has been discussed in the previous section, Chap. 4.2.2. The particular solution of the inhomogeneous differential equation for the simple sinusoidal force is obtained with the ansatz

$$x_{\text{part}}(t) = A \cos \omega t + B \sin \omega t .$$

The method of the 'variation of the constant' could be applied for the determination of the parameters A and B but this is not necessary in the present case. It is sufficient to insert the ansatz into the differential equation using

$$\dot{x}_{\text{part}} = -A\omega \sin \omega t + B\omega \cos \omega t \quad \ddot{x}_{\text{part}} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t .$$

Comparison of the coefficients in the result

$$\begin{aligned} \cos \omega t \{ -A\omega^2 + 2\beta B\omega + A\omega_0^2 \} + \sin \omega t \{ -B\omega^2 - 2\beta A\omega + B\omega_0^2 \} \\ = \gamma \cos \omega t \end{aligned}$$

leads to a system of linear equations

$$\begin{aligned} (-2\beta\omega)A + (\omega_0^2 - \omega^2)B &= 0 \\ (\omega_0^2 - \omega^2)A + (2\beta\omega)B &= \gamma . \end{aligned}$$

The solution is

$$A = \frac{\gamma(\omega_0^2 - \omega^2)}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]} \quad B = \frac{2\beta\gamma\omega}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]} . \quad (4.55)$$

It is often useful to employ the form

$$x_{\text{part}} = a \cos(\omega t - \varphi)$$

(amplitude a and phase φ) for the special solution. The standard conversion formulae

$$\begin{aligned} a &= [A^2 + B^2]^{1/2} \\ \cos \varphi &= \frac{A}{\sqrt{A^2 + B^2}} \quad \sin \varphi = \frac{B}{\sqrt{A^2 + B^2}} \quad \tan \varphi = \frac{B}{A} \end{aligned}$$

yield directly

$$a = \frac{\gamma}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}} \quad (4.56)$$

$$\tan \varphi = \frac{2\beta\omega}{(\omega_0^2 - \omega^2)} . \quad (4.57)$$

The phase is independent of the strength of the external force.

The following statements apply to the total solution

$$x(t) = x_{\text{hom}}(C_1, C_2, t) + x_{\text{part}}(t) .$$

- The integration constants C_1 and C_2 of the total solution are determined by the initial conditions. The final form of the solution can turn out to be quite complicated.
- The second term, the particular solution, will dominate for large times if the homogeneous solution describes an oscillation, which is damped exponentially

$$\lim_{t \text{ large}} x(t) = x_{\text{part}}(t) \quad \text{if} \quad \lim_{t \text{ large}} x_{\text{hom}}(t) = 0 .$$

The motion follows the external force

$$x_{\text{part}}(t) = a \cos(\omega t - \varphi)$$

after a more complicated transient process. This result shows that the oscillation does not follow the external force directly, but lags behind with the phase difference φ .

The particular solution is given by

$$a = \frac{\gamma}{|\omega_0^2 - \omega^2|} \quad \text{and} \quad \varphi = 0$$

for the case of a forced oscillation without damping. The absolute value of the denominator has to be used, as the amplitude is a positive quantity. The general solution can then be written as

$$x(t) = a_0 \cos(\omega_0 t - \delta_0) + a(\gamma, \omega, \omega_0) \cos \omega t .$$

The quantities a_0 and δ_0 are the integration constants. The solution represents a superposition of two harmonic oscillations, one with the eigenfrequency ω_0 , the other with the frequency ω of the external force. A particular situation is found for the initial conditions

$$x(0) = 0 \quad \dot{x}(0) = 0 .$$

The mass would not move in this case without an external force. The integration constants are obtained from

$$a_0 \cos \delta_0 + a = 0$$

$$\{-a_0 \omega_0 \sin(\omega_0 t - \delta_0) - a \omega \sin \omega t\}_{t=0} = a_0 \omega_0 \sin \delta_0 = 0 ,$$

as

$$\delta_0 = 0 \quad \text{and} \quad a_0 = -a$$

so that the total solution is

$$x(t) = a (\cos \omega t - \cos \omega_0 t) .$$

4.2.3.1 Examples of forced oscillations. The multitude of patterns, which can be obtained by the superposition of two cosinusoidal oscillations, is illustrated with two examples.

- The amplitude has the value $a = \gamma/3\omega$, if the eigenfrequency of the system is twice as large as the frequency of the external force $\omega_0 = 2\omega$. The grey curves in Fig. 4.22 show the individual oscillations: the curve $\cos \omega t$ and a curve, which oscillates with double the frequency. Addition leads to the black curve. The resulting motion is periodic (the pattern is repeated) but

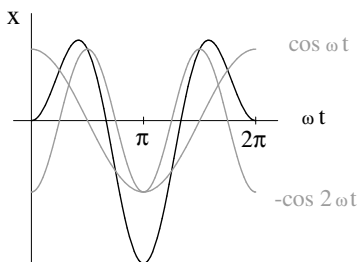


Fig. 4.22. Forced oscillation with a harmonic force: a particular solution for $\omega_0 = 2\omega$

not harmonic. The maximum displacement in the negative x -direction is twice as large as the maximum displacement in the positive direction.

- The eigenfrequency differs only slightly from the frequency of the external force

$$\omega_0 = \omega + \Delta\omega \quad \text{with} \quad \omega \gg \Delta\omega > 0 .$$

The amplitude is

$$a = \frac{\gamma}{(2\omega\Delta\omega + \Delta\omega^2)} \approx \frac{\gamma}{2\omega\Delta\omega}$$

in this case. The amplitude can be very large if $\Delta\omega$ is very small. The difference of the two cosine functions can be approximated by

$$\begin{aligned} \cos \omega t - \cos \omega_0 t &= \cos \omega t - \cos \omega t \cos \Delta\omega t + \sin \omega t \sin \Delta\omega t \\ &\approx \cos \omega t - \cos \omega t + (\Delta\omega t) \sin \omega t \\ &= (\Delta\omega t) \sin \omega t + O((\Delta\omega t)^2) \end{aligned}$$

as long as the condition $\Delta\omega t < 1$ is satisfied. The result is then

$$x(t) \approx \frac{\gamma}{2\omega} t \sin \omega t \quad \text{valid for} \quad \Delta\omega t < 1 .$$

This equation describes (Fig. 4.23a) a sinusoidal oscillation between the envelopes $\pm \gamma t/(2\omega)$. The amplitude can, for a given initial situation, attain very large values. This phenomena is called a *resonance*.

This behaviour is a consequence of the form of the amplitude

$$a(\omega) = \frac{\gamma}{|\omega_0^2 - \omega^2|}.$$

The dependence of this function on the external frequency ω is shown in Fig. 4.23b. The function $a(\omega)$ is singular for $\omega = \omega_0$. This singularity is caused

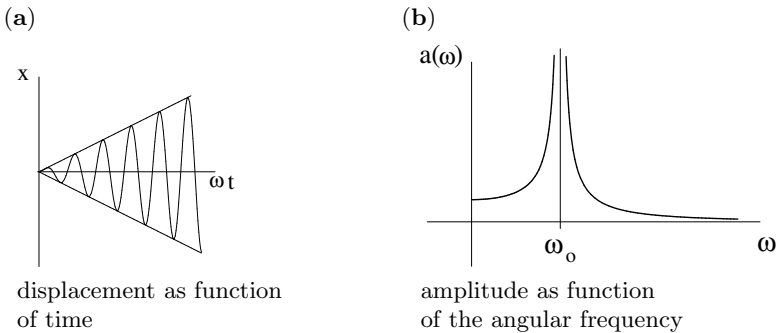


Fig. 4.23. Resonance catastrophe: undamped oscillator

by the strong coupling of the natural oscillation and the external oscillation. The system is able to follow the enforced oscillation the better the closer the external frequency ω approaches the eigenfrequency ω_0 . The amplitude is (formally) infinite, if the two frequencies are equal. However, the functional form of the restoring force according to Hooke's law is only valid for small displacements. A more realistic representation of the restoring force, valid for larger displacements, will have to be used for a proper description of 'resonance catastrophes'.

Resonance catastrophes of mechanical systems, which have been subjected to periodic forces, do indeed occur. A very dramatic example is the collapse of the Tacoma bridge in the vicinity of Seattle (USA) during the forties of the 20th century. There exists a short film, which shows that the bridge (a suspension bridge) is brought into a resonance condition by periodic gusts of wind. The displacements of the bridge (and the resulting torsion) were so strong that it collapsed.

The behaviour of the amplitude function $a(\omega)$ well beyond the resonance frequency is $a \rightarrow 0$ for $\omega \gg \omega_0$. The system is, due to its inertia, not able to follow the wild external oscillations, and does not respond to them.

4.2.3.2 Detailed discussion of the resonance phenomenon. The amplitude function (4.56) for a damped oscillator ($\beta \neq 0$)

$$a(\omega) = \frac{\gamma}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}} \quad (4.58)$$

is plotted in (Fig. 4.24). It starts with the value γ/ω_0^2 for $\omega = 0$. The amplitude grows with ω also in this case, but remains finite. The function $a(\omega)$

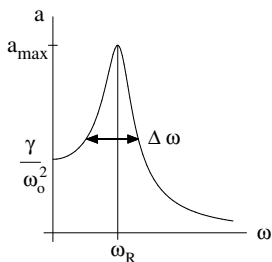


Fig. 4.24. Resonance: amplitude as function of angular frequency (damping)

has a maximum at the position

$$\omega_R = [\omega_0^2 - 2\beta^2]^{1/2} \quad (4.59)$$

which can be determined from the condition

$$\frac{da}{d\omega} = -\frac{2\omega\gamma}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{3/2}} \{\omega_0^2 - \omega^2 - 2\beta^2\} = 0.$$

The position of the resonance is shifted to lower values compared to the undamped case. The function $a(\omega)$ decreases beyond the point of resonance, finally as $1/\omega^2$. The structure of the resonance behaviour can (roughly) be characterised by the following features:

1. the **resonance frequency** ω_R .
2. the maximal amplitude

$$a(\omega_R) = a_{\max} = \frac{\gamma}{2\beta(\omega_0^2 - \beta^2)^{1/2}}. \quad (4.60)$$

3. the **half width** or **full width at half maximum**

$$\Delta\omega_{1/2} = \omega \left(\frac{a_{\max}}{2} \right)_{\text{above}} - \omega \left(\frac{a_{\max}}{2} \right)_{\text{below}}. \quad (4.61)$$

This corresponds to an interval around the resonance position ω_R between points for which the amplitude reaches half the maximum value (Fig. 4.24). An explicit expression for the half width as a function of β can be given (● D.tail 4.3). Usually an estimate is sufficient. For instance, the approximation $\Delta\omega_{1/2} \approx 2\sqrt{3}\beta$ can be used in the case of weak damping ($\omega_0 > \beta$).

The variation of the resonance curves with the degree of damping is illustrated in Fig. 4.25. The relevant parameter is the **quality factor**

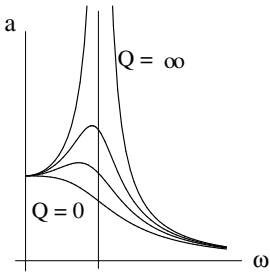


Fig. 4.25. Variation of the resonance curves with the quality factor

$$Q = \frac{\omega_R}{2\beta} = \frac{\sqrt{\omega_0^2 - 2\beta^2}}{2\beta}. \quad (4.62)$$

The value $Q \rightarrow \infty$ corresponds to vanishing damping ($\beta \rightarrow 0$). The maximum is shifted slowly towards smaller values of ω , as the damping increases. The maximum value of the amplitude decreases and the resonance structure becomes wider. The resonance position is hardly recognisable for $Q = 1$, it cannot be noticed at all for $Q = 0$ (corresponding to $\omega_0 = \sqrt{2}\beta$).

The phase function (4.57)

$$\tan \varphi(\omega) = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \quad \text{respectively} \quad \varphi(\omega) = \arctan \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right)$$

also shows a characteristic resonance pattern (Fig. 4.26). The function

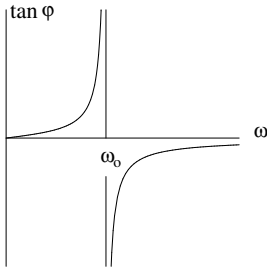


Fig. 4.26. The phase function $\tan \varphi(\omega)$

$\tan \varphi(\omega)$ begins at the value 0 for $\omega = 0$. It increases with increasing ω and becomes infinite for $\omega \rightarrow \omega_0$. Beyond this point, the function jumps to the value $-\infty$ and approaches the value $\tan \varphi = 0$ from below for larger ω -values. The corresponding behaviour of the phase angle $\varphi(\omega)$ is indicated in Fig. 4.27. The angle $\varphi(\omega)$ increases from zero to the value $\pi/2$ for $\omega = \omega_0$ and reaches the value π for large values of ω . The variation of the phase function with the quality factor Q can be described as follows: the transition through the position ω_0 is rather flat for strong friction (Q small). The larger Q becomes (the smaller the friction becomes) the steeper is the step from $\varphi = 0$ to $\varphi = \pi$. In the limit $Q \rightarrow \infty$ (no friction) the behaviour is given by a step function.

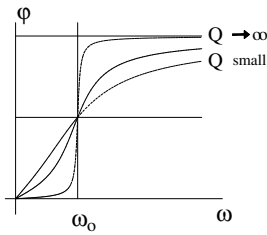


Fig. 4.27. Variation of φ with the quality factor

The total solution

$$x(t) = x_{\text{hom}}(C_1, C_2, t) + a(\omega) \cos(\omega t - \varphi(\omega))$$

(e.g. for the initial condition $x(0) = 0$, $\dot{x}(0) = 0$, see ● Probl. 4.15) is rather complicated in the transient phase. Damping effects (if present) finally lead to a solution of the form

$$\lim_{t \text{ large}} x(t) = a \cos(\omega t - \varphi) .$$

The mass point oscillates with the same frequency as the external force. The amplitude a expresses how strongly the system reacts to the external excitation. It is always proportional to γ , that is the strength of the external force. The phase describes in how far the mass point can follow the excitation. If the frequency ω is small, so is $\varphi(\omega)$. The oscillation follows (after the built-up) the external force without a substantial retardation. The phase difference is $\pi/2$ (independent of the degree of damping) for $\omega = \omega_0$. The mass oscillates for this value of ω with a sine function if it is driven by a cosine function. The phase difference is π for large values of ω . The mass oscillates (with a small amplitude) in a direction opposing the external force.

The solution of the forced, damped oscillator problem plays also a role in electrodynamics (see Vol. 2). An electric circuit (Fig. 4.28) consisting of an AC source (U), a coil (with the inductance L), a condenser (with the capacity C) and a resistance (R) is an *oscillating electric circuit*. The current

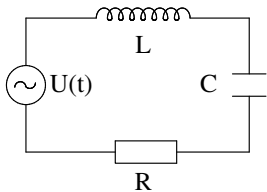


Fig. 4.28. Oscillating electric circuit

$i(t)$, which flows through such a circuit, is characterised by the differential equation

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = \frac{dU}{dt} .$$

The solutions of this differential equation correspond, independent of the interpretation of the individual terms, exactly to those of the mechanical oscillator discussed above.

4.2.4 Forced oscillations: general excitations

The discussion of the response of a mass-spring system to a harmonic excitation has to be continued with the question: how can a solution of the differential equation

$$a\ddot{x} + b\dot{x} + cx = F(t) \quad (4.63)$$

be obtained for a general external force $F(t)$? The answer to this question is based on the principle of superposition which can be formulated in the following way

If the external force can be decomposed in the form

$$F(t) = \sum_{n=0}^N F_n(t)$$


and if $x_n(t)$ is a particular solution of the differential equation

$$a\ddot{x}_n + b\dot{x}_n + cx_n = F_n(t),$$

then $x_{\text{part}}(t) = \sum_{n=0}^N x_n(t)$ is a particular solution of the differential equation

$$a\ddot{x} + b\dot{x} + cx = F(t).$$

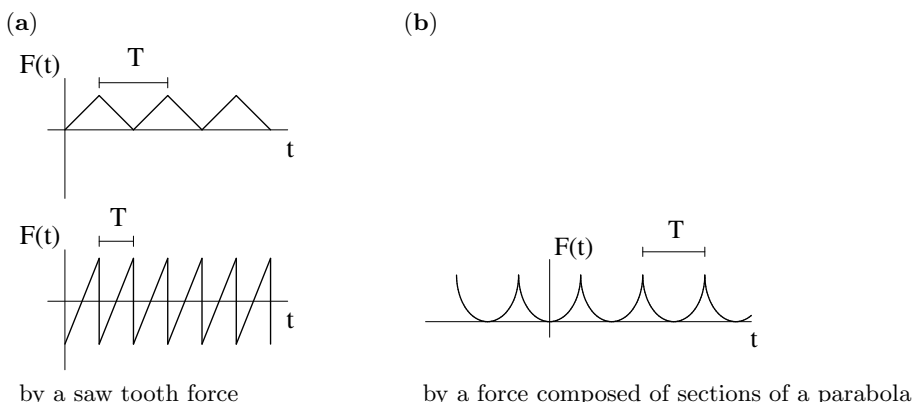
The proof of this statement is simple. Insert x_{part} into the complete differential equation, sort the right hand and the left hand side and use the assumptions. The principle of superposition allows the combination of particular solutions of linear, inhomogeneous differential equations with a more complicated inhomogeneous term from simpler constituents.

The theorem is (under appropriate conditions, see  Math.Chap. 1.3.4) also valid in the limit $N \rightarrow \infty$ so that

$$F(t) = \sum_{n=0}^{\infty} F_n(t) \quad x_{\text{part}}(t) = \sum_{n=0}^{\infty} x_n(t).$$

One condition, which has to be satisfied, says: the two series of functions have to converge absolutely and uniformly (in an interval).

This form of the theorem offers the possibility to consider more exotic, periodic excitations, as e.g. different 'saw tooth' forces (Fig. 4.29a) or a force that is composed of sections of parabolae (Fig. 4.29b). The periodic func-



by a saw tooth force

by a force composed of sections of a parabola

Fig. 4.29. Periodic excitations (with period T)

tions in these examples are continuous but not continuously differentiable. Excitations of such forms are used e.g. to model pulses in electro-technical applications.

For a representation of such periodic functions the theory of **Fourier series** has to be used. Periodic functions are represented in these series by a superposition of sine or/and cosine functions with periods chosen so that they fit into a basic interval of the independent variable. A naive approach to the construction of a Fourier series for the saw tooth force in Fig. 4.30 can be described as follows:

- The aim is a representation of the function $F(t) = t$ in the basic interval $-a \leq t \leq a$ with trigonometric functions so that the periodic continuation into neighbouring intervals $F(t \pm 2a) = F(t)$ is automatically satisfied. The representation of this odd periodic function by a trigonometric series is given by

$$F_1(t) = \frac{2a}{\pi} \sin\left(\frac{\pi}{a}t\right)$$

in a first approximation. The factor π/a in the argument of the sine function assures that the sine curve fits exactly into the basic interval. The amplitude of the sine function is determined by an optimal adaptation of the over- and under shooting (Fig. 4.30b).

- To this approximation a sine function with twice the frequency and an appropriate amplitude and sign is added. The amplitude is chosen so that the deviation of the first approximation from the exact function $F(t) = t$ is reduced in an optimal fashion. The function

$$F_2(t) = \frac{2a}{\pi} \left(\sin \frac{\pi}{a}t - \frac{1}{2} \sin \frac{2\pi}{a}t \right)$$

satisfies these requirements. The representation of the saw tooth function in this approximation ($F \approx F_1 + F_2$) is presented in Fig. 4.30c.

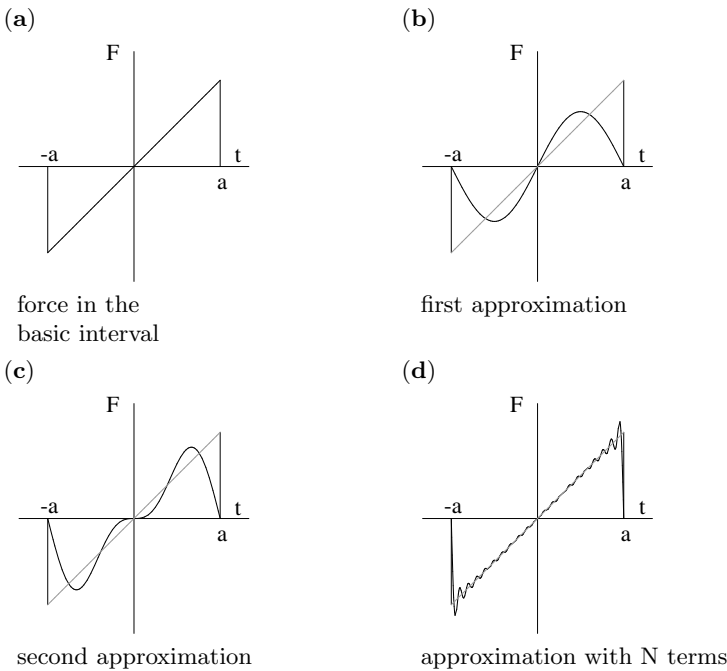


Fig. 4.30. Fourier representation of a saw tooth force

- The continuation of this method is in principle simple. In the next approximation a contribution with three oscillations in the basic interval is used, etc. The amplitude of each additional contribution is adapted so that the deviation from the function $F(t) = t$ is minimized more and more. An approximation with N terms is indicated in Fig. 4.30d. Finally (in the limit $N \rightarrow \infty$) the Fourier representation of the saw tooth function is obtained, which allows a periodic continuation of the basic function shown in Fig. 4.30a.

$$F(t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin\left(\frac{n\pi t}{a}\right). \quad (4.64)$$

The function is manifestly periodic. It satisfies the necessary condition $F(t) = F(t \pm 2a)$. The representation in the basic interval is repeated in all neighbouring intervals.

The verbal indication of a Fourier series given here is put on a proper foundation in **Math.Chap. 1.3.4**. An explicit derivation of the representation (4.64) of the saw tooth function is also presented in that chapter.

The answer to the question posed in the beginning is therefore: the general solution of the forced oscillator problem is

$$x(t) = x_{\text{hom}}(C_1, C_2, t) + \sum_{n=0}^N x_n(t)$$

if the external force can be represented in the form

$$F(t) = \sum_{n=0}^N F_n(t)$$

with a finite value of N and if the particular solutions $x_n(t)$ of the individual differential equations

$$a\ddot{x}_n(t) + b\dot{x}_n(t) + cx_n(t) = F_n(t)$$

are known. This statement still holds, if the limit $N \rightarrow \infty$ is considered, as for example in the representation of a periodic function by a Fourier series, provided the relevant mathematical conditions are satisfied.

5 General Formulation of the Mechanics of Point Particles

The second axiom of Newton implies, that the time development of the motion of a point particle or a system of point particles can be calculated, if the forces acting on the particle are known. Besides the fact that the solution of the equations of motion is not necessarily a simple matter, difficulties can arise from different quarters. It is possible that the forces (in the form of a force field or as a function of time or ...) are not known explicitly. The motion can be restricted by constraints, which are expressed in the form of geometrical conditions. A simple example of this kind of restriction is the motion on an inclined plane. The pressure, which an object (a mass point) exerts on the surface of the plane, generates a counter pressure which compensates in part the effect of gravitation. This constraining force can be determined by simple means in the case of the inclined plane. A more effective approach is, however, required in the general case. The discussion of problems of motion with constraints was initiated by J.-L. de Lagrange. The set of equations of motion, in which an ansatz for the constraining forces is introduced explicitly, is known as the Lagrange equations of the first kind (for short Lagrange I). A formal foundation of these equations is furnished by d'Alembert's principle, which can be understood as a concise extension of the second axiom if constraints for the motion are present.

D'Alembert's principle also serves as a basis for the derivation of the Lagrange equations of the second kind (for short Lagrange II). In these equations specific classes of constraining conditions are treated by a choice of optimal coordinates (the generalised coordinates). These equations of motion offer economy in the formulation as well as flexibility in applications. They can be considered as the heart of 'higher mechanics'.

An alternative approach to the problem of constrained motion is due to W. Hamilton (5.4.1, in particular 5.4.1.4). The basis of this formulation, Hamilton's principle, is a counterpart of d'Alembert's principle in the sense that Hamilton's principle is based on integrals while d'Alembert's principle is based on differentials.

The present chapter starts with a discussion of the equations of motion, in which the constraining forces are used in explicit form, that is the Lagrange equations of the first kind.

5.1 Lagrange I: the Lagrange equations of the first kind

The topic, which is treated in this section, is best introduced with the aid of some examples. The examination of these examples makes it possible to gather sufficient information on the properties of the constraining or restraining forces. This in turn allows the formulation of equations of motion including the constraints in a general fashion.

5.1.1 Examples for the motion under constraints

A point particle is supposed to move on an arbitrary surface in space under the influence of (simple) gravity (Fig. 5.1). Possible frictional effects are neglected. The initial conditions have to be chosen, so that the subsequent motion really takes place on the surface. The mass point is located on the surface at time $t = 0$ and its initial velocity \mathbf{v}_0 is tangential to the surface. It

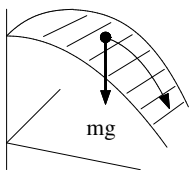


Fig. 5.1. Motion with constraints

is quite obvious that gravity $\mathbf{F} = m\mathbf{g}$ cannot be the only force that acts on the mass. If this were the case, it would be uniformly accelerated and move, depending on the initial conditions, along a parabola or along a straight line. In addition to gravity, forces due to the constraining surface have to be taken into account. These constraining forces arise from effects on an atomic level. The pressure exerted on the surface displaces the atoms of the surface layers downward from their equilibrium positions. They react with a restoring force on the object that initiated the 'disturbance'. Fortunately it is possible to characterise the restoring forces in an adequate manner without recourse to atomic or solid state physics. The following simple argument (Fig. 5.2) shows why this is the case: decompose the gravitational force \mathbf{F} for each point

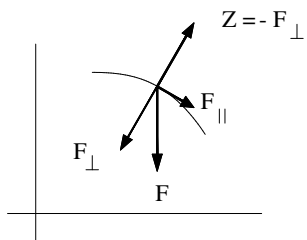


Fig. 5.2. Motion with constraints: sorting of the forces

of the surface into components tangential and perpendicular to the surface. The tangential component is responsible for the motion along the surface. As there exists no (macroscopic) motion of the mass in the normal direction, the component of the gravitation in this direction (\mathbf{F}_\perp) has to be compensated by a **constraining force** \mathbf{Z}

$$\mathbf{F}_\perp + \mathbf{Z} = \mathbf{0} . \quad (5.1)$$

An adept use of this compensation is the key to the Lagrangian equations of the first kind.

In the second example a mass point (realised by a small, pierced bead) is supposed to slide (down) on a stiff guiding wire (Fig. 5.3) under the action of gravitation. This is an example for a constrained motion along a curve in

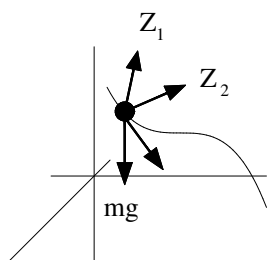


Fig. 5.3. Motion along a space curve with constraints: constraining forces

space. The situation concerning the constraining forces is more complicated. It is not sufficient to compensate one part of the gravitational force (\mathbf{Z}_1). In order to obtain a motion along the space curve, a second constraining force in the direction of the instantaneous centre of curvature (\mathbf{Z}_2) is required. The two constraining forces (and hence their vectorial sum, the total constraining force) are characterised by the fact that they are perpendicular to the instantaneous direction of motion for all times

$$\mathbf{Z}(t) \cdot d\mathbf{r} = (\mathbf{Z}_1(t) + \mathbf{Z}_2(t)) \cdot d\mathbf{r} = 0 . \quad (5.2)$$

This relation ensures, that even if constraining forces are present, only the normal, the openly acting forces \mathbf{F} contribute to the energy balance. The equations of motion for a mass point

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{Z}$$

with conservative forces \mathbf{F} and the (initially in detail unknown) constraining forces \mathbf{Z} lead via line integration to the law of energy conservation in the form

$$\frac{m}{2} v^2 + U(\mathbf{r}) = E_0 .$$

The potential energy U is solely determined by the open forces $\int \mathbf{F} \cdot d\mathbf{r}$, the line integral $\int \mathbf{Z} \cdot d\mathbf{r}$ vanishes. Constraining forces do not contribute in view of the condition (5.2).

The motion along a spatial curve, a problem of motion with one degree of freedom, is completely characterised by *one* differential equation. The energy principle is therefore sufficient for the discussion of conservative problems, even though the situation is more complicated with respect to the constraining forces. A relevant quantity for the description of the sole degree of freedom is the arc length along the specified curve. The length of an infinitesimal section of a curve can be represented in Cartesian coordinates as

$$ds = [dx^2 + dy^2 + dz^2]^{1/2} . \quad (5.3)$$

The arc length of a section of a curve, which is described in terms of a parameter q

$$x = x(q) \quad y = y(q) \quad z = z(q)$$

between points characterised by q_0 and q , is given by

$$s(q, q_0) = \int_{q_0}^q dq' \left[\left(\frac{dx}{dq'} \right)^2 + \left(\frac{dy}{dq'} \right)^2 + \left(\frac{dz}{dq'} \right)^2 \right]^{1/2} . \quad (5.4)$$

For the discussion of a problem of motion time has to be introduced. Equation (5.3) is equivalent to

$$\frac{ds}{dt} = \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right]^{1/2} = v .$$

The independence of the line integral from the choice of a specific parameter can be gleaned from a comparison of this the equation with (5.4).

The law of energy conservation can be written as

$$\frac{m}{2} \dot{s}^2 + U(s) = E_0 , \quad (5.5)$$

if it is possible to express the potential energy in terms of the arc length ($U(\mathbf{r}) \rightarrow U(s)$). The solution of this one dimensional differential equation for $s = s(t)$ describes the motion along a specified space curve. Examples are:

- A mass point moves under the influence of gravitation along a straight line in the x - z plane. The slope is specified by the angle $-\alpha$ (Fig. 5.4). The

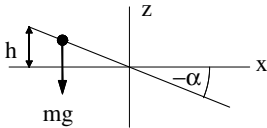


Fig. 5.4. Motion along a straight line under the influence of gravity

trajectory can be represented by the parametrisation

$$x = q \quad z = -(\tan \alpha) q \quad (y = 0) .$$

The initial conditions, in *accord* with the constraint, are

$$\left. \begin{array}{ll} x(0) = -\frac{h}{\tan \alpha} & z(0) = h \\ v_x(0) = 0 & v_z(0) = 0 \end{array} \right\} \quad \begin{array}{l} q(0) = -\frac{h}{\tan \alpha} \\ \dot{q}(0) = 0 \end{array} .$$

The law of energy conservation

$$\frac{m}{2} \dot{s}^2 + mgz = 0 + mgh$$

yields the differential equation

$$\dot{s} = \pm [2g(h - z)]^{1/2} .$$

The relation between s and z , that is needed in this example, can be taken from a simple geometrical argument (see Fig. 5.5, only the absolute value of the angle α is relevant)

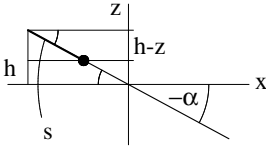


Fig. 5.5. Geometry: straight inclined line

$$\sin \alpha = \frac{h - z}{s} \quad (h - z) = s \sin \alpha .$$

The arc length s can be calculated alternatively via the integral (5.4), even if this is a bit more tedious

$$\begin{aligned} s &= \int_{q_0}^q dq' \left[\left(\frac{dx}{dq'} \right)^2 + \left(\frac{dz}{dq'} \right)^2 \right]^{1/2} = [1 + \tan^2 \alpha]^{1/2} \int_{-h/\tan \alpha}^q dq' \\ &= \frac{1}{\cos \alpha} \left(q + \frac{h}{\tan \alpha} \right) = \frac{1}{\sin \alpha} (q \tan \alpha + h) = \frac{1}{\sin \alpha} (-z + h) . \end{aligned}$$

The differential equation

$$\dot{s} = +\sqrt{2g \sin \alpha} \sqrt{s} \tag{5.6}$$

is obtained in each case. Only the positive sign is of interest, as the arc length increases with time. Direct integration (with a separation of variables and $s(0) = 0$) leads to

$$s(t) = \left(\frac{g}{2} \sin \alpha \right) t^2 .$$

The time dependence of the Cartesian coordinates can be determined with

$$q = s \cos \alpha - \frac{h}{\tan \alpha}$$

or more directly as

$$z = h - s \sin \alpha = h - \frac{g}{2} (\sin^2 \alpha) t^2$$

$$x = -\frac{z}{\tan \alpha} = -\frac{h}{\tan \alpha} + \frac{g}{2} (\cos \alpha \sin \alpha) t^2 .$$

This result is also valid for the free fall on an inclined plane provided the initial conditions are $y(0) = 0$, $\dot{y}(0) = 0$.

- The following example will turn out to be well known. A mass point moves (again) under the influence of gravity in the lower half of the x - z plane on a semicircle (radius R) about the origin (Fig. 5.6). The initial conditions

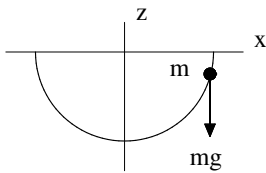


Fig. 5.6. Motion on a semicircle

are

$$\begin{aligned} x(0) &= R & z(0) &= 0 \\ v_x(0) &= 0 & v_z(0) &= 0 . \end{aligned}$$

The energy principle is not that different from the previous example

$$\frac{1}{2} \dot{s}^2 + gz = 0 .$$

The relation between z and s can be obtained in the following fashion: a parametric representation of the trajectory is

$$x = R \cos q \quad z = -R \sin q \quad (0 < q < \pi) .$$

This leads to

$$s = \int_0^q dq' [R^2 \sin^2 q' + R^2 \cos^2 q']^{1/2} = R \int_0^q dq' = Rq .$$

The result could have been obtained directly with the statement: arc length is radius times angle.

Introducing s (instead of q) into the parametric representation of the z -coordinate

$$z = -R \sin \frac{s}{R}$$

gives for the energy principle

$$\dot{s}^2 - 2gR \sin \frac{s}{R} = 0 .$$

The law of energy conservation becomes

$$\dot{\varphi}^2 - \frac{2g}{R} \cos \varphi = 0$$

if the angle φ , which is measured from the lowest point of the trajectory Fig. 5.7, is used instead of the arc length s

$$s = R \left(\frac{\pi}{2} - \varphi \right) \quad \text{and} \quad \dot{s} = -R\dot{\varphi}.$$

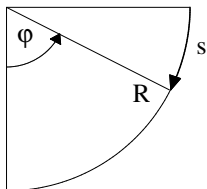


Fig. 5.7. Relation between arc length and angle of deflection

This result expresses energy conservation for the mathematical pendulum (with the initial conditions $\varphi(0) = \pi/2$, $\dot{\varphi} = 0$), which has been discussed in Chap. 4.2.1. It is irrelevant whether the constraining force is due to a rigid rod, a guiding wire or any other arrangement. Only the geometry of the guiding curve is important.

- The last example demonstrates that the representation of the potential energy through the arc length can lead to computational difficulties, even for relatively simple geometric situations. The guiding curve is in this case the parabola $z = x^2$. Otherwise the situation is similar to the previous example. The representation of the coordinates, $x = q$, $z = q^2$, allows the calculation of the arc length

$$\begin{aligned} s &= \int_{q_0}^q dq' \sqrt{1 + 4q'^2} \\ &= \frac{1}{4} \left\{ 2q' \sqrt{1 + 4q'^2} + \ln \left[2q' + \sqrt{1 + 4q'^2} \right] \right\} \Big|_{q_0}^q. \end{aligned}$$

It can be recognised immediately, that inversion in the form $z = z(s)$ will meet with difficulties after the insertion of $q = \sqrt{z}$, respectively $q_0 = \sqrt{z_0}$, into this result.

The following statements can be extracted from these examples: the energy principle is sufficient for a discussion of the motion of a mass point under the influence of a conservative, openly acting force along a given spatial curve. However, the principle does not provide an optimal access to the discussion of such problems. It might be possible to circumvent some of the difficulties by a more flexible choice of the coordinates, as in the Lagrange equations of motion of the second kind. The principle of energy (one differential equation) is under no circumstances sufficient for the discussion of the motion of a mass point on a given surface (a two dimensional problem). A more general

access to problems of motion, in which constraining forces are incorporated explicitly, is required.

5.1.2 Lagrange I for one point particle

The formulation, that is known under the name Lagrange I, is an extension of Newton's equations of motion. It offers the following advantages:

- (1) All problems with constraints are treated in a consistent manner.
- (2) The constraining forces can in all cases be calculated explicitly.

The last point may, under given circumstances, be desirable. The knowledge of the stress on the material of technical devices (machines), in other words the constraining forces acting in the device, is without any doubt important for their construction. A drawback of the equations of motion in the form Lagrange I is, as indicated above, a certain lack of flexibility.

5.1.2.1 Motion on a surface. The equations of motion for the motion on a specified surface can be obtained with the following argument: A surface embedded into three dimensional space (Fig. 5.8) can be characterised by an implicit equation of the form $f(x, y, z) = 0$ (● Math.Chap. 4.1.1). Examples are

surface of a sphere: $x^2 + y^2 + z^2 - R^2 = 0$

plane: $z - ax - b = 0$.

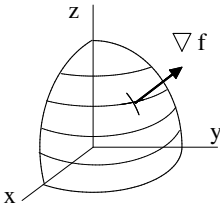


Fig. 5.8. Gradient vector on a surface

The total differential of the function $f = 0$ is

$$df = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0$$

or in vectorial form

$$df = \nabla f \cdot d\mathbf{r} = 0 \text{ .}$$

The infinitesimal displacement vector $d\mathbf{r}$ is tangential for each point of the surface. The equation expresses therefore the fact, that the gradient vector ∇f is always perpendicular to the surface $f = 0$. As the constraining forces have exactly the same property, the ansatz

$$\mathbf{Z} = \lambda(x, y, z) \nabla f \quad (5.7)$$

suggests itself. The factor of proportionality takes care of the variation of the strength of the constraining force with the position on the surface. This factor is not known initially. It has to be determined during the process of solving the problem of motion. This factor λ is called the **Lagrange multiplier**. The equations of motion with apparent and constraining forces can, according to the second axiom, be written in the form

$$\boxed{m\ddot{\mathbf{r}} = \mathbf{F} + \lambda \nabla f} \quad (5.8)$$

or in detail as

$$\begin{aligned} m\ddot{x} &= F_x + \lambda(x, y, z) \frac{\partial f}{\partial x} & m\ddot{y} &= F_y + \lambda(x, y, z) \frac{\partial f}{\partial y} \\ m\ddot{z} &= F_z + \lambda(x, y, z) \frac{\partial f}{\partial z} . \end{aligned} \quad (5.9)$$

These three equations do not suffice for the determination of the four unknown functions

$$x(t), y(t), z(t) \text{ and } \lambda(x(t), y(t), z(t)) .$$

A fourth equation is needed, which assures that the mass point moves on the surface. This is given by the equation of the surface, which has to be satisfied by the solution

$$\boxed{f(x(t), y(t), z(t)) = f(\mathbf{r}(t)) = 0 .} \quad (5.10)$$

The set of four (framed) equations for the four unknown functions are the **Lagrange equations of the first kind** (for the motion of a mass point on a given surface $f = 0$). These equations allow, in principle, the determination of the time development of the motion and of the strength of the constraining forces for each point of the trajectory.

5.1.2.2 Motion along space curves. The representation of a curve in space by the intersection of two surfaces

$$f_1(x, y, z) = 0 \quad f_2(x, y, z) = 0$$

is a useful tool for the discussion of the motion of a point particle on a space curve. Examples are:

- An ellipse with an arbitrary orientation in space can be represented by the intersection of a plane with a cylinder. An example is the ellipse described by the intersection of an inclined plane, that contains the y -axis

$$f_1 = z - x = 0 \quad \text{or} \quad z = x$$

with a cylinder about the z -axis (see Fig. 5.9a)

$$f_2 = x^2 + y^2 - R^2 = 0$$

(see ● D.tail 5.1).

- A slightly more exotic curve results from the intersection of an inclined surface resembling a sheet of corrugated iron

$$f_1 = z + x - \sin x = 0$$

with the x - z plane

$$f_2 = y = 0.$$

The result is a roller-coaster curve with a uniform average decline (Fig. 5.9b).

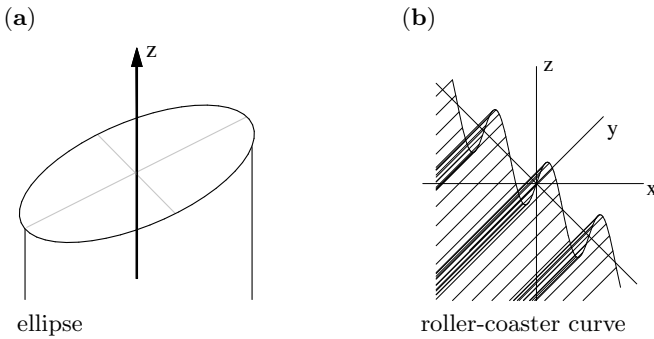


Fig. 5.9. Space curves as intersections of surfaces

The constraining forces are characterised by the postulate that they should be perpendicular to the curve for each point of the curve. A vector with these properties can be written as

$$\mathbf{Z} = \lambda_1(\mathbf{r}) \nabla f_1(\mathbf{r}) + \lambda_2(\mathbf{r}) \nabla f_2(\mathbf{r}). \quad (5.11)$$

The sum of the (independent) gradient vectors, each perpendicular to the corresponding surface and multiplied with a Lagrange multiplier, yields a vector with the desired properties.

For the determination of the five quantities $x(t)$, $y(t)$, $z(t)$, $\lambda_1(t)$, $\lambda_2(t)$ five equations are available

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}) + \lambda_1 \nabla f_1(\mathbf{r}) + \lambda_2 \nabla f_2(\mathbf{r})$$

as well as

$$f_1(\mathbf{r}) = 0 \text{ and } f_2(\mathbf{r}) = 0.$$

(5.12)

The three equations of motion for the components of the position vector and the equations representing the two surfaces are the Lagrange equations

of the first kind for the motion on a spatial curve. Energy conservation is valid for both the motion on a curve as well as on a surface

$$\frac{m}{2}v^2 + U(\mathbf{r}) = E_0$$

if the explicit forces \mathbf{F} are conservative. The constraining forces do not contribute to the energy balance for displacements along the curve or on the surface.

The central assumptions that were used for the formulation of the Lagrange equations of the first kind are:

- (a) The validity of the second axiom with a total force, which is the sum of the openly acting forces (also called **active forces**) and the constraining forces.
- (b) The constraining forces are, for all times, perpendicular to the specified curve or surface.

It is necessary to examine the validity of the second assumption by a comparison of theory and experiment. The formulation of the Lagrange equations of the first kind requires the use of assumptions or experiences which are not contained in Newton's axioms. The question has to be asked for this reason: is it possible to combine the axioms and the 'additional experience' in a more general principle of mechanics? This question will be answered with d'Alembert's principle (the principle of virtual work), but only in Chap. 5.2.1.

5.1.2.3 Time changing curves and surfaces. The constraints, which have been considered so far, have been generated by static curves or surfaces. The next question is obviously something like: how can the discussion of the motion of a point particle which moves under the influence of gravity (and without friction) along an oscillating (or otherwise moving) wire be handled? A surface or a curve, that changes with time, is characterised by the following set of equations

$$\begin{array}{ll} \text{surface :} & f(x, y, z, t) = 0 \\ \text{curve :} & f_1(x, y, z, t) = 0 \quad f_2(x, y, z, t) = 0 . \end{array} \quad (5.13)$$

Some explicit examples are:

- An moving inclined plane is described by the function

$$z - h(t) + (\tan \alpha)x = 0 .$$

The function $h(t) = v_0 t$ characterises e.g. a plane that moves uniformly up or down, the function $h(t) = a_0 t^2$ describes a plane that is uniformly accelerated or $h(t) = a_0 \cos \omega_0 t$ a plane that oscillates periodically.

- A sinusoidal wave front, which moves uniformly 'to the right' (the x -direction) can be represented by the function (Fig. 5.10a)

$$z - A \sin \left(\frac{2\pi}{L}x - \omega_0 t \right) = 0 .$$

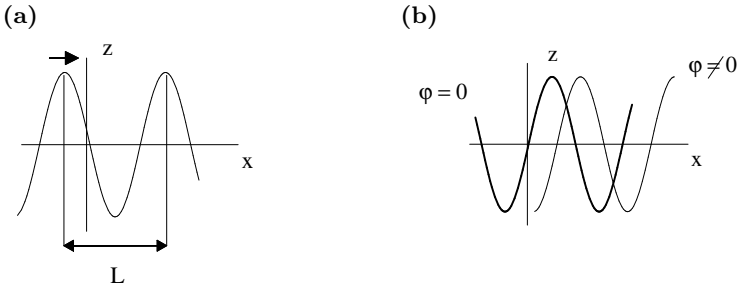


Fig. 5.10. Moving wave front

The function represents a stationary sinusoidal surface in three dimensional space for $\omega_0 = 0$. The periodicity of the surface is characterised by the wavelength L . The time dependent phase $\varphi = \omega_0 t$ describes the uniform displacement of the sinusoidal surface with respect to a reference surface with $\varphi = 0$ (Fig. 5.10b). The uniformly moving surface can e.g. serve as a simplified model of a water wave.

- The example indicated in Fig. 5.11 represents a straight piece of wire in the x - z plane, which rotates about the y -axis. This curve can be characterised by the intersection of the (time changing) surfaces

$$z - x \tan \alpha(t) = 0 \quad \text{and} \quad y = 0.$$

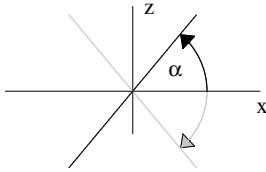


Fig. 5.11. Rotating straight piece of wire

The wire rotates uniformly about the y -axis for $\alpha(t) = \omega_0 t$, it oscillates with the frequency $f_0 = \omega_0/2\pi$ between the maximal angles $\pm\alpha_0$ for $\alpha(t) = \alpha_0 \sin \omega_0 t$.

5.1.2.4 Classification of constraints. All constraints (whether time dependent or not) can alternatively be specified in differential form. A surface $f(x, y, z, t) = 0$ (or the functions $f_1(x, y, z, t) = f_2(x, y, z, t) = 0$ in the case of a space curve) can e.g. be characterised by the total differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial t} dt = 0. \quad (5.14)$$

The constraint is termed **holonomic**, if the functions a_i in the general differential form

$$a_1(x, y, z, t)dx + a_2(x, y, z, t)dy + a_3(x, y, z, t)dz + a_0(x, y, z, t)dt = 0 \quad (5.15)$$

are, as in (5.14), the partial derivatives of a function $f(x, y, z, t)$ with respect to the appropriate coordinates. Holonomic signifies 'whole' or 'complete'. The terminology refers to the statement, that the differential form represents a total (that is complete) differential, from which the function f could be obtained by line integration.

The theorem of Schwarz (● Math.Chap. 4.2.2) for a function $f(x, y, z, t)$, which can be differentiated twice, can be used in order to formulate the definition of a holonomic constraint as:

A constraint of the form

$$a_1(x, y, z, t)dx + a_2(\dots)dy + a_3(\dots)dz + a_0(\dots)dt = 0$$

is called **holonomic**, if the conditions

$$\frac{\partial a_1}{\partial y} = \frac{\partial a_2}{\partial x}, \quad \dots \quad \frac{\partial a_0}{\partial x} = \frac{\partial a_1}{\partial t}, \quad \dots$$

are satisfied.

The two possibilities, which have been addressed so far, are a static or a moving surface (or curve) with

$$a_0 = \frac{\partial f}{\partial t} = 0 \quad \text{or} \quad a_0 = \frac{\partial f}{\partial t} \neq 0.$$

They are referred to as **scleronomic** (rigid), respectively **rheonomic** (flowing).

A suitable ansatz for the constraining forces in the case of *rheonomic* constraints has to be formulated now. It is necessary to rely once more on experience (if available) for this purpose. Experience tells us, that the constraining forces are perpendicular to the surfaces or curves, even if the constraints are rheonomic. This means, that the constraining forces can still be represented as

$$\mathbf{Z} = \lambda \nabla f(x, y, z, t) \quad (\text{surface}) \quad (5.16)$$

$$\mathbf{Z} = \lambda_1 \nabla f_1(x, y, z, t) + \lambda_2 \nabla f_2(x, y, z, t) \quad (\text{curve}). \quad (5.17)$$

The dependence of the Lagrange multipliers on the position *and* the time has been suppressed. The Lagrange equations have the same form for rheonomic-holonomic and scleronomic-holonomic constraints. On the other hand, the time dependence implies, that the constraining forces do work on a point particle, which is subjected to rheonomic-holonomic constraints (in other words: which moves on a time changing curve/surface).

The statement concerning the work follows from the argument: consider a point particle, for which the position on a surface $f(t)$ at time t is marked by the vector $\mathbf{r}(t)$. During the interval dt the point particle moves on the surface, which itself is in motion. The position of the particle on the surface $f(t+dt)$ at the infinitesimally neighbouring time $t+dt$ can be described by a vector $\mathbf{r} + d\mathbf{r}$ (Fig. 5.12). The constraint itself corresponds to the statement

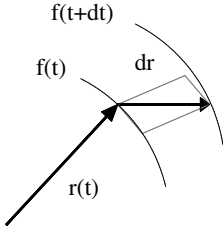


Fig. 5.12. Illustration of the situation for rheonomic constraints

$$\nabla f \cdot d\mathbf{r} = -\frac{\partial f}{\partial t} dt. \quad (5.18)$$

This equation indicates that the infinitesimal work of constraining forces (proportional to $\nabla f \cdot d\mathbf{r}$) in the presence of rheonomic constraints is not equal to zero (see right hand side).

Constraints of the form (5.15), which do not correspond to a total differential, are termed **nonholonomic**

$$a_1(x, y, z, t)dx + a_2(x, y, z, t)dy + a_3(x, y, z, t)dz + a_0(x, y, z, t)dt = 0$$

with

$$\frac{\partial a_1}{\partial y} \neq \frac{\partial a_2}{\partial x} \quad \frac{\partial a_1}{\partial z} \neq \frac{\partial a_3}{\partial y} \quad \dots$$

It is sufficient that one of the inequalities is satisfied. In this case the following nomenclature is used:

1. The constraint is termed nonholonomic-rheonomic, if $a_0 \neq 0$ and at least one of the coefficients is not a partial derivative of a function $f(x, y, z, t)$ with respect to the appropriate coordinate.
2. The constraint is termed nonholonomic-scleronomic, if at least one of the coefficients is not a partial derivative of a function $f(x, y, z, t)$ and if $a_0 = 0$.

Nonholonomic constraints arise, if coordinates and velocities are combined in some fashion. The condition

$$\frac{dx}{dt} - g(x, y, z) = 0$$

can, for instance, be expressed in the nonholonomic form

$$dx - g(x, y, z)dt = 0 .$$

More explicit verbal examples are

- the condition for the rolling of a wheel (see ● Probl. 5.1),
- the condition for guiding the runner of ice skates.

A last variant for the specification of constraints are inequalities. The constraint for the motion of a mathematical pendulum supported by a string (rather than a rod) is

$$x^2 + y^2 + z^2 \leq l^2 .$$

The pendulum can collapse, it can, however, not be further away from the point of suspension than the length of the string l , provided the string cannot be stretched or tears. Such conditions are called **one-sided constraints**. They cannot be incorporated in a simple manner into the equations of motion.

The difference between scleronomic and rheonomic constraints (in the form of a one-sided constraint) can be observed in the game of tennis. The ball will be reflected, but does not change its kinetic energy (neglecting side effects like the generation of noise), if the racket is just placed into the trajectory of the ball (scleronomic). Work is performed and the kinetic energy of the reflected ball is increased, if the ball is hit in the usual fashion (rheonomic constraints).

5.1.2.5 Examples for the solution of the equations of motion (Lagrange I). The pattern for the solution of problems of motion with explicit constraints can be demonstrated very well for the example of the free fall on an inclined plane. This example is a simple holonomic-scleronomic problem that can also be discussed in an elementary fashion. The results are therefore not (essentially) different from the treatment of the problem on p. 188 ff. The plane is characterised by the equation

$$f = z + (\tan \alpha)x = 0$$

(see Fig. 5.4 for a cut through the plane), so that the gradient of f is

$$\nabla f = (\tan \alpha, 0, 1) . \quad (5.19)$$

The equations of motion are therefore

$$\left. \begin{array}{l} m\ddot{x} = \lambda' \tan \alpha \\ m\ddot{y} = 0 \\ m\ddot{z} = -mg + \lambda' \end{array} \right\} \longrightarrow \left. \begin{array}{l} \ddot{x} = \lambda \tan \alpha \\ \ddot{y} = 0 \\ \ddot{z} = -g + \lambda \end{array} \right\} \quad (5.20)$$

It is useful to incorporate the mass into the definition of the multiplier: $\lambda'/m = \lambda$.

The first step towards the solution of this system of equations of motion is: differentiate the equation for the plane twice with respect to time

$$\ddot{z} + \tan \alpha \ddot{x} = 0$$

and insert this relation into the last of the equations (5.20)

$$-(\tan \alpha)\ddot{x} = -g + \lambda .$$

This has to be resolved with respect to λ , which is then inserted into the first of the equations (5.20)

$$\ddot{x} = -(\tan^2 \alpha)\ddot{x} + g \tan \alpha .$$

This expression can be sorted using

$$1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha}$$

to obtain a differential equation for the x - coordinate

$$\ddot{x} = g \sin \alpha \cos \alpha .$$

The general solution is

$$x(t) = x(0) + v_x(0)t + \frac{1}{2}g(\sin \alpha \cos \alpha)t^2 .$$

The solution of the differential equation (5.20) for the y - coordinate is simple

$$y(t) = y(0) + v_y(0)t .$$

The point particle can have a specified position and (constant) velocity in the y - direction. The equation of the plane $z(t) = -(\tan \alpha)x(t)$ should be used for the determination of the function $z(t)$. This gives

$$z(t) = -(\tan \alpha)x(0) - (\tan \alpha)v_x(0)t - \frac{1}{2}g(\sin^2 \alpha)t^2 .$$

No independent initial conditions can be specified for the z - component. The constraint assures that the mass point is located on and moves with the plane.

The Lagrange multiplier can e.g. be obtained from the last of the equations (5.20)

$$\lambda = g + \ddot{z} = g(1 - \sin^2 \alpha) = g \cos^2 \alpha ,$$

so that the constraining force is given by

$$\mathbf{Z} = m\lambda \nabla f = mg(\cos \alpha \sin \alpha, 0, \cos^2 \alpha) .$$

This corresponds exactly to the decomposition of the vector

$$\mathbf{F}_\perp = mg \cos \alpha \mathbf{e}_\perp$$

into Cartesian components.

A problem with rheonomic-holonomic constraints is the free fall on a moving inclined plane, which can be specified by

$$f = z - h(t) + (\tan \alpha)x = 0 \quad \nabla f = (\tan \alpha, 0, 1) .$$

(A more exhaustive discussion is called for in ● Probl. 5.2). The function $h(t)$ describes the motion of the plane in the z - direction. The apparent force is

once more simple gravitation. The equations of motion are therefore initially the same as in the corresponding scleronomic problem

$$\ddot{x} = \lambda \tan \alpha \quad \ddot{y} = 0 \quad \ddot{z} = -g + \lambda .$$

Differentiation of the constraint results in

$$\ddot{z} = \ddot{h} - \ddot{x} \tan \alpha ,$$

so that

$$\lambda = g + \ddot{h} - \ddot{x} \tan \alpha \quad \text{and} \quad \ddot{x} = (g + \ddot{h}) \cos \alpha \sin \alpha$$

follows. The general solution of the differential equation for x is naturally different from the scleronomic counterpart

$$x(t) = C_1 + C_2 t + \left(\frac{g}{2} t^2 + h(t) \right) \cos \alpha \sin \alpha . \quad (5.21)$$

This result is used to calculate the multiplier

$$\lambda = (g + \ddot{h}) \cos^2 \alpha$$

and the z -coordinate

$$z(t) = -(C_1 + C_2 t) \tan \alpha - \frac{g}{2} t^2 \sin^2 \alpha + h(t) \cos^2 \alpha . \quad (5.22)$$

A special solution, e.g. with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ of the x -coordinate, is

$$\begin{aligned} x(t) &= \left[(h(t) - h(0)) + \left(\frac{g}{2} t^2 - \dot{h}(0)t \right) \right] \sin \alpha \cos \alpha \\ z(t) &= - \left[(h(t) - h(0)) + \left(\frac{g}{2} t^2 - \dot{h}(0)t \right) \right] \sin^2 \alpha + h(t) . \end{aligned}$$

The motion in the two coordinate directions is composed of a free fall motion on a static plane (the term depending on g) plus a 'push' due to the motion of the plane. The initial values for the z -coordinate are, as they must, correlated with the initial conditions for the function $h(t)$

$$z(0) = h(0) \quad \dot{z}(0) = \dot{h}(0) .$$

The energy situation can be discussed as follows: the constraining force is given by

$$\mathbf{Z} = m(g + \ddot{h}(t))(\sin \alpha \cos \alpha, 0, \cos^2 \alpha) ,$$

so that the total force acting on the mass is

$$\mathbf{F} + \mathbf{Z} = -m g \mathbf{e}_z + \mathbf{Z} .$$

The total force is simple, if the plane is horizontal ($\alpha = 0$). It has only a z -component in this case, which depends on the second derivative of the function $h(t)$.

Integration of the equation of motion (line integration) along the trajectory between starting and end point

$$\int_i^f m \ddot{\mathbf{r}} \cdot d\mathbf{r} = \int_i^f \mathbf{F} \cdot d\mathbf{r} + \int_i^f \mathbf{Z} \cdot d\mathbf{r}$$

leads to

$$\left(\frac{m}{2} v_f^2 + U_f \right) - \left(\frac{m}{2} v_i^2 + U_i \right) = \Delta A .$$

The potential energy difference of the mass due to the gravitational field of the earth is

$$U_i - U_f = - \int_i^f \mathbf{F} \cdot d\mathbf{r} .$$

ΔA is the work supplied to the mass by the constraining force. The scalar product of \mathbf{Z} and $d\mathbf{r}$ does not vanish for the moving plane.

The kinetic energy, evaluated for the special solution of the equations of motion, is found to be

$$T(t) = \frac{m}{2} (\dot{x}^2 + \dot{z}^2) = \frac{m}{2} \left[(\dot{h}(0) - gt)^2 \sin^2 \alpha + \dot{h}^2(t) \cos^2 \alpha \right] ,$$

the corresponding potential energy in the gravitational field of the earth

$$U(t) = mgz = mg \left\{ - \left[(h(t) - h(0)) + \left(\frac{g}{2} t^2 - \dot{h}(0)t \right) \right] \sin^2 \alpha + h(t) \right\} .$$

These relations yield for the total energy of the mass at the time t

$$E(t) = (T(t) + U(t)) = \frac{m}{2} \left\{ \left[(\dot{h}(t)^2 - \dot{h}(0)^2) + 2g(h(t) - h(0)) \right] \cos^2 \alpha + (2gh(0) + \dot{h}(0)^2) \right\} .$$

The energy, which the moving plane transfers in the interval $[0, t]$ to the mass, is therefore

$$\Delta A(t) = E(t) - E(0) = \frac{m}{2} \left[(\dot{h}(t)^2 - \dot{h}(0)^2) + 2g(h(t) - h(0)) \right] \cos^2 \alpha .$$

This result can also be obtained by evaluation of the (in general path dependent) line integral with the constraining force, for which the time can be used as a parameter

$$\Delta A = \int_i^f (Z_x dx + Z_z dz) = \int_0^t (Z_x \dot{x} + Z_z \dot{z}) dt' .$$

The total energy $E = T + U$ changes in the case of a rheonomic constraint as the mass is lifted or lowered by the action of the constraining force. For the scleronomic case ($h(t)$, $\dot{h}(t) = 0$) energy is conserved

$$E(t) = E(0) = 0 .$$

The last example will demonstrate explicitly that computational difficulties can prevent an analytical solution of the Lagrangian equations of the

first kind. The example is again the motion along a section of a parabola, or alternatively in a parabolic tub, which is described by (Fig. 5.13)

$$f = z - x^2 = 0. \quad (5.23)$$

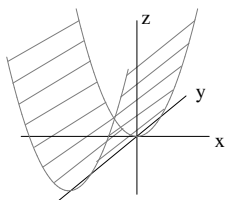


Fig. 5.13. Motion in a parabolic tub

This surface (a quadratic instead of a linear form) is not really complicated. The gradient of the function is

$$\nabla f = (-2x, 0, 1)$$

so that the Lagrange equations of motion are found to be

$$\ddot{x} = -2\lambda x \quad \ddot{y} = 0 \quad \ddot{z} = \lambda - g \quad \lambda = \lambda'/m. \quad (5.24)$$

The first step is again the elimination of the unknown multiplier. The equation of the surface (5.23) is differentiated with respect to time for this purpose

$$\dot{z} = 2x\dot{x} \quad \ddot{z} = 2\dot{x}^2 + 2x\ddot{x}$$

and \dot{x} in the second equation is replaced using the first of these equations

$$\ddot{z} - \frac{\dot{z}^2}{2z} = 2x\ddot{x}. \quad (5.25)$$

In the next step the equation for the x -component (5.24) is multiplied by $2x$

$$2x\ddot{x} = -4\lambda x^2 = -4\lambda z. \quad (5.26)$$

A comparison of (5.25) and (5.26) yields the relation

$$\ddot{z} - \frac{\dot{z}^2}{2z} = -4\lambda z. \quad (5.27)$$

In the final step the equation (5.24) for the z -component is multiplied by $-4z$

$$-4z\ddot{z} = -4\lambda z + 4gz$$

and λ is eliminated with the aid of (5.27)

$$(4z + 1)\ddot{z} - \frac{\dot{z}^2}{2z} + 4gz = 0. \quad (5.28)$$

This differential equation of second order for $z(t)$ cannot be solved analytically (in a simple fashion). Numerical methods or a power series expansion

(see ● Math.Chap. 6.3 and 6.4) are required. The application of the Lagrange equations of the first kind to the problem of motion in a parabolic tub presents the same difficulties, that were found for the problem of motion along a section of parabolic wire (which has been discussed via the energy principle on p. 191). The Lagrange multiplier $\lambda = \ddot{z} + g$ and the time development of the x -coordinate $x = \pm\sqrt{z}$ can only be determined, once a solution of the differential equation (5.28) is obtained with whatever means¹.

The situation indicated is symptomatic. As soon as the constraints are more complicated, the equations of motion with (unknown) constraining forces cannot be solved analytically for many situations. This is sufficient motivation to look for a more flexible formulation, the Lagrange equations of motion of the second kind. Two additional topics have to be addressed beforehand: the discussion of d'Alembert's principle and the formulation of the Lagrange equations of the first kind for systems of point particles.

5.2 D'Alembert's principle

The direct approach to the formulation of the Lagrange equations of the first kind for one point particle, that has been outlined in the previous sections, cannot be generalised in a simple fashion to the discussion of *systems* of point particles. In order to address such systems, it is useful to consider an alternative, though somewhat unusual approach, which is summarised in d'Alembert's principle. This principle is first formulated for the case of one mass point.

5.2.1 D'Alembert's principle for one mass point

The argument begins by reconsidering Newton's equations of motion without constraints

$$m\ddot{\mathbf{r}} - \mathbf{F} = \mathbf{0} . \quad (5.29)$$

The quantity $m\ddot{\mathbf{r}}$ (a force) is referred to as the **inertia resistance** for this purpose and (5.29) is interpreted in the following fashion: the inertia resistance and the acting forces are in an equilibrium.

5.2.1.1 The virtual displacement and virtual work. The advantage of the interpretation suggested is at first quite minimal. A dynamical problem has been formally reduced to a static problem: two forces, one real and one fictive, are in equilibrium. This interpretation will, however, turn out to be useful.

¹ The reader is invited to check the claim that consideration of the differential equation for the x -coordinate does not save the situation.

The next step is the definition of the concept of a **virtual displacement**. This is a possible, but only imagined infinitesimal displacement of the mass point. This displacement is characterised by the symbols

$$\delta \mathbf{s} = (\delta x, \delta y, \delta z) \quad \text{with} \quad \delta t = 0. \quad (5.30)$$

It differs from a real displacement

$$d\mathbf{s} = (dx, dy, dz) \quad \text{with} \quad dt \neq 0.$$

A real displacement always takes some time. The virtual displacement is instantaneous. The speed with which a virtual displacement is executed is infinite. Even if this definition sounds slightly absurd, its usefulness will soon become apparent. On formal grounds nothing can be said against this definition. Any concept can be defined, even instantaneous displacements.

The equilibrium condition (5.29) can be reexpressed with the aid of this concept in the form of a statement of the work involved

$$\delta A = (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \delta \mathbf{s} = 0. \quad (5.31)$$

In words: the work done on the system in formal equilibrium by a virtual displacement is equal to zero. This statement constitutes d'Alembert's principle, which is also termed the **principle of virtual work**, in its simplest (though not yet useful) form. In the case of one point particle without constraints it just reformulates the second axiom, in the sense, that the principle follows from the axiom

$$\text{equations of motion} \quad \longrightarrow \quad \text{d'Alembert's principle}.$$

The inverse statement

$$\text{d'Alembert's principle} \quad \longrightarrow \quad \text{equations of motion}$$

follows with the argument: The virtual displacements in the three directions $(\delta x, \delta y, \delta z)$ can be chosen arbitrarily if no constraints are present. It follows, that the virtual work can only vanish, if the equations of motion are valid

$$m\ddot{\mathbf{r}} - \mathbf{F} = \mathbf{0}.$$

The relations (5.29) and (5.31) are completely equivalent.

5.2.1.2 Formulation of the principle of d'Alembert. The statements gain in substance if a situation with constraints is considered. The virtual displacements $\delta x, \delta y, \delta z$ can *not* be independent of each other if the mass point is supposed to move along a curve or on a surface. The displacements have to be adapted to the constraints.

The constraint for the motion on a surface, a holonomic constraint, is in differential form

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = 0.$$

While this condition is true for a real displacement, only the relation

$$\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z = 0$$

is admitted for a virtual displacement as $\delta t = 0$. The statement (5.31) concerning the virtual work is valid for scleronomic as well as holonomic conditions. The purpose of the peculiar definition of a virtual displacement can be recognised at this stage. The definition reflects the fact, that constraining forces are perpendicular to the constraining surface in both scleronomic as well as rheonomic situations, in a concise fashion. This can be expressed by

$$\nabla f(x, y, z, t) \cdot \delta \mathbf{s} = 0. \quad (5.32)$$

The three components of the virtual displacement depend on each other in a definite way according to the surface f specified. The virtual constraint (5.32) can be combined with the relation (5.31) defining the virtual work in two different ways².

- (1) Represent one of the displacements through the other displacements. The virtual displacement in the x -direction can e.g. be expressed in terms of the others displacements if the partial derivative with respect to x does not vanish

$$\frac{\partial f}{\partial x} \equiv f_{[x]} \neq 0 \implies \delta x = - \left(\frac{f_{[y]}}{f_{[x]}} \right) \delta y - \left(\frac{f_{[z]}}{f_{[x]}} \right) \delta z.$$

The expression for δx can be inserted into equation (5.31)

$$\begin{aligned} & \left[- (m\ddot{x} - F_x) \frac{f_{[y]}}{f_{[x]}} + (m\ddot{y} - F_y) \right] \delta y + \\ & \left[- (m\ddot{x} - F_x) \frac{f_{[z]}}{f_{[x]}} + (m\ddot{z} - F_z) \right] \delta z = 0. \end{aligned}$$

The two remaining virtual displacements are independent. The last relation can therefore only be valid if

$$\begin{aligned} & \left[- (m\ddot{x} - F_x) \frac{f_{[y]}}{f_{[x]}} + (m\ddot{y} - F_y) \right] = 0 \\ & \left[- (m\ddot{x} - F_x) \frac{f_{[z]}}{f_{[x]}} + (m\ddot{z} - F_z) \right] = 0. \end{aligned}$$

These equations constitute the equations of motion for the y - and z -components, if the variable x is eliminated with the aid of the condition $f(x, y, z, t) = 0$.

- (2) The result of the second argumentation is completely equivalent. It leads directly to the Lagrange equations of the first kind for the situation considered. Equation (5.32) is multiplied by $-\lambda$ (the Lagrangian multiplier) and added to (5.31) with the result

² Partial derivatives will be written in shorthand as $f_{[x]}$, in order to distinguish them from the components of vectors as in F_x .

$$\begin{aligned} [(m\ddot{x} - F_x - \lambda f_{[x]}) \delta x + (m\ddot{y} - F_y - \lambda f_{[y]}) \delta y + \\ (m\ddot{z} - F_z - \lambda f_{[z]}) \delta z] = 0 . \end{aligned}$$

The quantity λ , which can be chosen freely, is then adjusted so that one of the terms in round brackets, e.g.

$$m\ddot{x} - F_x - \lambda f_{[x]} = 0$$

vanishes. There remains

$$[(m\ddot{y} - F_y - \lambda f_{[y]}) \delta y + (m\ddot{z} - F_z - \lambda f_{[z]}) \delta z] = 0 .$$

As two of the three virtual displacements can be chosen freely, the accompanying factors have to vanish. The final result is the same (eliminate λ from the equations of motion for the y - and z - coordinates with the equation of motion for the z - coordinate) as the one that has been obtained on the basis of a direct geometrical argument in Chap. 5.1

$$m\ddot{\mathbf{r}} = \mathbf{F} + \lambda \nabla f . \quad (5.33)$$

5.2.1.3 Comparison of Lagrange I and d'Alembert. A comparison of the discussion of d'Alembert's principle and of the Lagrange equations of the first kind (for the case of holonomic constraints) might help the understanding.

- (i) In order to set up the Lagrange equation the argument (which can be verified experimentally) is used: the constraining force is at all times perpendicular to the given surface f . This corresponds to the ansatz

$$\mathbf{Z} = \lambda \nabla f ,$$

because the strength of the constraining force is not known. As a consequence, the second axiom with a total force, which is composed of an active and a constraining force, leads to the equation of motion

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{Z} .$$

- (ii) The system is tested with virtual displacements. The starting point is d'Alembert's principle of virtual work in the form

$$\delta A = (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \delta \mathbf{s} = 0 .$$

In words: the virtual work of the inertial resistance and the of active forces is equal. For the case of a free point particle, this statement is identical with the second axiom

$$\delta A = 0 \quad \longleftrightarrow \quad m\ddot{\mathbf{r}} = \mathbf{F} .$$

For the case of a point particle with a holonomic constraint a second statement has to be added: the virtual displacement should take place on the given surface. Therefore the displacements in the three space directions are related by the condition

$$\nabla f \cdot \delta \mathbf{s} = 0 .$$

Combination of the two statements yields

$$m\ddot{\mathbf{r}} = \mathbf{F} + \lambda \nabla f .$$

The two arguments lead to the same result. This means that they are equivalent. The difference consists in the fashion in which the experience concerning the (direction of the) constraining force is introduced. In the first case a rather direct ansatz is used, in the second the experience concerning constraining forces is expressed in the definition of the virtual displacement in the form

$$\lambda \nabla f \cdot \delta \mathbf{s} = \mathbf{Z} \cdot \delta \mathbf{s} = 0 .$$

The virtual work of the constraining forces vanishes (always). By contrast the real work dA vanishes only in the case of (holonomic) scleronomic constraints.

The second point of view is more flexible. It can also be applied to the case of nonholonomic constraints (as long as it is not a one-sided constraint) and it can be extended to the discussion of systems of point particles. The argumentation with d'Alembert's principle is formal, but it achieves its goal more directly.

5.2.1.4 Variants and extensions. Two additional remarks are necessary to round off the discussion for the case of one point particle:

1. There exists an alternative (though slightly antiquated) formulation of d'Alembert's principle. Three 'forces' have been considered, which are connected by the vectorial equation

$$\mathbf{F} - m\ddot{\mathbf{r}} = -\mathbf{Z} .$$

The negative of the constraining force is that part of the active force, which does not lead to motion. This part is termed the **lost force**. The alternative formulation of the principle states therefore

$$\mathbf{Z} \cdot \delta \mathbf{s} = (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \delta \mathbf{s} = 0 \quad (5.34)$$

or in verbal form

A point particle moves in such a fashion, that the virtual work of the lost force vanishes for all times.

2. The Lagrange equations for the motion along a space curve can be derived from d'Alembert's principle with similar arguments as in the case of motion on a surface. Two constraints have to be considered

$$\nabla f_1 \cdot \delta \mathbf{s} = 0 \quad \text{and} \quad \nabla f_2 \cdot \delta \mathbf{s} = 0$$

besides the principle of virtual work

$$(m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \delta \mathbf{s} = 0 .$$

Only one of the virtual displacements can be chosen freely. Each of the constraints is multiplied with a suitable multiplier and these statements are subtracted from the principle of virtual work. The explicit result is

$$(m\ddot{x} - F_x - \lambda_1 f_{1x} - \lambda_2 f_{2x}) \delta x + (m\ddot{y} - F_y - \lambda_1 f_{1y} - \lambda_2 f_{2y}) \delta y \\ + (m\ddot{z} - F_z - \lambda_1 f_{1z} - \lambda_2 f_{2z}) \delta z = 0 .$$

Three quantities (two multipliers, and one displacement) can be chosen freely. If the multipliers are chosen, so that two of the expressions in brackets (say the first two) vanish, then the free choice of the displacement δz leads to three equations of motion which can be summarised in vector form as

$$m\ddot{\mathbf{r}} - \mathbf{F} - \lambda_1 \nabla f_1 - \lambda_2 \nabla f_2 = 0 . \quad (5.35)$$

With the concepts of virtual displacement and of virtual work the equations of motion gained by the geometric argument are recovered.

5.2.2 D'Alembert's principle for systems of point particles

D'Alembert's principle for systems of point particles constitutes a basis, on which classical mechanics can be founded. It will be formulated here in a compact notation. From this principle both the Lagrange equations of the first as well as of the second kind can be derived. Two additional examples are added to illustrate the solution of simple problems of motion with the aid of the Lagrange equations of the first kind.

5.2.2.1 Formulation. If all the constraining forces in a system of N point particles were known, one would write with Newton

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{Z}_i \quad (i = 1, 2, \dots, N) .$$

Internal as well as external forces are included in the active force \mathbf{F}_i on the i -th point particle of the system. The virtual work of the lost force for each of the point particles can be calculated as

$$\delta A_i = (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{s}_i = 0 \quad (i = 1 \dots N)$$

if the virtual displacement of each of the point particles in the system is defined by

$$\delta \mathbf{s}_i = (\delta x_i, \delta y_i, \delta z_i) .$$

As the work is a scalar quantity, the individual contributions can be added to yield the total virtual work

$$\sum_{i=1}^N (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{s}_i = 0 . \quad (5.36)$$

It is useful for the discussion, that follows, to adjust the notation by a consecutive numbering of the coordinates and the masses using the correspondence

$$\begin{array}{ccccccc}
 x_1 & y_1 & z_1 & x_2 & y_2 & z_2 & \dots & x_N & y_N & z_N \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & x_{3N-2} & x_{3N-1} & x_{3N} \\
 \\
 m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & \dots & m_N & m_N & m_N \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
 m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & & m_{3N-2} & m_{3N-1} & m_{3N}
 \end{array}$$

and the same pattern for the components of the force

$$\begin{array}{ccccccc}
 F_{1x} & F_{1y} & F_{1z} & F_{2x} & F_{2y} & F_{2z} & \dots & F_{Nx} & F_{Ny} & F_{Nz} \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
 F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & & F_{3N-2} & F_{3N-1} & F_{3N}
 \end{array}$$

D'Alembert's principle for a system of mass points can then be written as

$$\boxed{\sum_{i=1}^{3N} (m_i \ddot{x}_i - F_i) \delta x_i = 0} \quad (5.37)$$

Individual particles in the system can be bound on surfaces or curves, or distances between masses can have fixed values. For example, two point particles could have a constant separation l

$$N = 2 \longrightarrow (x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2 - l^2 = 0.$$

Assume that there exist r constraining conditions for the system of point particles, which are (except one-sided constraints) specified explicitly by

$$\begin{aligned}
 & a_{k,1} (x_1 \dots x_{3N}, t) dx_1 + a_{k,2} (x_1 \dots x_{3N}, t) dx_2 + \\
 & \dots + a_{k,3N} (x_1 \dots x_{3N}, t) dx_{3N} + a_{k,0} (x_1 \dots x_{3N}, t) dt = 0 \\
 & (k = 1, 2, \dots, r).
 \end{aligned}$$

The following cases can be distinguished as before:

$a_{k,i}$	$a_{k,0}$	
$\frac{\partial f_k(x_1, \dots, x_{3N}, t)}{\partial x_i}$	0	holonomic – scleronomic
$\frac{\partial f_k(x_1, \dots, x_{3N})}{\partial x_i}$	$\frac{\partial f_k(x_1, \dots, x_{3N}, t)}{\partial t}$	holonomic – rheonomic
arbitrary	0	nonholonomic – scleronomic
arbitrary	arbitrary	nonholonomic – rheonomic

The virtual displacements δx_i (all are instantaneous) are restricted by the corresponding conditions

$$\sum_{i=1}^{3N} a_{k,i} (x_1 \dots x_{3N}, t) \delta x_i = 0 \quad (k = 1, 2, \dots, r). \quad (5.38)$$

The argument for the derivation of the equations of motion is exactly the same as has been used before: multiply each of the constraints with a suitable Lagrange multiplier and add them to d'Alembert's principle

$$\sum_{i=1}^{3N} \{m_i \ddot{x}_i - F_i - \lambda_1 a_{1,i} - \lambda_2 a_{2,i} \dots \lambda_r a_{r,i}\} \delta x_i = 0.$$

Choose r multipliers so that r of the expressions in brackets vanish. The remaining $3N - r$ displacements can be chosen freely, so that the remaining expressions in brackets have to vanish as well. The result are the Lagrange equations of the first kind for a system of N point particles, for instance for r holonomic constraints

$$m_i \ddot{x}_i = F_i + \sum_{k=1}^r \lambda_k \frac{\partial f_k}{\partial x_i} \quad (i = 1, 2, \dots, 3N)$$

$$f_k (x_1 \dots x_{3N}, t) = 0 \quad (k = 1, 2, \dots, r). \quad (5.39)$$

The corresponding equations are

$$m_i \ddot{x}_i = F_i + \sum_{k=1}^r \lambda_k a_{k,i} (x_1 \dots x_{3N}, t) \quad (i = 1, 2, \dots, 3N) \quad (5.40)$$

$$\sum_{i=1}^{3N} a_{k,i} (x_1 \dots x_{3N}, t) \frac{dx_i}{dt} = -a_{k,0} (x_1 \dots x_{3N}, t) \quad (k = 1, 2, \dots, r),$$

if the constraints are specified by r nonholonomic differential forms. Situations with a mixed set of constraints (holonomic *and* nonholonomic) could also be considered.

The problem posed in each case is: find the $3N$ functions $x_i(t)$ and the r Lagrange multipliers λ_k . The components of the constraining forces

$$Z_i = \sum_{k=1}^r \lambda_k (x_1 \dots x_{3N}, t) a_{k,i} (x_1 \dots x_{3N}, t) \quad (i = 1, \dots, 3N)$$

can be calculated once the system of equations of motion has been solved (in general no easy task).

D'Alembert's principle, the principle of virtual work

$$\sum_{i=1}^{3N} (m_i \ddot{x}_i - F_i) \delta x_i = 0,$$

and the explicit incorporation of the constraints lead to the Lagrange equations of the first kind. It is, however, possible to use the same information in a different way. The Lagrange equations of the second kind are obtained if the constraints are eliminated by a suitable choice of coordinates. The number of equations, that have to be discussed, is reduced. In contrast to Lagrange I there are only $3N - r$ (instead of $3N + r$) equations to be considered in the case of Lagrange II.

5.2.2.2 Applications. As the Lagrange equations of the second kind are in general more easily applied, only two additional examples for the solution of problems of motion in terms of Lagrange I will be presented.

Atwood's machine can be described in the following fashion: Two masses m_1 and m_2 are connected by a strong wire that runs over a pulley (Fig. 5.14). The mass of the pulley and of the wire is neglected (not a necessity). The

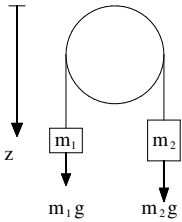


Fig. 5.14. Atwood's machine

system of two masses starts to move under the influence of gravity. The question to be answered is: how does it move?

It is sufficient to consider one coordinate, say the z -coordinate, which is oriented in the direction of the gravitational force. The holonomic constraint, which expresses the fact that the wire cannot be stretched, is

$$z_1 + z_2 = \text{const.}$$

This constraint leads to the Lagrange equations for the two masses

$$m_1 \ddot{z}_1 = m_1 g + \lambda \quad m_2 \ddot{z}_2 = m_2 g + \lambda .$$

The multiplier can easily be eliminated

$$m_1 \ddot{z}_1 - m_1 g = m_2 \ddot{z}_2 - m_2 g .$$

As the constraint is equivalent to $\ddot{z}_2 = -\ddot{z}_1$, this equation can be rewritten as

$$\ddot{z}_1 = \frac{m_1 - m_2}{m_1 + m_2} g .$$

The mass m_1 moves with uniform acceleration (e.g. downward, provided $m_1 > m_2$). The acceleration is reduced with respect to g . An explicit example with $m_1 = 16$ $m_2 = 14$ yields $\ddot{z}_1 = g/15$ and illustrates the purpose

of the machine. The accelerated motion is slowed down and hence easier to observe. The constraining force is the same for the two masses. It has the value

$$Z_1 = Z_2 = \lambda = \left\{ \begin{array}{l} m_1 \ddot{z}_1 - m_1 g \\ m_2 \ddot{z}_2 - m_2 g \end{array} \right\} = -\frac{2m_1 m_2}{(m_1 + m_2)} g .$$

This force is equal for both masses and directed upwards. Another example of a free-fall apparatus is addressed in ● Probl. 5.3.

The next example illustrates the application of d'Alembert's principle to static problems. A system of N stationary mass points is characterised by the *static* principle of virtual work

$$\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{s}_i = 0 . \quad (5.41)$$

This relation allows the derivation of the principle of the **lever** (compare Chap. 3.2.2). The lever is a planar system of two masses m_1 and m_2 with a fixed separation l_1 and l_2 from a common axis of rotation (Fig. 5.15). In

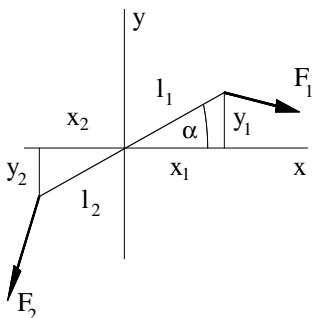


Fig. 5.15. The lever: coordinates

applying the static principle of virtual work the constraints

$$x_1^2 + y_1^2 = l_1^2 \quad \text{and} \quad x_2^2 + y_2^2 = l_2^2$$

have to be incorporated. An additional constraint is due to the fact, that the two masses are rotated by the same angle

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{1}{\tan \alpha} .$$

The three conditions can be expressed in virtual form (5.38)

$$x_1 \delta x_1 + y_1 \delta y_1 = 0 \quad x_2 \delta x_2 + y_2 \delta y_2 = 0$$

$$y_2 \delta x_1 - x_2 \delta y_1 - y_1 \delta x_2 + x_1 \delta y_2 = 0 .$$

The stationary Lagrange equations (5.39) for the lever with the (planar) active forces \mathbf{F}_1 and \mathbf{F}_2 are therefore

$$\begin{aligned}
F_{1,x} + \lambda_1 x_1 &+ \lambda_3 y_2 = 0 \\
F_{1,y} + \lambda_1 y_1 &- \lambda_3 x_2 = 0 \\
F_{2,x} &+ \lambda_2 x_2 - \lambda_3 y_1 = 0 \\
F_{2,y} &+ \lambda_2 y_2 + \lambda_3 x_1 = 0 .
\end{aligned}$$

Elimination of λ_1 from the first two and of λ_2 from the last two equations yields

$$\begin{aligned}
F_{1,x}y_1 - F_{1,y}x_1 + \lambda_3(y_2y_1 + x_2x_1) &= 0 \\
F_{2,x}y_2 - F_{2,y}x_2 - \lambda_3(y_2y_1 + x_2x_1) &= 0 .
\end{aligned}$$

By adding these two equations the lever condition

$$\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = \mathbf{0} ,$$

which has been discussed before (3.46), can be extracted. The sum of the moments of the forces applied vanishes.

5.3 The Lagrange equations of the second kind (Lagrange II)

The application of the Lagrange equations of the first kind, requires the solution of $3N + r$ equations for a system of N point particles with r constraints. The increased number of equations is the price that has to be paid for the explicit inclusion of the constraining forces. The goal of the formulation of the Lagrange equations of the second kind, is:

1. The constraining forces should not appear explicitly in the equations of motion.
2. The number of equations, that have to be solved, should correspond to the number of the remaining degrees of freedom, that is $3N - r$.

The simplest case of Lagrange equations of the second kind describes the motion of one point particle, which therefore offers itself for the introduction of this topic.

5.3.1 Lagrange II for one point particle

The method, on which Lagrange II is based, is the introduction of generalised coordinates. This method has already been used, be it in a small way. In Chap. 4 the motion of the mathematical pendulum or the Kepler problem have not been discussed in terms of Cartesian but rather in terms of polar coordinates. The question, that will be addressed now, is: how can an optimal set of coordinates for a given problem of motion (with or without constraints) be chosen?

5.3.1.1 Generalised coordinates. A problem of motion can be formulated in terms of any suitable set of coordinates $q_1(t)$, $q_2(t)$, $q_3(t)$ instead of just Cartesian coordinates $x(t)$, $y(t)$, $z(t)$. A general, explicit form for transformations between the two sets of coordinates is

$$\begin{aligned}x(t) &= x(q_1(t), q_2(t), q_3(t), t) \\y(t) &= y(q_1(t), q_2(t), q_3(t), t) \\z(t) &= z(q_1(t), q_2(t), q_3(t), t)\end{aligned}\tag{5.42}$$

or in abbreviation

$$x_i = x_i(q_1, q_2, q_3, t) \quad (i = 1, 2, 3) .$$

The possibility that the transformation depends explicitly on time is envisaged from the beginning. It will also be assumed that the inverse transformation exists

$$q_\mu = q_\mu(x_1, x_2, x_3, t) \quad (\mu = 1, 2, 3) .\tag{5.43}$$

Two examples may serve as an illustration of the possibilities.

- The coordinate transformation

$$x_1 = q_1 \sin q_2 \cos q_3 \quad x_2 = q_1 \sin q_2 \sin q_3 \quad x_3 = q_1 \cos q_2$$

was used in equation (2.74), p. 62. The generalised coordinates, that are addressed here, are the spherical coordinates

$$q_1 \rightarrow r \quad q_2 \rightarrow \theta \quad q_3 \rightarrow \phi .$$

- The motion on a plane, which rotates uniformly with the angular velocity ω about the y -axis (Fig. 5.16), can be formulated in terms of the following generalised coordinates

$$\begin{aligned}q_1 &= z - x \tan \omega t && \text{equation of the rotating plane} \\q_2 &= \sqrt{x^2 + z^2} && \text{distance from the } y\text{-axis} \\q_3 &= y && y\text{-coordinate} .\end{aligned}\tag{5.44}$$

The coordinates state that the point particle is moving on the rotating plane (q_1) and that it has the distance q_2 from the q_3 -axis. The inverse

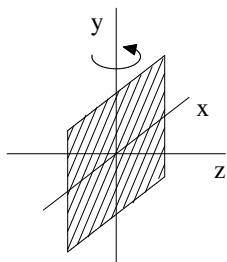


Fig. 5.16. Plane, rotating uniformly about the y -axis

transformation looks reasonably complicated. For instance the x -coordinate is given by

$$x = \cos \omega t \left(-q_1 \sin \omega t + \sqrt{q_2^2 - q_1^2 \cos^2 \omega t} \right).$$

The set of generalised coordinates is nonetheless very useful as the generalised coordinate q_1 expresses the constraint. This implies

$$q_1 = 0 \quad \text{and therefore also} \quad \dot{q}_1 = \ddot{q}_1 = 0.$$

The coordinate q_1 can be ignored in the discussion that follows. It is an **ignorable coordinate**. The inverse transformation turns out to be quite simple, if $q_1 = 0$ is used

$$x = q_2 \cos \omega t \quad y = q_3 \quad z = q_2 \sin \omega t.$$

The three Cartesian coordinates are represented (due to the constraint) by two generalised coordinates (and the time development specified). The number of generalised coordinates corresponds to the number of degrees of freedom of the problem, the motion on a specified (though rotating) plane.

5.3.1.2 From d'Alembert's principle to equations of motion in terms of generalised coordinates. The derivation of a set of equations of motion in terms of generalised coordinates will be based on d'Alembert's principle (Chap. 5.2) and the transformation between Cartesian and generalised coordinates. A number of preparatory steps are required for the actual derivation. They are all based on the application of the rules for partial differentiation.

1. The total derivative of the Cartesian coordinates with respect to time can, with the aid of the chain rule, be expressed in terms of the derivatives of the generalised coordinates with time

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{\mu=1}^3 \frac{\partial x_i}{\partial q_{\mu}} \dot{q}_{\mu} + \frac{\partial x_i}{\partial t}. \quad (5.45)$$

The derivative of a generalised coordinate with respect to time, \dot{q}_{μ} , is named a **generalised velocity**, or more precisely a **component of the generalised velocity**.

2. The kinetic energy

$$T = \frac{m}{2} \sum_{i=1}^3 \dot{x}_i^2$$

can be differentiated directly with respect to the generalised coordinates and the generalised velocities

$$\frac{\partial T}{\partial q_\nu} = m \sum_{i=1}^3 \frac{\partial \dot{x}_i}{\partial q_\nu} \dot{x}_i \quad (5.46)$$

$$\frac{\partial T}{\partial \dot{q}_\nu} = m \sum_{i=1}^3 \frac{\partial \dot{x}_i}{\partial \dot{q}_\nu} \dot{x}_i . \quad (5.47)$$

3. The relation between the partial derivatives

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_\nu} = \frac{\partial x_i}{\partial q_\nu} \quad (5.48)$$

can be extracted from (5.45). Insertion of (5.48) into (5.47) leads to

$$\frac{\partial T}{\partial \dot{q}_\nu} = m \sum_{i=1}^3 \frac{\partial x_i}{\partial q_\nu} \dot{x}_i . \quad (5.49)$$

4. The total derivative of the expression (5.49) with respect to time has to be calculated next

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) = m \sum_{i=1}^3 \left\{ \dot{x}_i \frac{\partial x_i}{\partial q_\nu} + \dot{x}_i \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_\nu} \right) \right\} . \quad (5.50)$$

5. The order of the differentiation can be interchanged in the last term on the right hand side of (5.50), if the equations representing the original transformation is twice continuously differentiable. This assertion can be demonstrated with the steps

$$\frac{d}{dt} \left(\frac{\partial x_i}{\partial q_\nu} \right) = \sum_{\mu=1}^3 \frac{\partial^2 x_i}{\partial q_\mu \partial q_\nu} \dot{q}_\mu + \frac{\partial^2 x_i}{\partial t \partial q_\nu} .$$

Interchange of the sequence of the partial differentiation according to the assumption gives

$$\frac{d}{dt} \left(\frac{\partial x_i}{\partial q_\nu} \right) = \frac{\partial}{\partial q_\nu} \left\{ \sum_{\mu=1}^3 \frac{\partial x_i}{\partial q_\mu} \dot{q}_\mu + \frac{\partial x_i}{\partial t} \right\} .$$

The expression in the brackets is just \dot{x}_i (see (5.45)), so that

$$= \frac{\partial \dot{x}_i}{\partial q_\nu}$$

follows.

6. The second term in (5.50) can be written with this result according to (5.46) as

$$m \sum_{i=1}^3 \dot{x}_i \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_\nu} \right) = m \sum_{i=1}^3 \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_\nu} = \frac{\partial T}{\partial q_\nu} . \quad (5.51)$$

The final goal of this preparatory discussion can, after sorting the result of (5.50), be given in the form

$$m \sum_{i=1}^3 \ddot{x}_i \frac{\partial x_i}{\partial q_\nu} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \frac{\partial T}{\partial q_\nu} . \quad (5.52)$$

D'Alembert's principle can be cast into a suitable form with the aid of (5.52). The starting point is the Cartesian expression (5.37)

$$\sum_{i=1}^3 (m\ddot{x}_i - F_i) \delta x_i = 0 . \quad (5.53)$$

A relation between the virtual displacements in the Cartesian and the generalised coordinates is needed. The total differential of the transformation (5.42)

$$dx_i = \sum_{\mu=1}^3 \frac{\partial x_i}{\partial q_\mu} dq_\mu + \frac{\partial x_i}{\partial t} dt$$

can be used for this purpose. A replacement of the real (d) displacements by the virtual (δ) displacements gives

$$\delta x_i = \sum_{\mu=1}^3 \frac{\partial x_i}{\partial q_\mu} \delta q_\mu , \quad (5.54)$$

as $\delta t = 0$ by definition. D'Alembert's principle is, in this fashion, transcribed into

$$\sum_{\mu} \left\{ \sum_i \left(m\ddot{x}_i \frac{\partial x_i}{\partial q_\mu} - F_i \frac{\partial x_i}{\partial q_\mu} \right) \right\} \delta q_\mu = 0 . \quad (5.55)$$

At this point the purpose of the preparative steps 1 to 6 should become apparent. The first term in (5.55) can be expressed by the derivatives of the kinetic energy, using (5.52). The second term corresponds to a transformation of the Cartesian components of the force F_i to **generalised force components** Q_μ

$$Q_\mu = \sum_{i=1}^3 F_i \frac{\partial x_i}{\partial q_\mu} \quad (\mu = 1, 2, 3) . \quad (5.56)$$

The final result of the argument is

$$\sum_{\mu=1}^3 \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu \right\} \delta q_\mu = 0 \quad (5.57)$$

if the definition of Q_μ is used to abbreviate matters and if the expression (5.52) is inserted for the first term. Three different cases can be distinguished:

1. The point particle moves freely. There are no constraints. In this case the three virtual displacements δq_μ are independent. D'Alembert's principle can only be satisfied if each of the expressions in curly brackets vanishes

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu = 0 \quad (\mu = 1, 2, 3) . \quad (5.58)$$

2. There exists one constraint. The simplest case is a holonomic constraint, $f(x, y, z, t) = 0$, so that one of the generalised coordinates can be chosen to be

$$q_1 = f(x, y, z, t) = 0 .$$

The corresponding virtual displacement vanishes ($\delta q_1 = 0$), while the remaining displacements δq_2 and δq_3 can be chosen freely. This leads to the equations of motion

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu = 0 \quad (\mu = 2, 3) . \quad (5.59)$$

3. A suitable choice of the generalised coordinates for a situation with two holonomic constraints is

$$q_1 = f_1(x_1, x_2, x_3, t) = 0 \quad q_2 = f_2(x_1, x_2, x_3, t) = 0 .$$

Only one equation of motion remains, which depends on the coordinate q_3 .

The equations of motion

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu = 0 \quad (\mu = 1, \dots)$$

(5.60)

are the **Lagrange equations of the second kind for the motion of one point particle** with, respectively without, constraints. The goal stated in the beginning has been reached: Constraining forces do not appear explicitly. The number of equations corresponds to the number of degrees of freedom. These equations, a basic set of equations of mechanics, represent according to their genesis a variant of d'Alembert's principle which is oriented towards practical applications.

5.3.1.3 Solution of the equations of motion (Lagrange II). The discussion of these equations will be initiated with two illustrative examples. In order to apply the Lagrange equations (5.60) a canonical set of standard steps is usually followed.

- Begin with the choice of generalised coordinates

$$x_i = x_i(q_1, q_2, q_3, t) , \quad (i = 1, 2, 3)$$

and calculate the time derivatives

$$\dot{x}_i = \sum_{\mu} \frac{\partial x_i}{\partial q_\mu} \dot{q}_\mu + \frac{\partial x_i}{\partial t} = v_i(q_1, \dots, \dot{q}_1, \dots, t) .$$

The right hand side of these equations contains only expressions that depend on the generalised coordinates q_μ (possibly in part ignorable), on the generalised velocities \dot{q}_μ and on the time.

- The expression for the Cartesian components of the velocity are inserted into the expression for the kinetic energy

$$T = \frac{m}{2} \sum_{i=1}^3 \dot{x}_i^2.$$

The result is an expression for the kinetic energy of the form

$$T = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t).$$

- The derivatives of T with respect to the generalised coordinates and the generalised velocities

$$\frac{\partial T}{\partial q_\mu}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right)$$

are required for the actual assembly of the equations of motion. In addition, the components of the generalised force Q_μ have to be calculated according to Eq. (5.56).

- The set of equations obtained with these steps represent a set of differential equations for the nonignorable functions $q_\mu(t)$. The time dependence of the Cartesian coordinates $x_i(t)$ can, if desired, be calculated with the transformation (5.42) after the solution of these equations.

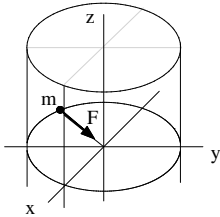
5.3.1.4 First examples for the solution. The first example for the illustration of this scheme is the following problem: calculate the motion of a point particle m , which moves on the surface of a cylinder

$$x^2 + y^2 - R^2 = 0$$

under the influence of a harmonic central force

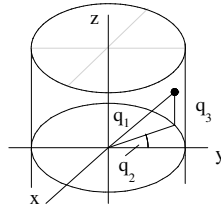
$$\mathbf{F} = -k\mathbf{r} = -k(x, y, z).$$

(a)



the force vector

(b)



generalised coordinates

Fig. 5.17. Point particle m on the surface of a cylinder under the influence of a harmonic central force

The force vector points at each instant of time in the direction of the origin of the coordinate system (see Fig. 5.17a).

The first step is the choice of the generalised coordinates. There is some freedom, but it is best to proceed according to the rule: the more appropriate the choice, the simpler are the differential equations which will finally have to be solved. The most adept choice uses, in most of the cases, the symmetry of the problem and (if given) the constraints. The choice is therefore

$$q_1 = x^2 + y^2 - R^2 = 0 \quad (\text{the constraint})$$

and because of the cylinder symmetry (Fig. 5.17b)

$$\begin{aligned} q_2 &= \arctan \frac{y}{x} & (\text{angle}) \\ q_3 &= z & (\text{coordinate, height above the } x\text{-}y \text{ plane}) \end{aligned} \quad (5.61)$$

in the present example. The inverse transformation is

$$x = R \cos q_2 \quad y = R \sin q_2 \quad z = q_3 \quad (5.62)$$

because of $q_1 = 0$. The time derivatives of the Cartesian coordinates are

$$\dot{x} = -R \dot{q}_2 \sin q_2 \quad \dot{y} = R \dot{q}_2 \cos q_2 \quad \dot{z} = \dot{q}_3. \quad (5.63)$$

They are used to find the kinetic energy in terms of generalised coordinates

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(R^2 \dot{q}_2^2 + \dot{q}_3^2).$$

Next, the required (partial) derivatives for the kinetic energy are calculated

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial q_3} = 0$$

and

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_2} = mR^2 \dot{q}_2 &\longrightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) = mR^2 \ddot{q}_2 \\ \frac{\partial T}{\partial \dot{q}_3} = m\dot{q}_3 &\longrightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_3} \right) = m\ddot{q}_3. \end{aligned}$$

The calculation of the generalised force uses the relations

$$\begin{aligned} \frac{\partial x}{\partial q_2} &= -R \sin q_2 & \frac{\partial y}{\partial q_2} &= R \cos q_2 & \frac{\partial z}{\partial q_2} &= 0 \\ \frac{\partial x}{\partial q_3} &= 0 & \frac{\partial y}{\partial q_3} &= 0 & \frac{\partial z}{\partial q_3} &= 1. \end{aligned}$$

The component Q_1 vanishes, as the transformation (5.61) does not contain the coordinate q_1 . The other components are

$$\begin{aligned} Q_2 &= F_x \frac{\partial x}{\partial q_2} + F_y \frac{\partial y}{\partial q_2} + F_z \frac{\partial z}{\partial q_2} \\ &= -k\{(R \cos q_2)(-R \sin q_2) + (R \sin q_2)(R \cos q_2) + 0\} \end{aligned}$$

$$= 0$$

$$\begin{aligned} Q_3 &= F_x \frac{\partial x}{\partial q_3} + F_y \frac{\partial y}{\partial q_3} + F_z \frac{\partial z}{\partial q_3} \\ &= -kq_3 . \end{aligned}$$

The Lagrange equations of motion for the problem posed are

$$\begin{aligned} m\ddot{q}_2 &= 0 \\ m\ddot{q}_3 + kq_3 &= 0 . \end{aligned}$$

They are astonishingly simple. The general solution can be given directly

$$\begin{aligned} q_2(t) &= C_1 + C_2 t \\ q_3(t) &= C_3 \cos \omega_0 t + C_4 \sin \omega_0 t \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}} . \end{aligned} \quad (5.64)$$

The quantity q_2 is the azimuthal angle, the quantity q_3 is the z -coordinate of a point on the surface of the cylinder. The angular motion is uniform, the motion in the z -direction is a harmonic oscillation. The superposition of these two forms of motion corresponds to a trigonometric curve (the oscillation) which is spread on the surface of the cylinder (the uniform rotation) (Fig. 5.18).

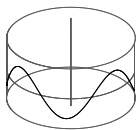


Fig. 5.18. Motion on the surface of a cylinder: solution

The motion can be studied more closely with a choice of a special set of initial conditions, as e.g.

$$\begin{aligned} x(0) &= R & y(0) &= 0 & z(0) &= h \\ v_x(0) &= 0 & v_y(0) &= v_0 & v_z(0) &= 0 . \end{aligned}$$

The initial position is (naturally) on the cylinder, with a height h above the x -axis. The initial velocity is a vector in the y -direction. The transformation (5.61) gives the initial values of the generalised coordinates as

$$q_2(0) = 0 \quad q_3(0) = h .$$

and the initial generalised velocities

$$\dot{q}_2(0) = v_0/R \quad \dot{q}_3(0) = 0 .$$

The special solution, which follows after the determination of the constants of integration with these specifications, are

$$q_2(t) = \frac{v_0}{R} t \quad q_3(t) = h \cos \omega_0 t . \quad (5.65)$$

The corresponding form of the solution in Cartesian coordinates is

$$x(t) = R \cos\left(\frac{v_0}{R}t\right) \quad y(t) = R \sin\left(\frac{v_0}{R}t\right) \quad z(t) = q_3(t). \quad (5.66)$$

A cosine curve is spread out on the surface of the cylinder. As the angular frequency of the rotation $\omega = v_0/R$ (corresponding to a time $T = 2\pi R/v_0$ for one rotation) and the angular frequency of the oscillation $\omega_0 = \sqrt{k/m}$ are not necessarily matched, the cosine curve on the cylinder does not close in general. Some special cases are:

The solution (5.65) represents a uniform circular motion of the mass in the x - y plane for $h = 0$. The solution describes a harmonic oscillation in the z -direction about the point $(x, y, z) = (R, 0, h)$ for $v_0 = 0$.

The next example is a problem which could not be solved easily via the Lagrange equations of the first kind. The task is the calculation of the motion of a point particle m under the influence of gravity $\mathbf{F} = (0, 0, -mg)$. The particle moves on a helix which is lifted or lowered uniformly in the z -direction. A stationary helix (Fig. 5.19) is characterised by the equations

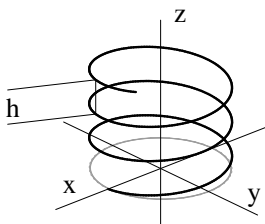


Fig. 5.19. The helix

$$x^2 + y^2 - R^2 = 0 \quad (5.67)$$

$$z = \frac{h}{2\pi}\phi = \frac{h}{2\pi} \arctan \frac{y}{x} \quad (5.68)$$

(see Chap. 2, p. 36). The first equation (5.67) describes a cylindrical surface, the second (5.68) the helix on this surface. The angle ϕ is increased by 2π after a full rotation. The z -coordinate changes by the pitch of the screw h for each turn. The second equation has to be replaced by

$$z = \frac{h}{2\pi} \left(\arctan \left(\frac{y}{x} \right) \pm \omega t \right) \quad \omega = \frac{2\pi}{T} \quad (5.69)$$

for a uniformly rotating helix. The positive sign corresponds to a helix, which is lifted, the negative sign to a helix, which is lowered.

An obvious choice of the ignorable coordinates is

$$q_1 = x^2 + y^2 - R^2 = 0 \quad (5.70)$$

$$q_2 = z - \frac{h}{2\pi} \left(\arctan \left(\frac{y}{x} \right) \pm \omega t \right) = 0. \quad (5.71)$$

These coordinates will not appear in the equations of motion. For the characterisation of the actual motion only the coordinate

$$q_3 = z \quad (5.72)$$

plays a role. In order to obtain the inverse of this transformation between the Cartesian and the generalised coordinates, the relation (5.71) for the coordinate q_2 has to be resolved in terms of the angle $\phi(t)$

$$\phi(t) = \arctan \frac{y(t)}{x(t)} = \frac{2\pi}{h} q_3(t) \mp \omega t. \quad (5.73)$$

This expression is then inserted into the relations between cylinder and Cartesian coordinates ((2.66), p. 61)

$$x(t) = R \cos \phi(t) \quad y(t) = R \sin \phi(t) \quad z(t) = q_3(t). \quad (5.74)$$

For a problem with two constraints only one generalised coordinate has to be considered. One of the holonomic constraints is rheonomic, so that the equation for the transformation depend explicitly on time. In order to express the kinetic energy as a function of the generalised coordinates and velocities the time derivatives

$$\dot{x} = -R\dot{\phi} \sin \phi \quad \dot{y} = R\dot{\phi} \cos \phi \quad \dot{z} = \dot{q}_3$$

have to be used. The kinetic energy is transformed into

$$T = \frac{m}{2} \left[R^2 \left(\frac{2\pi}{h} \dot{q}_3 \mp \omega \right)^2 + \dot{q}_3^2 \right] \quad (5.75)$$

because of

$$\dot{\phi} = \frac{2\pi}{h} \dot{q}_3(t) \mp \omega$$

The only component of the generalised force, which does not vanish in this example, is

$$Q_3 = F_z \frac{\partial z}{\partial q_3} = -mg, \quad \text{as} \quad F_x = F_y = 0.$$

The derivatives

$$\frac{\partial T}{\partial q_3} = 0 \quad \frac{\partial T}{\partial \dot{q}_3} = m \left[\frac{2\pi R^2}{h} \left(\frac{2\pi}{h} \dot{q}_3 \mp \omega \right) + \dot{q}_3 \right]$$

and

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_3} \right) = m \ddot{q}_3 \left(\frac{4\pi^2 R^2}{h^2} + 1 \right)$$

are needed in order to set up the equation of motion

$$m\ddot{q}_3 \left(\frac{4\pi^2 R^2}{h^2} + 1 \right) + mg = 0. \quad (5.76)$$

A simple rearrangement gives

$$\ddot{q}_3 = -\frac{gh^2}{4\pi^2 R^2 + h^2} = -g_{\text{eff}}.$$

The motion on a uniformly rotating helix under the influence of gravity is a uniformly accelerated falling motion. The acceleration is reduced with respect to the free fall. The reduction is larger for a larger radius. The reduction is decreased with increasing pitch. The effective acceleration can also be expressed in the form

$$g_{\text{eff}} = \frac{g}{\left(1 + \frac{4\pi^2 R^2}{h^2}\right)}.$$

The angular velocity, which characterises the uniform rotation, does not occur in the equation of motion. The time dependent constraining forces do not do any work while the mass point moves on a uniformly rotating helix (compare the problem of a uniformly moving inclined plane in Chap. 5.1, p. 201 for the case $h(t) = v_0 t$).

The general solution of the equation of motion (5.76) is

$$q_3(t) = C_1 + C_2 t - \frac{1}{2} g_{\text{eff}} t^2.$$

Initial conditions for the generalised coordinates, as e.g.

$$q_3(0) = 0 \quad \dot{q}_3(0) = 0,$$

correspond to the initial conditions

$$\begin{aligned} x(0) &= R & y(0) &= 0 & z(0) &= 0 \\ \dot{x}(0) &= 0 & \dot{y}(0) &= \mp R\omega & \dot{z}(0) &= 0 \end{aligned}$$

for the Cartesian coordinates. The mass point starts on the helix and moves with it. The initial conditions require $C_1 = C_2 = 0$, so that the explicit solution in Cartesian coordinates turns out to be

$$\begin{aligned} x(t) &= +R \cos \left[\frac{\pi h g}{4\pi^2 R^2 + h^2} t^2 \pm \omega t \right] \\ y(t) &= -R \sin \left[\frac{\pi h g}{4\pi^2 R^2 + h^2} t^2 \pm \omega t \right] \\ z(t) &= -\frac{1}{2} \left[\frac{gh^2}{4\pi^2 R^2 + h^2} \right] t^2. \end{aligned} \quad (5.77)$$

The fact, that this solution is in agreement with the original choice of the generalised coordinates, that is $q_1(t) = q_2(t) = 0$, can be verified directly.

5.3.1.5 Constraining forces and Lagrange II. The constraining forces do not feature in the Lagrange equations of the second kind. These forces can, however, be calculated with a direct procedure after the solution of these equations has been obtained: set up the problem according to the Lagrange equations of the first kind, insert the solution found with the aid of Lagrange II, determine the Lagrange multipliers and hence the constraining forces.

A calculation of the constraining forces for the two examples just discussed provides an example for the application of this procedure.

- The constraint of the first problem (see p. 220) $x^2 + y^2 - R^2 = 0$ leads with (5.33) to the (Lagrange I) equations of motion

$$m\ddot{x} = -kx + 2\lambda x \quad m\ddot{y} = -ky + 2\lambda y \quad m\ddot{z} = -kz. \quad (5.78)$$

The first equation can be resolved in the form

$$\lambda = \frac{1}{2} \left(\frac{m\ddot{x}}{x} + k \right).$$

Insertion of the solution (5.65) with the initial conditions specified above

$$x = R \cos \left(\frac{v_0 t}{R} \right) \quad y = R \sin \left(\frac{v_0 t}{R} \right) \quad z = h \cos \omega_0 t,$$

yields the result

$$\lambda = \frac{1}{2} \left(k - m \left(\frac{v_0}{R} \right)^2 \right)$$

for the multiplier λ . The second equation of motion is satisfied automatically. The constraining force is

$$\mathbf{Z} = \left(\left(k - m \frac{v_0^2}{R^2} \right) R \cos \left(\frac{v_0 t}{R} \right), \left(k - m \frac{v_0^2}{R^2} \right) R \sin \left(\frac{v_0 t}{R} \right), 0 \right).$$

The constraining force changes periodically with the angular coordinate $q_2(t)$. It is directed towards the interior or the exterior of the cylinder depending on the relative magnitude of the force constant and the velocity (in relation to the mass and the radius).

- The constraints

$$x^2 + y^2 - R^2 = 0 \quad \text{and} \quad z - \frac{h}{2\pi} (\arctan \frac{y}{x} \pm \omega t) = 0$$

of the second example (p. 223) and the ansatz

$$\mathbf{Z} = \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2$$

for the constraining forces lead to equations of motion (5.35) of the form

$$\begin{aligned} m\ddot{x} &= 2\lambda_1 x + \frac{h}{2\pi R^2} y \lambda_2 & m\ddot{y} &= 2\lambda_1 y - \frac{h}{2\pi R^2} x \lambda_2 \\ m\ddot{z} &= -mg + \lambda_2. \end{aligned} \quad (5.79)$$

The solution of the problem of motion is given in (5.77)

$$x = R \cos \phi \quad y = R \sin \phi \quad z = -\frac{1}{2} g_{\text{eff}} t^2$$

with

$$\phi = -\left(\frac{\pi}{h} g_{\text{eff}} t^2 \pm \omega t\right) .$$

The Lagrange multiplier λ_2 can be extracted from the third equation of motion in a direct fashion

$$\lambda_2 = m(g - g_{\text{eff}}) = \frac{4\pi^2 R^2}{4\pi R^2 + h^2} m g = \frac{4\pi^2 R^2}{h^2} m g_{\text{eff}} .$$

Any of the first two equations of motion in (5.79) then yields for λ_1

$$\lambda_1 = -\frac{m}{2} \dot{\phi}^2 = -\frac{m}{2} \left(\frac{2\pi}{h} g_{\text{eff}} t \pm \omega \right)^2 .$$

5.3.1.6 Conservative systems. A very useful reformulation of the Lagrange equations of the second kind can be given provided the constraints are holonomic and the acting forces are conservative. The Cartesian components of the active force can be represented in this case as a gradient of the potential energy U

$$\mathbf{F} = -\nabla U(x, y, z) .$$

The components of the generalised force can, using the chain rule for partial differentiation, be written in the form

$$Q_\mu = \sum_i F_i \frac{\partial x_i}{\partial q_\mu} = - \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\mu} = - \frac{\partial U}{\partial q_\mu} \quad (\mu = 1, \dots) . \quad (5.80)$$

This equation shows that the components of the conservative, generalised force can be calculated by first expressing the potential energy in terms of the generalised coordinates

$$U(x_1, x_2, x_3) \longrightarrow U(q_1, q_2, q_3, t)$$

and then calculating the force components as generalised gradients.

The potential energy of the first example (p. 220) discussed in the previous sections is

$$U = \frac{k}{2} (x^2 + y^2 + z^2) .$$

With the generalised coordinates (5.62)

$$x = R \cos q_2 \quad y = R \sin q_2 \quad z = q_3$$

this is transformed into

$$U = \frac{k}{2} (R^2 + q_3^2) .$$

This yields, as before, the components of the generalised force

$$Q_1 = Q_2 = 0 \quad Q_3 = -kq_3 .$$

The relation between the components of the generalised force (5.80) and the potential energy can also be employed if the potential energy depends explicitly on time after the transformation to generalised coordinates. A variant of the present example is the calculation of the motion of the point particle on a 'pulsating' cylinder (Fig. 5.20). The constraint is in this case

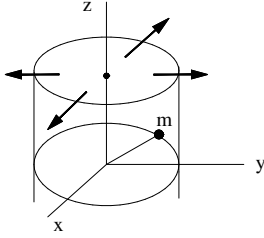


Fig. 5.20. Pulsating cylinder

$$x^2 + y^2 - R(t)^2 = 0 ,$$

an example for the specification of the pulsation could be

$$R(t) = R_0 + a \sin bt .$$

The potential energy has the same form as before

$$U = \frac{k}{2}(R(t)^2 + q_3^2) ,$$

so that, in spite of the time dependence of the potential energy, the general force comprises again only one component

$$\mathbf{Q} = (Q_1, Q_2, Q_3) = (0, 0, -kq_3) .$$

The time dependence of the transformation leads, however, via the kinetic energy, to a modified set of equations of motion. The time derivatives of the transformations are

$$\dot{x} = \dot{R} \cos q_2 - R\dot{q}_2 \sin q_2 \quad \dot{y} = \dot{R} \sin q_2 + R\dot{q}_2 \cos q_2 \quad \dot{z} = \dot{q}_3$$

in the case of the pulsating cylinder instead of (5.63). This gives

$$T = \frac{m}{2}(\dot{R}^2 + R^2\dot{q}_2^2 + \dot{q}_3^2)$$

and

$$\frac{\partial T}{\partial \dot{q}_2} = mR^2\dot{q}_2 \quad \text{and} \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) = m(2R\dot{R}\dot{q}_2 + R^2\ddot{q}_2) .$$

The differential equations for the motion on the pulsating cylinder are

$$\ddot{q}_2 + 2\frac{\dot{R}}{R}\dot{q}_2 = 0 \quad \ddot{q}_3 + \frac{k}{m}q_3 = 0 .$$

The equations of motion for the two generalised coordinates are still uncoupled. The angular motion is, however, not uniform. A kind of 'frictional term' appeared due to the time dependence of the constraint. The differential equation for q_2 can be solved with standard methods (● Probl. 5.7, see also ● Math.Chap. 6.3)

$$\begin{aligned} q_2(t) &= C_1 + C_2 \int^t dt_1 \exp \left[-2 \int^{t_1} \frac{\dot{R}(t_2)}{R(t_2)} dt_2 \right] \\ &= C_1 + C_2 \int^t dt_1 \frac{1}{R(t_1)^2} . \end{aligned}$$

If the radius of the cylinder does not change with time ($\dot{R} = 0$), the solution of the original problem (p. 220) is recovered

$$q_2(t) = C_1 + \frac{C_2}{R^2}t = C_1 + C_2' t .$$

5.3.1.7 The Lagrange function. A standard version of the Lagrange equation is obtained if the components of the generalised force can be represented in terms of a potential function. For this purpose a **Lagrange function** (also termed **Lagrangian**) is defined

$$L = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t) - U(q_1, q_2, q_3, t) . \quad (5.81)$$

This quantity has the dimension of an energy, but is *not* identical with the energy of the systems. The quantity, which is more closely associated with the energy, is the Hamiltonian (or Hamilton function, see (5.102) and Chap. 5.4). The Lagrangian is nonetheless a basic element of theoretical physics. It plays a prominent role from mechanics to the formulation of quantum field theories.

The following derivatives of the Lagrange function (5.81) are of interest: the potential energy depends only on the generalised coordinates and the time, so that the derivative with respect to the generalised velocities vanishes

$$\frac{\partial L}{\partial \dot{q}_\mu} = \frac{\partial T}{\partial \dot{q}_\mu} \quad \frac{\partial U}{\partial \dot{q}_\mu} = 0 .$$

The derivatives with respect to the generalised coordinates are

$$\frac{\partial L}{\partial q_\mu} = \frac{\partial T}{\partial q_\mu} - \frac{\partial U}{\partial q_\mu} = \frac{\partial T}{\partial q_\mu} + Q_\mu .$$

The original version (5.60) of the Lagrange equations of the second kind

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu = 0 \quad (\mu = 1, \dots)$$

can be recast in the 'standard form'

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) - \frac{\partial L}{\partial q_\mu} = 0 \quad (\mu = 1, \dots)} \quad (5.82)$$

if these relations are used. The original form of the equations of motion (5.60) is valid for holonomic constraints and arbitrary forces. The second form (5.82) can only be applied for holonomic constraints and conservative forces.

5.3.1.8 Generalised potentials. It is possible to define a Lagrange function under certain conditions, even if the forces are not conservative. The corresponding potential function is referred to as a generalised potential. One example, that may serve as an illustration of this point, is a situation, in which, besides conservative contributions to \mathbf{F} , dissipative parts, as e.g. a velocity dependent force with components $F_i^{\text{dis}} = -\kappa_i \dot{x}_i$, are present. The dissipative part of the generalised forces, calculated according to the definition, is

$$Q_\mu^{\text{dis}} = \sum_{i=1}^3 F_i^{\text{dis}} \frac{\partial x_i}{\partial q_\mu} = - \sum_{i=1}^3 \kappa_i \dot{x}_i \frac{\partial x_i}{\partial q_\mu}$$

in this case. The relation (5.48)

$$\frac{\partial x_i}{\partial q_\mu} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\mu},$$

which follows from the transformation (5.42), can be used here to write

$$Q_\mu^{\text{dis}} = - \sum_{i=1}^3 \kappa_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_\mu},$$

or by sorting the partial derivative

$$Q_\mu^{\text{dis}} = - \frac{\partial}{\partial \dot{q}_\mu} \left[\sum_{i=1}^3 \frac{\kappa_i}{2} \dot{x}_i^2 \right].$$

This result suggests the definition of a function, which is called (**Rayleigh's dissipation function**)

$$R = \sum_{i=1}^3 \frac{\kappa_i}{2} \dot{x}_i^2 = R(q_1, \dots, \dot{q}_1, \dots, t). \quad (5.83)$$

The Lagrange equations of the second kind for a conservative and a dissipative contribution can be written with this function as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) - \frac{\partial L}{\partial q_\mu} + \frac{\partial R}{\partial \dot{q}_\mu} = 0. \quad (5.84)$$

The argument can be generalised: the Lagrange equations can be used in the standard abbreviated form (5.82) with the Lagrange function

$$L = T - U^* , \quad (5.85)$$

if it is possible to define a potential function

$$U^* = U^*(q_1, \dots, \dot{q}_1, \dots, t), \quad (5.86)$$

so that the generalised forces can be expressed as

$$Q_\mu = - \left(\frac{\partial U^*}{\partial q_\mu} - \frac{d}{dt} \left(\frac{\partial U^*}{\partial \dot{q}_\mu} \right) \right). \quad (5.87)$$

The quantity U^* is the **generalised potential**. This potential can, as indicated, depend on the generalised coordinates, the generalised velocities and on time.

It is not possible to choose ignorable generalised coordinates, if nonholonomic constraints (or holonomic plus nonholonomic constraints) are present, as there exists no direct functional relationship between the Cartesian coordinates. The constraints in virtual form (5.38)

$$\sum_{i=1}^3 a_{k,i}(x_1, x_2, x_3, t) \delta x_i = 0, \quad (k = 1, \dots)$$

can, however, be expressed in terms of generalised coordinates

$$\sum_{\mu=1}^3 \sum_{i=1}^3 a_{k,i}(q_1, q_2, q_3, t) \frac{\partial x_i}{\partial q_\mu} \delta q_\mu = \sum_{\mu=1}^3 A_{k,\mu}(q_1, q_2, q_3, t) \delta q_\mu.$$

These conditions cannot be incorporated directly into the equations of the type Lagrange II, but have to be added with Lagrange multipliers, so that a mixed representation of the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu - \sum_k \lambda_k A_{k,\mu} = 0 \quad (5.88)$$

is obtained.

5.3.2 Lagrange II and conservation laws for one point particle

Conservation laws play a special role in Newton's formulation of mechanics. For this reason the question, in which way do conservation laws enter into the Lagrange formulation, ought to be asked and answered.

5.3.2.1 Generalised momenta in theory and practical applications.

It will be assumed in this section that only conservative forces act on the point particle and that only holonomic constraints apply. The equations of motion are then (see (5.82))

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) - \frac{\partial L}{\partial q_\mu} = 0 \quad (\mu = 1, \dots). \quad (5.89)$$

The relation

$$\frac{\partial L}{\partial q_\nu} = 0$$

is valid according to the definition of the partial derivative, if the Lagrange function does *not* depend on the generalised coordinate q_ν . If this is the case, one finds

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\nu} \right) = 0 \quad \text{or} \quad \frac{\partial L}{\partial \dot{q}_\nu} = \text{const.}$$

The partial derivative of the Lagrange function with respect to the corresponding generalised velocity is constant in time, in other words a conserved quantity. This partial derivative of the Lagrange function is the **generalised momentum** p_ν

$$p_\nu = \frac{\partial L}{\partial \dot{q}_\nu} . \quad (5.90)$$

The name implies a generalisation of the Cartesian momentum. The derivative with respect to one of the velocity components for the Lagrange function

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U(x_1, x_2, x_3)$$

is indeed

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i .$$

The generalised momentum and the standard momentum are identical in the case of Cartesian coordinates.

A coordinate q_ν , which does not appear in the Lagrange function, is called a **cyclical coordinate**. The conservation law, which follows on the basis of the definitions stated above, can be formulated as follows:

The generalised momentum is a conserved quantity if the corresponding generalised coordinate is cyclical

or in mathematical form

$$\frac{\partial L}{\partial q_\nu} = 0 \quad \longrightarrow \quad p_\nu = \frac{\partial L}{\partial \dot{q}_\nu} = \text{const.} \quad (5.91)$$

An example is the discussion of the motion of a point particle without constraints in terms of spherical coordinates. The transformation between the Cartesian coordinates and the coordinates r, θ, φ , (using the original notation, see (2.74), p. 62), is

$$x = r \cos \varphi \sin \theta \quad y = r \sin \varphi \sin \theta \quad z = r \cos \theta .$$

The simplest way to express the kinetic energy in terms of these coordinates is to use the result ((2.81), p. 64) for the appropriate generalised velocities and

consider the scalar product of the velocity vector with itself. The resulting Lagrange function

$$L = T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2 \theta) - U(r, \theta, \varphi) \quad (5.92)$$

leads to the generalised momenta

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \sin^2 \theta. \quad (5.93)$$

The interpretation of these generalised momenta is based on dimensional considerations: A generalised momentum has the usual dimension $[ML/T]$ if the generalised coordinate corresponds to a length. The generalised momentum corresponds to an angular momentum with the dimension $[ML^2/T]$, if the generalised coordinate is an angle. The dimensional statement is in general

$$p_\nu = \frac{\partial L}{\partial \dot{q}_\nu} \longrightarrow [p_\nu \cdot q_\nu] = [E T]$$

on the basis of the definition (5.91). The quantity 'energy multiplied with time' is an **action**, so that the dimension of the product of a generalised coordinate with a generalised momentum is

$$[p \cdot q] = \left[\frac{ML^2}{T} \right] \longrightarrow \text{action}.$$

The nature of the generalised momentum associated with a given coordinate can be gleaned from this relation.

The Lagrange function (5.92) is, for instance, relevant for the discussion of the spherical pendulum. A mass point m is attached to a rigid (weightless) rod, so that it can move on a surface of a sphere under the influence of simple gravity (Fig. 5.21). The distance of the mass from the point of suspension is

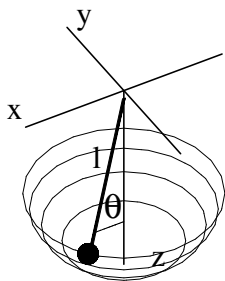


Fig. 5.21. The spherical pendulum: geometry

an ignorable coordinate in this problem as $r = l$ and $\dot{r} = 0$. The potential energy is

$$U = mgz = mgl \cos \theta.$$

The coordinate system is chosen as in Fig. 5.21, so that $U(z = 0) = 0$. The Lagrange function of the spherical pendulum can therefore be written as

$$L = \frac{ml^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) - mgl \cos \theta. \quad (5.94)$$

Clearly the angle φ is a cyclical coordinate, so that the angular momentum

$$p_\varphi = ml^2 \dot{\varphi}(t) \sin^2 \theta(t) = C \quad (5.95)$$

is conserved. The following modes of motion are possible:

- The initial condition $\dot{\varphi}(0) = 0$ leads to $\dot{\varphi}(t) = 0$. The pendulum swings in a plane which is characterised by the initial value of the azimuthal angle, that is by $\varphi(0)$. This motion corresponds (note the change of the notation for the coordinates) to the motion of the mathematical pendulum, which has been discussed in Chap. 4.2.1 in terms of the Newtonian formulation.
- The angular velocity $\dot{\varphi}(t)$ is not equal to zero, if the initial angular velocity $\dot{\varphi}(0)$ is not and if the initial values of the angle θ are not equal to 0 or π

$$\dot{\varphi}(t) \neq 0 \quad \text{if} \quad \dot{\varphi}(0) \neq 0 \quad \text{and} \quad \theta(0) \neq 0, \pi.$$

The equation of motion for the angle θ differs from that of the mathematical pendulum in this case.

The Lagrange equation for the coordinate θ with the replacement of $\dot{\varphi}(t)$ by the conservation law (5.95) takes the form

$$ml^2 \ddot{\theta} - \frac{C^2}{ml^2} \frac{\cos \theta}{\sin^3 \theta} - mgl \sin \theta = 0.$$

This equation corresponds to the time derivative of the conserved energy

$$\frac{ml^2}{2} \left(\dot{\theta}^2 + \frac{C^2}{m^2 l^4 \sin^2 \theta} \right) + mgl \cos \theta = E_0. \quad (5.96)$$

The energy within the Lagrangian formulation is, as outlined in the next section, under certain conditions identical with the Hamiltonian. For the moment the only point of interest is the solution of the equation (5.96). This differential equation takes the form

$$\left(\frac{dq}{dt} \right)^2 = \frac{2}{ml^2} (E_0 - mglq) (1 - q^2) - \frac{C^2}{m^2 l^4} \equiv U_{\text{eff}}(q)$$

after the substitution of the variable q

$$q = \cos \theta \quad \dot{q} = -\dot{\theta} \sin \theta.$$

It can be treated further with the method of separation of variables. Using the abbreviations

$$a = \frac{2E_0}{ml^2} \quad b = \frac{2g}{l} \quad c = \frac{C^2}{m^2 l^4}$$

one finds the 'solution'

$$t = \pm \int_{q(0)}^q dq' \frac{1}{\sqrt{U_{\text{eff}}(q')}} = \pm \int_{q(0)}^q \frac{dq'}{[(a - bq')(1 - q'^2) - c]^{1/2}}. \quad (5.97)$$

The effective potential in (5.97) is a polynomial of third degree. The consequence is: the integral on the right hand side of (5.97) is an elliptic integral.

It is possible to analyse just the potential $U_{\text{eff}}(q)$ if a more qualitative discussion is thought to be sufficient. The quantity U_{eff} has to be positive in the interval $-1 < q < 1$ or a part thereof. The effective potential $U_{\text{eff}}(q)$ is negative for large negative values of q , as the parameter b is greater than zero. The function $U_{\text{eff}}(q)$ is sketched in Fig. 5.22. In the relevant interval one

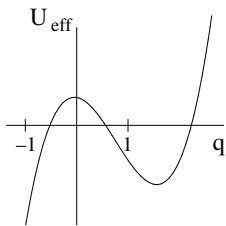


Fig. 5.22. The spherical pendulum: effective potential

finds two zeros of the polynomial (q_1 and q_2), which corresponds to circles of latitude with

$$\theta_1 = \arccos q_1 \quad \theta_2 = \arccos q_2$$

between which the pendulum will move.

The conservation law (5.95) involving the angle φ can be rewritten as

$$\frac{d\varphi}{dq} = \frac{d\varphi}{dt} \frac{dt}{dq} = \frac{C}{m l^2 (1 - q^2)} \frac{1}{\sqrt{U_{\text{eff}}(q)}}.$$

Integration also leads to an elliptic integral (of the third kind)

$$\varphi(q) - \varphi(q(0)) = \frac{C}{m l^2} \int_{q(0)}^q \frac{dq'}{\sqrt{U_{\text{eff}}(q')}} \frac{1}{(1 - q'^2)}. \quad (5.98)$$

While the pendulum swings between the two circles of latitude, the azimuthal angle changes. The pendulum performs a **precessional motion**. The motion of the spherical pendulum is periodic in the sense that it returns (similar to the pattern found for the mathematical pendulum in Chap. 4.2.1) with the sequence

$$\theta_1 \xrightarrow{\tau} \theta_2 \xrightarrow{\tau} \theta_1 \xrightarrow{\tau} \theta_2 \xrightarrow{\tau} \theta_1$$

to the original position. The four time intervals indicated are equal as demanded by (5.97). The period is therefore given by

$$T = 4 \int_{q_1}^{q_2} \frac{dq'}{\sqrt{U_{\text{eff}}(q')}}. \quad (5.99)$$

The corresponding angle $\Delta\varphi$, by which the pendulum precesses during this time interval, can be calculated with the aid of (5.98).

5.3.2.2 Energy and Hamilton's function. As the Lagrange function does *not* represent the total energy, energy conservation has to be discussed explicitly. The Lagrangian for one mass point is in general a function of up to 7 variables

$$L = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t) .$$

The total derivative of this function with respect to time is

$$\frac{dL}{dt} = \sum_{\mu=1}^3 \left[\frac{\partial L}{\partial q_\mu} \dot{q}_\mu + \frac{\partial L}{\partial \dot{q}_\mu} \ddot{q}_\mu \right] + \frac{\partial L}{\partial t} . \quad (5.100)$$

The factor of \dot{q}_μ can be replaced with the Lagrange equation (5.82)

$$\frac{\partial L}{\partial q_\mu} \dot{q}_\mu = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) \right) \dot{q}_\mu$$

and the terms in the square brackets in (5.100) can be condensed, so that the relation

$$\frac{dL}{dt} = \frac{d}{dt} \left[\sum_{\mu=1}^3 \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) \dot{q}_\mu \right] + \frac{\partial L}{\partial t}$$

follows. Introduction of the generalised momentum in this equation gives the central result

$$\frac{d}{dt} \left[\sum_{\mu=1}^3 p_\mu \dot{q}_\mu - L \right] = - \frac{\partial L}{\partial t} . \quad (5.101)$$

The expression in the brackets is another central element of classical mechanics. The quantity

$$H = \sum_{\mu=1}^3 p_\mu \dot{q}_\mu - L \quad (5.102)$$

is the **Hamiltonian** or **Hamilton function**. Two properties of this function can be stated directly:

- the Hamiltonian has (as the Lagrangian) the dimension of energy.
- Equation (5.101) implies that the Hamiltonian is a conserved quantity if the Lagrangian does not depend explicitly on time

$$\frac{\partial L}{\partial t} = 0 \rightarrow \frac{dH}{dt} = 0 \rightarrow H(t) = H(0) . \quad (5.103)$$

Not quite as obvious is the statement

- The Hamiltonian is identical with the total energy of the system, if the system is conservative and if the transformation between the Cartesian and the generalised coordinates does not depend explicitly on time

$$H \equiv E \quad \text{if} \quad \frac{\partial U}{\partial \dot{q}_\mu} = 0$$

$$\text{and} \quad x_i = x_i(q_1, q_2, q_3) \longrightarrow \frac{\partial x_i}{\partial t} = 0 . \quad (5.104)$$

This statement follows from one of the properties of the kinetic energy

$$T = \frac{m}{2} \sum_i \dot{x}_i^2 .$$

With the assumptions stated and with the derivative of the transformation (5.45)

$$\dot{x}_i = \sum_\mu \frac{\partial x_i}{\partial q_\mu} \dot{q}_\mu$$

the kinetic energy can be written in the form

$$T = \frac{m}{2} \sum_i \sum_{\nu, \mu} \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} \dot{q}_\nu \dot{q}_\mu .$$

The kinetic energy is a **homogeneous function** of second degree in the generalised velocities. For homogeneous functions of the m -th degree, which are characterised by

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^m f(x_1, \dots, x_n) ,$$

the **theorem of Euler**

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = m f$$

applies. This theorem gives for the kinetic energy

$$\sum_\mu \dot{q}_\mu \frac{\partial T}{\partial \dot{q}_\mu} = 2T .$$

The generalised momentum (5.90) is characterised by the derivative of the kinetic energy alone

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} = \frac{\partial T}{\partial \dot{q}_\mu}$$

as a consequence of the assumption $\partial U / \partial \dot{q}_\mu = 0$. This leads to

$$\sum_\mu \dot{q}_\mu \frac{\partial T}{\partial \dot{q}_\mu} = \sum_\mu p_\mu \dot{q}_\mu = 2T$$

and the Hamiltonian is, under the assumptions stated, identical with the energy of the point particle

$$H = \sum_{\mu} p_{\mu} \dot{q}_{\mu} - L = 2T - T + U = T + U = E . \quad (5.105)$$

Some additional remarks concerning the Hamiltonian are:

1. The assumptions, which have been stated, are sufficient but not necessary. The Hamiltonian can represent the total energy even if the forces are not conservative and if the transformation depends on time.
2. The question, whether the Hamiltonian represents the total energy is independent of the question, whether the Hamiltonian is a constant of motion. The Hamiltonian can be a conserved quantity without representing the total energy.
3. The relation (5.102) between the Hamiltonian and the Lagrangian is known as a Legendre transformation³.

5.3.2.3 Examples concerning conservation laws. The general remarks on conservation laws within the framework of Lagrangian theory are illustrated by a few explicit examples.

The first example is the one dimensional harmonic oscillator with

$$T = \frac{m}{2} \dot{x}^2 \quad U = \frac{k}{2} x^2 \quad L = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2 .$$

The generalised momentum is $p_x = m\dot{x}$ so that the Hamiltonian (5.102) can be written as

$$H = (m\dot{x})\dot{x} - \frac{m}{2} \dot{x}^2 + \frac{k}{2} x^2 = \frac{m}{2} \dot{x}^2 + \frac{k}{2} x^2 = E .$$

The generalised momentum is not conserved, the Hamiltonian represents the conserved total energy.

The problem of motion of a point particle under the influence of a central force has been addressed in previous chapters (see e.g. Chap. 2.2.1). A more detailed account is given below. The generalised coordinates r , θ and φ are used for the discussion of this problem with the Lagrangian (see 5.92)

$$L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\varphi}^2 \sin^2 \theta) - U(r) .$$

The generalised momenta are (as calculated before (5.93))

$$p_r = m\dot{r} \quad p_{\theta} = mr^2 \dot{\theta} \quad p_{\varphi} = mr^2 \dot{\varphi} \sin^2 \theta$$

so that the Hamiltonian takes the form

$$\begin{aligned} H &= m(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\varphi}^2 \sin^2 \theta) - L \\ &= \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\varphi}^2 \sin^2 \theta) + U(r) = E . \end{aligned}$$

The angle φ is a cyclical coordinate (as remarked before)

³ See Chap. 5.4.2 for additional remarks on the concept of Legendre transformations.

$$p_\varphi = mr^2\dot{\varphi} \sin^2 \theta = \text{const.} \quad \longrightarrow \quad p_\varphi(t) = p_\varphi(0) .$$

It is possible to choose the coordinate system in such a way, that a point particle is found on the z -axis at the time $t = 0$. The initial conditions for the polar angle θ and the generalised momentum p_φ are

$$\theta(0) = 0 \longrightarrow \sin \theta(0) = 0 \longrightarrow p_\varphi(t) = p_\varphi(0) = 0$$

in this case. The generalised momentum can only vanish for times $t > 0$, if

- the azimuthal velocity vanishes ($\dot{\varphi} = 0$) or
- the mass point does not moves at all ($r(t) = 0$) or
- the mass point moves only along the z -axis ($\theta(t) = 0$).

The first case leads to the more general situation: the mass point moves in a plane, which contains the z -axis and the straight line $y = x(\tan \varphi)$ (Fig. 5.23) if the azimuthal velocity vanishes. This indicates that the vector of the areal velocity has a fixed direction. The Lagrangian (5.92) simplifies for this choice

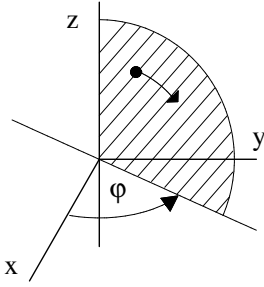


Fig. 5.23. Lagrange II: law of areas in the central force problem

of the coordinate system

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

and it is found that the angle θ is also a cyclical coordinate. The associated generalised momentum (an angular momentum) is a conserved quantity

$$p_\theta \equiv l = mr^2\dot{\theta} = \text{const.}$$

This statement demonstrates that the magnitude of the vector of the areal velocity is also conserved. The law of areas is valid but has to be made apparent by a suitable choice of the coordinate system.

The Lagrange equation of motion for the remaining radial coordinate

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial U(r)}{\partial r} = 0$$

can, after a replacement of $\dot{\theta}$ by the angular momentum l , be written as

$$m\ddot{r} - \frac{l^2}{2mr^2} + \frac{\partial U(r)}{\partial r} = 0 .$$

This is exactly the equation of motion which has been used in Chap. 4.1 for the discussion of the 'Kepler problem'. The associated Hamiltonian represents the total energy and is conserved as

$$\frac{\partial U}{\partial t} = 0 \quad \text{and} \quad \frac{\partial L}{\partial t} = 0 .$$

The Lagrangian for the motion of a point particle on the surface of a fixed cylinder under the influence of a harmonic restoring force (see p. 221) depends on the coordinates φ and z

$$L = \frac{m}{2}(R^2\dot{\varphi}^2 + \dot{z}^2) - \frac{k}{2}(R^2 + z^2) = T - U .$$

It does not depend on φ , so that the associated generalised momentum $p_\varphi = mR^2\dot{\varphi}$ is conserved. This reflects the fact that the circular motion is uniform, as long the radius R does not change with time. The Hamiltonian

$$H = (mR^2\dot{\varphi})\dot{\varphi} + (m\dot{z})\dot{z} - L = T + U = E$$

is conserved and represents the energy. The sufficient conditions are satisfied.

The situation is different, if the surface of the cylinder changes with time. In the rheonomic situation the Lagrangian takes the form

$$L = \frac{m}{2}(\dot{R}(t)^2 + R(t)^2\dot{\varphi}^2 + \dot{z}^2) - \frac{k}{2}(R(t)^2 + z^2) = T - U ,$$

where $R(t)$ is a given function of time. The coordinates φ is still cyclical

$$p_\varphi = mR(t)^2\dot{\varphi}(t) = \text{const.} ,$$

but the motion along the surface is not uniform, as it is influenced by the time dependence of $R(t)$. The Hamiltonian is not a conserved quantity. The partial derivative of the Lagrange function with respect to time does not vanish

$$\frac{\partial L}{\partial t} = m\dot{R}\ddot{R} + (m\dot{\varphi}^2 - k)R\dot{R} \neq 0 ,$$

due to the time dependent constraint. The (time dependent) Hamiltonian, calculated according to the definition

$$H = (mR^2\dot{\varphi})\dot{\varphi} + (m\dot{z})\dot{z} - L = T + U - \frac{m}{2}\dot{R}^2 ,$$

is a quantity, which does not correspond to the total energy of the point particle, which is given by $T + U$.

The only relevant degree of freedom for the motion of a mass point on a helix (see p. 223) is the height z . The Lagrangian (compare (5.75)) for a rising helix is given by

$$L = \frac{m}{2} \left[R^2 \left(\frac{2\pi}{h} \dot{z} + \omega(t) \right)^2 + \dot{z}^2 \right] - mgz = T - U$$

for the case of an arbitrary rotation. The Hamiltonian can be calculated via the generalised momentum

$$p_z = m \left[\frac{2\pi}{h} R^2 \omega + \dot{z} \left(1 + \left(\frac{2\pi}{h} R \right)^2 \right) \right]$$

as

$$H = p_z \dot{z} - L = \frac{m}{2} \left[\dot{z}^2 \left(1 + \left(\frac{2\pi}{h} R \right)^2 \right) - R^2 \omega(t)^2 \right] + mgz \neq T + U .$$

The Hamiltonian does not represent the total energy, but is a conserved quantity for a uniform rotation as the condition $\partial L / \partial t = 0$ is satisfied in this case. The partial derivative of the Lagrangian does not vanish, if the rotation is not uniform. The Hamiltonian is not a constant of motion in this case.

5.3.3 Lagrange II for a system of mass points

The discussion of the Lagrange equations of the second kind for a system of point particles does not differ greatly from the considerations for the case of one point particle. The situation, that is addressed in this case, can be sketched as follows: the forces which act in the system of N masses $\{m_1, \dots, m_N\}$ are classified as external \mathbf{F}_i and internal \mathbf{f}_{ji} forces. The effect of constraining forces has to be considered, if additional (geometrical) constraints are present. The constraining force acting on the i -th mass is, without further specification, denoted by \mathbf{Z}_i . The Newtonian equations of motion for this system are therefore

$$m_i \ddot{\mathbf{r}}_i = \mathbf{K}_i + \mathbf{Z}_i \quad (i = 1, 2, \dots, N)$$

$$\text{with} \quad \mathbf{K}_i = \mathbf{F}_i + \sum_{j=1}^N \mathbf{f}_{ji} \quad (\mathbf{f}_{ii} = \mathbf{0}) .$$

D'Alembert's principle is formulated, as before, with the aid of a consecutive numbering of the coordinates, masses and force components from 1 bis $3N$. The principle can then be stated in the form

$$\sum_{i=1}^{3N} (m_i \ddot{x}_i - K_i) \delta x_i = 0 .$$

The constraining forces are generated by a set of constraints. It will be assumed that the system is subjected to r holonomic (rheonomic or scleronomous) constraints

$$f_1(x_1, \dots, x_{3N}, t) = 0 \quad \dots \quad f_r(x_1, \dots, x_{3N}, t) = 0 .$$

The steps for the assembly of the Lagrange equations of the second kind correspond exactly to the steps used in Chap. 5.3.1 for the case of one point particle:

- (1) The first step is the choice of a suitable transformation between the Cartesian and the generalised coordinates

$$x_i = x_i(q_1, \dots, q_{3N}, t) \quad (i = 1, 2, \dots, 3N)$$

with the inverse

$$q_\mu = q_\mu(x_1, \dots, x_{3N}, t) \quad (\mu = 1, 2, \dots, 3N) .$$

The r constraints are incorporated by choosing e.g.

$$q_{3N-r+1} = f_1(x_1, \dots, x_{3N}, t) = 0$$

\vdots

$$q_{3N} = f_r(x_1, \dots, x_{3N}, t) = 0$$

for the last r generalised coordinates. These coordinates are ignorable, they can be excluded from the discussion.

- (2) The kinetic energy

$$T = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{x}_i^2$$

is expressed in the second step in terms of the remaining generalised coordinates using the time derivative of the transformation

$$\dot{x}_i = \sum_{\mu=1}^{3N-r} \frac{\partial x_i}{\partial q_\mu} \dot{q}_\mu + \frac{\partial x_i}{\partial t} .$$

The kinetic energy is now a function of the generalised coordinates, velocities and, if applicable, the time.

- (3) With this expression for the kinetic energy the derivatives

$$\frac{\partial T}{\partial q_\mu} = \sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_\mu}$$

and

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) = \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\mu} + \frac{\partial T}{\partial q_\mu} \quad (\mu = 1, 2, \dots, 3N - r)$$

are calculated. The derivation of these relations follows step by step the calculation for the case of one point particle. The only difference is: the sum over the indices of the generalised coordinates runs up to $(3N - r)$ instead of $(3 - r)$.

- (4) The relation between the virtual displacements in terms of the two sets of coordinates are

$$\delta x_i = \sum_{\mu=1}^{3N-r} \frac{\partial x_i}{\partial q_\mu} \delta q_\mu .$$

(5) The transformation of d'Alembert's principle leads to

$$\sum_{i=1}^{3N} \sum_{\mu=1}^{3N-r} \left[m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\mu} - K_i \frac{\partial x_i}{\partial q_\mu} \right] \delta q_\mu = 0 .$$

(6) Generalised forces are introduced as before with

$$Q_\mu = \sum_{i=1}^{3N} K_i \frac{\partial x_i}{\partial q_\mu}$$

and

$$K_i = F_i + \sum_{j=1}^{3N} f_{ji} .$$

The consecutive numbering of the components leads to a specific pattern for the structure of the matrix f_{ji} . The pattern arises from the following properties

- The condition $\mathbf{f}_{kk} = \mathbf{0}$ with $(k = 1, \dots, N)$ is incorporated.
- The decomposition of the interaction \mathbf{f}_{kl} between pairs of particles with $(k, l = 1, \dots, N)$ into components leads to the vanishing of a number of entries for f_{ji} in the matrix pattern.

It is found, as indicated in the diagram of the matrix f_{ji} , that non-vanishing contributions occur only for

$$1 \leq j = i \pm 3n \leq 3N \quad (n = 1, 2, \dots, N-1) .$$

j \ i	1	2	3	4	5	6	7	...	3N-2	3N-1	3N
1	o	o	o	x	o	o	x		x	o	o
2	o	o	o	o	x	o	o	...	o	x	o
3	o	o	o	o	o	x	o		o	o	x
4	x	o	o	o	o	o	x		x	o	o
5	o	x	o	o	o	o	o	...	o	x	o
6	o	o	x	o	o	o	o		o	o	x
7	x	o	o	x	o	o	o		x	o	o
⋮		⋮			⋮			⋮		⋮	

(7) D'Alembert's principle can be recast in the form

$$\sum_{\mu=1}^{3N-r} \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu \right\} \delta q_\mu = 0$$

with the use of steps (3) and (6). As all $3N - r$ generalised coordinates are independent, the set of equations of motion is, as before

$$\boxed{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} - Q_\mu = 0 \quad (\mu = 1, 2, \dots, 3N - r) .} \quad (5.106)$$

This set of equations of motion is valid for conservative as well as nonconservative active forces.

The external and the internal forces can be represented in terms of potential energies (U respectively V) followed by the introduction of a suitable Lagrangian for a conservative system (conservative internal as well as external forces). With the consecutive numbering of the coordinates the total potential energy of the system can be written as

$$\begin{aligned} U + V = & U_1(x_1, x_2, x_3) + U_2(x_4, x_5, x_6) + \dots \\ & U_N(x_{3N-2}, x_{3N-1}, x_{3N}) + V_{12}(x_1 x_2 x_3, x_4 x_5 x_6) + \\ & V_{13}(x_1 x_2 x_3, x_7 x_8 x_9) + \dots \\ & + V_{N-1,N}(x_{3N-5} x_{3N-4} x_{3N-3}, x_{3N-2} x_{3N-1} x_{3N}) . \end{aligned}$$

The coordinates are divided into sets of three for the potential energy of the external forces U . The individual terms of the potential energy of the internal forces V depend on all admissible combinations of sets of three coordinates. The Cartesian components of the forces can be regained with

$$K_i = -\frac{\partial U}{\partial x_i} - \frac{\partial V}{\partial x_i} ,$$

the generalised force components are

$$Q_\mu = -\sum_{i=1}^{3N} \frac{\partial(U+V)}{\partial x_i} \frac{\partial x_i}{\partial q_\mu} = -\frac{\partial(U+V)}{\partial q_\mu} \quad (\mu = 1, 2, \dots, 3N - r) .$$

These can be calculated in a different fashion, if the total potential energy is first expressed in term of the generalised coordinates

$$U + V = U(q_1, \dots, q_{3N-r}, t) + V(q_1, \dots, q_{3N-r}, t)$$

and the generalised gradients of these functions are evaluated afterwards. With the definition of the *extended Lagrangian*

$$L = T - U - V \quad (5.107)$$

and the statement, that the total potential energy does not depend on the generalised velocities, one arrives at the 'standard form' of the Lagrange equations of the second kind for a system of N point particles with conservative forces, which is subjected to r holonomic constraints

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) - \frac{\partial L}{\partial q_\mu} = 0 \quad (\mu = 1, 2, \dots, 3N - r) .} \quad (5.108)$$

These equations constitute an optimal starting point for the discussion of arbitrary (but holonomic and conservative) problems of motion. A selection of such problems is discussed in Chap. 6.

The unchanged form of the Lagrange equations allows the transcription of previous statements relating to possible conservation laws.

1. The generalised momentum is a conserved quantity, if the corresponding coordinate is cyclical

$$\frac{\partial L}{\partial q_\mu} = 0 \quad \Rightarrow \quad p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} = \text{const.}$$

2. The Hamiltonian of the system is defined by

$$H = \sum_{\mu=1}^{3N-r} p_\mu \dot{q}_\mu - L .$$

3. The total time derivative of the Hamiltonian equals the negative value of the partial time derivative of the Lagrangian

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t} .$$

Hence follows the statement: H is a conserved quantity if L does not depend explicitly on time

$$\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{dH}{dt} = 0 \quad \Rightarrow \quad H(t) = H(0) .$$

4. The Hamiltonian is identical with the total energy of the system

$$H = T + U + V = E$$

if the relations

$$\frac{\partial(U + V)}{\partial \dot{q}_\mu} = 0 \quad \text{as well as} \quad \frac{\partial x_i}{\partial t} = 0$$

are valid.

The tools for the discussion of problems of motion in mechanics are assembled with the Lagrange equations of the second kind for systems of point particles. An alternative foundation of mechanics is due to W. Hamilton. Hamilton's formulation does open new avenues for the discussion of problems of motion (e.g. under the heading 'canonical transformations'). Its real significance lies, however, to a lesser degree in the application to specific problems of mechanics but rather in the possibility to transfer basic concepts from mechanics into more advanced theories. Hamilton's equation of motion (in a modified form) constitute a pragmatic access to statistical mechanics (and hence thermodynamics) and to quantum mechanics.

5.4 Hamilton's formulation of mechanics

The Lagrange equations of motion can also be derived from Hamilton's principle, which will be detailed in this chapter. This principle can be regarded as an alternative possibility for an axiomatic foundation of mechanics. The foundation of mechanics can therefore be approached from two different perspectives. While d'Alembert's principle is a principle based on differentials, Hamilton's principle is based on integrals. The differential principle is more flexible. For systems (e.g. conservative, holonomic), for which a Lagrangian can be defined, the two principles are equivalent. The variational calculus on which Hamilton's principle is based, can, on the other hand, be applied more easily in areas of physics beyond mechanics.

After a brief look at the motivation from the point of view of physics, the variational calculus will be introduced and illustrated with a few examples. The variational calculus allows a detailed discussion of Hamilton's principle. Hamilton's equations of motion are presented in the second part of this chapter. The central point for the discussion of these equations of motion is the Hamiltonian, which has to be regarded as a function of the generalised coordinates and generalised momenta in the present context. The chapter closes with an introduction to phase space concepts and their relevance for chaotic motion.

5.4.1 Hamilton's principle

The formulation of this principle in the context of mechanics is:

Consider the motion of point particles in a (conservative) system between the times t_1 and t_2 . The motion proceeds in such a fashion that the integral

$$I = \int_{t_1}^{t_2} L(q_1(t) \dots q_n(t), \dot{q}_1(t) \dots \dot{q}_n(t), t) dt$$

(5.109)

has an extremal value for the 'actual path'.

The concept of a 'path' has to be interpreted as follows: the n generalised coordinates of the system span an n -dimensional space. The n -tuple of points in this space $\{q_1(t_0) \dots q_n(t_0)\}$ describes the instantaneous configuration of the system. This space is referred to as the **configuration space**. A curve in configuration space with the parametric representation $\{q_1(t) \dots q_n(t)\}$ with $t_1 \leq t \leq t_2$ is a 'path'. The statement made with Hamilton's principle is therefore: of all the possible curves, which can connect a given initial configuration with a given final configuration, exactly one is distinguished. It is the curve, for which the integral above is extremal (minimal or maximal). This curve describes the actual time development of the system. In order to

find the curve describing the 'actual path', it can be imagined that suitable neighbouring curves are tested until the one with the extremum is found. The mathematical formulation of this 'trial' is the content of the variational calculus, which will be introduced with a simple example.

5.4.1.1 The Euler-Lagrange variational equations. Given is a function $f(t, x, \dot{x})$ of three (!) variables t, x, \dot{x} . The variable t is the independent variable, $x(t)$ the dependent variable and \dot{x} the derivative of x with respect to t . The functional relationship between x and t is not known. Rather the question is: determine $x(t)$ by demanding

1. The curve $x(t)$ should pass through two specified points (t_1, x_1) and (t_2, x_2) in the x - t plane.
2. The definite integral

$$I = \int_{t_1}^{t_2} f(t, x, \dot{x}) dt$$

should be extremal.

A mathematical relation, which associates a number with a function, is termed a **functional**. I is a functional of x in the sense of this definition

$$t, x(t) \longrightarrow \text{relation : } \int f(t, x, \dot{x}) dt \longrightarrow I = I[x].$$

Roughly speaking, a functional is a function of a function, where (in the present example) each function $x(t)$ is associated with a number I . An integral over a function is only one example for a functional. The relations, which define functionals, can be more complicated.

The solution of the problem posed can be approached with the following argumentation: under the assumption, that $x(t)$ is the correct function, arbitrary variations about this curve can be considered. These variations can be expressed in the form

$$x_v(t) = x(t) + \epsilon \varphi(t).$$

The function $\varphi(t)$ can be chosen freely with the exception of one restriction. It should be continuous and, in order to satisfy requirement (1), it should vanish for t_1 and t_2

$$\varphi(t_1) = \varphi(t_2) = 0.$$

The parameter ϵ is a constant, which would be arbitrarily small, if the trial function and the true function coincide. The ansatz can be differentiated

$$\dot{x}_v(t) = \dot{x}(t) + \epsilon \dot{\varphi}(t)$$

and inserted into the functional

$$\int_{t_1}^{t_2} f(t, x_v, \dot{x}_v) dt = I([x_v, \varphi], \epsilon) \equiv I(\epsilon).$$

The quantity I is a functional of x_v and φ , but a function of ϵ . The notation $I([x_v, \varphi], \epsilon)$ emphasises this point. A closer look at the function $I(\epsilon)$ shows, that it has to have an extremum for $\epsilon = 0$, if $x_v(t)$ is identical with the desired solution $x(t)$. The condition to be used for the determination of $x(t)$ is therefore

$$\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0.$$

In order to apply this condition, the integrand is expanded in a Taylor series about $\epsilon = 0$

$$\begin{aligned} f(t, x_v, \dot{x}_v) &= f(t, x, \dot{x}) + \epsilon \left\{ \frac{\partial f}{\partial x_v} \frac{\partial x_v}{\partial \epsilon} + \frac{\partial f}{\partial \dot{x}_v} \frac{\partial \dot{x}_v}{\partial \epsilon} \right\}_{\epsilon=0} \\ &\quad + \frac{\epsilon^2}{2!} \{ \dots \}_{\epsilon=0} + \dots \end{aligned}$$

and this expansion is inserted into the integral

$$\begin{aligned} I(\epsilon) &= \int_{t_1}^{t_2} f(t, x, \dot{x}) dt + \epsilon \int_{t_1}^{t_2} \left\{ \frac{\partial f}{\partial x} \varphi + \frac{\partial f}{\partial \dot{x}} \dot{\varphi} \right\} dt \\ &\quad + \frac{\epsilon^2}{2!} \int_{t_1}^{t_2} \{ \dots \} dt + \dots \end{aligned}$$

Differentiation gives

$$\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \int_{t_1}^{t_2} \left\{ \frac{\partial f}{\partial x} \varphi + \frac{\partial f}{\partial \dot{x}} \dot{\varphi} \right\} dt \stackrel{!}{=} 0.$$

Partial integration of the second term in the brackets yields

$$\int_{t_1}^{t_2} \frac{\partial f}{\partial \dot{x}} \dot{\varphi} dt = \left. \frac{\partial f}{\partial \dot{x}} \varphi \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) \right] \varphi dt.$$

The first term on the right hand side vanishes because of the condition $\varphi(t_1) = \varphi(t_2) = 0$. Hence there remains

$$\int_{t_1}^{t_2} dt \left\{ \frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) \right\} \varphi(t) = 0. \quad (5.110)$$

The variational function can, with the exception that it should be continuous and have the value zero for t_1 and t_2 , be chosen freely. The condition (5.110) can only be satisfied if the expression within the brackets vanishes

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) = 0. \quad (5.111)$$

The result is a differential equation, which has exactly the structure of the Lagrange equation of the second kind. The name of the more general equation (5.111) is **Euler-Lagrange variational equation**.

5.4.1.2 Examples from the variational calculus. One of the many options for the application of the variational calculus is the solution of geometrical problems. Some of the classical examples of this type of problem are quite remarkable.

Specify two points in the x - t plane. The task is: determine the equation of the (section of the) curve, which yields the shortest connection of the two points (Fig. 5.24). The answer is actually well known. The variational

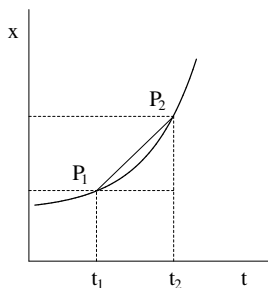


Fig. 5.24. Illustration of the simple variational problem

approach indicated above can be applied in order to verify its correctness. The ansatz looks as follows: the length of arc for an arbitrary curve through the two points P_1 and P_2 can be expressed as

$$S[x] = \int_{t_1}^{t_2} [dx^2 + dt^2]^{1/2} = \int_{t_1}^{t_2} [1 + \dot{x}^2]^{1/2} dt .$$

The function f to be discussed is therefore

$$f = [1 + \dot{x}^2]^{1/2} .$$

For setting up the variational equation, which determines the extremum of the functional $S[x]$, the derivatives

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \dot{x}} = \frac{\dot{x}}{[1 + \dot{x}^2]^{1/2}}$$

are needed. The Euler-Lagrange equation to be solved is

$$\frac{d}{dt} \left[\frac{\dot{x}}{[1 + \dot{x}^2]^{1/2}} \right] = 0 .$$

A first integration gives

$$\dot{x} = C_1 [1 + \dot{x}^2]^{1/2} ,$$

which can be resolved with respect to \dot{x}

$$\dot{x} = \pm \left[\frac{C_1^2}{1 - C_1^2} \right]^{1/2} = C_2 .$$

A second integration yields the expected result, a section of a straight line

$$x(t) = C_3 + C_2 t .$$

The two constants are determined by the conditions $x(t_1) = x_1$ and $x(t_2) = x_2$.

The next problem is known under the name 'problem of the brachystochrone' (problem of the shortest time). It is one of the first variational problems, that has been discussed in the literature (J. Bernoulli, 1696): Two points in the x - y plane are connected via a fictitious wire. The question is: which form of the wire is required so that a mass point moving under the influence of gravity takes the shortest possible time to get from point P_1 to point P_2 (Fig. 5.25) if it starts from rest?

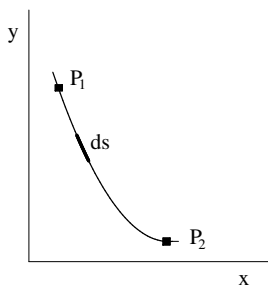


Fig. 5.25. Problem of the brachystochrone

The time required for the motion along an infinitesimal section ds of the curve is

$$dt = \frac{ds}{v(t)} ,$$

if the motion is characterised by the velocity $v(t)$. The time to traverse the complete curve between the points P_1 and P_2 is therefore

$$T = \int_1^2 \frac{ds}{v(t)} .$$

This integral should be made extremal by the choice of a suitable function $y(x)$. The infinitesimal arc length ds has to be expressed in terms of its Cartesian components for this purpose

$$ds = [dx^2 + dy^2]^{1/2}$$

and a relation between the instantaneous velocity and the coordinate has to be specified. This relation follows from energy conservation (using $v_1 = 0$)

$$\frac{m}{2} v^2 + mgy = E_1 (= mgy_1)$$

in the form

$$v(t) = \left[\frac{2E_1}{m} - 2gy \right]^{1/2} = [C_0 - 2gy]^{1/2} .$$

The ansatz for the calculation of the total time for the motion between the two points is therefore

$$T = \int_1^2 \frac{[dx^2 + dy^2]^{1/2}}{[C_0 - 2gy]^{1/2}} .$$

At this point it has not yet been decided, which of the variables is the dependent or the independent one. The variational equations are simpler, if y is chosen as the independent and x as the dependent variable. This choice leads to

$$T = \int_{y_1}^{y_2} \frac{[1 + x'^2]^{1/2}}{[C_0 - 2gy]^{1/2}} dy \quad x' = \frac{dx}{dy} .$$

Application of the variational calculus with the function f

$$f(y, x, x') = \left[\frac{1 + x'^2}{C_0 - 2gy} \right]^{1/2}$$

requires the derivatives

$$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial x'} = \frac{x'}{[(1 + x'^2)(C_0 - 2gy)]^{1/2}} .$$

A first integration of the Euler-Lagrange variational equation

$$\frac{d}{dy} \left(\frac{\partial f}{\partial x'} \right) = 0$$

gives the result

$$\frac{x'}{[(1 + x'^2)(C_0 - 2gy)]^{1/2}} = C_1 .$$

This equation has to be resolved with respect to x'

$$x' = \pm \left[\frac{A - y}{B + y} \right]^{1/2} \quad \text{with} \quad A = \frac{C_0}{2g} \quad B = \frac{1 - C_1^2 C_0}{2C_1^2 g} .$$

The solution of this linear differential equation for the function $x(y)$ can be obtained by a direct integration

$$x_2 - x_1 = \pm \int_{y_1}^{y_2} \left[\frac{A - y}{B + y} \right]^{1/2} dy .$$

The integral can be evaluated with the substitution

$$y = A - R(1 - \cos \alpha) \quad R = \frac{1}{2}(A + B) \quad (5.112)$$

with

$$dy = -R \sin \alpha \, d\alpha \quad B + y = R(1 + \cos \alpha) = \frac{R \sin^2 \alpha}{1 - \cos \alpha}.$$

The resulting integral

$$x = \pm R \int (1 - \cos \alpha) d\alpha$$

is elementary so that the x -coordinate is found to be (before insertion of appropriate upper and lower limits)

$$x = x_0 \pm R(\alpha - \sin \alpha). \quad (5.113)$$

The expressions $y = y(\alpha)$ and $x = x(\alpha)$ are the parametric representation of a **cycloid**⁴ (Fig. 5.26). The trajectory, for which the time for the motion

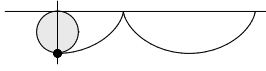


Fig. 5.26. A cycloid

between the two points is extremal (minimal), is a cycloid, or rather a section of a cycloid.

5.4.1.3 A short derivation of the variational equations. The central statement of the variational calculus for the case of one function $x(t)$ is:

<p>The functional</p> $I[x] = \int_{t_1}^{t_2} f(t, x, \dot{x}) dt$ <p>has an extremum, if the function $x(t)$ is determined via the variational equation</p> $\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) = 0.$	(5.114)
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This statement is often derived in an abbreviated fashion. The variation of

$$I = \int_{t_1}^{t_2} f(t, x, \dot{x}) dt$$

is specified in the form

$$\delta I = \delta \left[\int_{t_1}^{t_2} f(t, x, \dot{x}) dt \right] = \int_{t_1}^{t_2} \delta f(t, x, \dot{x}) dt.$$

The variational symbol δ can be taken under the integral sign, as the limits are not affected by the variation. One may abbreviate the derivation of the variational equations with the following rules:

⁴ The cycloid is a curve which is obtained if a wheel (radius R) is rolling along a straight line and the trajectory of a point on the rim of the wheel is traced. The parametric representation is also discussed in ● Probl. 4.13.

(a) The rule $\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x}$ corresponds to the chain rule

$$\frac{\partial f}{\partial \epsilon} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \epsilon}.$$

(b) Variation and differentiation with respect to time can be interchanged

$$\delta \dot{x} = \frac{d}{dt}(\delta x)$$

as one has

$$\delta \dot{x} = \frac{\partial}{\partial \epsilon} \left(\frac{dx}{dt} \right) d\epsilon = \left[\frac{d}{dt} \left(\frac{\partial x}{\partial \epsilon} \right) \right] d\epsilon = \frac{d}{dt} \left[\left(\frac{\partial x}{\partial \epsilon} \right) d\epsilon \right] = \frac{d}{dt}(\delta x).$$

Variation using the two rules gives

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \frac{d}{dt}(\delta x) \right) dt = 0.$$

Partial integration of the second term with $\delta x = 0$ at the integration limits yields

$$\delta I = \int_{t_1}^{t_2} \left[\left(\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) \right) \delta x \right] dt = 0.$$

The symbolic manipulation indicated can be extended to cover all aspects of the variational calculus. Nonetheless, the precise mathematical formulation should be employed if in doubt.

The Lagrange equation of the second kind for the case of one degree of freedom follows directly from Hamilton's principle. It is only necessary to replace the function f by the Lagrangian L and the coordinate x by the generalised coordinate q in (5.114). The extension to the case of several degrees of freedom is the next topic.

5.4.1.4 Formulation of Hamilton's principle. The extension of the discussion to the case of several degrees of freedom, $q_1 \dots q_n$, is based on a functional of n functions

$$I[q_1 \dots q_n] = \int_{t_1}^{t_2} L(t, q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n) dt. \quad (5.115)$$

The variation of each of the functions q_μ uses the ansatz

$$q_{\mu, v}(t) = q_\mu(t) + \epsilon \varphi_\mu(t) \quad (\mu = 1, 2, \dots, n)$$

with

$$\varphi_\mu(t_1) = \varphi_\mu(t_2) = 0.$$

Expansion of L in a Taylor series around $\epsilon = 0$

$$L(t, q_{1, v} \dots q_{n, v}, \dot{q}_{1, v} \dots \dot{q}_{n, v}) = L(t, q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$$

$$+ \epsilon \left\{ \left[\frac{\partial L}{\partial q_1} \varphi_1 + \frac{\partial L}{\partial \dot{q}_1} \dot{\varphi}_1 \right] + \dots + \left[\frac{\partial L}{\partial q_n} \varphi_n + \frac{\partial L}{\partial \dot{q}_n} \dot{\varphi}_n \right] \right\} + \dots$$

leads to the condition for an extremum of the functional

$$\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \int_{t_1}^{t_2} \sum_{\mu=1}^n \left\{ \frac{\partial L}{\partial q_\mu} \varphi_\mu + \frac{\partial L}{\partial \dot{q}_\mu} \dot{\varphi}_\mu \right\} dt = 0.$$

Partial integration of the second term in each of the contributions to the sum using $\varphi_\mu(t_1) = \varphi_\mu(t_2) = 0$ yields

$$\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \int_{t_1}^{t_2} \left[\sum_{\mu=1}^n \left\{ \frac{\partial L}{\partial q_\mu} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) \right\} \varphi_\mu \right] dt = 0. \quad (5.116)$$

As all the functions φ_μ can be chosen freely the variational equations

$$\frac{\partial L}{\partial q_\mu} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) = 0 \quad (\mu = 1, 2, \dots, n)$$

are found. This shows that the Lagrange equations follow from Hamilton's principle (if a Lagrange function of the system can be defined). The quantity [Lagrangian times time] has the dimension of an action. For this reason Hamilton's principle is often referred to as the **action principle** or (as the extremum corresponds in mechanics in most cases to a minimum) the **principle of least action**.

The Lagrangian, a function of the generalised coordinates, the generalised velocities and time, is the central quantity for the Lagrange form of the equations of motion. It is the Hamiltonian, which plays the main role in Hamilton's form of the equations of motion, which will now be discussed in detail. The Hamiltonian represents, under the conditions stated before (see p. 236), the total energy of the system. This fact emphasises the central role of this quantity in nearly all areas of physics.

5.4.2 Hamilton's equation of motion

The Hamiltonian is connected to the Lagrangian

$$L = L(q_1(t) \dots q_n(t), \dot{q}_1(t) \dots \dot{q}_n(t), t)$$

by a Legendre transformation

$$H = \sum_{\mu} p_{\mu} \dot{q}_{\mu} - L.$$

A set of equations of motion, which is based on the Hamiltonian, can be gained with the following argument. Consider the first differential of the Hamiltonian, which follows from the definition

$$\begin{aligned}
dH &= \sum_{\mu} p_{\mu} d\dot{q}_{\mu} + \sum_{\mu} \dot{q}_{\mu} dp_{\mu} - \sum_{\mu} \frac{\partial L}{\partial q_{\mu}} dq_{\mu} - \sum_{\mu} \frac{\partial L}{\partial \dot{q}_{\mu}} d\dot{q}_{\mu} - \frac{\partial L}{\partial t} dt \\
&= \sum_{\mu} \dot{q}_{\mu} dp_{\mu} - \sum_{\mu} \frac{\partial L}{\partial q_{\mu}} dq_{\mu} - \frac{\partial L}{\partial t} dt .
\end{aligned}$$

The first and the fourth terms cancel due to the definition of the generalised momentum

$$p_{\mu} = \frac{\partial L}{\partial \dot{q}_{\mu}} .$$

Use in addition the Lagrange equation in the form

$$\dot{p}_{\mu} = \frac{\partial L}{\partial q_{\mu}}$$

and find

$$dH = \sum_{\mu} \dot{q}_{\mu} dp_{\mu} - \sum_{\mu} \dot{p}_{\mu} dq_{\mu} - \frac{\partial L}{\partial t} dt . \quad (5.117)$$

This form of the total differential shows explicitly that H is a function of the generalised coordinates, the generalised momenta and (possibly) the time

$$H = H(q_1(t) \dots q_n(t), p_1(t) \dots p_n(t), t) .$$

The Legendre transformation allows a transition from a function of the generalised coordinates and velocities (L) to a function of the generalised coordinates and momenta (H). Using this property of the Hamiltonian $H = H(\mathbf{q}, \mathbf{p}, t)$, a variant of the total differential can be written down

$$dH = \sum_{\mu} \frac{\partial H}{\partial q_{\mu}} dq_{\mu} + \sum_{\mu} \frac{\partial H}{\partial p_{\mu}} dp_{\mu} + \frac{\partial H}{\partial t} dt . \quad (5.118)$$

The differentials of the basic variables are independent of each other. Therefore a comparison of the statements (5.116) and (5.118) yields the set of equations of motion

$$\boxed{\dot{p}_{\mu} = -\frac{\partial H}{\partial q_{\mu}} \quad \dot{q}_{\mu} = \frac{\partial H}{\partial p_{\mu}} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (\mu = 1, 2, \dots, n) .} \quad (5.119)$$

These differential equations are **Hamilton's equations of motion**.

A well used example is the simple, linear harmonic oscillator. The Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2} x^2$$

leads to the equations of motion

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \quad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \frac{\partial H}{\partial t} = 0 . \quad (5.120)$$


These are two differential equations of first order for the two functions $x(t)$ and $p(t)$. The practical way to the solution is, however, obtained by differentiating the second equation with respect to time

$$\ddot{x} = \frac{1}{m}\dot{p}$$

and replacing \dot{p} with the first equation. The result is

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

that is the Lagrangian (or Newtonian) equation of motion.

The statement, that is indicated by this example, is valid in general: The Lagrange equations and the Hamilton equations are completely equivalent. Hamilton's equations can, as Lagrange equations, also be derived directly from Hamilton's principle. The formal difference between the two sets of equations is: the Lagrange equations of motion represent a set of n differential equations of second order for the generalised coordinates $q_\mu(t)$. The Hamilton equations of motion represent a set of $2n$ differential equations of first order for the generalised coordinates q_μ and the generalised momenta p_μ . The equivalence of the two sets of equations of motion follows from a general theorem of the theory of differential equations, which says: a system of n differential equations of second order for n functions can always be recast as a system of $2n$ differential equations for $2n$ functions of first order (see  Math.Chap. 6.1).

The impression, that not much has been gained by the introduction of a second set of equations, is not correct. Hamilton's equations do open new possibilities:

- As H corresponds in most cases to the energy (and as energy is a central concept of physics) the new equations constitute an appropriate starting point for the extension of mechanics (e.g. to quantum mechanics).
- There exist independent methods for the solution of Hamilton's equations, which are based on the concept of canonical transformations.

Before a discussion of this method two new concepts, which play a special role in Hamiltonian mechanics, have to be introduced.

5.4.2.1 The phase space. The time development of a system is described in terms of the $2n$ -tuple

$$\{q_1(t) \dots q_n(t), p_1(t) \dots p_n(t)\}$$

in the Hamiltonian formulation. This suggests the introduction of a $2n$ -dimensional space, which is spanned by the (generalised) coordinates and momenta. This space is the phase space. Every point in phase space characterises a possible instantaneous state of the system. The time development of the system is represented by a curve (a one dimensional manifold)

in this ($2n$ -dimensional) space. Such curves are called **phase space trajectories**. Hamilton's equations of motion with initial conditions determine unique phase trajectories⁵. The phase trajectories of the harmonic oscillator are ellipses, which are specified by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2 = E_0 .$$

The energy value determines the size of the ellipses, the quantity ($k m$) the eccentricity. For a given energy all possible states of the system are described by one of the ellipses. The fact, that the phase space trajectories of this example are closed, expresses the oscillatory character of the motion.

5.4.2.2 The Poisson brackets. The Poisson brackets are an aid, with which the time change of physical quantities can be expressed in a compact fashion. They provide, at a later stage (see Vol. 3), an important link between classical and quantum mechanics. The Poisson brackets for two arbitrary quantities, which depend on the generalised coordinates, the generalised momenta and the time

$$\begin{aligned} u &= u(q_1 \dots q_n p_1 \dots p_n t) \\ v &= v(q_1 \dots q_n p_1 \dots p_n t) , \end{aligned}$$

are defined as

$$\{u, v\} = \sum_{\mu=1}^n \left(\frac{\partial u}{\partial q_{\mu}} \frac{\partial v}{\partial p_{\mu}} - \frac{\partial u}{\partial p_{\mu}} \frac{\partial v}{\partial q_{\mu}} \right) . \quad (5.121)$$

A number of rules for the use of these brackets can be established. The definition (5.121) shows directly, that the brackets are antisymmetric with respect to the change of the sequence of the two quantities

$$\{u, v\} = -\{v, u\} .$$

In addition, the brackets satisfy the **Jacobi identity**

$$\{u\{v, w\}\} + \{v\{w, u\}\} + \{w\{u, v\}\} = 0 . \quad (5.122)$$

This relation can be proven (be it in a slightly tedious fashion) by insertion of the definition. Direct evaluation also provides the proof of the product rule

$$\{vw, u\} = w\{v, u\} + v\{w, u\} . \quad (5.123)$$

Of particular interest are the fundamental Poisson brackets

$$\{q_{\mu}, p_{\nu}\} = \delta_{\mu\nu} \quad \{q_{\mu}, q_{\nu}\} = \{p_{\mu}, p_{\nu}\} = 0 \quad (\mu, \nu = 1, \dots, n) , \quad (5.124)$$

as they constitute one of the links between mechanics and quantum mechanics.

The total change of a quantity $u(\mathbf{q}, \mathbf{p}, t)$ in time is calculated with the chain rule as

⁵ Additional remarks on the phase space can be found in Chap. 5.4.3.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{\mu} \left(\frac{\partial u}{\partial q_{\mu}} \frac{dq_{\mu}}{dt} + \frac{\partial u}{\partial p_{\mu}} \frac{dp_{\mu}}{dt} \right) .$$

Use of the Hamilton equations

$$\dot{q}_{\mu} = \frac{\partial H}{\partial p_{\mu}} \quad \dot{p}_{\mu} = -\frac{\partial H}{\partial q_{\mu}}$$

for the elimination of the time derivatives of the generalised coordinates and momenta gives

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{\mu} \left(\frac{\partial u}{\partial q_{\mu}} \frac{dH}{dp_{\mu}} - \frac{\partial u}{\partial p_{\mu}} \frac{dH}{dq_{\mu}} \right) .$$

The standard abbreviation of this equation is

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial t} + \{u, H\}} . \quad (5.125)$$

Equation (5.125), which characterises the time development of a function $u(\mathbf{q}, \mathbf{p}, t)$, can be used in many ways, as for instance:

- The time development of the system is governed by the Poisson bracket alone, if the function u does not depend explicitly on time

$$\frac{\partial u}{\partial t} = 0 \quad \longrightarrow \quad \frac{du}{dt} = \{u, H\} .$$

The function is a constant of motion, if the Poisson bracket with the Hamiltonian vanishes in addition

$$\{u, H\} = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} = 0 \quad \longrightarrow \quad \frac{du}{dt} = 0 .$$

- On the other hand, the total derivative and the partial derivative with respect to time will be equal, if the partial derivative does not vanish, but the bracket vanishes

$$\{u, H\} = 0 \quad \longrightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial t} .$$

- A symmetric form of Hamilton's equation of motion (5.119) is obtained for $u = q_{\mu}$ or $u = p_{\mu}$

$$\dot{q}_{\mu} = \{q_{\mu}, H\} \quad \dot{p}_{\mu} = \{p_{\mu}, H\} . \quad (5.126)$$

Additional Poisson brackets, which are of use in practical applications, are

$$\{u, q_{\mu}\} = \sum_{\nu=1}^n \left(\frac{\partial u}{\partial q_{\nu}} \frac{\partial q_{\mu}}{\partial p_{\nu}} - \frac{\partial u}{\partial p_{\nu}} \frac{\partial q_{\mu}}{\partial q_{\nu}} \right) = -\frac{\partial u}{\partial p_{\mu}}$$

and correspondingly

$$\{u, p_{\mu}\} = \frac{\partial u}{\partial q_{\mu}} .$$

These relations can also be used to obtain the fundamental Poisson brackets (5.124).

5.4.2.3 Concerning canonical transformations. There exists an independent method for the solution of Hamilton's equations of motion. This method will, however, only be sketched for the simple situation involving only one degree of freedom. The relevant equations are

$$H = H(q, p, t) \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \dot{q} = \frac{\partial H}{\partial p} .$$

The basic idea is to find a transformation of the phase space coordinates

$$\{q, p\} \longrightarrow \{Q, P\}$$

so that as many of the transformed coordinates as possible are cyclical. The transformation to be considered in the case of one generalised coordinate is

$$q = f(Q, P) \quad p = g(Q, P) . \quad (5.127)$$

This transformation is not the most general, as an additional time dependence could be included. However, this form is sufficient for the illustration intended. From the general class of transformations in phase space those are of primary interest, which maintain the form of the equations of motion. This means: after insertion of the transformation into the Hamiltonian

$$\begin{aligned} H(q, p, t) &= H(f(Q, P), g(Q, P), t) \\ &\equiv K(Q, P, t) \end{aligned}$$

the new phase space variables Q and P are expected to satisfy the equations of motion

$$\dot{P} = -\frac{\partial K}{\partial Q} \quad \dot{Q} = \frac{\partial K}{\partial P} . \quad (5.128)$$

Transformations in phase space, which guarantee the form invariance of the equations of motion, are called **canonical transformations**. The question, which has to be answered first, is: how can it be verified, whether a given transformation is canonical? The question can be answered as follows: differentiate the transformation (5.127) with respect to time using the chain rule

$$\dot{q} = \frac{\partial f}{\partial Q} \dot{Q} + \frac{\partial f}{\partial P} \dot{P} \quad \dot{p} = \frac{\partial g}{\partial Q} \dot{Q} + \frac{\partial g}{\partial P} \dot{P} .$$

This system of equations can be resolved with respect to \dot{Q} , respectively with respect to \dot{P} . The result is

$$\frac{\partial g}{\partial P} \dot{q} - \frac{\partial f}{\partial P} \dot{p} = \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) \dot{Q} \quad (5.129)$$

$$\frac{\partial g}{\partial Q} \dot{q} - \frac{\partial f}{\partial Q} \dot{p} = - \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) \dot{P} . \quad (5.130)$$

One then considers

$$\frac{\partial K}{\partial P} = \frac{\partial H}{\partial q} \frac{\partial f}{\partial P} + \frac{\partial H}{\partial p} \frac{\partial g}{\partial P}$$

and inserts the equation of motion in the original coordinates

$$\begin{aligned} \frac{\partial K}{\partial P} &= \frac{\partial g}{\partial P} \dot{q} - \frac{\partial f}{\partial P} \dot{p} \\ &= \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) \dot{Q} . \end{aligned}$$

The second line follows from equation (5.129). The corresponding relation for the derivative with respect to the coordinate can be calculated in the same fashion

$$\begin{aligned} \frac{\partial K}{\partial Q} &= \frac{\partial H}{\partial q} \frac{\partial f}{\partial Q} + \frac{\partial H}{\partial p} \frac{\partial g}{\partial Q} = \frac{\partial g}{\partial Q} \dot{q} - \frac{\partial f}{\partial Q} \dot{p} \\ &= - \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) \dot{P} . \end{aligned}$$

The conclusion to be extracted from these calculations is: form invariance of the equations of motion is guaranteed if the relation

$$\left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q} \right) = 1 \quad (5.131)$$

is satisfied. The expression on the left hand side is reminiscent of the Poisson bracket. It is abbreviated in the form

$$\{q, p\}_{Q, P} = \frac{\partial q}{\partial Q} \frac{\partial p}{\partial P} - \frac{\partial q}{\partial P} \frac{\partial p}{\partial Q} \quad (5.132)$$

and referred to as a **Lagrange bracket**. A canonical transformation can therefore be recognised by the fact that the Lagrange bracket has the value 1.

The next question to be answered is: what is the use of the canonical transformations? An answer to this question is obtained with the following argument: assume that a transformation can be found so that the transformed Hamiltonian does not depend on the coordinate Q

$$K = K(P, t) .$$

The consequence is

$$\begin{aligned} \dot{P} = - \frac{\partial K}{\partial Q} &= 0 \quad \longrightarrow \quad P = \text{const.} \\ \dot{Q} = \frac{\partial K}{\partial P} &= G(P, t) = G(C, t) \quad \longrightarrow \quad Q(t) = \int^t G(C, t') dt' . \end{aligned}$$

The variable Q is cyclical, the integration of the equation of motion is now relatively simple.

The harmonic oscillator with

$$H = \frac{p^2}{2m} + \frac{k}{2}x^2$$

can serve once more to illustrate the point. The transformation

$$x = \left[\frac{2P}{m\omega} \right]^{1/2} \sin Q \quad \omega = \sqrt{\frac{k}{m}}$$

$$p = [2m\omega P]^{1/2} \cos Q$$

is canonical, as one finds

$$\frac{\partial x}{\partial Q} \frac{\partial p}{\partial P} - \frac{\partial x}{\partial P} \frac{\partial p}{\partial Q} = \cos^2 Q + \sin^2 Q = 1.$$

The transformed Hamiltonian is

$$K = \frac{1}{2m} (2m\omega P) \cos^2 Q + \frac{k}{2} \left(\frac{2P}{m\omega} \right) \sin^2 Q = \omega P$$

so that the corresponding equations of motion

$$\begin{aligned} \dot{P} &= 0 & P &= \text{const.} = \frac{E_0}{\omega} \\ \dot{Q} &= \frac{\partial K}{\partial P} = \omega & Q &= \omega t + \delta \end{aligned}$$

are simple. The well established solutions in Cartesian coordinates with the constants of integration E_0 and δ are regained simply

$$\begin{aligned} x(t) &= \left[\frac{2E_0}{m\omega^2} \right]^{1/2} \sin(\omega t + \delta) \\ p(t) &= [2mE_0]^{1/2} \cos(\omega t + \delta). \end{aligned}$$

The transfer of these arguments to the case of an n -dimensional configuration space can be carried out directly, if the quantities q and p are interpreted as vectors in an n -dimensional space and corresponding sums are used in the application of the chain rule.

An answer to the question: "Do there exist methods, which allow the determination of canonical transformations which lead to a cyclical structure of the transformed Hamiltonian?" is, to a certain degree, provided by the *theory of Hamilton-Jacobi*⁶. This theory offers, independent insight into the structure of mechanical problems. It does, however, not offer any advantages over the Lagrangian formulation from a practical point of view. For this reason the formal aspects of theoretical mechanics are concluded with these remarks.

⁶ Appropriate chapters of two text books are quoted in the list of references under [6].

5.4.3 A cursory look into phase space

The study of phase space trajectories is the basis for the analysis of integrable as well as chaotic forms of motion. Chaotic motion is distinguished by the fact, that long term predictions of the motion can, for all practical purposes, not be obtained with sufficient accuracy. This uncertainty is due to an extreme sensitivity of the system to the initial conditions. This sensitivity is the reason that the solutions of the equations of motion differ exponentially for infinitesimally close initial conditions on a sufficiently large time scale. One consequence is, for instance, observed for the rotational motion of one of the moons of Saturn. The moon Hyperion executes a chaotic tumbling motion. Knowledge of the initial orientation with an accuracy of 10 digits is not enough to predict the orientation after a few years with sufficient accuracy.

5.4.3.1 Basic concepts. A complete discussion of the dynamics of nonlinear systems is not possible in the present context. The discussion has to be restricted to the principal features and some of the basic concepts for the simplest situation, that is one-dimensional motion. The equation

$$p = m \dot{q} = \pm [2m(E_0 - U(q))]^{1/2} \quad (5.133)$$

defines a **phase space trajectory** for each value of the initial energy E_0 , if the one dimensional system is characterised by the Hamiltonian

$$H = \frac{p^2}{2m} + U(q) = E_0 .$$

The totality of phase space trajectories is named a **phase space portrait** of the system.

The phase space portrait of the harmonic oscillator, a family of concentric ellipses, constitutes an example, for which every phase space trajectory is restricted to a finite section of phase space. The phase space of the mathematical pendulum (see Chap. 4.2.1), an example of a nonlinear system, shows definitely more structure. The phase space portrait of this system⁷ is indicated in Fig. 5.27a for different values of the total energy. The structure of the phase space trajectories of the mathematical pendulum depends on the initial conditions. They are restricted to an interval $-\pi < q < \pi$ for a proper oscillation of the pendulum or they cover the complete range of the variable $-\pi \leq q \leq \pi$ for a looping motion. A motion with looping can be represented in a more realistic fashion, if the phase space portrait is continued periodically (Fig. 5.27b) or if the trajectories are spread out on the surface of a cylinder which is connected at the points $q = \pm\pi$.

A more qualitative discussion of the situation in the phase space characterized by the differential equation (5.133) can be based on singular points and the **separatrix** (Fig. 5.27), This curve separates the phase space sector

⁷ Plotted is the angular velocity $\dot{q} \equiv \dot{\varphi}$ in units of s^{-1} for a given set of parameters versus the angle $q \equiv \varphi$ instead of p versus q .

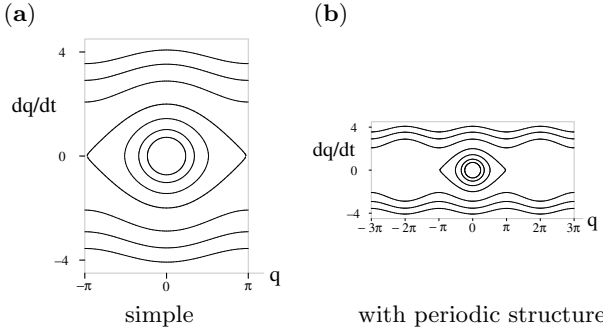


Fig. 5.27. Phase space portrait of the mathematical pendulum

of the 'open' trajectories from that of the 'closed' ones. It runs through specific **equilibrium points**. Equilibrium points are in general characterised by the properties

$$\dot{q}_\nu = \frac{p_\nu}{m_\nu} = 0 \quad \dot{p}_\nu = \frac{\partial U(\dots)}{\partial q_\nu} = 0$$

or specifically for the case of one degree of freedom

$$p_{\text{eq}} = 0 \quad \left. \frac{dU(q)}{dq} \right|_{\text{eq}} = 0.$$

The velocity has the value zero and the potential energy has an extremum. These conditions can be implemented by expanding both sides of the equation for the phase space trajectory (5.133) in the vicinity of an equilibrium point

$$p - p_{\text{eq}} + \dots = \pm (2m)^{1/2} [E_0 - H(q_{\text{eq}}, p_{\text{eq}}) - \frac{1}{2} U''(q_{\text{eq}}, p_{\text{eq}}) (q - q_{\text{eq}})^2 + \dots]^{1/2}.$$

A consistent reorganisation to second order gives an equation

$$(p - p_{\text{eq}})^2 + m U''(q_{\text{eq}}, p_{\text{eq}}) (q - q_{\text{eq}})^2 = 2m (E_0 - H(q_{\text{eq}}, p_{\text{eq}})) \quad (5.134)$$

which describes straight lines, ellipses and hyperbolae. The following possibilities in relation to the parameters involved can be recognized:

- This relation reduces to

$$(p - p_{\text{eq}}) = \pm \sqrt{m U''(q_{\text{eq}}, p_{\text{eq}})} (q - q_{\text{eq}}),$$

if the Hamiltonian and the initial energy coincide at the equilibrium point

$$H(q_{\text{eq}}, p_{\text{eq}}) = E_0$$

and if the potential energy is maximal

$$U''(q_{\text{eq}}, p_{\text{eq}}) < 0.$$

The two intersecting straight lines are part of the separatrix (Fig. 5.28a).

- One obtains hyperbolae (Fig. 5.28a) if the second derivative of the potential energy is smaller than zero

$$U''(q_{\text{eq}}, p_{\text{eq}}) < 0$$

and if the two energy values do not coincide

$$H(q_{\text{eq}}, p_{\text{eq}}) \neq E_0 .$$

This equilibrium point is termed a **hyperbolic singularity**. The motion in the vicinity of the maximum of the potential is (obviously) unstable.

- An equilibrium point with

$$U''(q_{\text{eq}}, p_{\text{eq}}) > 0$$

is called an **elliptic singularity** (Fig. 5.28b). The motion in the vicinity of such equilibrium points is stable.

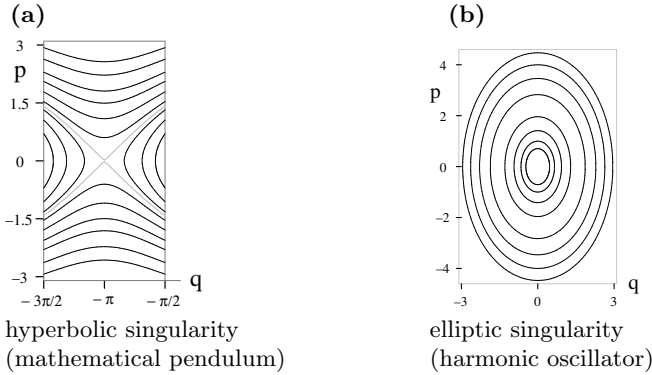


Fig. 5.28. Phase space portraits

The singular points and the separatrix of the mathematical pendulum can be calculated directly. The Hamiltonian (see (4.38))

$$H = \frac{m}{2} l^2 \dot{q}^2 - mgl \cos q = E_0$$

yields the equilibrium points

$$p_{\text{eq}} = ml^2 \dot{q}_{\text{eq}} = 0 \quad \sin q_{\text{eq}} = 0 \quad \longrightarrow \quad q_{\text{eq}} = k\pi \quad (k = 0, \pm 1, \pm 2, \dots) .$$

These points have the following properties:

- The second derivative of the potential energy is $U''(q) = +mgl \cos q$. This shows that elliptic singularities occur for even values of k and hyperbolic singularities for odd values.

- The separatrix includes the points $(q, \dot{q}) = (\pm\pi, 0)$, the corresponding energy is $H(q_{\text{eq}}, p_{\text{eq}}) = E_0 = mgl$. The separatrix is therefore, on the basis of (5.133), characterised by the differential equation

$$\dot{q} = \pm \sqrt{\frac{g}{l}} [2(1 + \cos q)]^{1/2} = \pm 2\omega \cos\left(\frac{q}{2}\right). \quad (5.135)$$

The phase space trajectories are closed curves for $E_0 < mgl$. These curves represent the periodic solutions which have been discussed in Chap. 4.2.1. The pendulum rotates about the suspension point for values of the initial energy, which are larger than mgl .

The motion along the separatrix corresponds to the following sets of initial conditions

- The initial condition $q(0) = 0$, $\dot{q}(0) = 2\omega$ leads to a motion from the lowest to the highest point.
- The motion begins at the highest point and returns to the highest point for the initial condition $q(0) = \pm\pi$, $\dot{q} = 0$.

An explicit equation for the motion on the separatrix can be obtained by integration of the differential equation (5.135), which can be solved by separation of variables with the result

$$\pm\omega t = \ln \left(\tan \left(\frac{q'}{4} + \frac{\pi}{4} \right) \right) \bigg|_{q(0)}^q.$$

Resolution with respect to the angular coordinate gives for instance for the example with the initial condition $q(0) = 0$, $\dot{q}(0) = 2\omega$

$$q(t) = +4 \arctan(e^{\omega t}) - \pi.$$

This result demonstrates that the pendulum needs an infinite amount of time in order to move from a position with $q = 0$ to the highest point $q = \pi$, as the values of the arcustangent are $\arctan 0 = 0$ and $\arctan \infty = \pi/2$. The angular velocity of the motion on the separatrix is obtained by differentiation of the equation for $q(t)$

$$\dot{q} = \frac{2\omega}{\cosh \omega t}.$$

The angular velocity approaches, in accord with the time development of the coordinate, the limiting value zero.

The phase space portrait of the damped oscillator, a dissipative system (see Chap. 4.2.2), can be discussed directly by considering the solutions of the differential equation

$$m\ddot{q} + b\dot{q} + kq = 0.$$

For weak damping the phase space trajectories are spirals (Fig. 5.29a) which approach the equilibrium point $(q, p) = (0, 0)$ asymptotically. A phase space

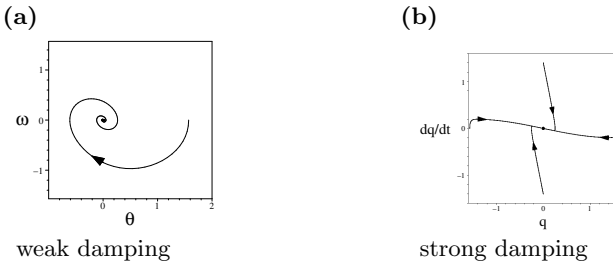


Fig. 5.29. Phase space portrait of the damped oscillator

portrait with spirals, which do not cross,⁸ is called a **focus**, the equilibrium point is a **point attractor**. A point attractor 'attracts' all trajectories in the surrounding phase space (the so called **attractor basin**). For strong damping the phase space trajectories approach the equilibrium point without oscillations (Fig. 5.29b). The corresponding phase space portrait is termed a **knot**.

5.4.3.2 Nonlinear systems and chaos. The situation changes dramatically, if nonlinear systems, which are subjected to driving forces, are considered. A relevant example is the driven mathematical pendulum with damping. The corresponding equation of motion

$$\ddot{q} + b\dot{q} + \omega^2 \sin q = d \sin \Omega t \quad (5.136)$$

contains, besides the terms of the usual mathematical pendulum, a frictional Stokes term and a periodic driving force. This differential equation cannot be solved analytically, however, extensive numerical investigations are possible. Two examples for the numerical solution of the differential equation (5.136) are shown in Fig. 5.30. The example of Fig. 5.30a, which has been obtained with the parameters

$$b = 0.5, \quad \omega = 1, \quad \Omega = 2/3, \quad d = 0.5$$

shows, after a transient period, a periodic, oscillatory motion. The same set of parameters, *except* for $d = 1.2$, have been used for the calculation of the time development in the second example (Fig. 5.30b). The result is obviously not periodic. Quite different structures of the solutions can be obtained for different values of the parameter d .

It should be emphasised that exactly the same result is obtained, if integration of the differential equation is repeated with exactly the same initial values (and the same numerical method). This is a consequence of the deterministic character of the problem. Solutions with close neighbouring values for the initial conditions drift apart in an exponential fashion for the second

⁸ A crossing of phase space trajectories would indicate that a unique solution of the equations of motion does not exist.

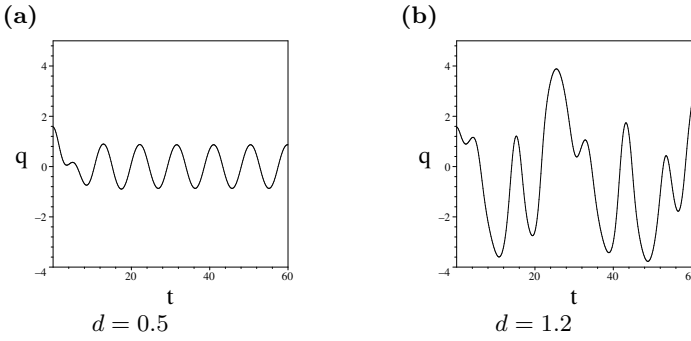


Fig. 5.30. The time development $q(t)$ of the damped, driven pendulum for different parameters (d)

type of solution. This phenomenon is illustrated by the phase space trajectories in Fig. 5.31. The phase space trajectories, which are compared in this figure, are obtained with the same set of parameters (with $d = 1.2$) but a difference generated by a factor of 1.00001 in the initial angle.

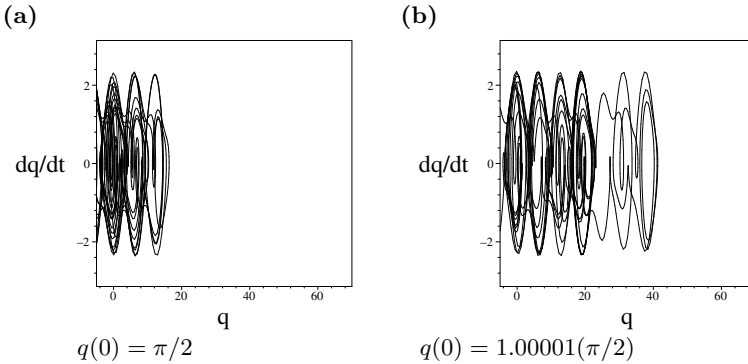


Fig. 5.31. Phase space trajectories of the damped, driven pendulum ($d = 1.2$) for two infinitesimally different initial conditions

Figure 5.32 shows the difference of the two solutions on a semi-logarithmic scale. On the average the two solutions drift apart according to an exponential law. Such a behaviour is called **chaotic**. It is impossible to predict the time development of a realistic, chaotic physical system exactly, if the (possible experimental) uncertainty in the initial values is taken into account.

A representation of the motion in a phase space plot as in Fig. 5.31a,b does not provide sufficient insight into the structure of such solutions. A more distinct representation is possible with **Poincaré cuts**, in which only phase space points for times with $t_k = 2\pi k/\omega$ are included. The Poincaré cut for

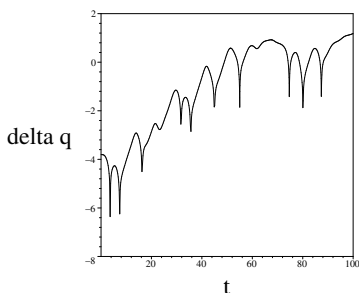


Fig. 5.32. Deviation of the solutions in Fig. 5.31 as a function of time

the trajectory of a *periodic* motion with the frequency ω of the driving force would contain only a single point in such a stroboscopic representation. The pendulum returns to the same point after the same time interval Δt with the same velocity. In Fig. 5.33a the phase space plot of another oscillatory solution (with the parameters $b = 0.5$, $\omega = 1$, $\Omega = 2/3$, $d = 1.1$) of the differential equation (5.136) is illustrated. The Poincaré cut with $\Delta t = 2\pi/3\omega$ contains 6 points in this example, which are 'illuminated' during the time development of the motion in the same sequence (Fig. 5.33b).

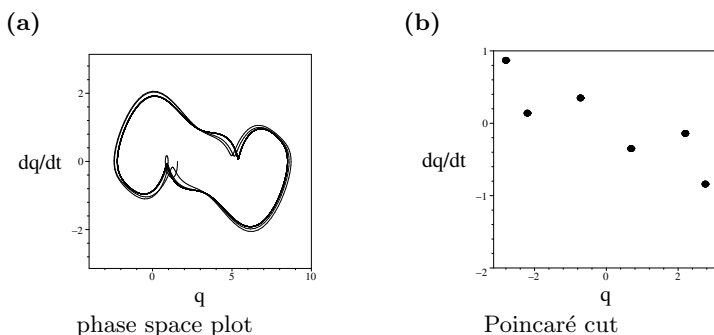


Fig. 5.33. Phase space plot and Poincaré cut of a solution for the driven pendulum (5.136)

Such regular repetitions do not occur for a chaotic motion. The Poincaré cuts can, however, display remarkable structures (see Fig. 5.34). The analysis of such structures is one of the tasks of the dynamics of nonlinear systems.

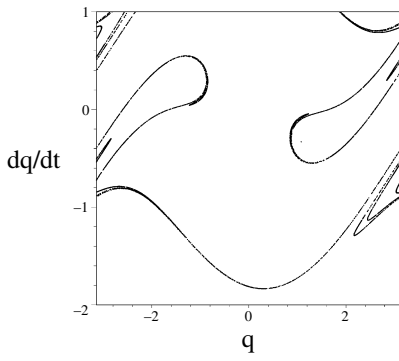


Fig. 5.34. Poincaré cut for a chaotic solution

6 Application of the Lagrange Formalism

Only a few selected examples of the many possibilities to apply the Lagrange equations of the second kind can be presented here. Two subjects will be treated more extensively besides the discussion of systems of coupled harmonic oscillators. These are rotating coordinate systems, a prime example for noninertial systems, and the motion of rigid bodies, the theory of spinning tops.

6.1 Coupled harmonic oscillators

The simplest system of coupled linear oscillators consists of two masses which are connected by a spring. The masses are attached to a suitable support by two additional springs (Fig. 6.1). A linear oscillator chain can be constructed by the addition of further units consisting of a mass and a spring. Such systems are capable of oscillating in the longitudinal (along the chain) as well as in the transverse direction (perpendicular to the chain). In the follow-

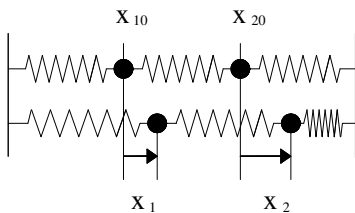


Fig. 6.1. Coupled oscillators: equilibrium position and displacement

ing sections the longitudinal oscillations of such systems, beginning with the simplest system of two masses, will be investigated. As long as it is assumed that the displacements from the equilibrium positions are not too large, the discussion is based on equations of motion, which are linear (corresponding to Hooke's law) in the displacements. This property allows the introduction of appropriate generalised coordinates, the normal coordinates, with the aid of linear transformations, and therefore the analysis of relatively complicated oscillating systems, which can serve as basic models in solid state physics.

The structure of the Lagrange equations is quite similar for transverse oscillations so that the methods and the results from the discussion of longitudinal oscillations can be transferred directly. The aim in this case is solely the derivation of an equation of motion for the transverse oscillations of a string. The string is modelled by a transversely oscillating chain of N point particles followed by the limiting process $N \rightarrow \infty$. The partial differential equation, which describes the oscillations of the string, is an example of a wave equation.

6.1.1 Coupled oscillating system: two masses and three springs

The simplest coupled oscillating system consists of two equal masses (m) and three equal springs (with the spring constant k). In the equilibrium situation the springs are relaxed. Of interest are here the oscillations of the system in the direction of the chain, the x -direction (Fig. 6.1). The displacements from the equilibrium positions x_{10} , x_{20} will be denoted by x_1 and x_2 . Formal constraints are $y_i = z_i = 0$ with $i = 1, 2$. The Lagrange function is specified by the kinetic energy

$$T = \frac{m}{2} \dot{x}_1^2 + \frac{m}{2} \dot{x}_2^2$$

and the potential energy. The two external springs act like external forces. The external potential energy is therefore

$$U = \frac{k}{2}(x_1^2 + x_2^2) .$$

The central spring gives rise to a coupling of the two masses. It acts like an internal force. The spring will be extended or compressed by the amount $|x_1 - x_2|$, if the mass m_1 is displaced by the amount x_1 (positive or negative) and the mass m_2 by the amount x_2 (positive or negative). The potential energy

$$V = \frac{k}{2}(x_1 - x_2)^2$$

can be associated with this spring. The Lagrange function of this mass-spring system

$$L = T - U - V = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 - x_1x_2 + x_2^2) \quad (6.1)$$

leads to the equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = m\ddot{x}_2 - kx_1 + 2kx_2 = 0 .$$

The two differential equations are coupled. A direct solution is not simple. An alternative is an investigation of the possibility to introduce appropriate generalised coordinates (q_1 and q_2) which leads to decoupled equations of motion.

6.1.1.1 Eigenmodes. The observation which suggests a choice of the generalised coordinates is the fact, that the internal potential energy depends on the difference of the Cartesian coordinates. It seems therefore worthwhile to try the sum and the difference of the Cartesian coordinates as an ansatz for the two generalised coordinates

$$q_1 = x_1 + x_2 \quad q_2 = x_1 - x_2 .$$

The inverse of this linear transformation is

$$x_1 = \frac{1}{2}(q_1 + q_2) \quad x_2 = \frac{1}{2}(q_1 - q_2) . \quad (6.2)$$

These relations lead to a transformed Lagrangian in the generalised coordinates with the terms

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) = \frac{m}{8}(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2 + \dot{q}_1^2 - 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) \\ &= \frac{m}{4}(\dot{q}_1^2 + \dot{q}_2^2) \\ U + V &= \frac{k}{8}(q_1^2 + 2q_1q_2 + q_2^2 + 4q_2^2 + q_1^2 - 2q_1q_2 + q_2^2) \\ &= \frac{k}{4}(q_1^2 + 3q_2^2) . \end{aligned}$$

The Lagrangian

$$L = \frac{m}{4}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{k}{4}(q_1^2 + 3q_2^2) \quad (6.3)$$

indicates already that the equations of motion in the generalised coordinates will be decoupled. One derives directly

$$\frac{m}{2}\ddot{q}_1 + \frac{k}{2}q_1 = 0 \longrightarrow \ddot{q}_1 + \omega_1^2 q_1 = 0 \quad \text{with} \quad \omega_1 = \sqrt{\frac{k}{m}} \quad (6.4)$$

$$\frac{m}{2}\ddot{q}_2 + \frac{3k}{2}q_2 = 0 \longrightarrow \ddot{q}_2 + \omega_2^2 q_2 = 0 \quad \text{with} \quad \omega_2 = \sqrt{\frac{3k}{m}} .$$

These differential equations characterise two harmonic oscillators with different frequencies. The general solution of the differential equations (6.4) can be written down immediately

$$q_1(t) = A_1 \cos(\omega_1 t + \delta_1) \quad q_2(t) = A_2 \cos(\omega_2 t + \delta_2) . \quad (6.5)$$

The coordinates q_1 and q_2 are the **normal coordinates** or **normal (or eigen) modes** of the system. The corresponding frequencies ω_1 and ω_2 are the **normal frequencies or eigenfrequencies**.

An interpretation of the normal coordinates can be obtained, if special sets of initial conditions are considered.

The *symmetric* normal mode corresponds for instance to the initial conditions

$$x_1(0) = x_2(0) = A \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 .$$

Both masses are initially displaced by the same amount to the right (Fig. 6.2). The left spring is stretched, the right one is compressed. The central spring is relaxed. The initial conditions for the normal coordinates are

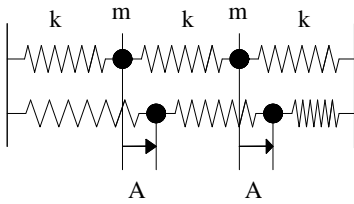


Fig. 6.2. Symmetric normal mode: initial conditions for the Cartesian coordinates

$$q_1(0) = 2A \quad q_2(0) = 0 \quad \dot{q}_1(0) = \dot{q}_2(0) = 0 ,$$

so that the special solution

$$q_1(t) = 2A \cos \omega_1 t \quad q_2(t) = 0$$

is obtained. The motion of the individual masses is described in Cartesian coordinates by the equations

$$x_1(t) = \frac{1}{2}q_1(t) = A \cos \omega_1 t \quad x_2(t) = \frac{1}{2}q_1(t) = A \cos \omega_1 t .$$

The two masses oscillate synchronously in the same direction with the frequency ω_1 .

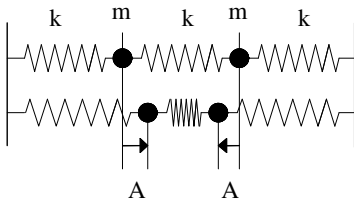


Fig. 6.3. Antisymmetric normal mode: initial conditions for the Cartesian coordinates

The *antisymmetric* normal mode is determined by the initial conditions

$$x_1(0) = -x_2(0) = A \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 .$$

Both external springs are stretched in this case, the central spring is compressed (Fig. 6.3).

The corresponding statements for the time development of the motion are

$$q_1(0) = 0 \quad q_1(t) = 0 \quad \dot{q}_1(0) = 0$$

$$q_2(0) = 2A \quad q_2(t) = 2A \cos \omega_2 t \quad \dot{q}_2(0) = 0$$

$$x_1(t) = A \cos \omega_2 t \quad x_2(t) = -A \cos \omega_2 t .$$

The two masses oscillate with the frequency ω_2 in opposite directions.

The antisymmetric normal oscillation has a higher frequency

$$\omega_1 = \sqrt{\frac{k}{m}} < \omega_2 = \sqrt{\frac{3k}{m}} .$$

This difference in the frequencies can be associated directly with the amount of energy which is originally stored by the system in the two cases. The initial energies of the symmetric (s) and the antisymmetric (a) normal oscillations are, respectively

$$\begin{aligned} E_s(0) &= T_s(0) + U_s(0) + V_s(0) \\ &= 0 + \frac{k}{2}A^2 + \frac{k}{2}A^2 + 0 = kA^2 = \omega_1^2(mA^2) \end{aligned}$$

$$\begin{aligned} E_a(0) &= T_a(0) + U_a(0) + V_a(0) \\ &= 0 + \frac{k}{2}A^2 + \frac{k}{2}A^2 + \frac{k}{2}(2A)^2 = 3kA^2 = \omega_2^2(mA^2) . \end{aligned}$$

The mode with the larger amount of potential energy, initially stored in the springs, exhibits a higher frequency.

6.1.1.2 General oscillatory modes of two mass system. Any additional pattern for the oscillations of the mass-spring system can be obtained by a superposition of the two basic oscillations. Consider e.g. the initial conditions for the Cartesian coordinates

$$x_1(0) = A \quad x_2(0) = 0 \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 . \quad (6.6)$$

The mass m_1 is initially displaced by the amount A to the right. The second mass is at rest in the equilibrium position. One of the external springs is stretched, the central spring is compressed. The initial conditions for the normal coordinates are

$$q_1(0) = q_2(0) = A \quad \dot{q}_1(0) = \dot{q}_2(0) = 0 .$$

This leads to

$$q_1(t) = A \cos \omega_1 t \quad q_2(t) = A \cos \omega_2 t$$

and hence to

$$x_1(t) = \frac{A}{2} (\cos \omega_1 t + \cos \omega_2 t) \quad x_2(t) = \frac{A}{2} (\cos \omega_1 t - \cos \omega_2 t) .$$

The result is sketched in Fig. 6.4. The two normal modes are shown in

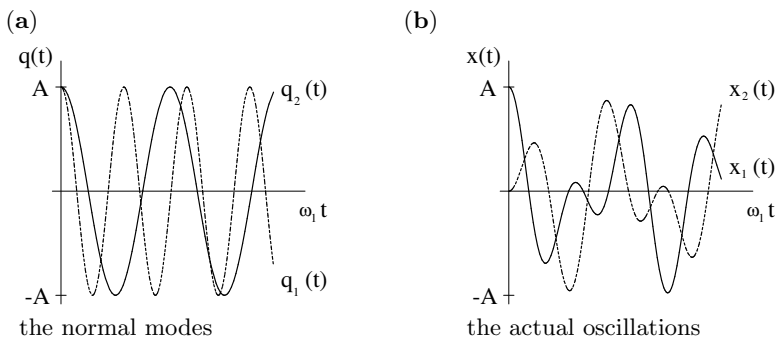


Fig. 6.4. Coupled oscillator (two equal masses, three equal springs) with the initial conditions (6.6)

Fig. 6.4a: the slower symmetric oscillation $q_1(t)$ and the faster (by a factor $\sqrt{3} \approx 1.7$) antisymmetric oscillation $q_2(t)$. The superposition of the two basic oscillations does not yield a harmonic pattern for x_1 or x_2 , but a more complex oscillatory (though periodic) motion (Fig. 6.4b). Whenever the displacement of m_2 is (cum grano salis) large, small displacements are found for m_1 and vice versa. This is a consequence of energy conservation. Such a pattern, which shows up in this example, is of certain technical interest. The continuous interchange of the amplitudes of the two oscillations is referred to as a beat.

6.1.2 Beats

Beats are developed more fully, if the springs in the system are different, for instance if the central spring has a different spring constant (k_2) as compared to the two exterior springs (k_1). The masses are again taken to be equal (Fig. 6.5). The Lagrangian for this system is

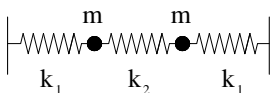


Fig. 6.5. Coupled oscillator (two equal masses, different springs)

$$L = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{k_1}{2} (x_1^2 + x_2^2) - \frac{k_2}{2} (x_1 - x_2)^2 . \quad (6.7)$$

An uncoupled system of equations of motion is also found in this example if the generalised coordinates

$$q_1 = x_1 + x_2 \quad q_2 = x_1 - x_2$$

are used. The transformed Lagrangian

$$L = \frac{m}{4} (\dot{q}_1^2 + \dot{q}_2^2) - \frac{k_1}{4} (q_1^2 + q_2^2) - \frac{k_2}{2} q_2^2 \quad (6.8)$$

leads to the equations of motion

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 &= 0 & \text{with} & \quad \omega_1 = \sqrt{\frac{k_1}{m}} \\ \ddot{q}_2 + \omega_2^2 q_2 &= 0 & \text{with} & \quad \omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}}. \end{aligned} \quad (6.9)$$

The normal oscillations are per definition harmonic oscillations. The results for the frequencies correspond to the previous results if $k_1 = k_2$ (Chap. 6.1.1). The actual motion of the individual masses for the initial conditions (6.6) is once more described by

$$x_1(t) = \frac{A}{2} (\cos \omega_1 t + \cos \omega_2 t) \quad x_2(t) = \frac{A}{2} (\cos \omega_1 t - \cos \omega_2 t) . \quad (6.10)$$

The motion of the two masses is found to couple¹ weakly, if the central spring is very soft ($k_2 \ll k_1$). The approximation

$$\omega_2 = \sqrt{\frac{k_1}{m} + \frac{2k_2}{m}} = \sqrt{\frac{k_1}{m}} \sqrt{1 + \frac{2k_2}{k_1}} \approx \omega_1 + 2\Delta \quad \left(\Delta = \frac{1}{2} \omega_1 \frac{k_2}{k_1} \right) \quad (6.11)$$

can be derived in this case with the aid of the binomial expansion. The relations

$$\begin{aligned} \cos \alpha + \cos \beta &= 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \\ \cos \alpha - \cos \beta &= -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right), \end{aligned}$$

which are obtained directly from the sum and difference formulae of the trigonometric functions, can be used to rewrite the sum and the difference of the cosine functions. The arguments of the trigonometric functions on the right hand side, with $\alpha = \omega_1 t$ and $\beta = \omega_2 t$, are

$$\frac{1}{2} (\omega_1 - \omega_2) t \approx -\Delta t \quad \frac{1}{2} (\omega_1 + \omega_2) t \approx \omega_1 t .$$

The difference of the two frequencies Δ is (by assumption) small and can be neglected in comparison with ω_1 . The time development of the Cartesian coordinates is therefore given by

¹ A spring with a smaller spring constant experiences a larger extension for a given force.

$$x_1(t) \approx [A \cos \Delta t] \cos \omega_1 t \quad x_2(t) \approx -[A \sin \Delta t] \sin \omega_1 t. \quad (6.12)$$

The corresponding motion can be described as follows: The term $\cos \Delta t$ represents a cosine function that changes slowly with time, the term $\cos \omega_1 t$ corresponds to a fast oscillation. The product of the amplitude A and the slow component can be regarded as an amplitude varying slowly with time for small values of Δ . This factor constitutes the envelope for the second cosine function which oscillates much faster (Fig. 6.6a). The motion of the mass m_2 is described by a corresponding function with a sinusoidal oscillation within a sinusoidal envelope (Fig. 6.6b).

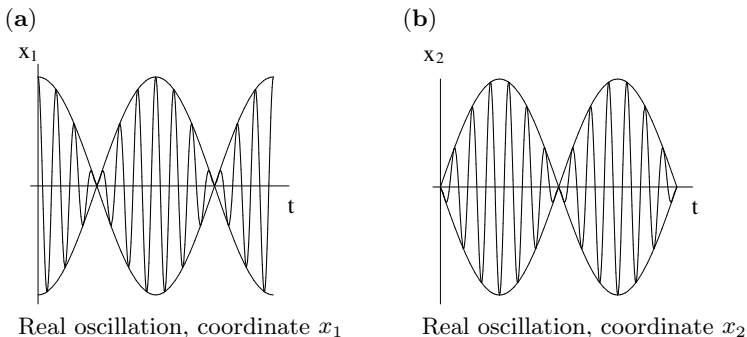


Fig. 6.6. Beats (for $k_2 < k_1$)


A comparison of the two solutions invites the comment: the initial conditions were chosen, so that the mass m_1 is initially displaced and that the mass m_2 is at rest. The amplitude of the second mass increases slowly as a consequence of the weak coupling to the motion of the first mass, the amplitude of this mass decreases. The first mass is nearly at rest after some time, while the second one oscillates at full force. The exchange of energy between the two masses is then reversed and repeated periodically. A periodic exchange of energy (with the period Δ) between the two masses takes place. This phenomenon of beats occurs also for different exterior springs and different masses. The exchange of energy is, however, not complete in this case.

The phenomenon of beats was first (about 1900) observed in electrical engineering. A circuit of two alternating current generators (electrical oscillators), connected in parallel, developed current and voltage oscillations, which follow the pattern indicated for the mechanical equivalent. They were (naturally) not very welcome in this case.

6.1.3 The linear oscillator chain

A more general problem will be discussed after these simpler examples: the linear oscillator chain of N masses. This problem constitutes a classic example

for the application of the Lagrange formalism to more complicated problems of motion. The linear oscillator chain has played a role in the foundation of the (classical) theory of solids. The discussion centres, from a mathematical point of view, on the algebraic **eigenvalue problem**.

Eigenvalue problems are met in different forms in physics. Examples are the discussion of the solution of differential equations with boundary conditions or the application of linear transformations, which lead to homogeneous, linear systems of algebraic equation. Algebraic eigenvalue problems are reviewed in  Math.Chap. 3.2.4.

In the simplest chain only nearest neighbours are connected by springs. It is composed of N different masses and $N + 1$ different springs (Fig. 6.7). The task is the calculation of possible longitudinal oscillations of this chain. The

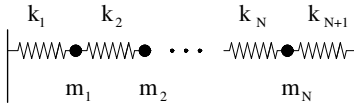


Fig. 6.7. The linear oscillator chain with different masses and different springs with a coupling of nearest neighbours

Lagrangian of this system in terms of Cartesian coordinates, which describe the displacement from the equilibrium positions, is

$$L = \frac{m_1}{2} \dot{x}_1^2 + \dots + \frac{m_N}{2} \dot{x}_N^2 \quad (6.13)$$

$$- \left\{ \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (x_3 - x_2)^2 \right.$$

$$\left. + \dots + \frac{1}{2} k_N (x_N - x_{N-1})^2 + \frac{1}{2} k_{N+1} x_N^2 \right\} .$$

A mathematically more transparent alternative emerges, if all the terms of the potential energy are multiplied out and sorted in the form of products. The Lagrangian then reads

$$L = \sum_{i=1}^N \frac{m_i}{2} \dot{x}_i^2 - \frac{1}{2} \sum_{i=1}^N \sum_{l=1}^N B_{il} x_i x_l . \quad (6.14)$$

The coefficients B_{il} are related to the spring constants by

$$B_{ii} = k_{i+1} + k_i$$

$$B_{i+1,i} = B_{i,i+1} = -k_{i+1} \quad (6.15)$$

$$B_{il} = 0 \quad \text{if } l \neq i \text{ or } l \neq i \pm 1$$

in the case of a coupling of nearest neighbours. This form turns out to be convenient, even if the coupling is more complicated than the coupling of nearest neighbours.

The Lagrangian (6.14) leads (in the general case) to the equations of motion

$$m_i \ddot{x}_i + \sum_{l=1}^N B_{il} x_l = 0 \quad (i = 1, 2, \dots, N) . \quad (6.16)$$

6.1.3.1 Determination of the eigenmodes. This system of coupled, homogeneous linear differential equations of second order (with constant coefficients) is to be solved, respectively to be discussed. The examples with two masses suggest, that the differential equations can be decoupled by a linear transformation to normal coordinates

$$x_i(t) = \sum_{\mu=1}^N a_{i\mu} q_{\mu}(t) . \quad (6.17)$$

The form

$$q_{\mu}(t) = A_{\mu} \cos(\omega_{\mu} t + \delta_{\mu}) \quad (6.18)$$

is expected for the normal coordinates. The amplitudes A_{μ} and the phases δ_{μ} are, as in the examples with two masses, to be determined via the initial conditions. The following quantities have to be calculated

- (1) The eigenfrequencies ω_{μ} .
- (2) The expansion coefficients $a_{i\mu}$. The fact that a linear transformation between the coordinates q_{μ} and x_i is adequate, follows from the linearity of the differential equations (6.16) for the coordinates x_i .
- (3) In addition, the consistency of the ansatz (6.18) should be checked in the end.

Insertion of the ansatz (6.17) into the system of differential equations (6.16) and use of the relation

$$\ddot{q}_{\mu}(t) = -\omega_{\mu}^2 q_{\mu} ,$$

in accordance with (6.18), leads to the equations


$$\sum_{\mu=1}^N \left\{ \sum_{l=1}^N B_{il} a_{l\mu} - \omega_{\mu}^2 m_i a_{i\mu} \right\} q_{\mu}(t) = 0 \quad (i = 1, 2, \dots, N) .$$

The following argument can be given if it is assumed ² that the eigenfrequencies are different

$$\omega_1 < \omega_2 < \dots < \omega_{\mu} .$$

As the functions $q_{\mu}(t)$ are linearly independent³ each of the equations above can only be satisfied if the coefficients of $q_{\mu}(t)$ in each of the equations vanishes. As a consequence a system of equations of the form

² The case that some of the frequencies are equal can also be discussed. The discussion is, however, more involved.

³ The concise definition of linear independence is addressed in  Math.Chap. 3.2.4.

$$\begin{array}{rcl}
(B_{11} - m_1\omega_\mu^2)a_{1\mu} + & B_{12}a_{2\mu} + \dots & B_{1N}a_{N\mu} = 0 \\
B_{21}a_{1\mu} + (B_{22} - m_2\omega_\mu^2)a_{2\mu} + \dots & & B_{2N}a_{N\mu} = 0 \\
\vdots & \dots & \\
B_{N1}a_{1\mu} + & B_{N2}a_{2\mu} + \dots & (B_{NN} - m_N\omega_\mu^2)a_{N\mu} = 0,
\end{array}$$

which is summarised as

$$\sum_{l=1}^N B_{il}a_{l\mu} - \omega_\mu^2 m_i a_{i\mu} = 0 \quad (i = 1, 2, \dots, N), \quad (6.19)$$

is found for each index μ ($\mu = 1, \dots, N$). This homogeneous system of linear equations for the coefficients $a_{1\mu} \dots a_{N\mu}$ has then and only then a nontrivial solution if the determinant of the coefficients of the system vanishes.

This theorem of linear algebra is discussed in [● Math.Chap. 3.2.4](#).

This condition presents a possibility to determine all the frequencies (the eigenfrequencies) of the chain. For this purpose the index μ in the determinant of the coefficients is dropped temporarily so that the determinant of the system (6.19) reads

$$\det |B_{il} - \omega^2 m_i \delta_{il}| = 0 \quad (6.20)$$

or in full detail

$$\left| \begin{array}{cccccc}
(B_{11} - m_1\omega^2) & B_{12} & B_{13} & \dots & B_{1N} \\
B_{21} & (B_{22} - m_2\omega^2) & B_{23} & \dots & B_{2N} \\
\vdots & & & \dots & \\
B_{N1} & B_{N2} & B_{N3} & \dots & (B_{NN} - m_N\omega^2)
\end{array} \right| = 0.$$

The evaluation of this determinant yields an equation of N -th degree for ω^2

$$\alpha_1(\omega^2)^N + \alpha_2(\omega^2)^{N-1} + \dots + \alpha_N(\omega^2) + \alpha_{N+1} = 0. \quad (6.21)$$

This equation is the **characteristic** or **secular equation**. Its N roots, the squares of the eigenfrequencies ω_μ^2 (also called the characteristic frequencies), are the solutions of (6.20), for which the transformation between the generalised coordinates q_μ and the Cartesian coordinates x_i is nontrivial. Real (positive) values for ω_μ , are only obtained, if the relation

$$\omega_\mu^2 \geq 0$$

is satisfied for all μ . The fact, that the solutions of the secular equation correspond to the squares of positive definite frequencies, follows from a second theorem of linear algebra: The roots of the secular equation are real and positive, if the matrix of the coefficients (B_{ik}) is real and symmetric. This property is expressed by (6.15). It is actually a consequence of the third axiom for the forces between the springs.

After the calculation of the eigenfrequencies (respectively their squares) the determination of the expansion coefficients $\{a_{i\mu}\}$ for each value of μ is the next task. As each system of linear equations (6.19) is homogeneous, it is only possible to determine ratios of the coefficients, as e.g.

$$\frac{a_{1\mu}}{a_{N\mu}}, \frac{a_{2\mu}}{a_{N\mu}}, \dots, \frac{a_{N-1\mu}}{a_{N\mu}} \quad \text{for } a_{N\mu} \neq 0.$$

This indetermination has no physical consequence. As the inequalities

$$\sum_i m_i a_{i\mu}^2 > 0$$

are valid, each of these sums of positive quantities can be scaled so that

$$\sum_i m_i a_{i\mu}^2 = 1 \quad (\mu = 1, 2, \dots, N). \quad (6.22)$$

The **normalisation** determines all coefficients $a_{i\mu}$ uniquely.

It is possible to prove another property of the coefficients $\{a_{i\mu}\}$ which allows a direct geometrical interpretation. Consider (6.19) for a frequency ω_μ

$$\omega_\mu^2 m_i a_{i\mu} = \sum_l B_{il} a_{l\mu} \quad (i, \mu = 1, 2, \dots, N) \quad (6.23)$$

and the same relation for a different frequency ω_ν ($\nu \neq \mu$)

$$\omega_\nu^2 m_i a_{i\nu} = \sum_l B_{il} a_{l\nu} \quad (i, \nu = 1, 2, \dots, N). \quad (6.24)$$

The i -th equation from the set (6.23) is multiplied by $a_{i\nu}$ and summed over i

$$\omega_\mu^2 \sum_i m_i a_{i\mu} a_{i\nu} = \sum_{il} B_{il} a_{l\mu} a_{i\nu}.$$

The i -th equation of the set (6.24) is multiplied by $a_{i\mu}$ in the same manner and summed over i . Subtraction of the two resulting equations yields the relation

$$(\omega_\mu^2 - \omega_\nu^2) \sum_i m_i a_{i\mu} a_{i\nu} = \sum_{il} (B_{il} a_{i\nu} a_{l\mu} - B_{il} a_{i\mu} a_{l\nu}).$$

The right hand side vanishes due to the symmetry of the coefficients B_{il}

$$\sum_{il} (B_{il} a_{i\nu} a_{l\mu} - B_{il} a_{i\mu} a_{l\nu}) = \sum_{il} a_{i\nu} a_{l\mu} (B_{il} - B_{li}) = 0.$$

As it was assumed that $\omega_\mu \neq \omega_\nu$, it follows, that

$$\sum_i m_i a_{i\mu} a_{i\nu} = 0 \quad \text{for } (\mu \neq \nu, \quad \mu, \nu = 1, 2, \dots, N). \quad (6.25)$$

The normalisation (6.22) and the property of the solution just demonstrated allow the following interpretation: the set of coefficients

$$(\sqrt{m_1} a_{1\mu}, \sqrt{m_2} a_{2\mu}, \dots, \sqrt{m_N} a_{N\mu}) = \mathbf{a}_\mu$$

can be interpreted as the components of a vector in an N -dimensional (abstract) vector space (see **Math.Chap. 3.1.3**). The vector \mathbf{a}_μ is called the **eigenvector** associated with the eigenvalue (eigenfrequency) ω_μ . The results obtained for the eigenvectors can be summarised in the form

$$\mathbf{a}_\mu \cdot \mathbf{a}_\nu = \delta_{\mu\nu} . \quad (6.26)$$

Such a scalar product of eigenvectors is termed an **orthonormality relation**. The N eigenvectors have the length 1 and are perpendicular with respect to each other.

It remains to check the consistency of the ansatz (6.18). The Lagrangian has to be expressed in terms of the normal coordinates for this purpose. The transformation (6.17) gives

$$\dot{x}_i(t) = \sum_n a_{i\mu} \dot{q}_\mu(t)$$

as the expansion coefficients $a_{i\mu}$ are independent of time. The kinetic energy

$$T = \sum_i \frac{m_i}{2} \dot{x}_i^2 = \frac{1}{2} \sum_{i,\mu,\nu} (m_i a_{i\mu} a_{i\nu}) \dot{q}_\mu \dot{q}_\nu$$

is simplified with the orthogonality relation (\sum_i) as

$$T = \frac{1}{2} \sum_{\mu,\nu} \delta_{\mu,\nu} \dot{q}_\mu \dot{q}_\nu = \frac{1}{2} \sum_\mu \dot{q}_\mu^2 .$$

The potential energy in (6.14)

$$U + V = \frac{1}{2} \sum_{il} B_{il} x_i x_l = \frac{1}{2} \sum_{il\mu\nu} (B_{il} a_{i\mu} a_{l\nu}) q_\mu q_\nu$$

is reformulated with the equations of motion (6.19)

$$\sum_{il} (B_{il} a_{l\nu}) a_{i\mu} = \omega_\nu^2 \sum_i m_i a_{i\nu} a_{i\mu} = \omega_\nu^2 \delta_{\mu\nu}$$

so that

$$U + V = \frac{1}{2} \sum_\mu \omega_\mu^2 q_\mu^2$$

is obtained. The Lagrangian in terms of the normal coordinates takes the form

$$L = \frac{1}{2} \sum_\mu (\dot{q}_\mu^2 - \omega_\mu^2 q_\mu^2) . \quad (6.27)$$

This corresponds to the equations of motion

$$\ddot{q}_\mu + \omega_\mu^2 q_\mu = 0 \quad (\mu = 1, 2, \dots, N) . \quad (6.28)$$

The ansatz (6.18) is consistent.

6.1.3.2 Summary of the solution for the linear chain. The algebraic eigenvalue problem is found in many areas of physics. It is therefore opportune to summarise the rather detailed argumentation for the solution of the problem of the linear oscillator chain in the form of a recipe, which can be applied in other circumstances.

Starting point is the Lagrangian (6.14)

$$L = \sum_{i=1}^N \frac{m_i}{2} \dot{x}_i^2 - \frac{1}{2} \sum_{il=1}^N B_{il} x_i x_l .$$

The following steps have to be executed in order to find the solution:

Step 1: Use the ansatz for the normal modes (6.18)

$$q_\mu(t) = A_\mu \cos(\omega_\mu t + \delta_\mu) .$$

The parameters A_μ and δ_μ are to be determined from the initial conditions. Set up the characteristic equation (secular equation) (6.20) or (6.21)

$$\det |B_{il} - \omega^2 m_i \delta_{il}| = 0$$

and determine the eigenfrequencies ω_μ .

Step 2: For each of the eigenfrequencies solve the system of linear equations (6.19)

$$\sum_{l=1}^N (B_{il} - \omega_\mu^2 m_i \delta_{il}) a_{l\mu} = 0 \quad (i = 1, 2, \dots, N \quad \text{for each } \mu)$$

using the normalisation condition (6.22)

$$\sum_{i=1}^N m_i a_{i\mu}^2 = 1$$

to obtain the eigenvectors. The transformation between the Cartesian and the normal coordinates is

$$x_i(t) = \sum_{\mu=1}^N a_{i\mu} q_\mu(t) .$$

Step 3: Calculate the integration constants $\{A_1, \delta_1, \dots, A_N, \delta_N\}$ using the initial conditions

$$\{x_1(0), \dot{x}_1(0), \dots, x_N(0), \dot{x}_N(0)\} .$$

These steps are illustrated with some explicit examples.

6.1.3.3 Examples of linear oscillator chains.

- The first example is the short chain, however with two different masses and three different springs. Starting with the Lagrangian

$$L = \frac{1}{2} (m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2) - \frac{1}{2} (k_1 x_1^2 + k_2 (x_1 - x_2)^2 + k_3 x_2^2) \quad (6.29)$$

one finds for the squares of the eigenfrequencies (see ● Probl. 6.1)

$$\begin{aligned} (\omega^2)_{1,2} &= \frac{1}{2} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2} \right) \\ &\pm \frac{1}{2} \left[\left(\frac{k_1 + k_2}{m_1} - \frac{k_2 + k_3}{m_2} \right)^2 + \frac{4k_2^2}{m_1 m_2} \right]^{1/2} > 0. \end{aligned} \quad (6.30)$$

The eigenfrequencies are the positive square roots of these positive quantities. The results for the simpler cases, which have been discussed before, are contained in (6.30).

The determination of the eigenvectors is more time consuming. The systems of equations are simple for the special case of equal masses and equal springs

$$\begin{aligned} +ka_{11} - ka_{21} &= 0 \\ -ka_{11} + ka_{21} &= 0 \end{aligned} \quad \text{for } \omega_1 = \sqrt{\frac{k}{m}}$$

and

$$\begin{aligned} -ka_{12} - ka_{22} &= 0 \\ -ka_{12} - ka_{22} &= 0 \end{aligned} \quad \text{for } \omega_2 = \sqrt{\frac{3k}{m}}.$$

The solution corresponds (up to the question of normalisation) to the ansatz (6.2), which has been guessed in Chap. 6.1.1.1

$$\begin{aligned} x_1 &= a_{11} q_1 + a_{12} q_2 = \frac{1}{\sqrt{2m}} (q_1 + q_2) \\ x_2 &= a_{21} q_1 + a_{22} q_2 = \frac{1}{\sqrt{2m}} (q_1 - q_2). \end{aligned} \quad (6.31)$$

- The next example is a chain with three equal masses and four equal springs characterised by the Lagrangian

$$\begin{aligned} L &= \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \\ &\quad - \frac{1}{2} (2kx_1^2 - kx_1x_2 - kx_2x_1 + 2kx_2^2 - kx_2x_3 - kx_3x_2 + 2kx_3^2). \end{aligned} \quad (6.32)$$

The secular equation is here

$$\begin{vmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 2k - m\omega^2 \end{vmatrix} = 0.$$

Evaluation of the determinant leads to

$$(2k - m\omega^2)[(2k - m\omega^2)^2 - 2k^2] = 0.$$

The square roots of the solutions of this cubic equation in ω^2 are

$$\omega_1 = \sqrt{(2 - \sqrt{2})\frac{k}{m}} \quad \omega_2 = \sqrt{2\frac{k}{m}} \quad \omega_3 = \sqrt{(2 + \sqrt{2})\frac{k}{m}}. \quad (6.33)$$

The eigenvectors have to be determined next and the initial conditions (see ● D.tail 6.1) have to be incorporated. Three linear systems of equations have to be solved for the determination of the expansion coefficients. The structure is in every case

$$\begin{aligned} (2k - \omega_\mu^2 m)a_{1\mu} - ka_{2\mu} &= 0 \\ -ka_{1\mu} + (2k - \omega_\mu^2 m)a_{2\mu} - ka_{3\mu} &= 0 \\ -ka_{2\mu} + (2k - \omega_\mu^2 m)a_{3\mu} &= 0. \end{aligned}$$

Solution of these systems of equations for the three eigenvalues (6.33) yields the normalised eigenvectors

$$\begin{aligned} \mathbf{a}_1(t) &= \left(\frac{1}{2}, \quad \frac{1}{\sqrt{2}}, \quad \frac{1}{2} \right) \frac{1}{\sqrt{m}} \\ \mathbf{a}_2(t) &= \left(\frac{1}{\sqrt{2}}, \quad 0, \quad -\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{m}} \\ \mathbf{a}_3(t) &= \left(\frac{1}{2}, \quad -\frac{1}{\sqrt{2}}, \quad \frac{1}{2} \right) \frac{1}{\sqrt{m}}. \end{aligned}$$

The determination of a special solution for a given set of initial values requires the solution of a system of six equations for the amplitudes and the phases of the normal modes (6.18). The set of initial values

$$x_1(0) = x_0 \quad x_2(0) = x_3(0) = 0 \quad \dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$$

indicates that the first mass is initially displaced to the right, the others remain at the equilibrium position. The parameters of the normal modes are found to be

$$A_3 = A_1 \quad A_2 = \sqrt{2}A_1 \quad A_1 = \frac{x_0\sqrt{m}}{2}$$

and

$$\delta_1 = \delta_2 = \delta_3 = 0$$

so that the explicit Cartesian solution for this example is

$$\begin{aligned} x_1(t) &= \frac{x_0}{2} \left(\frac{1}{2} \cos \omega_1 t + \cos \omega_2 t + \frac{1}{2} \cos \omega_3 t \right) \\ x_2(t) &= \frac{x_0}{2\sqrt{2}} (\cos \omega_1 t - \cos \omega_3 t) \end{aligned} \quad (6.34)$$

$$x_3(t) = \frac{x_0}{2} \left(\frac{1}{2} \cos \omega_1 t - \cos \omega_2 t + \frac{1}{2} \cos \omega_3 t \right) .$$

The resulting, quite complex pattern of oscillations is illustrated in Fig. 6.8.

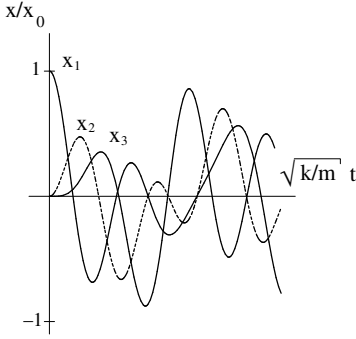


Fig. 6.8. Coupled oscillator: three equal masses and four equal springs

- The linear oscillator chain with a large number of masses oscillating in the longitudinal direction can serve as a model for a one dimensional crystal. The simplest possible model is a chain with N equal masses and $N + 1$ equal springs between next neighbours (respectively the boundary). The eigenfrequencies and the transformation between the Cartesian and the normal coordinates can be calculated in an analytical fashion for this model because of the simple structure of the matrix $[B]$ consisting of the diagonal and a symmetric band next to the diagonal. The specifications of the boundary conditions for the Lagrangian (6.13) with the masses $m_i = m$ and the spring constants $k_i = k$ is complemented by addition of two boundary points

$$x_0(t) = 0 \quad \text{and} \quad x_{N+1}(t) = 0 . \quad (6.35)$$

The equations of motion, that can be extracted from the Lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{N+1} \{ m \dot{x}_i + k(x_i - x_{i-1})^2 \} ,$$

are

$$m \ddot{x}_l - k(x_{l-1} - 2x_l + x_{l+1}) = 0 \quad (l = 1, \dots, N) . \quad (6.36)$$

The ansatz (6.17) and a representation of the normal modes (6.18) by linearly independent sine and cosine functions (instead of one trigonometric function with amplitude and phase)

$$x_l(t) = \sum_{\mu=1}^N (a_{l\mu} \cos \omega_\mu t + b_{l\mu} \sin \omega_\mu t) \quad (6.37)$$

leads, after insertion into the equation of motion for each $l = 1, \dots, N$, to the expression

$$\sum_{\mu=1}^N \{ (-ka_{l-1,\mu} + (2k - m\omega_\mu^2)a_{l\mu} - ka_{l+1,\mu}) \cos \omega_\mu t + (-kb_{l-1,\mu} + (2k - m\omega_\mu^2)b_{l\mu} - kb_{l+1,\mu}) \sin \omega_\mu t \} = 0. \quad (6.38)$$

The boundary condition (6.35) requires

$$a_{l\mu} = b_{l\mu} = 0 \text{ for } l = 0, (N+1) \text{ and for all } \mu = 1, \dots, N.$$

The equations (6.38) can only be satisfied if the individual factors of the sine and cosine function vanish⁴. The resulting system of linear equations for the coefficients $a_{l\mu}$ and $b_{l\mu}$ are identical. It is sufficient to consider one of these systems. A nontrivial solution of the system for $a_{l\mu}$ (suppress the index μ)

$$-ka_{l-1} + (2k - m\omega^2)a_l - ka_{l+1} = 0 \quad (6.39)$$

can only be obtained if the $N \times N$ determinant of the coefficient vanishes

$$\begin{vmatrix} 2k - m\omega^2 & -k & 0 & 0 & \cdots & \cdots & 0 \\ -k & 2k - m\omega^2 & -k & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdots & \cdots & \cdots & \cdot & \cdot & -k \\ 0 & \cdots & \cdots & \cdots & 0 & -k & 2k - m\omega^2 \end{vmatrix} = 0.$$

The evaluation of the characteristic equation of N -th degree by direct means is rather cumbersome for large values of N . A more elegant evaluation is offered by the ansatz

$$a_l = ae^{i(l\alpha - \beta)}. \quad (6.40)$$

The use of the complex exponential function abbreviates the argumentation. Actually, only the real part is of interest.

Simple functions of complex variables are discussed in **Math.Chap. 7**.

The condition

⁴ A formal proof of this statement is based on the use of the integral

$$\int_{-\infty}^{\infty} dt e^{i(\omega_1 - \omega_2)t} = 2\pi \delta(\omega_1 - \omega_2)$$

and the decomposition of the integral into real and imaginary parts. The delta function, which occurs here, will, however, only be introduced in Vol. 2. A less formal argument uses the fact that the trigonometric functions are linearly independent, so that the relation (6.38) can only be satisfied for all values of the variable t , if the coefficients of the functional series vanish.

$$-e^{-i\alpha} + \left(2 - \frac{m}{k}\omega^2\right) - e^{i\alpha} = 0$$

results, if this ansatz is inserted into each of the equations of the system (6.39) and if common factors are eliminated. Resolution with respect to ω^2 yields

$$\begin{aligned}\omega^2 &= \frac{k}{m} (2 - e^{-i\alpha} - e^{i\alpha}) \\ &= 2\frac{k}{m} (1 - \cos \alpha) = 4\frac{k}{m} \sin^2 \frac{\alpha}{2} .\end{aligned}\quad (6.41)$$

In the last step a special form of the sum formula for trigonometric functions has been used.

The relation (6.41) is valid for each of the N roots of the characteristic equation. The explicit form, after reintroduction of the index μ , is

$$a_{l\mu} = a_\mu e^{i(l\alpha_\mu - \beta_\mu)} \quad (l = 0, 1, \dots, (N+1); \quad \mu = 1, \dots, N)$$

and

$$\omega_\mu = 2\sqrt{\frac{k}{m}} \sin \frac{\alpha_\mu}{2} \quad (\mu = 1, \dots, N) . \quad (6.42)$$

The parameters α_μ and β_μ can be determined from the boundary conditions, which demand

$$a_{0\mu} = a_{N+1,\mu} = 0 . \quad (6.43)$$

The condition $a_{0\mu} = 0$ gives

$$0 = a_{0\mu} = a_\mu \cos \beta_\mu$$

if (6.40) is used and if it is assumed (without restricting the generality) that a_μ is real (obviously the expansion coefficients for the expansion of the Cartesian coordinates in terms of trigonometric functions have to be real). This condition requires $\beta_\mu = \pi/2$ (modulo π times an odd number). The second condition in (6.43)

$$0 = a_{N+1,\mu} = a_\mu \cos \left[(N+1)\alpha_\mu - \frac{\pi}{2} \right] = a_\mu \sin [(N+1)\alpha_\mu]$$

then gives

$$\alpha_\mu = \frac{\mu\pi}{N+1} .$$

These results lead to a general formula for the eigenfrequencies

$$\omega_\mu = 2\sqrt{\frac{k}{m}} \sin \left[\frac{\mu\pi}{2(N+1)} \right] \quad (6.44)$$

and, on the basis of a similar calculation for the contribution of the sine functions, to the expansion of the Cartesian coordinates in terms of the generalised coordinates

$$x_l(t) = \sum_{\mu=1}^N \sin\left(\frac{l\mu\pi}{N+1}\right) (a_\mu \cos \omega_\mu t + b_\mu \sin \omega_\mu t). \quad (6.45)$$

The expansion coefficients a_μ , and b_μ (corresponding to A_μ and φ_μ in the previous ansatz for the normal modes) are fixed by the initial conditions (see ● D.tail 6.2)

$$\{x_l(0), \dot{x}_l(0)\} \quad (l = 1, \dots, N).$$

In consequence of the periodic structure of the formula (6.44) for the frequencies only a finite number of eigenvalues occurs. The sum formula for the sine function gives

$$\sin\left(\frac{(N+1+r)\pi}{2(N+1)}\right) = \sin\left(\frac{(N+1-r)\pi}{2(N+1)}\right)$$

so that

$$\omega_{N+1+r} = \omega_{N+1-r} \quad (6.46)$$

follows. The frequency spectrum is repeated, be it in inverse order, if values with $\mu > (N+1)$ are considered. There exist exactly $(N+1)$ eigenfrequencies, if the modes with $\mu = 0$ and $\mu = (N+1)$ are included as a single equivalent **zero mode** (see (6.45)).

The formulae, which have been obtained here, reproduce the results of the previous examples with equal masses and equal springs. The frequency formula (6.44) reduces for instance for the example with $N = 2$ to

$$\omega_1 = 2\sqrt{\frac{k}{m}} \sin \frac{\pi}{6} = \sqrt{\frac{k}{m}} \quad \omega_2 = 2\sqrt{\frac{k}{m}} \sin \frac{\pi}{3} = \sqrt{\frac{3k}{m}}.$$

Besides longitudinal oscillations of the linear oscillator chain transverse oscillations are of interest. In particular, the differential equation for an oscillating string can be derived by investigation of transverse modes. This partial differential equation of second order in time and one coordinate, describes the propagation of transverse wave forms along the string.

6.1.4 The differential equation of an oscillating string

The basis for the model of an oscillating string is a linear oscillator chain in which a set of N equal mass points is attached to an elastic string in the x -direction (Fig. 6.9). The uniform distance of the particles is d in the

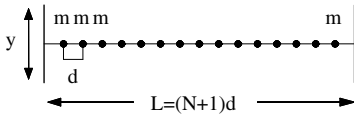


Fig. 6.9. Modelling a uniform, oscillating string

equilibrium situation. The total length of the unextended string is therefore $L = (N + 1)d$. Instead of longitudinal oscillations, transverse oscillations of the mass points, with small displacements in the y -direction, are considered in this model (Fig. 6.10a).

The restoring force acting on the k -th particle, which is caused by the tension of the string due to the position of next neighbours, can be determined in the following fashion: the forces on the k -th mass due to next neighbours are decomposed into an x - and a y -component (see Fig. 6.10b)

$$\mathbf{F}_{k-1,k} = -\tau \sin \theta_{k-1} \mathbf{e}_y - \tau \cos \theta_{k-1} \mathbf{e}_x$$

$$\mathbf{F}_{k+1,k} = -\tau \sin \theta_{k+1} \mathbf{e}_y + \tau \cos \theta_{k+1} \mathbf{e}_x .$$

τ is the string tension, which can be assumed to be equal for both neighbours provided the displacements are small. The following approximations are valid in this case

$$\sin \theta_{k-1} \approx \tan \theta_{k-1} = \frac{y_k - y_{k-1}}{d}$$

$$\sin \theta_{k+1} \approx \tan \theta_{k+1} = \frac{y_k - y_{k+1}}{d}$$

$$\cos \theta_{k-1} \approx \cos \theta_{k+1} \approx 1 .$$

The x -components of the forces due to the neighbours cancel each other. The restoring force acting on the k -th mass is therefore

$$\mathbf{F}_k = \mathbf{F}_{k-1,k} + \mathbf{F}_{k+1,k} = -\frac{\tau}{d} \{ (y_k - y_{k-1}) + (y_k - y_{k+1}) \} \mathbf{e}_y . \quad (6.47)$$

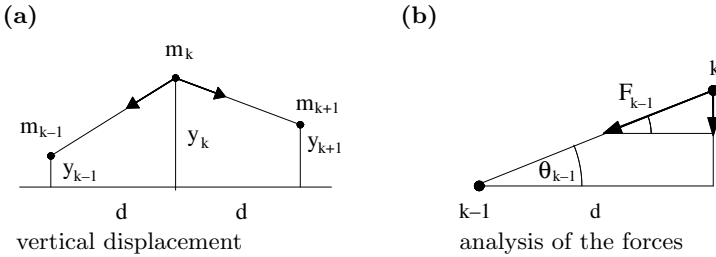


Fig. 6.10. Oscillating string

The boundary points can be included, if two particles, which do not move, are attached at these points

$$y_0(t) = y_{N+1}(t) = 0 , \quad \dot{y}_0(t) = \dot{y}_{N+1}(t) = 0 .$$

The potential energy of the chain oscillating in the transverse direction can be given as

$$U = \frac{1}{2} \frac{\tau}{d} \sum_{i=0}^N (y_{i+1} - y_i)^2 .$$

The derivative of U

$$\mathbf{F}_k = -\frac{\partial U}{\partial y_k} \mathbf{e}_y$$

reproduces the force (6.47). The Lagrangian of the chain, which oscillates in the transverse direction

$$L = \frac{m}{2} \sum_{i=0}^{N+1} \dot{y}_i^2 - \frac{1}{2} \frac{\tau}{d} \sum_{i=0}^N (y_{i+1} - y_i)^2, \quad (6.48)$$

is formally the same as the Lagrangian for longitudinal oscillations (with coupling between next neighbours and equal spring constants). It would be possible to discuss once more normal modes and other features.

In order to make the transition from this model to a realistic string, the limiting processes

$$\begin{aligned} N &\longrightarrow \infty, d \longrightarrow 0, \text{ so that } (N+1)d = L = \text{const.} \\ m &\longrightarrow 0, d \longrightarrow 0, \text{ so that } m/d = \rho = \text{const.} \end{aligned}$$

have to be considered. The limits $N \longrightarrow \infty$ and $d \longrightarrow 0$ have to be approached so that the product $(N+1)d$ remains constant. At the same time the limit $m \longrightarrow 0$ has to be taken in a fashion, so that the ratio m/d remains constant. This ratio is the linear mass density ρ of the (continuous) uniform⁵ string. The Lagrangian (6.48) yields, after division by d , the equations of motion

$$\frac{m}{d} \ddot{y}_i = \frac{F_i}{d} = \tau \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{d^2} \right] \quad (i = 1, \dots, N). \quad (6.49)$$

The transition to the continuous limit requires the replacement

$$y_i(t) \longrightarrow y(x, t),$$

where the position of the i -th mass is counted from the beginning of the chain. The time derivative in (6.49) has to be replaced by a partial derivative, as y is now considered to be a function of two variables. The resulting equation of motion for each line element of the string

$$\frac{m}{d} \frac{\partial^2 y(x, t)}{\partial t^2} = \tau \left[\frac{y(x+d, t) - 2y(x, t) + y(x-d, t)}{d^2} \right]$$

is treated further by carrying out the limiting process on the right hand side of this equation

$$\lim_{d \rightarrow 0} \left[\frac{y(x+d, t) - 2y(x, t) + y(x-d, t)}{d^2} \right] = \frac{\partial^2 y(x, t)}{\partial x^2}.$$

The expression in the square brackets corresponds exactly to the representation of the second partial derivative with respect to x by a difference quotient. In addition, m/d has to be replaced by ρ . The final result is a differential

⁵ The modelling of a nonuniform string with a function $\rho(x)$ is also possible.

equation for the change of the displacement y with time t and position x along the string

$$\frac{\partial^2 y(x, t)}{\partial t^2} - \frac{\tau}{\rho} \frac{\partial^2 y(x, t)}{\partial x^2} = 0 . \quad (6.50)$$

This is the **wave equation** for a continuous, one dimensional system. It is a **partial differential equation** of second order. The standard form of the wave equation contains the quantity

$$v = \sqrt{\frac{\tau}{\rho}} \quad (6.51)$$

with the dimension

$$[v] = \left[\frac{ML}{T^2} \cdot \frac{L}{M} \right]^{1/2} = \left[\frac{L}{T} \right]$$

of a velocity. This velocity actually corresponds to the **phase velocity** of the wave.

A discussion of partial differential equations, including the discussion of the wave equation and its solution is found in Vol. 2.

The standard form of the wave equation is therefore

$$\boxed{\frac{\partial^2 y(x, t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2} = 0 .}$$

A specification of boundary conditions as e.g.

$$y(0, t) = 0 , \quad y(L, t) = 0 \quad (6.52)$$

is required for a complete determination of an explicit solution in the case of a string of length L fixed at both ends. In addition initial conditions concerning the position and the time derivative of the function $y(x, t)$ at time $t = 0$, as e.g.

$$y(x, 0) = f(x) \quad \left. \frac{\partial y(x, t)}{\partial t} \right|_{t=0} = g(x) \quad (6.53)$$

are needed. The string has a definite form at time $t = 0$ and each 'point' has a definite velocity in the transverse direction. Different wave forms (Fig. 6.11) can be generated by this specification of the boundary and the initial conditions. The specification of the boundary conditions (6.52) leads to standing waves with frequencies (or wavelength), which are adapted to the length of the string. Arbitrary wave structures, which move along the string (Fig. 6.11b), are obtained for the more general initial conditions.

The next section deals with the subject of rotating coordinate systems. This subject is a good example for the demonstration of effects observed in noninertial systems. The nature and the structure of the apparent forces in

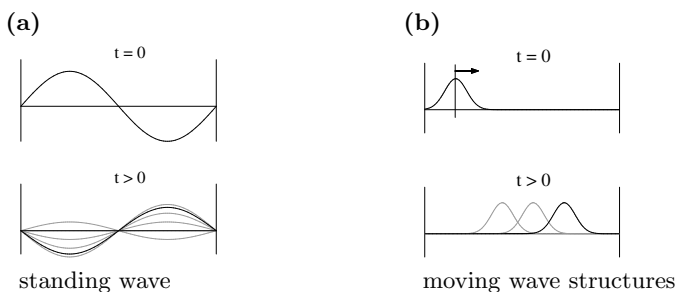


Fig. 6.11. Illustration of oscillating strings

rotating coordinate frames is investigated in the first part of the next section. The second part is devoted to a discussion of the effects of these noninertial forces on the rotating earth.

6.2 Rotating coordinate systems

A coordinate system attached to the earth represents a rotating coordinate system. This raises the question, how the description of motion from the point of view of the rotating earth (our point of view!) will be influenced by the accelerated motion of this frame of reference. The question can be put more precisely in the following fashion (Fig. 6.12): an 'observer' in an *inertial system* S (with the coordinates (x_1, x_2, x_3)) examines a physical process (e.g. the time development of the motion of an object) from the point of view of his/her coordinate system. A second 'observer' registers the same process, however from the point of view of a coordinate system S' (coordinates (q_1, q_2, q_3)), which rotates with respect to the inertial system about a given axis. The axis of rotation passes through the common origin of

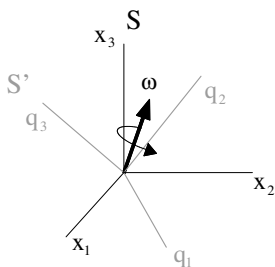


Fig. 6.12. Inertial coordinate system S and rotating system S'

the two coordinate systems. The practical question, that has to be posed, is: how can the description of the time development of the motion of an object from the point of view of S be transcribed into a description of this motion

from the point of view of S' and vice versa? In particular, it should be asked: how can the equations of motion used by the observer in the initial system be transformed into equations of motion for the observer in S' ?

A first answer to these questions is: the observer in S' experiences apparent forces. This fact can be demonstrated with a simple example, a uniform rotation of the system S' about a common 3- axis of the two systems.

6.2.1 Simple manifestation of apparent forces

From the point of view of S a point particle is observed, which is at rest in the $x_1 - x_2$ plane (Fig. 6.13a). This experiment is not really exciting, but it is sufficient for the intended illustration. The observer in S concludes, according to Newton's first axiom, that no forces act on the point particle.

From the point of view of S' the particle moves on a circle (Fig. 6.13b). A physicist using this system may therefore conclude (again with the first axiom): the point particle is neither at rest, nor does it move uniformly, hence it is subjected to a force. As inertial systems are, according to Newton

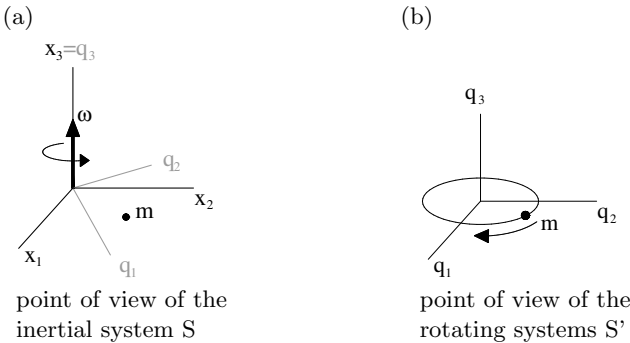


Fig. 6.13. Recognising apparent forces

(and Einstein), the proper reference frames for an appraisal of forces, the force registered by S' is solely due to the noninertial character of this system of reference.

The Lagrange formulation, in the Cartesian coordinates x_i for the inertial system and generalised coordinates q_μ for the rotating system, is an optimal basis for the quantitative investigation of these apparent forces. The transformation between the coordinates for the example of a rotation about the 3- axis can be noted as

$$\begin{aligned} x_1(t) &= q_1(t) \cos \alpha(t) - q_2(t) \sin \alpha(t) \\ x_2(t) &= q_1(t) \sin \alpha(t) + q_2(t) \cos \alpha(t) \\ x_3(t) &= q_3(t) . \end{aligned} \tag{6.54}$$

The fact that this set of equations describes a *counterclockwise* rotation of S' with respect to S can be checked with

$$\begin{aligned} \text{for } \alpha = 0 & : q_1 = x_1 & q_2 = x_2 , \\ \text{for } \alpha = \pi/2 & : q_1 = -x_2 & q_2 = x_1 . \end{aligned}$$

Further argumentation follows the standard pattern outlined in Chap. 5: the Lagrangian from the point of view of S is

$$L = \frac{m}{2} \sum_i \dot{x}_i^2 - U(x_1, x_2, x_3) .$$

The Lagrangian in terms of the generalised coordinates (the coordinates of the rotating system) is obtained with the aid of the transformation (6.55) and

$$\begin{aligned} \dot{x}_1 &= \dot{q}_1 \cos \alpha(t) - \dot{q}_2 \sin \alpha(t) - q_1 \omega(t) \sin \alpha(t) - q_2 \omega(t) \cos \alpha(t) \\ \dot{x}_2 &= \dot{q}_1 \sin \alpha(t) + \dot{q}_2 \cos \alpha(t) + q_1 \omega(t) \cos \alpha(t) - q_2 \omega(t) \sin \alpha(t) \\ \dot{x}_3 &= \dot{q}_3 . \end{aligned}$$

The quantity $\omega(t) = \dot{\alpha}(t)$ is the angular velocity of the rotation. The Lagrangian expressed in terms of the generalised coordinates and velocities is, after direct calculation, found to be

$$\begin{aligned} L = \frac{m}{2} \{ \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\omega(q_1 \dot{q}_2 - q_2 \dot{q}_1) \\ + \omega^2 (q_1^2 + q_2^2) \} - U(q_1, q_2, q_3, \alpha) . \end{aligned} \quad (6.55)$$

The first three terms represent the kinetic energy of the point particle from the point of view of the rotating coordinate system

$$T_R = \frac{m}{2} (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) . \quad (6.56)$$

The remaining terms in the curly brackets and the potential (energy) U can be combined in the form of a generalised potential

$$U^* = -\frac{m}{2} \{ 2\omega(q_1 \dot{q}_2 - q_2 \dot{q}_1) + \omega^2 (q_1^2 + q_2^2) \} + U(q_1, q_2, q_3, \alpha) . \quad (6.57)$$

The additional terms in (6.57) represent the potential energy due to an (apparent) force. The calculation of the equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\mu} \right) - \frac{\partial L}{\partial q_\mu} = 0 \quad (\mu = 1, 2, 3)$$

requires the derivatives

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_1} &= m\dot{q}_1 - m\omega q_2 & \frac{\partial L}{\partial q_1} &= m\omega \dot{q}_2 + m\omega^2 q_1 - \frac{\partial U}{\partial q_1} \\ \frac{\partial L}{\partial \dot{q}_2} &= m\dot{q}_2 + m\omega q_1 & \frac{\partial L}{\partial q_2} &= -m\omega \dot{q}_1 + m\omega^2 q_2 - \frac{\partial U}{\partial q_2} \end{aligned}$$

(and trivial relations for q_3). They can be noted in the form

$$\begin{aligned}
m\ddot{q}_1 - m\dot{\omega}q_2 - 2m\omega\dot{q}_2 - m\omega^2q_1 + \frac{\partial U}{\partial q_1} &= 0 \\
m\ddot{q}_2 + m\dot{\omega}q_1 + 2m\omega\dot{q}_1 - m\omega^2q_2 + \frac{\partial U}{\partial q_2} &= 0 \\
m\ddot{q}_3 + \frac{\partial U}{\partial q_3} &= 0 .
\end{aligned} \tag{6.58}$$

Three types of forces can be recognised. The terms $m\ddot{q}_i$ represent the inertial forces from the point of view of the system S' . The active forces correspond to the partial derivatives of the potential function $-\partial U/\partial q_i$. The remaining terms must be interpreted as **apparent forces**, which arise due to the motion of the coordinate system with respect to an inertial system. The apparent forces, which are inspected more closely in section (6.2.2), correct – so to speak – the equations of motion for the observer in a noninertial system.

6.2.2 General discussion of apparent forces in rotating coordinate systems

The general discussion (an arbitrary axis of rotation through the common origin of the two coordinate systems) of the transformation and the equations of motion is best accomplished with a vectorial formulation. The decomposition of the position vector with respect to the trihedral of the inertial system (Fig. 6.14)

$$S : \quad \mathbf{r}(t) = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3 \tag{6.59}$$

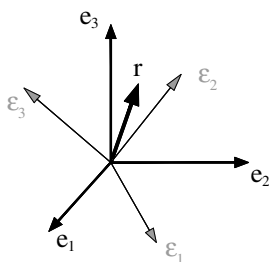


Fig. 6.14. Basis of the two coordinate systems

is referred to a set of basis vectors, which do (per definition) not depend on time. The decomposition of the position vector with respect to the rotating coordinate system is

$$S' : \quad \mathbf{r}(t) = q_1(t)\boldsymbol{\varepsilon}_1(t) + q_2(t)\boldsymbol{\varepsilon}_2(t) + q_3(t)\boldsymbol{\varepsilon}_3(t) . \tag{6.60}$$

The unit vectors of the rotating system change with time. Exceptions are e.g. the example of a rotation about the common 3- axis with

$$\varepsilon_3(t) = \mathbf{e}_3 .$$

The velocity of a point particle from the point of view of the inertial system S is as usual

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}_1(t)\mathbf{e}_1 + \dot{x}_2(t)\mathbf{e}_2 + \dot{x}_3(t)\mathbf{e}_3 .$$

The corresponding definition of the velocity from the point of view of S' has the same form

$$\mathbf{v}_R(t) = \dot{q}_1(t)\varepsilon_1(t) + \dot{q}_2(t)\varepsilon_2(t) + \dot{q}_3(t)\varepsilon_3(t) . \quad (6.61)$$

The rotating observer registers the change of the coordinates q_μ with time. He/she is not yet aware of the fact that the system rotates.

On the other hand, the result is

$$\begin{aligned} \dot{\mathbf{r}}(t) &= \dot{q}_1(t)\varepsilon_1(t) + \dot{q}_2(t)\varepsilon_2(t) + \dot{q}_3(t)\varepsilon_3(t) \\ &\quad + q_1(t)\dot{\varepsilon}_1(t) + q_2(t)\dot{\varepsilon}_2(t) + q_3(t)\dot{\varepsilon}_3(t) \\ &\text{or} \\ \mathbf{v}(t) &= \mathbf{v}_R(t) + \Delta\mathbf{v}(t) , \end{aligned} \quad (6.62)$$

if the full time derivative of the decomposition of the position vector in S' (6.60) is taken into account. The velocities, that are measured by the respective observers (\mathbf{v} for system S or \mathbf{v}_R for system S'), are not equal. They differ by an additional term. This term arises because the frame of reference of the rotating observer and not only the object under observation move during the measurement of the velocity (in the time interval between t and $t + dt$).

The argumentation is simpler for a rotation about the common 3- axis, which will be indicated first. The starting point is the transformation between the basis vectors of the two coordinate systems (compare (2.52) and (2.53))

$$\begin{aligned} \varepsilon_1(t) &= \mathbf{e}_1 \cos \alpha(t) + \mathbf{e}_2 \sin \alpha(t) \\ \varepsilon_2(t) &= -\mathbf{e}_1 \sin \alpha(t) + \mathbf{e}_2 \cos \alpha(t) \\ \varepsilon_3(t) &= \mathbf{e}_3 . \end{aligned} \quad (6.63)$$

The time derivative of this transformation

$$\begin{aligned} \dot{\varepsilon}_1 &= \omega(-\mathbf{e}_1 \sin \alpha + \mathbf{e}_2 \cos \alpha) = \omega\varepsilon_2 \\ \dot{\varepsilon}_2 &= -\omega(\mathbf{e}_1 \cos \alpha + \mathbf{e}_2 \sin \alpha) = -\omega\varepsilon_1 \\ \dot{\varepsilon}_3 &= \mathbf{0} \end{aligned}$$

shows that the additional velocity term has the form

$$\Delta\mathbf{v} = -\omega q_2 \varepsilon_1 + \omega q_1 \varepsilon_2 .$$

In order to express the additional term in vectorial form a vector for the angular velocity has to be defined. The definition in this simpler situation is

$$\boldsymbol{\omega} = \omega \mathbf{e}_3 = \omega \varepsilon_3$$

(the sense of rotation and the corresponding vector of the angular velocity are related by the right hand rule) so that

$$\Delta \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \\ 0 & 0 & \omega \\ q_1 & q_2 & q_3 \end{vmatrix} = -q_2 \omega \boldsymbol{\varepsilon}_1 + q_1 \omega \boldsymbol{\varepsilon}_2 .$$

The transformation between the Cartesian and the generalised velocity components for the simpler situation can be summarised as

$$\mathbf{v}(t) = \mathbf{v}_R(t) + \boldsymbol{\omega}(t) \times \mathbf{r}(t) . \quad (6.64)$$

This vector equation has to be read in the following fashion: the left hand side represents the decomposition of the velocity with respect to the system S . The right hand side has to be decomposed with respect to the system S' . Application of the transformation between the basis vectors to the left hand side gives the right hand side and vice versa. This vectorial relation is, as will be shown immediately, also valid in the case of a rotation about an arbitrary axis.

Before the general case is discussed, it is opportune to derive the equations of motion for the case of a rotation about the common 3- axis. The acceleration vectors from the point of view of the two coordinate systems are

$$S : \quad \mathbf{a}(t) = \ddot{x}_1(t) \mathbf{e}_1 + \ddot{x}_2(t) \mathbf{e}_2 + \ddot{x}_3(t) \mathbf{e}_3$$

$$S' : \quad \mathbf{a}_R(t) = \ddot{q}_1(t) \boldsymbol{\varepsilon}_1(t) + \ddot{q}_2(t) \boldsymbol{\varepsilon}_2(t) + \ddot{q}_3(t) \boldsymbol{\varepsilon}_3(t) .$$

The equations of motion in the rotating coordinate system can be obtained from

$$\frac{d v_R(t)}{d t} = a_R(t) + \dot{q}_1(t) \dot{\boldsymbol{\varepsilon}}_1(t) + \dot{q}_2(t) \dot{\boldsymbol{\varepsilon}}_2(t) .$$

The explicit result, already given in (6.58), can be summarised in the form

$$m \mathbf{a}_R(t) = -m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) - 2m(\boldsymbol{\omega} \times \mathbf{v}_R) - m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) + \mathbf{Q} . \quad (6.65)$$

The generalised force \mathbf{Q} corresponds to the gradient of the (transformed) potential U in (6.57) with respect to the coordinates q_μ

$$\mathbf{Q} = \left\{ -\frac{\partial U}{\partial q_1}, -\frac{\partial U}{\partial q_2}, -\frac{\partial U}{\partial q_3} \right\} .$$

A comparison of the individual terms in (6.65) with those in (6.58) can be given in a few lines:

$$\begin{aligned} \dot{\boldsymbol{\omega}} \times \mathbf{r} &= \begin{vmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \\ 0 & 0 & \dot{\omega} \\ q_1 & q_2 & q_3 \end{vmatrix} = (-\dot{\omega} q_2) \boldsymbol{\varepsilon}_1 + (\dot{\omega} q_1) \boldsymbol{\varepsilon}_2 \\ \boldsymbol{\omega} \times \mathbf{v}_R &= \begin{vmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \\ 0 & 0 & \omega \\ \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{vmatrix} = (-\omega \dot{q}_2) \boldsymbol{\varepsilon}_1 + (\omega \dot{q}_1) \boldsymbol{\varepsilon}_2 \end{aligned} \quad (6.66)$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \begin{vmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \\ 0 & 0 & \omega \\ -\omega q_2 & \omega q_1 & 0 \end{vmatrix} = (-\omega^2 q_1) \boldsymbol{\varepsilon}_1 + (-\omega^2 q_2) \boldsymbol{\varepsilon}_2 .$$

It turns out, that (6.65) is also valid in the case of a general rotation.

Three of the terms in (6.65) correspond to apparent forces. The first term, which only occurs if the rotation is accelerated, does not carry a name

$$\mathbf{F}_{\dot{\boldsymbol{\omega}}} = -m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) . \quad (6.67)$$

The term, which depends on the velocity of the point particle registered in the rotating frame, is called the **Coriolis force**

$$\mathbf{F}_C = -2m(\boldsymbol{\omega} \times \mathbf{v}_R) . \quad (6.68)$$

The term with the double cross product is the **centrifugal force**

$$\mathbf{F}_Z = -m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) . \quad (6.69)$$

The demonstration of the fact that the same result (in vector form) can be obtained with the Lagrangian formulation in the general case requires the use of a representation of a rotation in three space dimensions (compare Chap. 6.3.5). The corresponding calculation can be carried out but it is quite lengthy. It cannot be avoided, if the rotation of rigid bodies (see Chap. 6.3) is discussed. It is, fortunately, possible to shorten the discussion in the present context by appealing directly to the vectorial formulation. Any vector \mathbf{A} can be decomposed with respect to the two coordinate systems (inertial and rotating)

$$\begin{aligned} \mathbf{A}(t) &= a_1(t) \mathbf{e}_1 + a_2(t) \mathbf{e}_2 + a_3(t) \mathbf{e}_3 \\ &= A_1(t) \boldsymbol{\varepsilon}_1(t) + A_2(t) \boldsymbol{\varepsilon}_2(t) + A_3(t) \boldsymbol{\varepsilon}_3(t) . \end{aligned}$$

The time derivative is, as discussed above,

$$\begin{aligned} \dot{\mathbf{A}}(t) &= \dot{a}_1(t) \mathbf{e}_1 + \dot{a}_2(t) \mathbf{e}_2 + \dot{a}_3(t) \mathbf{e}_3 \\ &= \dot{A}_1(t) \boldsymbol{\varepsilon}_1(t) + \dot{A}_2(t) \boldsymbol{\varepsilon}_2(t) + \dot{A}_3(t) \boldsymbol{\varepsilon}_3(t) \\ &\quad + A_1(t) \dot{\boldsymbol{\varepsilon}}_1(t) + A_2(t) \dot{\boldsymbol{\varepsilon}}_2(t) + A_3(t) \dot{\boldsymbol{\varepsilon}}_3(t) . \end{aligned}$$

It is necessary, to find a general expression for the vectors $\dot{\boldsymbol{\varepsilon}}_\mu$. The vector of the angular velocity $\boldsymbol{\omega}$ is used as a marker for this purpose. This vector indicates the axis of rotation, its absolute value is the magnitude of the angular velocity. The unit vectors $\boldsymbol{\varepsilon}_\mu$ move on cones with the opening angle $2\Theta_\mu$ about this axis (Fig. 6.15).

The following elementary consideration leads to a relation for the time derivative of the vectors $\boldsymbol{\varepsilon}_\mu$: The end point of the vector $\boldsymbol{\varepsilon}_\mu(t)$ describes a circle about the $\boldsymbol{\omega}$ -axis (Fig. 6.15a). The magnitude of the orbital velocity (see (2.56), p. 55) of the end point is $|\dot{\boldsymbol{\varepsilon}}_\mu(t)| = \rho \omega(t)$. The radius vector $\boldsymbol{\rho}$, which is perpendicular to the $\boldsymbol{\omega}$ -axis, has the length $\rho = \sin \theta_\mu$, as $\boldsymbol{\varepsilon}_\mu(t)$ is a unit vector (Fig. 6.15b). The direction of the vector $\boldsymbol{\omega} \times \boldsymbol{\varepsilon}_\mu$ is according to the right hand rule identical with the direction of the vector $\dot{\boldsymbol{\varepsilon}}_\mu(t)$ (Fig. 6.15b).

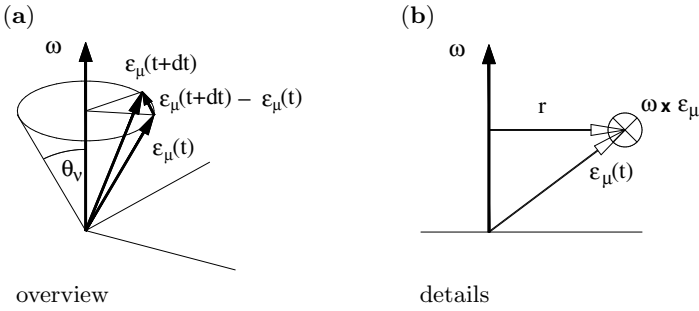


Fig. 6.15. The time derivative of the vectors ε_i

As the absolute value of the cross product ($\sin \theta_\mu \omega(t)$) is also identical with the magnitude of the vector $\dot{\varepsilon}_\mu(t)$, the relation

$$\dot{\varepsilon}_\mu = \boldsymbol{\omega} \times \varepsilon_\mu$$

must be valid in general. This result allows the formulation of a general relation between the time derivative of a vector $\mathbf{A}(t)$ from the point of view of the two coordinate systems

$$\left. \frac{d}{dt} \mathbf{A} \right|_I = \left. \frac{d}{dt} \mathbf{A} \right|_R + \boldsymbol{\omega} \times \mathbf{A} . \quad (6.70)$$

This relation is valid for every vector. For this reason it is often abbreviated with the short hand

$$\left. \frac{d}{dt} \right|_I = \left. \frac{d}{dt} \right|_R + \boldsymbol{\omega} \times , \quad (6.71)$$

where it is understood that this relation can be applied to any vector. Again the left hand side refers to the decomposition of any vector in the inertial system, the right hand side to the decomposition in the rotating system.

The following points can be noted in particular:

- (1) The velocity transformation (6.64) is valid in general.
- (2) The angular acceleration is the same for the two coordinate systems

$$\dot{\boldsymbol{\omega}}|_I = \dot{\boldsymbol{\omega}}|_R = \dot{\boldsymbol{\omega}} . \quad (6.72)$$

- (3) Differentiation of the velocity transformation yields the relation between the acceleration in the two reference systems

$$\begin{aligned} \left. \frac{d}{dt} (\mathbf{v}_I) \right|_I &= \left. \frac{d}{dt} (\mathbf{v}_R + (\boldsymbol{\omega} \times \mathbf{r})) \right|_R + \boldsymbol{\omega} \times (\mathbf{v}_R + (\boldsymbol{\omega} \times \mathbf{r}))_R \\ \mathbf{a}_I &= \mathbf{a}_R + [\dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_R + \boldsymbol{\omega} \times \mathbf{v}_R + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]_R . \end{aligned} \quad (6.73)$$

This relation, which is indexed explicitly for clarity, has already been obtained for the simpler example.

These general results will be illustrated by two additional, transparent examples demonstrating the role of the apparent forces.

6.2.2.1 Examples for apparent, noninertial forces. In the first example the relative motion of the reference systems is once more a uniform rotation ($\dot{\omega} = 0$) about the common 3- axis

$$\boldsymbol{\omega} = \omega \mathbf{e}_3 = \omega \boldsymbol{\varepsilon}_3 .$$

A mass point, as seen by the inertial system (Fig. 6.16), rotates uniformly on a circle in the x_1 - x_2 plane. This rotation can be described by a vectorial

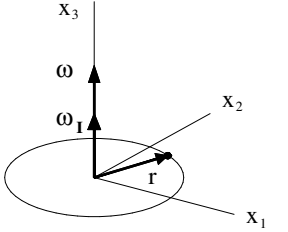


Fig. 6.16. Noninertial reference systems I: circular motion in S

angular velocity $\boldsymbol{\omega}_I = \omega_I \mathbf{e}_3$. The relation between the orbital velocity and the angular velocity is

$$\mathbf{v}_I = \boldsymbol{\omega}_I \times \mathbf{r}$$

(as $\boldsymbol{\omega}_I \perp \mathbf{r}$ in this example). This circular motion is due to a central force \mathbf{F} , for which the stability condition

$$\mathbf{F} = m \boldsymbol{\omega}_I \times (\boldsymbol{\omega}_I \times \mathbf{r})$$

(the mass multiplied by the central acceleration equals the acting force) applies.

The following observations describes the point of view of the rotating coordinate system:

- The velocity of the mass point is

$$\mathbf{v}_R = \mathbf{v}_I - \boldsymbol{\omega} \times \mathbf{r} = (\boldsymbol{\omega}_I - \boldsymbol{\omega}) \times \mathbf{r} .$$

The orbital velocity from the point of view of the rotating frame depends on the difference of the two angular velocities. The mass point is at rest from the point of view of the rotating system, if the mass and the coordinate system rotate in the same sense and if the angular velocities have the same magnitude $\boldsymbol{\omega} = \boldsymbol{\omega}_I$. The velocity is $\mathbf{v}_R = -\boldsymbol{\omega} \times \mathbf{r}$, if the mass point is at rest in the inertial system $\boldsymbol{\omega}_I = \mathbf{0}$. From the point of view of S' the mass point moves, as if it had the angular velocity $-\boldsymbol{\omega}$.

- The forces on the mass point as observed in the rotating system are

$$\begin{aligned} \mathbf{F}_R &= \mathbf{F} + \mathbf{F}_C + \mathbf{F}_Z \\ &= m \boldsymbol{\omega}_I \times (\boldsymbol{\omega}_I \times \mathbf{r}) - 2m \boldsymbol{\omega} \times ((\boldsymbol{\omega}_I - \boldsymbol{\omega}) \times \mathbf{r}) - m \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) . \end{aligned}$$

This expression can be condensed in the form

$$\mathbf{F}_R = m(\boldsymbol{\omega}_I - \boldsymbol{\omega}) \times [(\boldsymbol{\omega}_I - \boldsymbol{\omega}) \times \mathbf{r}] ,$$

if the two angular velocities are, as explicitly assumed, proportional to each other $\boldsymbol{\omega}_I = a \boldsymbol{\omega}$ (same or opposite direction). The rotating observer observes a circular motion, which is determined by the difference of the angular velocities. The centrifugal force and the Coriolis force combine to act as an apparent central force.

The same relative motion of the two reference systems is assumed in the next example (Fig. 6.17). The mass point moves, however, with a constant velocity along the x_1 - axis from the point of view of the inertial system. The

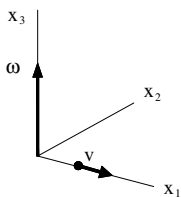


Fig. 6.17. Noninertial reference systems II: straight line motion in S

question to be answered is: how does the mass point move from the point of view of the rotating observer? The question will, this time, be answered by an explicit solution of the equations of motion in the rotating system. The equations of motion (6.65) for the coordinates q_1 and q_2 are

$$\ddot{q}_1 = 2\omega\dot{q}_2 + \omega^2 q_1 \quad (6.74)$$

$$\ddot{q}_2 = -2\omega\dot{q}_1 + \omega^2 q_2 . \quad (6.75)$$

The coordinate q_3 can be ignored. As there are no active forces in the inertial system, only the Coriolis and the centrifugal forces contribute in the rotating system. The initial conditions for the motion as seen in this system are $q_1(0) = q_2(0) = 0$, the mass point starts at the origin, and $\dot{q}_1(0) = v$, $\dot{q}_2(0) = 0$, the mass moves initially in the q_1 - direction which coincides at $t = 0$ with the x_1 - direction.

The two differential equations are coupled. In order to decouple them, the first equation is differentiated twice, the second equation once with respect to time

$$\dddot{q}_1 = 2\omega \ddot{q}_2 + \omega^2 \dot{q}_1 \quad (6.76)$$

$$\ddot{q}_2 = -2\omega \dot{q}_1 + \omega^2 q_2 . \quad (6.77)$$

The velocity \dot{q}_2 can be eliminated from (6.77) with (6.74)

$$\dot{q}_2 = \frac{1}{2\omega} \ddot{q}_1 - \frac{\omega}{2} q_1 .$$

The result is

$$\ddot{q}_2 = -2\omega\ddot{q}_1 + \frac{\omega}{2}\ddot{q}_1 - \frac{\omega^3}{2}q_1 = -\frac{3}{2}\omega\ddot{q}_1 - \frac{\omega^3}{2}q_1 .$$

Insertion into (6.76) leads to

$$\ddot{q}_1 + 2\omega^2\ddot{q}_1 + \omega^4q_1 = 0 . \quad (6.78)$$

The ansatz $q_1 = \exp(\lambda t)$ for the solution of this homogeneous linear differential equation of fourth order with constant coefficients yields the characteristic equation

$$\lambda^4 + 2\omega^2\lambda^2 + \omega^4 = 0 .$$

This equation has two double roots

$$\lambda_1 = \lambda_2 = i\omega \quad \lambda_3 = \lambda_4 = -i\omega .$$

The general solution is therefore

$$q_1(t) = (C_1 + C_2t)e^{i\omega t} + (C_3 + C_4t)e^{-i\omega t} . \quad (6.79)$$

The solution of a differential equation of fourth order contains, as expected, four integration constants. The initial conditions $q_1(0) = 0$ and $\dot{q}_1(0) = v$ lead to two additional statements, as \ddot{q}_1 as well as \ddot{q}_2 are determined by derivatives of lower order according to (6.74) and (6.75)

$$\begin{aligned} \ddot{q}_1 &= 2\omega\dot{q}_2 + \omega^2q_1 & \longrightarrow & \ddot{q}_1(0) = 0 \\ \ddot{q}_1 &= 2\omega\ddot{q}_2 + \omega^2\dot{q}_1 = -3\omega^2\dot{q}_1 + 2\omega^2q_2 & \longrightarrow & \ddot{q}_1(0) = -3\omega^2v . \end{aligned}$$

The implementation of the initial conditions is slightly tedious. The solution has to be differentiated three times and the resulting system of linear equations for the coefficients $C_1 \dots C_4$ at time $t = 0$ has to be solved. The final result (it must be real!) of this calculation is (see ● D.tail 6.3)

$$q_1(t) = vt \cos \omega t .$$

The calculation of $q_2(t)$ involves a suitable combination of the basic equations (6.74) and (6.75)

$$\left. \begin{aligned} q_2(t) &= \frac{1}{\omega^2}\ddot{q}_2(t) + \frac{2}{\omega}\dot{q}_1(t) \\ \ddot{q}_2(t) &= \frac{1}{2\omega}\ddot{q}_1 - \frac{\omega}{2}\dot{q}_1(t) \end{aligned} \right\} \rightarrow q_2 = \frac{1}{2\omega^3}\ddot{q}_1 + \frac{3}{2}\frac{1}{\omega}\dot{q}_1 = -vt \sin \omega t .$$

The trajectory, which is described by the two equations, is a spiral. The spiral is traversed as indicated in Fig. 6.18. The separation from the origin grows, as might have been expected, linearly with time.

The result, calculated with some labour, can be obtained in a much simpler way. The position vector in the inertial system with the initial conditions stated is

$$\mathbf{r}(t) = vte_1 .$$

The vector e_1 transforms in the present example as

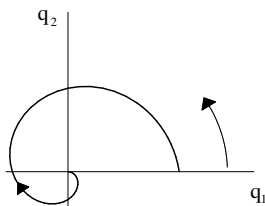


Fig. 6.18. Noninertial reference systems II: spiral motion in S'

$$\mathbf{e}_1 = \varepsilon_1(t) \cos \omega t - \varepsilon_2(t) \sin \omega t$$

so that the result from the point of view of S'

$$\mathbf{r}(t) = (vt \cos \omega t) \varepsilon_1 + (-vt \sin \omega t) \varepsilon_2 \quad (6.80)$$

follows directly.

The conclusion that can be extracted from the last example is: decide in all cases beforehand, whether it is simpler to solve the equation of motion in the rotating system or to transform the solution in the inertial system into the rotating frame. The second option is by far simpler in the last example.

6.2.3 Apparent forces and the rotating earth

The discussion of apparent forces on the rotating earth is really quite complicated, if the aim is a very accurate description of the situation. The earth undergoes a complicated tumbling motion which is composed of three individual rotations:

- (1) Rotation of the earth about the north-south axis (ω).
- (2) Rotation of the earth around the sun (ω_{ES}). The facts that the rotational axis of the earth is inclined with respect to the plane of the orbit around the sun (declination) and that the motion of the earth around the sun is accelerated have to be included.
- (3) Rotation of the complete planetary system about the centre of the spiral nebula milky way (ω_{S}).

The effect of rotations (2) and (3) can, however, quite safely be neglected in comparison with the effects of rotation (1). The angular velocity of the first rotation is

$$\omega = \frac{2\pi}{\text{day}} = \frac{2\pi}{24 \cdot 3600} \text{ s}^{-1} = 7.272 \cdot 10^{-5} \text{ s}^{-1} \quad \dot{\omega} \approx 0.$$

This number is not terribly impressive. On the other hand, a rather respectable velocity of

$$v_{\text{Äq}} = R_{\text{E}} \omega = (6.38 \cdot 10^3 \text{ km}) \omega \approx 1670 \frac{\text{km}}{\text{h}}$$

is found for the velocity of an object resting on the equator. The rotation of the earth is from west to the east (the sun rises in the east), the vector ω

points to the north. The rotation is considered to be uniform, as it can be assumed, that the relation $\dot{\boldsymbol{\omega}} = \mathbf{0}$ is satisfied to a good approximation.

From the point of view of an earth-bound system the following forces (Fig. 6.19) act on a mass m , which is at rest on the surface of the earth:

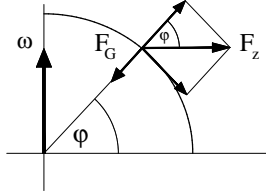


Fig. 6.19. Forces on a mass resting on the surface of the earth

- (1) The gravitation $\mathbf{F}_G = -m\mathbf{g}$, which is directed towards the centre of the earth.
- (2) The centrifugal force $\mathbf{F}_Z = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ $|\mathbf{r}| = R_E$.

The vector \mathbf{F}_Z is parallel to the equatorial plane and directed outward. The magnitude of this vector depends on the geographical latitude⁶ φ

$$F_Z = mR_E\omega^2 \cos \varphi .$$

This dependence follows from

$$|\boldsymbol{\omega} \times \mathbf{r}| = R_E\omega \sin(90^\circ - \varphi) = R_E\omega \cos \varphi$$

and the fact that the angle between $\boldsymbol{\omega} \times \mathbf{r}$ and $\boldsymbol{\omega}$ equals 90° . With the values for R_E and ω , the centrifugal acceleration is found to be

$$a_Z = 3.4 \cos \varphi \frac{\text{cm}}{\text{s}^2} .$$

The centrifugal acceleration is relatively weak in comparison with the gravitational acceleration

$$a_G = 980 \frac{\text{cm}}{\text{s}^2} .$$

It amounts only to about 0.35 % of the gravitation, but its effects can nonetheless be observed. The total acceleration in the vertical direction is

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \mathbf{a}_{ZV} = -(g - \omega^2 R_E \cos^2 \varphi) \mathbf{e}_r ,$$

where \mathbf{F}_Z has been decomposed into components in the direction of and a component perpendicular to the gravitational force (Fig. 6.19). The acceleration due to gravity is decreased slightly and depends on the geographical latitude. The reduction vanishes at the poles. The magnitude of the horizontal components is

⁶ The equator has the latitude 0° , the north pole $+90^\circ = \pi/2$, the south pole -90° .

$$a_{ZH} = \omega^2 R_E \cos \varphi \sin \varphi .$$

The direction is towards the south on the northern hemisphere and towards the north on the southern hemisphere. The dependence on the latitude (northern hemisphere) is illustrated in Fig. 6.20. Objects should start

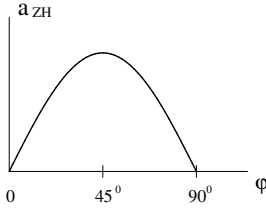


Fig. 6.20. Horizontal component of the centrifugal acceleration on the northern hemisphere

to move towards the equator under the influence of this component. This is not observed. One reason (besides friction) is the following: the earth is not a sphere but a geoid. The earth possesses plastic properties and has, in the course of its existence, adjusted to the centrifugal forces. The distribution of mass on the earth is approximately such, that the vector

$$\mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

is always perpendicular to a plane tangential to the surface of the earth.

6.2.3.1 Free fall on the rotating earth. The discussion of free fall from the point of view of the rotating earth has to include the apparent forces. An object, that falls towards the earth from a height h , experiences a deviation from the vertical due to the Coriolis force. This deviation will now be determined. A local coordinate system is (as in the actual experiment) used for this purpose (Fig. 6.21). The coordinate system is located on the

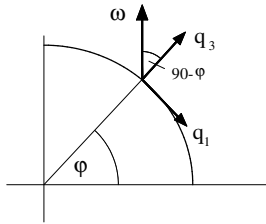


Fig. 6.21. Local trihedron, schematic

surface of the geoid in such a fashion, that the q_1 - direction corresponds to the north-south tangent, the q_2 - direction to the west-east tangent and the q_3 - direction to the vertical.

The equations of motion for the free fall on the rotating earth with $\dot{\boldsymbol{\omega}} = \mathbf{0}$ are

$$m\mathbf{a}_R = m\mathbf{g}_{\text{eff}} - 2m(\boldsymbol{\omega} \times \mathbf{v}_R) . \quad (6.81)$$

Centrifugal effects are included in \mathbf{g}_{eff} . The vectors have to be decomposed into components with respect to the local system. It can safely be assumed that the earth possesses a spherical shape for this purpose. The decomposition of the angular velocity in dependence of the latitude φ is

$$\boldsymbol{\omega} = -\omega \cos \varphi \boldsymbol{\varepsilon}_1 + \omega \sin \varphi \boldsymbol{\varepsilon}_3 , \quad (6.82)$$

the relevant factor of the Coriolis force

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{v}_R &= \begin{vmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \\ -\omega \cos \varphi & 0 & \omega \sin \varphi \\ \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{vmatrix} \\ &= (-\omega \dot{q}_2 \sin \varphi) \boldsymbol{\varepsilon}_1 \\ &\quad + (\omega \dot{q}_1 \sin \varphi + \omega \dot{q}_3 \cos \varphi) \boldsymbol{\varepsilon}_2 + (-\omega \dot{q}_2 \cos \varphi) \boldsymbol{\varepsilon}_3 . \end{aligned}$$

The equations of motion in terms of components are therefore

$$\begin{aligned} \ddot{q}_1 &= 2\omega \dot{q}_2 \sin \varphi \\ \ddot{q}_2 &= -2\omega \dot{q}_1 \sin \varphi - 2\omega \dot{q}_3 \cos \varphi \\ \ddot{q}_3 &= 2\omega \dot{q}_2 \cos \varphi - g_{\text{eff}} . \end{aligned} \quad (6.83)$$

The obvious symmetry of this system of equations is no accident but a consequence of the energy principle. The relation

$$\sum_{i=1}^3 \dot{q}_i \ddot{q}_i + g_{\text{eff}} \dot{q}_3 = 0$$

is obtained if the i -th equation is multiplied with \dot{q}_i and all three expressions are added. This result can be rewritten as

$$\frac{d}{dt} \left(\sum_i \frac{1}{2} \dot{q}_i^2 + g_{\text{eff}} q_3 \right) = \frac{1}{m} \left[\frac{d}{dt} (T + U_{\text{eff}}) \right] = 0 . \quad (6.84)$$

The Coriolis force does not do any work, as it is perpendicular to the instantaneous direction of motion for all times.

An approximate solution of the differential equations (6.83) for the fall on the rotating earth can be found with the following argument. It can be expected because of the small value of ω , that the deviations from the vertical are small. It is therefore possible to neglect \dot{q}_1 and \dot{q}_2 on the right hand side of the equations of motion in comparison with \dot{q}_3 . The approximate equations of motion are therefore

$$\begin{aligned} \ddot{q}_1 &= 0 \\ \ddot{q}_2 &= -2\omega \dot{q}_3 \cos \varphi \\ \ddot{q}_3 &= -g_{\text{eff}} . \end{aligned} \quad (6.85)$$

A 'standard free fall experiment' is characterised by the initial conditions

$$\begin{aligned} q_1(0) = q_2(0) = 0 & \quad q_3(0) = h \\ \dot{q}_i(0) = 0 & \quad (i = 1, 2, 3) . \end{aligned}$$

The object is initially at the height h above the surface of the earth and at rest with respect to the rotating earth. The solutions of the first and the third equations of the system (6.85) (for the initial conditions above) are

$$q_1(t) = 0 \quad q_3(t) = h - \frac{1}{2}g_{\text{eff}}t^2 . \quad (6.86)$$

The solution for q_3 can be inserted into the second differential equation. The result is (again for the initial conditions stated)

$$q_2 = \frac{1}{3}\omega t^3 g_{\text{eff}} \cos \varphi . \quad (6.87)$$

There exists a deviation from the vertical in the easterly direction on the northern as well as the southern hemisphere ($-\pi/2 \leq \varphi \leq \pi/2$). Insertion of definite values as e.g. $h = 100$ m and $g_{\text{eff}} = 9.8$ m/s² into the equation for q_3 , gives

$$q_3(T) = 0 = 100 - 4.9 T^2 ,$$

that is a time of $T = 4.52$ s for the duration of the fall. The easterly deviation at a geographical latitude φ of 45° is

$$q_2(T) \approx 1.6 \text{ cm}$$

in this case. The deviation is small, but it can be measured. The rotation of the earth can be verified by this experiment.

The steps for an exact solution of the differential equation (6.83) are given in **• D.tail 6.4** (part 1). The solution with the initial conditions for the free fall from rest is

$$\begin{aligned} q_1(t) &= \frac{ABg}{8\omega^4} \left\{ (\omega t)^2 + \frac{1}{2}(\cos 2\omega t - 1) \right\} \\ q_2(t) &= \frac{Bg}{4\omega^3} \left\{ (\omega t) - \frac{1}{2} \sin 2\omega t \right\} \\ q_3(t) &= h - \frac{1}{2}gt^2 + \frac{B^2g}{8\omega^4} \left\{ (\omega t)^2 + \frac{1}{2}(\cos 2\omega t - 1) \right\} . \end{aligned} \quad (6.88)$$

The quantities A and B stand for

$$A = 2\omega \sin \varphi \quad B = 2\omega \cos \varphi . \quad (6.89)$$

The functions $\sin \omega t$ and $\cos \omega t$ can be expanded in a power series. The approximation discussed beforehand is obtained in lowest order. Inclusion of additional terms of the expansion allows the derivation of more accurate approximations. In addition to the easterly deviation a southerly deviation is found on the northern hemisphere ($0 < \varphi < \pi/2$). This deviation

is $1.8 \cdot 10^{-4}$ cm at $\varphi = 45^\circ$ for an initial height of $h = 100$ m. It can not be measured in an actual experiment.

The occurrence of an easterly deviation can be understood in a simple, qualitative fashion: the orbital velocity of the point of impact from the point of view of an inertial system is $v_i = R_E \omega \cos \varphi$. The orbital velocity of the mass at height h is on the other hand $v_m = (R_E + h) \omega \cos \varphi$. The object about to fall has a larger velocity in the easterly direction as the point of impact. For this reason it will hit the earth east of the starting position.

6.2.3.2 Projectile motion on the rotating earth and related effects.

A set of initial condition as

$$\begin{aligned} q_1(0) &= q_2(0) = q_3(0) = 0 \\ \dot{q}_1(0) &= v_1 \quad \dot{q}_2(0) = 0 \quad \dot{q}_3(0) = v_3 > 0 \end{aligned}$$

describes projectile motion on the rotating earth with an initial velocity in the north-south ($v_1 > 0$) or south-north ($v_1 < 0$) direction. The exact solution of the differential equations (6.83) with these initial conditions is (see ● Probl. 6.4)

$$\begin{aligned} q_1(t) &= v_1 t + \frac{ABg}{8\omega^4} \left\{ (\omega t)^2 + \frac{1}{2}(\cos 2\omega t - 1) \right\} \\ &\quad + \frac{A}{4\omega^3} (Av_1 + Bv_3) \left\{ \frac{1}{2} \sin 2\omega t - (\omega t) \right\} \\ q_2(t) &= \frac{Bg}{4\omega^3} \left\{ (\omega t) - \frac{1}{2} \sin 2\omega t \right\} \\ &\quad + \frac{1}{4\omega^2} (Av_1 + Bv_3) \{ \cos 2\omega t - 1 \} \\ q_3(t) &= v_3 t - \frac{1}{2} g t^2 + \frac{B^2 g}{8\omega^4} \left\{ (\omega t)^2 + \frac{1}{2}(\cos 2\omega t - 1) \right\} \\ &\quad + \frac{B}{4\omega^3} (Av_1 + Bv_3) \left\{ \frac{1}{2} \sin 2\omega t - (\omega t) \right\} . \end{aligned} \tag{6.90}$$

The parameters A and B are as before

$$A = 2\omega \sin \varphi \quad B = 2\omega \cos \varphi .$$

An easterly or westerly deviation can also be found in this case. Initial velocities of $v_1 = 500$ m/s in a northerly direction and a vertical component of $v_3 = 100$ m/s lead to a projectile range of about 10 km. The target in the north is missed by about 15 m for this range.

Effects of the Coriolis force can in addition be observed in the following phenomena:

- The formation of cyclones.
- The course of the gulf stream.

The sense of rotation in the first two examples is due to the apparent forces. Cyclones in the northern hemisphere rotate, as does the gulf stream, in the sense of the clock.

- The course of rivers. A deviation towards the east can (*cum grano salis*) be observed for the Siberian rivers flowing to the north, as there are few geographical impediments acting against the Coriolis forces in the Northern Siberian plane.
- The formation of vortices, e.g. in connection with the famous 'bath tub problem'. The possibility of a different sense of rotation for the formation of vortices during the draining of bath tubs on the northern or the southern hemisphere has been discussed heatedly for some time. So far this question could not be resolved by experiment due to 'technical difficulties'.
- The Foucault pendulum. The Foucault pendulum is a mathematical pendulum, which is observed during a sufficiently long period on the rotating earth. It is perfectly suited to demonstrate the effects of the Coriolis force or in other words the rotation of the earth.

6.2.3.3 The Foucault pendulum. The appropriate equations of motion (and the arrangement of the local coordinate system) correspond to the equations (6.83) for the free fall on the rotating earth. The incorporation of the constraint of a fixed length of the rod or string of the pendulum

$$(q_1^2 + q_2^2 + q_3^2 - l^2) = 0$$

requires the formulation of equations of motion of the form Lagrange I

$$\begin{aligned} \ddot{q}_1 &= 2\omega\dot{q}_2 \sin \varphi && + \lambda q_1 \\ \ddot{q}_2 &= -2\omega\dot{q}_1 \sin \varphi && - 2\omega\dot{q}_3 \cos \varphi && + \lambda q_2 \\ \ddot{q}_3 &= 2\omega\dot{q}_2 \cos \varphi && - g_{\text{eff}} + \lambda q_3 . \end{aligned} \quad (6.91)$$

Small oscillations of the pendulum can be characterised by

$$\frac{q_1}{l}, \frac{q_2}{l} \ll 1$$

provided the length of the string is sufficiently large. The constraint

$$q_3 = \pm l \left[1 - \left(\frac{q_1}{l} \right)^2 - \left(\frac{q_2}{l} \right)^2 \right]^{1/2}$$

can be expanded in a binomial series in this case

$$q_3 = \pm l \left(1 - \frac{1}{2} \left[\left(\frac{q_1}{l} \right)^2 + \left(\frac{q_2}{l} \right)^2 \right] + \dots \right) .$$

The position of the pendulum is characterised by $q_3 = -l$ (note the orientation of the q_3 -axis) in lowest approximation. The quantities q_1/l and q_2/l can be regarded as quantities of first order. The dominant contribution in the

third equation of (6.91) are the gravity (use $g_{\text{eff}} \equiv g$) and the constraining force. The derivative \ddot{q}_3 is of second, the Coriolis force of first order. The parameter $\lambda = -(g/l)$ is of first order, so that the equations of motion can be approximated consistently to first order by

$$\begin{aligned}\ddot{q}_1 &= 2\omega\dot{q}_2 \sin \varphi - \frac{g}{l} q_1 \\ \ddot{q}_2 &= -2\omega\dot{q}_1 \sin \varphi - \frac{g}{l} q_2 .\end{aligned}\tag{6.92}$$

The term with \dot{q}_3 in the equation for q_2 can be neglected as being of higher order.

It is useful to introduce (see \bullet Math.Chap. 7) the complex variable $u = q_1 + i q_2$ (and the abbreviation $R = \omega \sin \varphi$) for the discussion of the system of equations (6.92). The resulting linear differential equation with constant coefficients

$$\ddot{u} + 2iR\dot{u} + \frac{g}{l}u = 0\tag{6.93}$$

can be solved with the standard ansatz. The roots of the characteristic equation are

$$\alpha_{1,2} = -R \pm \sqrt{R^2 + (g/l)} ,$$

so that the general solution can be noted as

$$u(t) = C_1 e^{i\alpha_1 t} + C_2 e^{i\alpha_2 t} .\tag{6.94}$$

The pendulum was initially displaced towards the South

$$q_1(t=0) = C \quad (C > 0) \quad q_2(t=0) = 0$$

and set into motion without a push

$$\dot{q}_1(0) = \dot{q}_2(0) = 0$$

in the original Foucault experiment in 1851. These initial conditions yield the equations

$$C_1 + C_2 = C \quad \alpha_1 C_1 + \alpha_2 C_2 = 0$$

for the constants of integration. The solution of this set of linear equations is

$$\begin{aligned}C_1 &= \frac{-C\alpha_2}{(\alpha_1 - \alpha_2)} = \frac{C}{2} \left(1 + \frac{R}{\sqrt{R^2 + (g/l)}} \right) \\ C_2 &= \frac{C\alpha_1}{(\alpha_1 - \alpha_2)} = \frac{C}{2} \left(1 - \frac{R}{\sqrt{R^2 + (g/l)}} \right) .\end{aligned}$$

The pendulum executes, from the point of view of the rotating earth, for these initial condition a pattern in the form of a rosette (Fig. 6.22). In order to demonstrate the correctness of this statement, the following steps are required: calculate the time derivative of the function $u(t)$

$$\begin{aligned}\dot{u} &= i\alpha_1 C_1 (e^{i\alpha_1 t} - e^{i\alpha_2 t}) \\ &= \frac{Cg}{l\sqrt{R^2 + (g/l)}} e^{-iRt} \sin\left(\left[R^2 + \left(\frac{g}{l}\right)\right]^{1/2} t\right).\end{aligned}\quad (6.95)$$

This derivative has the value zero, if the argument of the sine function is a multiple of π , that is if the relation

$$\left[R^2 + (g/l)\right]^{1/2} t = k\pi \quad (k = 0, \pm 1, \pm 2, \dots)$$

is satisfied. With the derivative of the complex function, the derivatives of the coordinates q_1 and q_2 have the value zero as well. This feature is responsible for the appearance of spikes in the trajectory at the times

$$\tau_k = \frac{k\pi}{\sqrt{R^2 + (g/l)}} \quad (k = 0, \pm 1, \pm 2, \dots).$$

These times correspond exactly to the turning points of the standard mathematical pendulum in harmonic approximation, if the Coriolis force is absent, that is if $R = 0$.

In the time interval from 0 to τ_1 the pendulum moves (from the point of view of the rotating earth) not exactly towards north, but reaches a point which is characterised by

$$u(\tau_1) = C_1 e^{-i(R\tau_1 - \pi)} + C_2 e^{-i(R\tau_1 + \pi)} = -C e^{-iR\tau_1}, \quad (6.96)$$

or respectively by (Fig. 6.22a)

$$q_1(\tau_1) = -C \cos(R\tau_1) \quad q_2(\tau_1) = C \sin(R\tau_1).$$

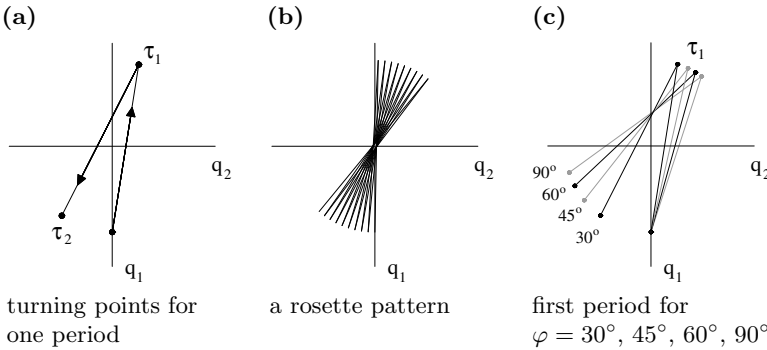


Fig. 6.22. The Foucault pendulum

The first turning point of the pendulum is found in the North-East (check this against the direction of the Coriolis force). This point is rotated by the angle

$$\Delta\phi_1 = \omega\tau_1 \sin\varphi$$

(it depends on the geographical latitude φ) against the north-south direction. In general the turning points are described by

$$q_1(\tau_k) = (-1)^k C \cos(R \tau_k) \quad q_2(\tau_k) = (-1)^{k+1} C \sin(R \tau_k) .$$

The second turning point (with $k = 2$) is found in the south-westerly quadrant. This rotation of the plane of oscillation, which continues in the pattern indicated (Fig. 6.22b), is solely due to the action of the Coriolis force (as it vanishes for $R = 0$). The experimental verification of the rosette pattern⁷ is another proof of the rotation of the earth (about the north-south axis). The variation of the rotation of the plane of oscillation with the geographical latitude can be gleaned from Fig. 6.22c, which shows the first period of oscillation for values of φ of 30° , 45° , 60° , 90° (from left to right).

The last section of the 'Application of the Lagrange formalism' is devoted to the discussion of the motion of rigid bodies. The main interest will be centred on the development of the theory of spinning tops. It will, however, only be possible to address the simplest of the many interesting aspects of this theory.

6.3 The motion of rigid bodies

The description of the motion of extended objects is more complicated than the description of the motion of point particles. A simplification is encountered if the body is rigid. A body is called rigid, if it is composed of a set of point particles for which the mutual distances do not change with time.

A rigid body possesses (with the exception of a pair of dumb-bells) six degrees of freedom. Three correspond to the translation of an arbitrary point of reference in the body (usually the centre of gravity) and three to a rotation of the body itself. The mathematical description of the rotational motion demands a certain amount of effort. This motion can be characterised (as shown in Chap. 6.3.2) by the components of the angular velocity ($\omega_1, \omega_2, \omega_3$) referred to a body-fixed frame of reference. The kinetic energy of the rotational motion takes the form

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu, \nu=1}^3 I_{\mu\nu} \omega_\mu \omega_\nu .$$

The inertial response of the body with respect to rotations is determined by a 3×3 matrix

$$\hat{\mathbf{I}} = [I_{\mu\nu}] .$$

Details concerning the topic 'inertial matrix' are presented in Chap. 6.3.2.

⁷ To be viewed e.g. in the Deutsche Museum in Munich. ● D.tail 6.4 (part 2) contains an applet for the illustration of the time development of the Foucault pendulum.

The fact that the inertial response is determined by a matrix (and not by a scalar as in the case of translations) is the reason why the relation between angular momentum and angular velocity has the form of a matrix equation (Chap. 6.3.3). As a consequence, a more complex pattern of motion has to be expected even in relatively simple situations (like the motion of a force-free symmetric top).

The three components of the angular velocity ω_μ do not correspond directly to a set of generalised velocities. The connection with the actual generalised coordinates used for the description of the rotational motion (normally the Euler angles) is, for this reason, much more involved. The resulting equations of motion (Chap. 6.3.4) are not easily applied. A small selection of examples will nonetheless be discussed after the formulation of the equations which describe the motion of a spinning top (Chap. 6.3.5).

6.3.1 Preliminaries

The characterisation of a rigid body given above implies that absolutely rigid bodies are not found in nature. This concept is, however, an excellent approximation in many circumstances. It will be used throughout this section.

Simple counting is sufficient to demonstrate that the number of degrees of freedom of a rigid body is always 6, provided the number of mass points making up the rigid body is equal to or larger than 3 (Fig. 6.23). A body with

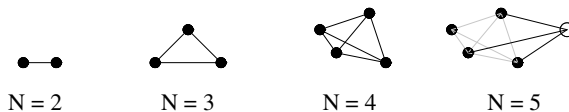


Fig. 6.23. Simple rigid bodies

two mass points ($N = 2$) is rigid, if the masses are connected by one 'rigid rod'. The number of degrees of freedom N_F is restricted by one constraint from six to five. For a rigid body consisting of three respectively of four mass points, three (triangle) respectively six (tetrahedron) rigid rods are needed in order to fix the relative position of the constituents. The superficial number of degrees of freedom $N_F = 3 \times N$ is reduced in both cases to $N_F = 6$. The situation can be characterised from this point on as follows: for each additional mass three additional constraints are needed, in order to fix the position of the additional point particle with respect to the original body. An additional mass could e.g. vibrate with respect to the remaining body if only two additional constraints were provided. It would not be rigid. The three new degrees of freedom are immediately 'frozen' by three constraints, so that the number of degrees of freedom is always

$$N_F = 6 \quad \text{for all } N \geq 3.$$

The assembly of the relevant equations (of motion) is achieved with the following steps:

- Begin with the Lagrangian for a system of N mass points with $(3N - 6)$ constraints. Choose nontrivial, generalised coordinates.
- Express the Lagrangian as a function of the generalised coordinates and derive the Lagrangian equations of motion.
- In order to model realistic objects with a *continuous* mass distribution (compare Chap. 3.2.4.1, p. 127) an additional step is required: make the transition from the discrete to a continuous mass distribution.

The Lagrangian for a system of N mass points is

$$L = \frac{1}{2} \sum_i m_i v_i^2 - \sum_i U(\mathbf{r}_i) - \frac{1}{2} \sum_{i \neq k} V(|\mathbf{r}_i - \mathbf{r}_k|). \quad (6.97)$$

The internal potential energy

$$V = \frac{1}{2} \sum_{i,k} V_{i,k} = \text{const.}$$

can be neglected for the discussion of a rigid body. This energy is constant as long as it depends only on the separation of the mass points and hence is not of interest. The discussion can be restricted to the Lagrangian $L = T - U$.

The choice of the $(3N - 6)$ ignorable coordinates is trivial

$$\begin{aligned} q_7 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - l_{12}^2 = 0 \\ &\vdots \quad \quad \quad \vdots \\ q_{3N} &= (x_N - x_{N-1})^2 + \dots + (z_N - z_{N-1})^2 - l_{N-1,N}^2 = 0. \end{aligned}$$

The quantities l_{ik} represent the fixed separation of the $(3N - 6)$ pairs of masses. A general assertion for the remaining six nontrivial coordinates is expressed by the theorem of Chasles:

The general motion of a rigid body is composed of a translation and a rotation of the whole system.

It is sufficient for an illustration of this theorem to consider three points of a rigid body (Fig. 6.24). The three points can be moved from an initial to a final situation by a rotation of the body about a suitable axis followed by a translation. The sequence of these steps can be interchanged.

6.3.2 The kinetic energy of rigid bodies

A suitable point of reference of the body has to be chosen for a description of the translation. The centre of gravity is well suited for this purpose

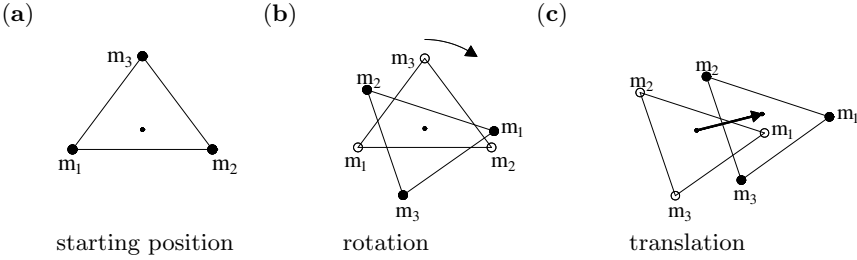


Fig. 6.24. Illustration of the theorem of Chasles

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{r}_i \quad M = \sum_i m_i .$$

The three coordinates of the centre of gravity can serve as three of the six generalised coordinates. The description of the rotation is more complicated. The generalised coordinates for the description of the rotation will only be chosen in Chap. 6.3.4. The motion of the rigid body is in the meantime characterised by the choice and the manipulation of two coordinate systems.

6.3.2.1 The coordinate systems. Coordinate system 1 is an inertial system. This **space-fixed coordinate system** is spanned by the coordinates (x, y, z) . It describes the point of view of an external observer. Coordinate system 2 with the coordinates (x_1, x_2, x_3) is the **body fixed coordinate system**. This is firmly attached to the body. The centre of mass can be chosen as the origin of this system. This choice is not necessary, it does, however, simplify the discussion (as will be shown). The body-fixed system is, in general, not an inertial system.

The position of each of the N masses of the body is

$$\mathbf{r}_i(t) = \mathbf{R}_{CM}(t) + \mathbf{r}_{i,CM}(t) . \quad (6.98)$$

The position of the i -th mass point from the point of view of the space-fixed frame \mathbf{r}_i is equal to the sum of the position of the centre of mass \mathbf{R}_{CM} and the position of the mass with respect to the centre of mass $\mathbf{r}_{i,CM}$ (Fig. 6.25). The velocities of the individual point particles can, in accordance with the

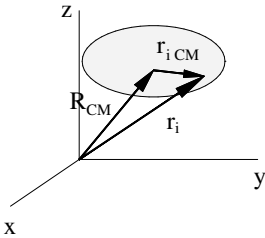


Fig. 6.25. Position of a mass element from the point of view of the body-fixed, centre of mass system

theorem of Chasles, be specified as

$$\mathbf{v}_i(t) = \mathbf{V}_{CM}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_{i,CM}(t) \quad \mathbf{V}_{CM} = \dot{\mathbf{R}}_{CM} . \quad (6.99)$$

The velocity of each mass from the point of view of the space-fixed system equals the velocity of the centre of mass plus a rotational contribution. The following independent argument can be used in order to justify this relation: place yourself into the centre of mass and attach an inertial system (I). The velocity of the i -th mass from the point of view of I is (compare (6.64), p. 299)

$$\mathbf{v}_i^{(I)} = \mathbf{v}_i^{(B)} + \boldsymbol{\omega} \times \mathbf{r}_{i,CM} .$$

The velocity $\mathbf{v}_i^{(B)}$ is the velocity of the mass as seen in the body-fixed frame. It is, however, zero as the body moves with the frame of reference $\mathbf{v}_i^{(B)} = \mathbf{0}$. Add now the possible translation of the centre of mass and find the relation (6.99)

$$\mathbf{v}_i(t) = \mathbf{V}_{CM}(t) + \mathbf{v}_i^{(I)}(t) = \mathbf{V}_{CM}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_{i,CM} .$$

Note that the vector $\mathbf{r}_{i,CM}$, which is defined with respect to the rotating frame, does not depend on time. It is the instantaneous axis of rotation (described by the vector $\boldsymbol{\omega}(t)$), which changes its direction (and its magnitude) with time.

6.3.2.2 The kinetic energy of rotational motion in a body-fixed frame referred to the centre of mass. The expression for the velocity of the i -th mass can be used to represent the kinetic energy of a rigid body in terms of the velocity of the centre of mass and the angular velocity after this choice of the frames of reference. The starting point is

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \mathbf{v}_i^2 \\ &= \frac{1}{2} \sum_i m_i \{ \mathbf{V}_{CM}^2 + 2\mathbf{V}_{CM} \cdot (\boldsymbol{\omega} \times \mathbf{r}_{i,CM}) \\ &\quad + (\boldsymbol{\omega} \times \mathbf{r}_{i,CM}) \cdot (\boldsymbol{\omega} \times \mathbf{r}_{i,CM}) \} . \end{aligned} \quad (6.100)$$

The first term describes the kinetic energy of the translation

$$T_{\text{trans}} = \frac{M}{2} \mathbf{V}_{CM}^2 . \quad (6.101)$$

The second term vanishes if the body-fixed system is referred to the centre of mass, because the definition of the centre of mass as the origin of this system implies

$$\sum_i m_i \mathbf{r}_{i,CM} = \mathbf{0} .$$

The third term, which will finally be interpreted as the kinetic energy of the rotational motion, requires a more extensive discussion. In a first step

a standard formula for the scalar product of two cross products (see \bullet Math.Chap. 3.1.2) has to be invoked

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

in order to write

$$T_{\text{rot}} = \frac{1}{2} \sum_i m_i \{ \mathbf{r}_{i,\text{CM}}^2 \boldsymbol{\omega}^2 - (\mathbf{r}_{i,\text{CM}} \cdot \boldsymbol{\omega})^2 \} .$$

A direct evaluation is possible with the decomposition of the two vectors $\boldsymbol{\omega}$ and $\mathbf{r}_{i,\text{CM}}$ with respect to the *body-fixed* system.

$$T_{\text{rot}} = \frac{1}{2} \sum_i m_i \left[(x_{1i}^2 + x_{2i}^2 + x_{3i}^2)(\omega_1^2 + \omega_2^2 + \omega_3^2) - (x_{1i}\omega_1 + x_{2i}\omega_2 + x_{3i}\omega_3)^2 \right] .$$

The expression for the rotational kinetic energy is in detail

$$\begin{aligned} T_{\text{rot}} = \frac{1}{2} \left\{ \left[\sum_i m_i (x_{2i} + x_{3i})^2 \right] \omega_1^2 + \left[- \sum_i m_i x_{1i} x_{2i} \right] \omega_1 \omega_2 \right. \\ + \left[- \sum_i m_i x_{1i} x_{3i} \right] \omega_1 \omega_3 + \left[- \sum_i m_i x_{1i} x_{2i} \right] \omega_1 \omega_2 \\ + \left[\sum_i m_i (x_{1i} + x_{3i})^2 \right] \omega_2^2 + \left[- \sum_i m_i x_{2i} x_{3i} \right] \omega_2 \omega_3 \\ + \left[- \sum_i m_i x_{1i} x_{3i} \right] \omega_1 \omega_3 + \left[- \sum_i m_i x_{2i} x_{3i} \right] \omega_2 \omega_3 \\ \left. + \left[\sum_i m_i (x_{1i} + x_{2i})^2 \right] \omega_3^2 \right\} \end{aligned}$$

if the factors of the individual products of the angular velocities are collected. These factors are independent of time. They depend only on the geometry and the mass distribution of the rigid body.

The factors can be interpreted as elements of a 3×3 matrix

$$I_{\mu\nu} = \sum_{i=1}^N m_i \left[\delta_{\mu\nu} \sum_{\lambda=1}^3 x_{\lambda i}^2 - x_{\mu i} x_{\nu i} \right] \quad (\mu, \nu = 1, 2, 3) \quad (6.102)$$

so that the expression for the rotational kinetic energy can be brought into a more compact form

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu\nu} I_{\mu\nu} \omega_\mu \omega_\nu .$$

The matrix elements in (6.102) corresponds in detail to

$$I_{11} = \sum_{i=1}^N m_i (x_{2i}^2 + x_{3i}^2) \quad I_{12} = - \sum_{i=1}^N m_i (x_{1i} x_{2i}) \quad \text{etc.}$$

The complete matrix

$$\hat{\mathbf{I}}^{(CM)} \equiv \hat{\mathbf{I}} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}_{\text{B}} \quad (6.103)$$

is called the **inertia matrix** or the **inertia tensor**⁸. The diagonal elements of the matrix are the moments of inertia with respect to the three axes of the body-fixed coordinate system. The remaining elements are called **centrifugal moments** or alternatively **products of inertia**. The matrix is symmetric $I_{\mu\nu} = I_{\nu\mu}$. There are six independent quantities, which describes the inertial response of a rigid body against a change of its rotation.

The total kinetic energy of a rigid body is

$$T = T_{\text{trans}} + T_{\text{rot}} = \frac{M}{2} V_{CM}^2 + \frac{1}{2} \sum_{\mu\nu} I_{\mu\nu} \omega_{\mu} \omega_{\nu} . \quad (6.104)$$

The form of the second term is only valid in a body-fixed system, for which the centre of mass is the origin. Quantities referred to this coordinate system will from now on be characterised consistently by Greek indices or be identified by the subscript B . The rotational contribution can be expressed in a more compact form using a matrix notation. The definition

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_{\text{B}}$$

and the matrix (6.103) allows to write

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} \boldsymbol{\omega}^T \hat{\mathbf{I}} \boldsymbol{\omega} \\ &= \frac{1}{2} (\omega_1, \omega_2, \omega_3)_{\text{B}} \begin{pmatrix} I_{11} & \dots & I_{13} \\ \vdots & \ddots & \vdots \\ I_{31} & \dots & I_{33} \end{pmatrix}_{\text{B}} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_{\text{B}} . \end{aligned} \quad (6.105)$$

6.3.2.3 Body-fixed systems with an arbitrary origin. The relation among the velocities (6.99) is still valid if an arbitrary point of the body-fixed system is chosen as the origin instead of the centre of mass. The relation for a point O (Fig. 6.26a) would read

$$\mathbf{v}_i = \mathbf{V}_O + (\boldsymbol{\omega} \times \mathbf{r}_{i,O}) . \quad (6.106)$$

⁸ The name tensor refers to the properties of $\hat{\mathbf{I}}$ with respect to linear transformation, as indicated briefly on p. 330. If not indicated otherwise, the elements of this tensor are referred to the body-fixed system.

The validity of this relation can be demonstrated, beginning with

$$\mathbf{v}_i = \mathbf{V}_{CM} + (\boldsymbol{\omega} \times \mathbf{r}_{i,CM}) .$$

The position of the centre of mass and of the new origin are related by (Fig. 6.26b)

$$\mathbf{R}_{CM}(t) = \mathbf{a}(t) + \mathbf{R}_O(t) . \quad (6.107)$$

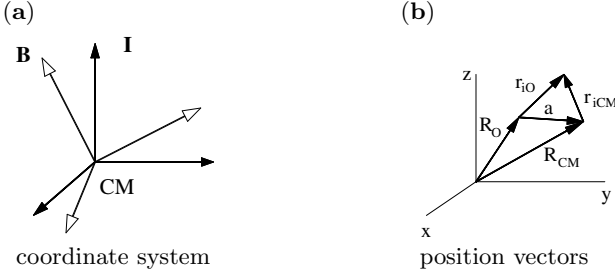


Fig. 6.26. Body-fixed system with arbitrary origin

The vector \mathbf{a} changes its direction with time, its magnitude is, however, independent of time ($|\mathbf{a}(t)| = \text{const.}$), as the initial and the final point of the vector are rigidly attached to the body. This implies the relation

$$\mathbf{r}_{i,O} = \mathbf{a} + \mathbf{r}_{i,CM} . \quad (6.108)$$

The velocity of the point O from the point of view of the space-fixed system is

$$\mathbf{V}_O = \mathbf{V}_{CM} - (\boldsymbol{\omega} \times \mathbf{a}) .$$

The point O is translated with the centre of mass and rotates around the centre with the angular velocity $\boldsymbol{\omega}$ (as any body-fixed point). The equations assembled allow the argument

$$\begin{aligned} \mathbf{v}_i &= \mathbf{V}_{CM} + (\boldsymbol{\omega} \times \mathbf{r}_{i,CM}) \\ &= \mathbf{V}_O + (\boldsymbol{\omega} \times \mathbf{a}) + \boldsymbol{\omega} \times (\mathbf{r}_{i,O} - \mathbf{a}) \\ &= \mathbf{V}_O + (\boldsymbol{\omega} \times \mathbf{r}_{i,O}) . \end{aligned}$$

The body-fixed system related to the centre of mass and the body-fixed system related to an arbitrary point are equivalent. The velocity \mathbf{v}_i of a point of the rigid body is composed of the velocity of the reference point in the body-fixed system plus a rotation about an (instantaneous) axis through this point. The same vector $\boldsymbol{\omega}$ features in (6.99) as well as in (6.106). This means that, the axes of rotation through the two reference points of the rigid body are parallel for all times.

The expression for the kinetic energy (6.104) is *not* valid, if an arbitrary origin is chosen in the body-fixed system. The sum $\sum_i m_i \mathbf{r}_{i,O}$ does not vanish.

For this reason the kinetic energy of a rigid body cannot be decomposed into a translational plus a rotational part (see Chap. 6.3.3.5 for details).

6.3.3 The structure of the inertia matrix

It is necessary to take a closer look at the inertia matrix before choosing generalised coordinates for the description of the rotation and the discussion of the resulting equations of motion

The total mass of a rigid body with a *continuous* mass distribution is given by

$$M = \sum_i m_i \rightarrow \iiint_V \rho(\mathbf{r}) dV .$$

The elements of the inertia matrix in the body-fixed system have to be calculated by

$$I_{\mu\nu} = \iiint_V \rho(\mathbf{r}) \left\{ \delta_{\mu\nu} \sum_{\lambda=1}^3 x_\lambda^2 - x_\mu x_\nu \right\} dV \quad (6.109)$$

for such a body. This calculation involves the evaluation of triple integrals (see ● Math.Chap. 4.3.3).

6.3.3.1 Examples for the calculation of the elements of the inertia matrix. A simple example is the inertia matrix of a sphere (radius R) with a homogeneous mass distribution (Fig. 6.27a, r_C denotes the separation of a point from the centre of the sphere)

$$\rho(\mathbf{r}) = \begin{cases} \rho & \text{for } r_C \leq R \\ 0 & \text{for } r_C > R . \end{cases}$$

The centre of mass is obviously the centre of the sphere. The orientation of the axis is irrelevant because of the symmetry of the body. Cylinder coordinates (r, φ, z – see however also ● Probl. 6.6) should be used for the evaluation of the relevant triple integrals, as the integrand depends on the distance from a coordinate axes (r) and not on the distance from the origin (r_C) (Fig. 6.27b). The calculation of e.g.

$$I_{33} = \iiint_V \rho(\mathbf{r}) (x_1^2 + x_2^2) dx_1 dx_2 dx_3$$

in cylinder coordinates

$$x_1 = r \cos \varphi \quad x_2 = r \sin \varphi \quad x_3 = z$$

and a homogeneous mass distribution requires the evaluation of the integral

$$I_{33} = \rho \int_{-R}^R dz \int_0^{\sqrt{(R^2 - z^2)}} r^3 dr \int_0^{2\pi} d\varphi .$$

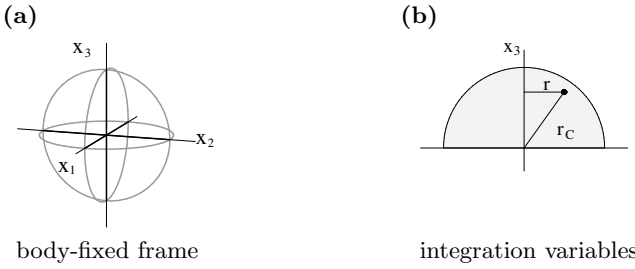


Fig. 6.27. Calculation of the inertia matrix of a sphere (homogeneous mass distribution)

The result of the simple calculation is

$$I_{33} = \frac{8}{15} \rho \pi R^5 = \frac{2}{5} M R^2 .$$

The expression for the mass of the sphere

$$M = \frac{4}{3} \pi \rho R^3$$

is used to simplify the final result. The calculation of the product of inertia follows the same pattern, as for instance

$$\begin{aligned} I_{23} &= - \iiint \rho(\mathbf{r}) x_2 x_3 dx_1 dx_2 dx_3 \\ &= - \rho \int_{-R}^R z dz \int_0^{\sqrt{R^2 - z^2}} r^2 dr \int_0^{2\pi} \sin \varphi d\varphi \\ &= 0 \quad (\text{because of } \int_0^{2\pi} \sin \varphi d\varphi = 0) . \end{aligned}$$

The complete inertia matrix can be written down directly because of the symmetry

$$\hat{\mathbf{I}}_{\text{hom. sphere}} = \begin{pmatrix} \frac{2}{5} M R^2 & 0 & 0 \\ 0 & \frac{2}{5} M R^2 & 0 \\ 0 & 0 & \frac{2}{5} M R^2 \end{pmatrix} . \quad (6.110)$$

All centrifugal moments vanish. The moments of inertia with respect to the three body-fixed axes, are, as is to be expected because of the symmetry of the object, equal.

The calculation of the moments of inertia could have been simplified with the following argument: it is sufficient to evaluate the sum

$$I_{11} + I_{22} + I_{33} = 3I = 2\rho \iiint r^2 dV = 8\rho \pi \frac{R^5}{5} = \frac{6}{5} M R^2 ,$$

as all the moments of inertia are equal because of the symmetry. The result can be read off immediately.

The second example is a homogeneous cube with an edge length a . The orientation of the axes plays a role in this case. If the axes are perpendicular to the centre of the lateral surfaces (Fig. 6.28a), the inertia matrix is found to be (as a special case of the following example, the cuboid)

$$\hat{\mathbf{I}}_{\text{hom. cube}} = \begin{pmatrix} \frac{1}{6}Ma^2 & 0 & 0 \\ 0 & \frac{1}{6}Ma^2 & 0 \\ 0 & 0 & \frac{1}{6}Ma^2 \end{pmatrix}. \quad (6.111)$$

An explicit calculation for an arbitrary orientation of the axes through the centre of gravity is rather cumbersome. It turns out, as explained in the following sections, that this result can be obtained by other means than the evaluation of integrals.

The calculation of the inertia matrix of a homogeneous cuboid with the sides

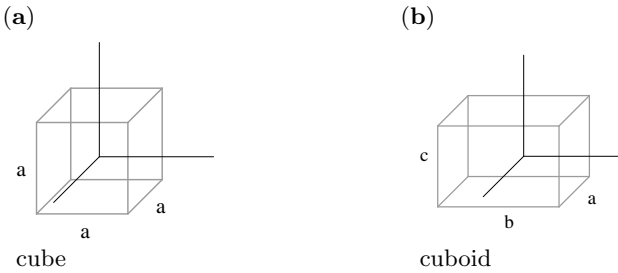


Fig. 6.28. Position of the axes for the calculation of the inertia matrix

a, b, c in the 1, 2, 3- directions is also simple, provided the axes through the centre of mass are perpendicular to the lateral surfaces. The result is (● Probl. 6.6)

$$\hat{\mathbf{I}}_{\text{hom. cuboid}} = \begin{pmatrix} \frac{1}{12}M(b^2 + c^2) & 0 & 0 \\ 0 & \frac{1}{12}M(a^2 + c^2) & 0 \\ 0 & 0 & \frac{1}{12}M(a^2 + b^2) \end{pmatrix}. \quad (6.112)$$

The results quoted for the cube and the cuboid, invite the remark: The question whether the inertia matrix is a diagonal matrix or not, seems to depend on the orientation of the axes of the body-fixed system. If the axes are symmetry axes, then the inertia matrix is diagonal, otherwise centrifugal moments

occur. A set of axes, for which a diagonal inertia matrix is obtained, are called **principal axes of inertia**. The corresponding moments of inertia (nomenclature I_μ) are called **principal moments of inertia**. The expression for the kinetic energy of rotation in a principal axes system is simpler

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu} I_{\mu} \omega_{\mu}^2 . \quad (6.113)$$

6.3.3.2 The principal axes theorem. The question could be raised, whether it is possible to find a body-fixed coordinate system for every (arbitrarily shaped) body, so that only principal moments of inertia occur. An answer is provided by the principal axes theorem:

At least one set of principal axes exists for every rigid body.

The proof of this theorem will be presented in some detail. The transition from an arbitrary body-fixed system (1, 2, 3) to a potential principal axes system ($\tilde{1}$, $\tilde{2}$, $\tilde{3}$) is achieved by a rotation about an axis through the common origin (Fig. 6.29). The components of the position vector with respect to the

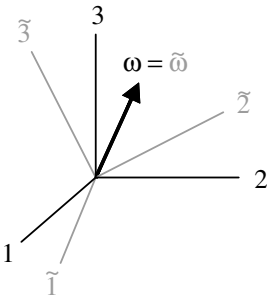


Fig. 6.29. Illustration of the principal axes theorem

two coordinate systems are related by a transformation matrix, the rotation matrix \hat{D} (see **Math.Chap. 3.2.3**),

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ,$$

symbolically

$$\tilde{\mathbf{r}} = \hat{D} \mathbf{r} .$$

The only statement concerning the rotational matrix, that is needed, is: rotations are orthogonal transformations. This property is characterised in matrix form by

$$\sum_i D_{ki} D_{il} = \delta_{kl} \iff \hat{D}^T \hat{D} = \hat{D} \hat{D}^T = \hat{E} \quad \text{or} \quad \hat{D}^T = \hat{D}^{-1} . \quad (6.114)$$

The transformation for the coordinates applies to any other vector, for instance the angular velocity⁹ (for all times)

$$\tilde{\omega}(t) = \hat{D} \omega(t) . \quad (6.115)$$

Insertion of the unit matrix in the expression for the kinetic energy and an appropriate use of the relation (6.114) gives

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} \omega^T \hat{I} \omega = \frac{1}{2} \omega^T \hat{E} \hat{E} \omega \\ &= \frac{1}{2} \omega^T \hat{D}^T [\hat{D} \hat{D}^T] \hat{D} \omega = \frac{1}{2} \tilde{\omega}^T [\hat{D} \hat{D}^T] \tilde{\omega} . \end{aligned}$$

The matrix $\hat{D} \hat{D}^T$ is expected to represent the inertia matrix in a principal axes system \tilde{I} , so that

$$T_{\text{rot}} = \frac{1}{2} \tilde{\omega}^T \tilde{I} \tilde{\omega} . \quad (6.116)$$

The kinetic energy, a scalar quantity, has the same form in any system of reference.

The relation

$$\hat{D} \hat{D}^T = \tilde{I} , \quad (6.117)$$

becomes after multiplication by \hat{D}^T from the left *or* by \hat{D} from the right

$$\hat{D} \hat{D}^T = \hat{D}^T \tilde{I} \quad \text{respectively} \quad \hat{D} \hat{I} = \tilde{I} \hat{D} .$$

It has the explicit form

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{pmatrix} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

provided the system marked by a tilde is a principal axes system. This matrix equation corresponds to 9 (linear) equations. The first column of the product matrix corresponds to the relations

$$I_{11}D_{11} + I_{12}D_{12} + I_{13}D_{13} = I_1 D_{11}$$

$$I_{21}D_{11} + I_{22}D_{12} + I_{23}D_{13} = I_1 D_{12}$$

$$I_{31}D_{11} + I_{32}D_{12} + I_{33}D_{13} = I_1 D_{13} .$$

A similar set of equations is obtained with $D_{2\mu}$ and I_2 as well as with $D_{3\mu}$ and I_3 . These equations correspond to the second and the third columns of the product matrix. These equations have the same form, if the index of the principal moment of inertia and the first index of the rotation matrix are suppressed

⁹ Equation (6.115) relates the components of the vector ω , which is decomposed with respect to the two coordinate systems.

$$\begin{aligned}
(I_{11} - I)D_1 + I_{12}D_2 + I_{13}D_3 &= 0 \\
I_{21}D_1 + (I_{22} - I)D_2 + I_{23}D_3 &= 0 \\
I_{31}D_1 + I_{32}D_2 + (I_{33} - I)D_3 &= 0.
\end{aligned} \tag{6.118}$$

This system of equations represents an algebraic eigenvalue problem given the matrix $[I_{\mu\nu}]$. To be determined are the eigenvalues I_κ and the corresponding eigenvectors $(D_{\kappa 1}, D_{\kappa 2}, D_{\kappa 3})$. The matrix of eigenvectors mediates the transformation between the original body-fixed system and the principal axes system. The principal axes theorem is proven, if there exists a physically meaningful solution of this eigenvalue problem.

The discussion follows the pattern, which has been used for the problem of coupled oscillators (compare (6.19)). The condition for the existence of a nontrivial solution of the eigenvalue problem is

$$\det(\hat{\mathbf{I}} - I \hat{\mathbf{E}}) = 0. \tag{6.119}$$

The corresponding secular equation is a cubic equation

$$I^3 + aI^2 + bI + c = 0$$

with three roots. The roots are real if the matrix is symmetric. This condition ($I_{\mu\nu} = I_{\nu\mu}$) is satisfied due to the definition of the centrifugal moments. The roots are positive ($I_\mu > 0$) if the condition

$$I_{11} + I_{22} > I_{33} \quad (\text{and all cyclic transpositions}) \tag{6.120}$$

is satisfied. The validity of the condition, that the sum of two diagonal elements of the matrix $\hat{\mathbf{I}}$ is larger than the third, can be demonstrated directly as e.g. by

$$\begin{aligned}
I_{11} + I_{22} &= \iiint \rho(\mathbf{r}) \{x_2^2 + x_3^2 + x_1^2 + x_3^2\} dV \\
&= I_{33} + 2 \iiint \rho(\mathbf{r}) x_3^2 dV > I_{33}.
\end{aligned}$$

The three real positive roots are the principal moments of inertia of the rigid body, which has been characterised initially by a full inertia matrix $[\hat{\mathbf{I}}_{\mu\nu}]$.

This argument completes the proof of the principal axes theorem: there exists always a set of body-fixed axes for which the inertia matrix is diagonal. The principal axes are the symmetry axes of the body if the body exhibits some symmetry.

The relative orientation of the two coordinate systems involved is determined by the rotation matrix $\hat{\mathbf{D}}$. This matrix is composed of the eigenvectors. The solution of the linear system of equations (6.118) with the eigenvalues I_μ yields the eigenvector $(D_{\mu 1}, D_{\mu 2}, D_{\mu 3})$, in detail

$$\begin{aligned}
I_1 &\longrightarrow (D_{11}, D_{12}, D_{13}) \\
I_2 &\longrightarrow (D_{21}, D_{22}, D_{23}) \\
I_3 &\longrightarrow (D_{31}, D_{32}, D_{33}).
\end{aligned}$$

6.3.3.3 Illustration of the principal axes theorem. An explicit example is the following problem: given is the inertia matrix (in suitable units)

$$\hat{I} = \begin{pmatrix} 9 & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & \frac{19}{2} & -\frac{1}{2} \\ -2\sqrt{2} & -\frac{1}{2} & \frac{19}{2} \end{pmatrix}.$$

The task is the determination of the principal moments of inertia and the orientation of the principal axes system with respect to the initial system (see ● D.tail 6.5). Evaluation of the determinant (6.119) leads to the secular equation

$$I^3 - 28I^2 + 245I - 650 = 0,$$

with the solution

$$I_1 = 5 \quad I_2 = 10 \quad I_3 = 13.$$

The order of the roots is arbitrary. A change of the order corresponds to a renaming of the axes.

For the determination of the rotation matrix three systems of linear equations have to be solved. The result is the rotation matrix

$$\hat{D} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The matrix \hat{D} describes the transformation of the original system into the principal axes system. The inverse matrix $\hat{D}^{-1} = \hat{D}^T$ describes the rotation, which would transform the principal axes system into the original one.

It can be shown with a little patience that the matrix \hat{D} factorises in the form

$$\hat{D} = \hat{D}_2 \hat{D}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

This implies that the first rotation (\hat{D}_1) is a rotation by an angle of -45° about the 1-axis (Fig. 6.30a), followed by a second rotation with an angle of 45° about the 2-axis of the intermediate coordinate system (Fig. 6.30b).

There remain three short remarks and one longer comment concerning the structure of the inertia matrix.

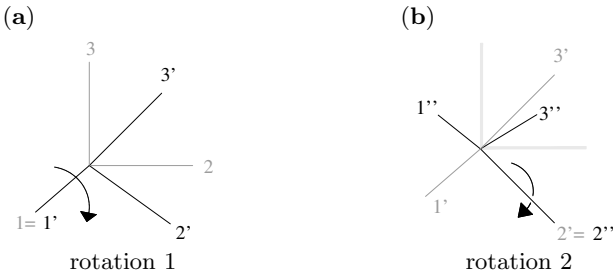


Fig. 6.30. Illustration of the principal axes theorem

6.3.3.4 Additional remarks. The short remarks are:

1. The specification of three numbers for the principal moments of inertia does not determine the shape of a body at all. Consider for example the principal moments of a cuboid (6.112)

$$I_1 = \frac{M}{12}(b^2 + c^2) \quad I_2 = \frac{M}{12}(a^2 + c^2) \quad I_3 = \frac{M}{12}(a^2 + b^2).$$

The principal moments of inertia of a homogeneous ellipsoid with the length a , b , c in the 1-, 2-, 3- direction (see ● Probl. 6.6) are

$$I_1 = \frac{M}{5}(b^2 + c^2) \quad I_2 = \frac{M}{5}(a^2 + c^2) \quad I_3 = \frac{M}{5}(a^2 + b^2). \quad (6.121)$$

An ellipsoid and a cuboid of the same mass will have the same principal moments of inertia, if the lengths involved satisfy the relations

$$a_C = \sqrt{\frac{12}{5}} a_E \quad \text{etc.}$$

Every rigid body possesses three principal moments of inertia and every set of three positive numbers can be represented by the formula for the ellipsoid (6.121). It is therefore possible to characterise every rigid body by an ellipsoid of inertia (possibly with 2 or 3 equal axes).

2. All expressions for the principal moments of inertia have the form

(geometrical factor) times (mass) times (the square of a characteristic distance from the axis of rotation).

The form of the body can (within limits) be recognised by the factor, e.g. $2/5$ for a sphere, $1/6$ for a cube etc.

3. The distinction of scalars, vectors and tensors refers to the behaviour of these quantities under linear coordinate transformations. A set of three quantities (x_1, x_2, x_3) which transforms according

$$x'_i = \sum_k a_{ik} x_k \quad \longrightarrow \quad \mathbf{x}' = \hat{\mathbf{A}} \mathbf{x} \quad \text{with} \quad \hat{\mathbf{A}} = [a_{ik}]$$

form a vector. A set of 9 quantities, which transform according to

$$y'_{ik} = \sum_{lm} a_{il} a_{km} y_{lm} \quad (6.122)$$

are called the components of a tensor of second rank. The corresponding matrix form of the transformation is

$$\hat{y}' = \hat{A} \hat{y} \hat{A}^T. \quad (6.123)$$

This is exactly the behaviour found for the inertia matrix (6.117) under linear coordinate transformations. The name 'tensor' is actually more appropriate than 'matrix' ¹⁰.

6.3.3.5 Steiner's theorem. The body-fixed system with an origin at the position of the centre of mass is nearly always a good choice for the discussion of rotations of a rigid body. There is one exception: a point of the rigid body, which is not the centre of mass, is at rest. It is preferable in this case to choose a body-fixed system referred to this point.

This statement can be demonstrated in the following fashion: The velocity transformation (6.106)

$$\mathbf{v}_i = \mathbf{V}_O + (\boldsymbol{\omega} \times \mathbf{r}_{i,O})$$

is, as shown, valid for every body-fixed point. This and the equation (6.108) for the relation between the position vectors referred to the point O and the centre of mass (the constant vector \mathbf{a} denotes the separation of the two points)

$$\mathbf{r}_{i,O} = \mathbf{r}_{i,CM} + \mathbf{a},$$

can be used to express the kinetic energy of a rigid body as

$$T = \frac{1}{2} \left[M \mathbf{V}_O^2 + 2 \sum_i m_i \mathbf{V}_O \cdot (\boldsymbol{\omega} \times \mathbf{r}_{i,O}) + \sum_i m_i (\boldsymbol{\omega} \times \mathbf{r}_{i,O}) \cdot (\boldsymbol{\omega} \times \mathbf{r}_{i,O}) \right].$$

The second terms does not vanish in general. The expression for the kinetic energy can be simplified to

$$T = T_{\text{rot}} = \frac{1}{2} \sum_i m_i (\boldsymbol{\omega} \times \mathbf{r}_{i,O})^2, \quad (6.124)$$

if the point O is at rest ($\mathbf{V}_O = 0$). The decomposition into components with respect to the system O yields as before for the centre of mass system

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu, \nu} I_{\mu\nu}^{(O)} \omega_{\mu}^{(O)} \omega_{\nu}^{(O)}. \quad (6.125)$$

¹⁰ Please note: A tensor of second rank over \mathcal{R}_3 is always a 3×3 matrix. A 3×3 matrix is not necessarily a tensor.

The inertia tensor and the components of the angular velocity are referred to a coordinate system for which O is the origin. The angular velocity components with respect to the system O and to the centre of mass system are equal, if the coordinate axes of the two systems are *chosen to be parallel*

$$\omega_{\mu}^{(O)} = \omega_{\mu}^{(CM)} = \omega_{\mu} .$$

The expression for the elements of the inertia tensor in the system O

$$I_{\mu\nu}^{(O)} = \sum_i m_i \left[\delta_{\mu\nu} \sum_{\lambda=1}^3 x_{\lambda i}^{(O)2} - x_{\mu i}^{(O)} x_{\nu i}^{(O)} \right]$$

can be reformulated with

$$x_{\mu i}^{(O)} = x_{\mu i}^{(CM)} + a_{\mu}$$

in the following manner

$$\begin{aligned} I_{\mu\nu}^{(O)} = & \sum_i m_i \left[\delta_{\mu\nu} \sum_{\lambda} x_{\lambda i}^{(CM)2} - x_{\mu i}^{(CM)} x_{\nu i}^{(CM)} \right] \\ & + \sum_i m_i \left[\delta_{\mu\nu} \sum_{\lambda} (2x_{\lambda i}^{(CM)} a_{\lambda} + a_{\lambda}^2) - (a_{\mu} x_{\nu i}^{(CM)} + a_{\nu} x_{\mu i}^{(CM)} + a_{\mu} a_{\nu}) \right] . \end{aligned}$$

All terms linear in $x_{\mu i}^{(CM)}$ vanish, as the sum

$$\sum_i m_i x_{\mu i}^{(CM)} = 0$$

corresponds to the coordinates of centre of mass in the centre of mass system. There remains

$$I_{\mu\nu}^{(O)} = I_{\mu\nu}^{(CM)} + M \left[\delta_{\mu\nu} \sum_{\lambda} a_{\lambda}^2 - a_{\mu} a_{\nu} \right] . \quad (6.126)$$

This set of equations is known as **Steiner's parallel axes theorem**. It allows the calculation of the inertia matrix for an arbitrary body fixed coordinate system if the inertia matrix is specified in the centre of mass system provided the axes of the two systems are parallel. An example for the application of this theorem is the following problem.

Calculate the inertia tensor of a homogeneous cube with edge length b for the case that the origin of the coordinate system is one of the corner points and that the axes are parallel to the edges. The moments of inertia for the centre of mass system (with axes parallel to the edges) are (see 6.111)

$$I_{\mu\nu} = \delta_{\mu\nu} \frac{M}{6} b^2 .$$

The vector \mathbf{a} from the centre of mass to the origin of O is e.g.

$$\mathbf{a} = \left(-\frac{1}{2}b, -\frac{1}{2}b, -\frac{1}{2}b \right) .$$

The theorem gives

$$I_{11}^{(O)} = \frac{1}{6}Mb^2 + M\left(\frac{3}{4}b^2 - \frac{1}{4}b^2\right) = \frac{2}{3}Mb^2$$

and in view of the symmetry

$$I_{11}^{(O)} = I_{22}^{(O)} = I_{33}^{(O)} .$$

The result for the centrifugal moments in the system O is

$$\begin{aligned} I_{12}^{(O)} &= 0 + M\left(-\frac{1}{4}b^2\right) = -\frac{1}{4}Mb^2 \\ &= I_{13}^{(O)} = I_{23}^{(O)} . \end{aligned}$$

A coordinate system whose origin is displaced by a constant vector is not necessarily a principal axes system even if this is the case for the centre of mass system. If desired, a principal axes system through O can be determined by solution of the relevant eigenvalue problem.

6.3.4 The angular momentum of a rigid body

The angular velocity is a basic quantity for the description of the motion of a rigid body. A related quantity is the angular momentum. The relation between these two quantities is not at all trivial. The angular momentum (of a system of mass points) depends (see Chap. 3.2.2) on the point of reference. A suitable point of reference is the centre of gravity, the usual origin of the body-fixed frame. The total angular momentum of a rigid body in this coordinate system is found to be

$$\mathbf{l}_{CM} = \sum m_i(\mathbf{r}_{i,CM} \times \mathbf{v}_{i,CM}) .$$

The velocity of the i -th mass for a rotational motion from the point of view of the centre of mass is

$$\mathbf{v}_{i,CM} = (\boldsymbol{\omega} \times \mathbf{r}_{i,CM}) .$$

Insertion leads to

$$\mathbf{l}_{CM} = \sum_i m_i(\mathbf{r}_{i,CM} \times (\boldsymbol{\omega} \times \mathbf{r}_{i,CM})) .$$

The reduction of the double vector product can be achieved with the identity (● Math.Chap. 3.1.2)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) = a^2\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a} .$$

The result is

$$\mathbf{l}_{CM} = \sum_i m_i [\mathbf{r}_{i,CM}^2 \boldsymbol{\omega} - (\mathbf{r}_{i,CM} \cdot \boldsymbol{\omega})\mathbf{r}_{i,CM}] .$$

The decomposition of this expression with respect to the body-fixed system (using standard notation) is

$$l_\mu = \sum_i m_i \left[\omega_\mu \sum_\lambda x_{\lambda i}^2 - \left(\sum_\nu x_{\nu i} \omega_\nu \right) x_{\mu i} \right]$$

or sorted differently

$$l_\mu = \sum_\nu \omega_\nu \left[\sum_i m_i \left\{ \delta_{\mu\nu} \sum_\lambda x_{\lambda i}^2 - x_{\nu i} x_{\mu i} \right\} \right].$$

The relation between the components of the angular momentum and the angular velocity in the body-fixed system can therefore be written as

$$l_\mu = \sum_{\nu=1}^3 I_{\mu\nu}^{(CM)} \omega_\nu \equiv \sum_{\nu=1}^3 I_{\mu\nu} \omega_\nu \quad (\mu = 1, 2, 3). \quad (6.127)$$

These three equations can be summarised as a matrix equation

$$\mathbf{l}_{CM} = \hat{\mathbf{I}} \boldsymbol{\omega}. \quad (6.128)$$

The following remarks apply directly:

1. The vectors \mathbf{l}_{CM} and $\boldsymbol{\omega}$ do not point in the same direction. This is not even the case for a rotation about one of the body-fixed coordinate axes, e.g. for $\boldsymbol{\omega} = (\omega, 0, 0)$

$$\mathbf{l}_{CM} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega \\ I_{21}\omega \\ I_{31}\omega \end{pmatrix} \neq \text{const. } \boldsymbol{\omega}.$$

2. The relation between angular momentum and angular velocity is simpler in a principal axes system

$$\mathbf{l}_{CM} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}.$$

The two vectors are still not collinear even in this case. They are collinear only for the case of a spherical top (a body with three equal principal moments of inertia $I_1 = I_2 = I_3 = I$) or for a rotation about one of the body fixed axes, e.g. the body-fixed 1-axis $\boldsymbol{\omega} = (\omega, 0, 0)$

$$\mathbf{l}_{CM} = I \boldsymbol{\omega} \quad \text{respectively} \quad \mathbf{l}_{CM} = I_1 \boldsymbol{\omega}.$$

The more complicated relation between the angular momentum and the angular velocity is the reason for the relative complexity of the rotational motion of a rigid body.

Some useful alternative expressions for the kinetic energy can be obtained in terms of the angular momentum, as e.g.

$$T_{\text{rot}} = \frac{1}{2} \sum_\mu \omega_\mu l_\mu = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{l}_{CM}, \quad (6.129)$$

if one of the sums (e.g. $\sum_\nu \dots$) in the previous formula

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu, \nu} I_{\mu\nu} \omega_\mu \omega_\nu$$

is replaced by the angular momentum (6.127).

There exist several options for a principal axes system

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu} I_{\mu} \omega_{\mu}^2 = \frac{1}{2} \sum_{\mu} l_{\mu} \omega_{\mu} = \frac{1}{2} \sum_{\mu} \frac{l_{\mu}^2}{I_{\mu}}. \quad (6.130)$$

The last form is particularly simple for a spherical top $I_{\mu} = I$

$$T_{\text{rot}} = \frac{1}{2I} l_{\text{CM}}^2. \quad (6.131)$$

This concludes the discussion of the coordinate systems which are necessary for the description of the motion of a rigid body as well as the introduction of the relevant kinematic quantities (angular velocity, inertia tensor, angular momentum). The next step, before the derivation of the equations of motion and their solution, is the specification of the generalised coordinates for the rotational motion.

6.3.5 The Euler angles

The Lagrangian of a rigid body is

$$L = T - U = \sum \frac{m_i}{2} v_i^2 - \sum U(\mathbf{r}_i).$$

The discussion up to this point concentrated on the reformulation of the kinetic energy

$$\begin{aligned} T &= T_{\text{trans}} + T_{\text{rot}} \\ &= \frac{M}{2} (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{2} \sum_{\mu, \nu} I_{\mu\nu} \omega_{\mu} \omega_{\nu}. \end{aligned} \quad (6.132)$$

Three of the six generalised coordinates are the coordinates of the centre of gravity X, Y, Z . It remains to specify the generalised coordinates for an explicit description of the rotation. The usual choice is a set of three angles, the **Euler angles** $\alpha(t), \beta(t), \gamma(t)$, which describe the time change of the orientation of the body-fixed frame with respect to a given space-fixed frame. The corresponding generalised velocities are $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$. It is necessary to represent the three components ω_{μ} of the angular velocity in the body-fixed system in terms of the generalised velocities (and coordinates) of the rotational motion. The Lagrangian then becomes

$$L = T(\dot{X}, \dot{Y}, \dot{Z}, \alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) - U(X, Y, Z, \alpha, \beta, \gamma).$$

Equations of motion can be derived on the basis of this Lagrangian following the standard steps.

The definition of the Euler angles is slightly awkward, but not really difficult. The two coordinate systems

space – fixed : (x, y, z) , body – fixed : (x_1, x_2, x_3)

are assumed to coincide initially¹¹. An arbitrary orientation of the body-fixed coordinate system with respect to the space-fixed system can be described by three rotations about different axes through the common origin, which are executed consecutively.

Rotation 1: The body-fixed system is rotated anticlockwise by an angle α about the z -axis (Fig. 6.31). This rotation is described by

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_I \quad (6.133)$$

The components of a vector (e.g. the position vector) in the space-fixed system and the new system (characterised by one dash) are related by a simple rotation matrix. The range of the angle α is $0 \leq \alpha \leq 2\pi$.

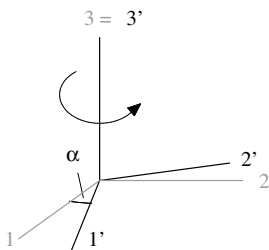


Fig. 6.31. The Euler angle α

Rotation 2: A rotation about the x'_1 -axis (known as the **line of nodes**) again anticlockwise by the angle β (Fig. 6.32a). The rotation matrix connecting the coordinates of these two systems (one and two dashes) is

$$\begin{pmatrix} x''_1 \\ x''_2 \\ x''_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \quad (6.134)$$

The range of the angle β is (see spherical coordinates) $0 \leq \beta \leq \pi$.

Rotation 3: The third rotation is an anticlockwise rotation about the x''_3 -axis by the angle γ (Fig. 6.32b). This rotation is described by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_B = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x''_1 \\ x''_2 \\ x''_3 \end{pmatrix} \quad (6.135)$$

The range of the angle γ is $0 \leq \gamma \leq 2\pi$.

¹¹ The translation of the body-fixed system with respect to the space-fixed system does not play any role in the discussion that follows.

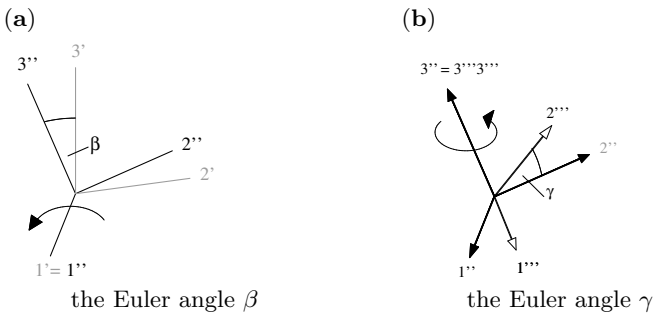


Fig. 6.32. Euler angles

The matrix for the total rotation, which arises by consecutive execution of the three individual rotations, is obtained by multiplication of the three transformation matrices

$$\mathbf{r}_B = \hat{D}_\gamma \hat{D}_\beta \hat{D}_\alpha \mathbf{r}_I = \hat{D} \mathbf{r}_I.$$

The sequence of the operations should *not* be interchanged. Evaluation of the matrix product yields the complete rotation matrix \hat{D}

$$\hat{D} = \begin{pmatrix} -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma, & \cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma, & \sin \beta \sin \gamma \\ -\sin \alpha \cos \beta \cos \gamma - \cos \alpha \sin \gamma, & \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma, & \sin \beta \cos \gamma \\ \sin \alpha \sin \beta, & -\cos \alpha \sin \beta, & \cos \beta \end{pmatrix} \quad (6.136)$$

The rotation matrix describes the relation between the components of a vector in the space-fixed (I) and the body-fixed (B) system. The inverse transformation is effected by the transposed matrix

$$\mathbf{r}_I = \hat{D}^T \mathbf{r}_B. \quad (6.137)$$

The three Euler angles are the generalised coordinates for the description of the rotational motion¹².

The next step is the derivation of a relation between the three components of the angular velocity and the time derivatives of the Euler angles. This relation can be obtained by representing the time derivative of each individual rotation by a vector, which is then decomposed into components with respect to the body-fixed system.

1. $\dot{\alpha}$ is a vector in the z -direction of the space fixed system. The components of this vector with respect to the body fixed system are obtained by transformation with the rotation matrix \hat{D} in (6.136). One finds for this reason

¹² Variants of the definition of the Euler angles are found in the literature. They differ by the sense of rotation and the choice of the line of nodes (for which the x'_2 -axis can be chosen). Sufficient care has to be exercised if equations (of motion or other) from the literature are used.

$$\begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\alpha}_3 \end{pmatrix}_B = \hat{D} \begin{pmatrix} 0 \\ 0 \\ \dot{\alpha} \end{pmatrix}_I = \begin{pmatrix} \dot{\alpha} \sin \beta \sin \gamma \\ \dot{\alpha} \sin \beta \cos \gamma \\ \dot{\alpha} \cos \beta \end{pmatrix}_B. \quad (6.138)$$

2. $\dot{\beta}$ is a vector in the direction of the line of nodes. The decomposition with respect to the body-fixed system is obtained by a rotation with \hat{D}_γ from equation (6.135) about the angle γ

$$\begin{pmatrix} \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix}_B = \hat{D}_\gamma \begin{pmatrix} \dot{\beta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\beta} \cos \gamma \\ -\dot{\beta} \sin \gamma \\ 0 \end{pmatrix}_B. \quad (6.139)$$

3. $\dot{\gamma}$ is a vector along the x_3 -axis of the body fixed system with the decomposition

$$\dot{\gamma} = \begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix}_B. \quad (6.140)$$

Addition of the components of the vectors of the individual angular velocities ($\dot{\alpha}, \dot{\beta}, \dot{\gamma}$) in the body-fixed frame yields the components of the total angular velocity ω_μ

$$\omega_\mu = \dot{\alpha}_\mu + \dot{\beta}_\mu + \dot{\gamma}_\mu \quad (\mu = 1, 2, 3).$$

The representation of the components of the angular velocity in the body fixed system in terms of the generalised velocities $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ is therefore given by

$$\begin{aligned} \omega_1 &= \dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma \\ \omega_2 &= \dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma \\ \omega_3 &= \dot{\alpha} \cos \beta + \dot{\gamma}. \end{aligned} \quad (6.141)$$

This relations have to be inserted into the expression for the rotational kinetic energy. The simplest is the principal axes form

$$T_{\text{rot}} = \frac{1}{2} \sum_{\mu} I_{\mu} \omega_{\mu}^2 = T_{\text{rot}}(\beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}). \quad (6.142)$$

6.3.6 The equations of motion for the rotation of a rigid body

The derivation of the equations of motion follows the standard steps. The (simple) equations of motion for the translational motion

$$M\ddot{X} = -\frac{\partial U}{\partial X} \quad M\ddot{Y} = -\frac{\partial U}{\partial Y} \quad M\ddot{Z} = -\frac{\partial U}{\partial Z} \quad (6.143)$$

can be obtained directly from the Lagrangian

$$\begin{aligned} L &= \frac{M}{2} (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + T_{\text{rot}}(\beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) \\ &\quad - U(X, Y, Z, \alpha, \beta, \gamma). \end{aligned} \quad (6.144)$$

The right hand side of (6.143) represents the components of the generalised force, which control the motion of the centre of mass.

The steps for the derivation of the equations of motion, which characterise the rotation, will not be detailed here in full¹³. The equation for the coordinate γ is the simplest. It follows from

$$\frac{d}{dt} \left(\frac{\partial T_{\text{rot}}}{\partial \dot{\gamma}} \right) - \frac{\partial T_{\text{rot}}}{\partial \gamma} = - \frac{\partial U}{\partial \gamma} . \quad (6.145)$$

The right hand side represents to the appropriate generalised force. As the coordinate is an angle, the generalised force represents a torque. For the evaluation of the left hand side (LS) the chain rule

$$LS = \frac{d}{dt} \left(\sum_{\mu} \frac{\partial T_{\text{rot}}}{\partial \omega_{\mu}} \frac{\partial \omega_{\mu}}{\partial \dot{\gamma}} \right) - \sum_{\mu} \frac{\partial T_{\text{rot}}}{\partial \omega_{\mu}} \frac{\partial \omega_{\mu}}{\partial \gamma}$$

has to be applied. The following derivatives are needed

$$\frac{\partial T_{\text{rot}}}{\partial \omega_{\mu}} = I_{\mu} \omega_{\mu}$$

$$\frac{\partial \omega_1}{\partial \dot{\gamma}} = \frac{\partial \omega_2}{\partial \dot{\gamma}} = 0 \quad \frac{\partial \omega_3}{\partial \dot{\gamma}} = 1$$

$$\frac{\partial \omega_1}{\partial \gamma} = \dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma = \omega_2$$

$$\frac{\partial \omega_2}{\partial \gamma} = -\dot{\alpha} \sin \beta \sin \gamma - \dot{\beta} \cos \gamma = -\omega_1$$

$$\frac{\partial \omega_3}{\partial \gamma} = 0 .$$

The left hand side can be summarised as

$$LS = \frac{d}{dt} (I_3 \omega_3) - (I_1 \omega_1 \omega_2 - I_2 \omega_2 \omega_1) ,$$

so that the equation of motion takes the form

$$I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 = - \frac{\partial U}{\partial \gamma} . \quad (6.146)$$

The explicit form of this equation of motion (rather than the summary given above) is obtained, if the angular velocities ω_{μ} are expressed in terms of the Euler angles. This result is

$$I_3 (\ddot{\alpha} \cos \beta - \dot{\alpha} \dot{\beta} \sin \beta + \ddot{\gamma}) - (I_1 - I_2) \left\{ [\dot{\alpha}^2 \sin^2 \beta - \dot{\beta}^2] \sin \gamma \cos \gamma + \dot{\alpha} \dot{\beta} \sin \beta (\cos^2 \gamma - \sin^2 \gamma) \right\} = - \frac{\partial U}{\partial \gamma} . \quad (6.147)$$

¹³ A detailed discussion of all the steps which are necessary for the derivation of the equations of motion in terms of the Euler angles are given in ● D.tail 6.6 and 6.7.

It should be noted that the equation of motion in terms of the Euler angles are coupled (in a rather complicated fashion). Any detailed discussion has to be based on this form. The form in terms of the components of the angular velocities in the body-fixed system is on the other hand sufficient for the treatment of some problems.

The Lagrange equations for the coordinates α and β are still more complicated. A lengthier calculation leads to the results (see ● D.tail 6.6)

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\alpha}} \right) - \frac{\partial T}{\partial \alpha} &= - \frac{\partial U}{\partial \alpha} \\
 &= \ddot{\alpha} \left[(I_1 \sin^2 \gamma + I_2 \cos^2 \gamma) \sin^2 \beta + I_3 \cos^2 \beta \right] \\
 &\quad + 2\dot{\alpha}\dot{\beta} \left[I_1 \sin^2 \gamma + I_2 \cos^2 \gamma - I_3 \right] \sin \beta \cos \beta \\
 &\quad + 2\dot{\alpha}\dot{\gamma} \left[I_1 - I_2 \right] \sin^2 \beta \sin \gamma \cos \gamma \\
 &\quad + \ddot{\beta} \left[I_1 - I_2 \right] \sin \beta \sin \gamma \cos \gamma \\
 &\quad + \dot{\beta}^2 \left[I_1 - I_2 \right] \cos \beta \sin \gamma \cos \gamma \\
 &\quad + \dot{\beta}\dot{\gamma} \left[(I_1 - I_2)(\cos^2 \gamma - \sin^2 \gamma) - I_3 \right] \sin \beta \\
 &\quad + \ddot{\gamma} \left[I_3 \cos \beta \right]
 \end{aligned} \tag{6.148}$$

and

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\beta}} \right) - \frac{\partial T}{\partial \beta} &= - \frac{\partial U}{\partial \beta} \\
 &= \ddot{\alpha} \left[I_1 - I_2 \right] \sin \beta \sin \gamma \cos \gamma \\
 &\quad + \dot{\alpha}^2 \left[I_3 - I_1 \sin^2 \gamma - I_2 \cos^2 \gamma \right] \sin \beta \cos \beta \\
 &\quad + \dot{\alpha}\dot{\gamma} \left[I_3 + (I_1 - I_2)(\cos^2 \gamma - \sin^2 \gamma) \right] \sin \beta \\
 &\quad + \ddot{\beta} \left[I_1 \cos^2 \gamma + I_2 \sin^2 \gamma \right] \\
 &\quad - 2\dot{\beta}\dot{\gamma} \left[I_1 - I_2 \right] \sin \gamma \cos \gamma .
 \end{aligned} \tag{6.149}$$

Terms involving the expression $I_\mu \dot{\omega}_\mu$ do not occur directly in these equations as might have been expected in view of the expression (6.132) for the kinetic energy. The reason is the more complicated relation (6.141) between the components of the angular velocity ω_μ and the time derivatives of the Euler angles. There exist no generalised coordinates which correspond directly to the quantities ω_μ .

The differential equations for the angular velocity components ω_1 and ω_2 , which correspond to (6.146), follow from (6.147), (6.148) and (6.149) for the Euler angles by taking suitable linear combinations

$$\begin{aligned}
 I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 &= \frac{1}{\sin \beta} \\
 &\quad \left[-\sin \gamma \frac{\partial U}{\partial \alpha} - \sin \beta \cos \gamma \frac{\partial U}{\partial \beta} + \cos \beta \sin \gamma \frac{\partial U}{\partial \gamma} \right]
 \end{aligned} \tag{6.150}$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_1 \omega_3 = \frac{1}{\sin \beta} \left[-\cos \gamma \frac{\partial U}{\partial \alpha} + \sin \beta \sin \gamma \frac{\partial U}{\partial \beta} + \cos \beta \cos \gamma \frac{\partial U}{\partial \gamma} \right]. \quad (6.151)$$

The expressions on the right hand side correspond to the components of the torque in the 1- and 2- direction of the body-fixed frame. The form of these quantities indicates, in which way the equations of motion in terms of generalised coordinates α , β and γ have to be combined (see ● D.tail 6.7).

The equations for the rotational motion of a rigid body are (in the short form)

$$\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 &= M_1 \\ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 &= M_2 \\ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 &= M_3. \end{aligned} \quad (6.152)$$

They can be summarised in the compact (but not very lucid) form

$$\sum_{\lambda} \epsilon_{\mu\nu\lambda} (I_{\lambda} \dot{\omega}_{\lambda} - M_{\lambda}) - (I_{\mu} - I_{\nu}) \omega_{\mu} \omega_{\nu} = 0 \quad \text{with } (\mu, \nu) = (1, 2), (2, 3), (3, 1) \quad (6.153)$$

because of their cyclic nature. The quantity $\epsilon_{\mu\nu\lambda}$ represents the Levi-Civita symbol (see ● Math.Chap. 3.1.2). The set of differential equations (6.152) (or (6.153)) is known as **Euler's equations**. The Euler equations do not contain any torques for a freely rotating body ($U = 0$).

The solution of the Euler equations (with or without the action of torques) or of the equations of motion in terms of the Euler angles describe the – in general rather complicated – time development of the rotational motion of a rigid body. The next section illustrates this remark.

6.3.7 Rotational motion of rigid bodies

The motion of the physical pendulum and of spinning tops can be calculated with the aid of Euler's equations (6.152) in the short or in the long form.

6.3.7.1 The physical pendulum. The physical pendulum is a rigid body which rotates about an axis through an arbitrary body-fixed point. The Euler equations reduce to

$$I_1^{(O)} \dot{\omega}_1 = M_1$$

if the axis of rotation corresponds to a principal axis (e.g. the 1- axis). The moment of inertia $I_1^{(O)}$ has to be calculated with the parallel axes theorem (6.126). The torque (due to gravitation) depends on the sine of the angle by which the pendulum is displaced, so that the results of the mathematical pendulum (Chap. 4.2.1) can be applied directly. The only change is the replacement of the quantity $[g/l]^{1/2}$ by $[Mgs/I_1^{(O)}]^{1/2}$, where M is the mass

of the rigid body and s is the shortest distance between the centre of mass and the axis of rotation.

6.3.7.2 Calculation of the rotation of a force-free symmetrical top.

The simplest example of a spinning top is the force-free spherical top with $I_1 = I_2 = I_3$. The axis of rotation and the angular velocity do not change with time. This follows from the simple equations of motion

$$\dot{\omega}_\mu = 0 \quad (\mu = 1, 2, 3)$$

with the solution $\omega_\mu(t) = \omega_\mu(t=0)$.

The situation is less trivial for the force-free symmetric top, for instance with the principal moments of inertia $I_1 = I_2 = I$, $I_3 \neq I$. The 3-axis is the **axis of symmetry** in this case. An ellipsoid of revolution (also referred to as a spheroid) can serve as a representative of a symmetric top. The following shapes have to be distinguished:

- $I > I_3$ The mass distribution is concentrated around the 3-axis. The top has, relative to the 3-axis, the form of a cigar (Fig. 6.33a). Such a top is called **prolate**.
- $I < I_3$ The shape is flattened in this case (Fig. 6.33b). This is an **oblate** top.

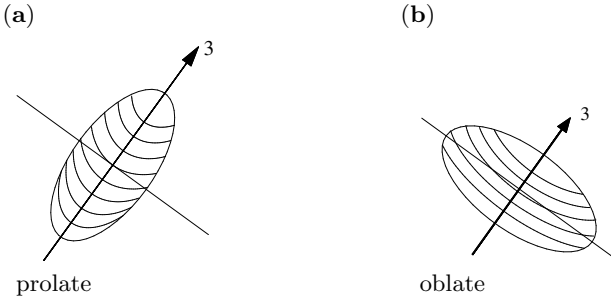


Fig. 6.33. Symmetric tops

The force-free symmetric top can be discussed with the Euler equation of the form (6.152). The equation of motion for the 3-component of the angular velocity of a symmetric top with $I_1 = I_2 = I$ is

$$I_3 \dot{\omega}_3 = 0 \quad \longrightarrow \quad \omega_3(t) = \omega_3(0) .$$

The component of the angular velocity, with respect to the 3-axis does not change with time. The equations of motion for the other components can be solved in a direct fashion. The equations

$$I \dot{\omega}_1 - (I - I_3) \omega_2 \omega_3 = 0$$

$$I \dot{\omega}_2 - (I_3 - I) \omega_3 \omega_1 = 0$$

reduce to a relative simple system of coupled differential equations

$$\dot{\omega}_1 + \Omega \omega_2 = 0 \quad \dot{\omega}_2 - \Omega \omega_1 = 0$$

if the definition

$$\Omega = \left(\frac{I_3 - I}{I} \right) \omega_3 = \text{const.} \quad (6.154)$$

is introduced. A solution of the coupled differential equations requires differentiation of e.g. the first equation with respect to time and insertion of the second equation. The result is

$$\ddot{\omega}_1 + \Omega \dot{\omega}_2 = \ddot{\omega}_1 + \Omega^2 \omega_1 = 0 .$$

This differential equation of an harmonic oscillator has the solution

$$\omega_1(t) = C_1 \cos \Omega t + C_2 \sin \Omega t . \quad (6.155)$$

The function $\omega_2(t)$ can be calculated directly

$$\omega_2 = -\frac{1}{\Omega} \dot{\omega}_1 = C_1 \sin \Omega t - C_2 \cos \Omega t . \quad (6.156)$$

The initial condition

$$\omega_1(0) = A \quad \omega_2(0) = 0$$

will be used for a more detailed discussion of the solution. The top rotates initially about the 1- axis (and about the 3- axis). The resulting special solution

$$\omega_1(t) = A \cos \Omega t \quad \omega_2(t) = A \sin \Omega t \quad (6.157)$$

describes the projection of the angular velocity vector onto the 1 – 2 plane of the body-fixed system (Fig. 6.34a). The projection of the endpoint of $\boldsymbol{\omega}$ onto the 1-2 plane rotates uniformly with the frequency Ω on a circle. The projection of the vector of the angular velocity on the 3- axis is constant. This implies that the vector $\boldsymbol{\omega}$ rotates with the same frequency on a cone about the symmetry axis. The motion of the $\boldsymbol{\omega}$ - vector is called **the regular precession**. Ω is the frequency of the precession. The sense of rotation is positive (counterclockwise) if Ω is positive.

The components of the angular momentum vector in the body fixed frame are according to (6.127)

$$\mathbf{l}_{\text{CM}} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}_{\text{B}} = \begin{pmatrix} I A \cos \Omega t \\ I A \sin \Omega t \\ I_3 \omega_3 \end{pmatrix}_{\text{B}} . \quad (6.158)$$

The angular momentum vector, as observed in the body-fixed system, rotates about the symmetry axis with the same frequency, however, on a cone with a different apex (opening) angle (Fig. 6.34b). The vectors $\boldsymbol{\omega}$ and \mathbf{l}_{S} span for all times a plane which includes the symmetry axis. The ratio of the amplitudes is

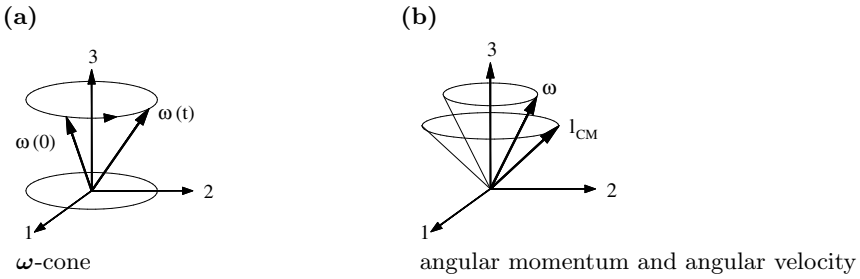


Fig. 6.34. Regular precession of a prolate, symmetrical top in the body-fixed system

$$\frac{I_3 \omega_3}{IA} < \frac{\omega_3}{A}$$

for a prolate top ($I > I_3$). The apex angle of the l -cone is larger than that of the ω -cone (Fig. 6.35a) for this shape. The l -cone is inside of the ω -cone (Fig. 6.35b) for an oblate top. The kinetic energy of the top is

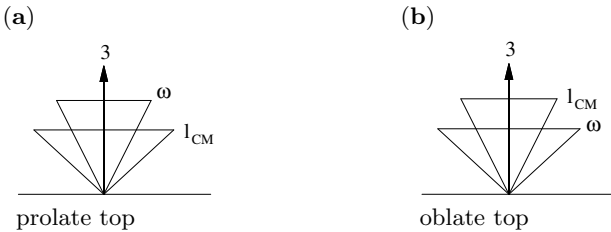


Fig. 6.35. Regular precession observed in the body-fixed system

$$T_{\text{rot}} = \frac{1}{2} l_{CM} \omega = \frac{1}{2} (IA^2 + I_3 \omega_3^2) = \text{const.}$$

As no forces act on the top, energy must be conserved.

6.3.7.3 Illustration of the rotation of a force-free symmetric top in the space-fixed frame. A study of the vectors ω and l from the point of view of the body-fixed system does not convey an impression of the actual motion of the symmetric top. The time development of the motion has to be viewed from the space-fixed inertial system for this purpose. A possible uniform translation is not of interest, so that the centre of mass can be chosen as the origin of both coordinate systems. The relation between the time derivatives in the space-fixed and the body-fixed coordinate systems (see Chap. 6.2, rotating coordinate systems) is valid for each vector, therefore also for the angular momentum

$$\dot{l}_I = \dot{l}_B + (\omega \times l_B) .$$

Insertion of the quantities required ($\mathbf{l}_B \equiv \mathbf{l}_{CM}$, see ● D.tail 6.8) on the right hand side yields, as expected,

$$\dot{\mathbf{l}}_I = \mathbf{0} . \quad (6.159)$$

The angular momentum in an inertial system is a conserved quantity for a force-free motion. The angular momentum is a vector fixed in space so that both the axis of symmetry and the vector of the angular velocity precess about it from the point of view of the space-fixed frame (Fig. 6.36).

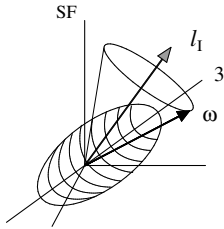


Fig. 6.36. Regular precession: Viewed from the space-fixed system

The actual motion can be illustrated with **Poinsot's construction**. For a prolate top the following situation (Fig. 6.37) is found: The ω - vector,

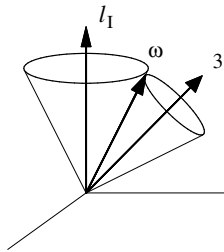


Fig. 6.37. Regular precession: Poinsot's construction for a prolate top

which marks the instantaneous axis of rotation, rotates on a cone about the angular momentum vector (which is fixed in space). This cone is the **space fixed cone** (herpolhode). A second cone, the **body fixed cone** (polhode), whose axis is the symmetry axis, rolls on the space fixed cone.

The rolling of the body fixed cone on the space fixed cone is such that the line of contact of the two cones corresponds to the ω - vector. The three vectors (angular momentum, angular velocity and symmetry axis) lie (as in the body-fixed frame) for all times in the same plane. The resulting motion of the symmetry axis and the rolling motion of the body fixed cone provide an impression of the rotational motion of a force-free rigid body with the characteristics of a symmetrical ellipsoid of rotation. The body fixed cone rolls, as can be checked with a similar construction, on the inside of the space fixed cone in the case of an oblate top (Fig. 6.38).

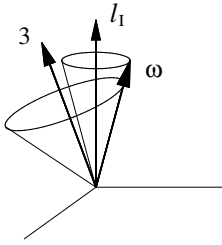


Fig. 6.38. Regular precession: Poincaré's construction for an oblate top

The theory of the force-free, symmetric top has some relevance for the discussion of the rotation of the earth. The earth is, in reasonable approximation, a force-free, oblate top with the axes $a = b = 6377$ km (equator) and $c = 6356$ km (pole). The symmetry axis (geometric north pole) differs from

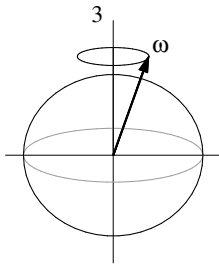


Fig. 6.39. Regular precession of the symmetry axis of the earth

the axis of rotation (ω). The kinematical north pole marks a circle about the geometric north pole (Fig. 6.39). The frequency of precession is

$$|\Omega| = \left| \frac{I_3 - I}{I} \omega_3 \right|.$$

The period of this precession can be calculated to be

$$T = \frac{2\pi}{\Omega} \approx 305 \text{ days} \approx 10 \text{ months},$$

if the moments of inertia of the earth (assuming a homogeneous distribution of mass) are obtained from the known data and if $\omega_3 = 2\pi/\text{day}$ is used.

The corresponding 'experimental' data are:

- The precession is not regular, there are fluctuations related to meteorological and geological events.
- The mean period is not 10 months but approximately 14 months (Chandler period). The discrepancy is explained by a detailed theory, which makes allowances for the fact that the earth is not completely rigid, but quasi fluid¹⁴.

¹⁴ The interested reader will find a relevant publication under [7] in the list of references.

- The radius of the circle of precession of the ω -cone on the surface of the earth is about 4 m (in the mean).

The discussion of the force-free top has to be replaced by the solution of the Euler equations in terms of the Euler angles if more detailed information is desired. This more involved approach is indicated in the next section.

6.3.7.4 The time development of the Euler angles for the force-free symmetric top. The discussion of the motion of a top can be put on a fully quantitative level, if the explicit calculation based on the equations of motion (6.147), (6.147) and (6.147) in terms of the Euler angles is carried out. Such a calculation will be presented for the example of a symmetric top ($I_1 = I_2 = I \neq I_3$). The expression (6.142) for the rotational kinetic energy simplifies in this case

$$T_{\text{rot}} = \frac{1}{2} \left[I(\dot{\beta}^2 + \dot{\alpha}^2 \sin^2 \beta) + I_3(\dot{\gamma} + \dot{\alpha} \cos \beta)^2 \right]. \quad (6.160)$$

A corresponding simplification can be expected for the Euler equations. It is, however, possible to approach the problem of the free symmetric top in a more direct fashion.

The Lagrangian of the free top $L_{\text{free}} \equiv T$ does not depend on the angles α and γ . The corresponding generalised momenta

$$p_\alpha = \frac{\partial T}{\partial \dot{\alpha}} = I \dot{\alpha} \sin^2 \beta + I_3(\dot{\gamma} + \dot{\alpha} \cos \beta) \cos \beta = C_1 \quad (6.161)$$

and

$$p_\gamma = \frac{\partial T}{\partial \dot{\gamma}} = I_3(\dot{\gamma} + \dot{\alpha} \cos \beta) = C_2 \quad (6.162)$$

are conserved quantities. The constants C_1 and C_2 are determined by the initial conditions which are to be specified. Insertion of the expression (6.141) for $\dot{\gamma}$ into (6.162) reproduces the statement

$$I_3 \omega_3 = C_2 \quad (6.163)$$

of the previous section. The projection of the vector of the angular velocity on the symmetry axis ω_3 is a conserved quantity. Combination of (6.161) and (6.162) yields the relation

$$I \dot{\alpha} \sin^2 \beta = C_1 - C_2 \cos \beta \quad (6.164)$$

which can be used together with (6.162) to eliminate the derivatives $\dot{\alpha}$ and $\dot{\gamma}$ from the Lagrangian (6.160). The result is an expression which depends only on β and its derivatives

$$L_{\text{free}} = \frac{1}{2} \left[I \dot{\beta}^2 + \frac{(C_1 - C_2 \cos \beta)^2}{I \sin^2 \beta} + \frac{C_2^2}{I_3} \right]. \quad (6.165)$$

This result could also be obtained by elimination of the angles α and γ from the Euler equation for β and a first integration.

The constant term C_2^2/I_3 can be incorporated into the conserved energy by using the energy value

$$E = E_0 - \frac{C_2^2}{2I_3} \quad (6.166)$$

instead of $E_0 = L_{\text{free}}(0)$.

The rescaled Lagrangian (6.165) corresponds to the differential equation

$$\dot{\beta} = \pm \left[\frac{2E}{I} - \frac{(C_1 - C_2 \cos \beta)^2}{I^2 \sin^2 \beta} \right]^{1/2} \quad (6.167)$$

for the function $\beta(t)$. The notation can be simplified with the abbreviations

$$a = 2E/I \quad a_1 = C_1/I \quad a_2 = C_2/I. \quad (6.168)$$

In addition, the variable β is replaced with the substitution

$$q = \cos \beta \quad \text{with} \quad \dot{q} = -\dot{\beta} \sin \beta$$

(compare the discussion of the spherical pendulum). The resulting differential equation can be solved by separation of variables

$$t = \int_{q(0)}^q \frac{dq'}{\sqrt{a(1 - q'^2) - (a_1 - a_2 q')^2}}. \quad (6.169)$$

The radicand in this integral is a quadratic function, so that the integral can be evaluated in an elementary fashion¹⁵.

The result for $q(t)$, respectively $\beta(t)$ can be inserted into the differential equations (6.161) and (6.162) (or (6.164)) for the functions $\alpha(t)$ and $\gamma(t)$. A direct integration (possibly numerically) of these equation yields the remaining ingredients for the construction of the time dependent rotation matrix (6.136), which describes the motion of the body-fixed system (that is the top) with respect to the (inertial) space-fixed coordinate system. For example, the projection of the symmetry axis onto the space-fixed system is obtained as one of the relations contained in the transformation (6.137). The result is

$$(x, y, z)_{\text{sym.ax}, I} = (\sin \alpha(t) \sin \beta(t), -\cos \alpha(t) \sin \beta(t), \cos \beta(t)) \quad (6.170)$$

if the symmetry axis is characterised in the body-fixed system by the vector $\mathbf{r}_B = (0, 0, 1)$. This result illustrates the following features:

- The angle $\beta(t)$ is the (time changing) angle between the vertical (the z -axis of the space-fixed system) and the axis of symmetry. This function describes the 'wobbling motion' of the top.

and

¹⁵ More details concerning the explicit calculation of the motion of the free, symmetric top are presented in ● D.tail 6.9.

- The projection of the axis of symmetry onto the x - y plane rotates counterclockwise in this plane for positively increasing values of the angle $\alpha(t)$. A snap shot of the situation is indicated in Fig. 6.40. The figure also shows the limits β_1 and β_2 for the possible variation of the angle β (black arrows) which is discussed below. The additional rotation of the top about the axis of symmetry (the rotation by the angle γ) is not described by (6.170).

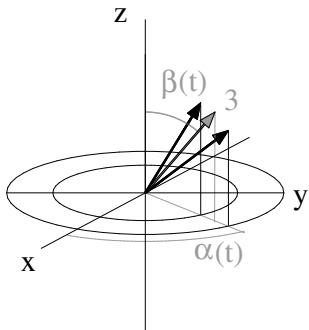


Fig. 6.40. Inertial system: The rotating symmetry axis of the free top

It is possible to extract some information concerning the form of the precession of the top without an explicit solution of the equations of motion. The Lagrangian (6.165) can be decomposed into a kinetic part for the angle β and an effective potential. Two possible shapes of this effective potential

$$U_{\text{eff}} = \frac{(C_1 - C_2 \cos \beta)^2}{2I \sin^2 \beta} \quad (6.171)$$

are illustrated in Fig. 6.41 for the relevant interval $0 \leq \beta \leq \pi$. This figure demonstrates that the angle β can only vary within a restricted range $\beta_1 \leq \beta \leq \beta_2$ which is determined by the initial energy E_0 .

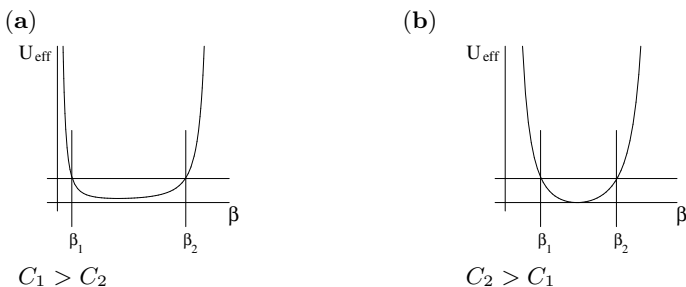


Fig. 6.41. Effective potential of the free top

6.3.7.5 The heavy symmetric top. A symmetric top, which is supported in a point of the axis of symmetry (in most cases the tip, which is not the centre of mass) is called a heavy symmetric top (Fig. 6.42). The (constant) gravitational force exerts a torque on the top in this case. The point of support (which is considered to be stationary) serves as the common origin of the space-fixed and the body-fixed coordinate systems. The total kinetic energy of the top is then only rotational (according to the relation (6.124)). The torque acting on the top is

$$\mathbf{M} = M_K(\mathbf{s} \times \mathbf{g}) ,$$

where M_K is the mass of the top and \mathbf{s} is the vector from the point of support to the centre of mass. The vector of the torque (see Fig. 6.42) is perpendicular to the vertical and the axis of symmetry. It points in the direction of the line of nodes (● D.tail 6.10) and hence leads to a rotation of the angular momentum vector. The calculation of the motion of the heavy top differs only slightly

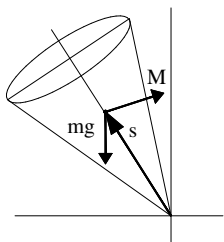


Fig. 6.42. Heavy top

from the case of the free top on a *formal level*. The Lagrangian of the free top has to be supplemented by the potential energy due to the torque

$$U_{\text{grav}}(\beta) = M_K g s \cos \beta . \quad (6.172)$$

The coordinates α and γ are still cyclical as the additional energy term depends only on the angle β . This implies that the steps between (6.161) and (6.169) can be repeated with the extended Lagrangian¹⁶

$$L_{\text{heavy}} = \frac{1}{2} \left[I(\dot{\beta}^2 + \dot{\alpha}^2 \sin^2 \beta) + I_3(\dot{\gamma} + \dot{\alpha} \cos \beta)^2 \right] - M_K g s \cos \beta . \quad (6.173)$$

The solution of the extended equation of motion for the angle β can be written down after the substitution $q = \cos \beta$ and use of the definitions (6.168) for the parameters

$$t = \int_{q(0)}^q \frac{dq'}{\sqrt{(a - bq')(1 - q'^2) - (a_1 - a_2 q')^2}} . \quad (6.174)$$

¹⁶ The moments of inertia are, however, referred to a body-fixed coordinate system with the point of support as the origin.

The additional parameter b is defined as $b = 2M_K g s / I$. The relation (6.174) for the heavy top differs from the corresponding relation (6.169) for the free top. The radicand is given by a cubic polynomial in the present case instead of a quadratic function in q . The integral in (6.174) corresponds to an **elliptic integral of the first kind**¹⁷. Further discussion along the lines indicated before involves therefore heavier mathematical tools.

It is, however, still possible to gain some insight into the motion of the heavy top by a discussion of the polynomial in (6.174) and an effective potential similar to the one given in (6.171). The effective potential, which can be extracted from the Lagrangian (6.173)

$$U_{\text{eff, heavy}}(\beta) = \frac{(C_1 - C_2 \cos \beta)^2}{2I \sin^2 \beta} + M_K g s \cos \beta, \quad (6.175)$$

is represented in Fig. 6.43. The angle β is also restricted to a finite interval

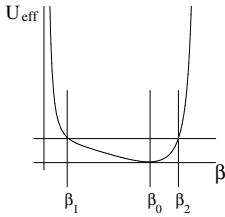


Fig. 6.43. Effective potential for a heavy top ($C_1 > C_2$)

$\beta_1 \leq \beta \leq \beta_2$ for a given initial energy. The potential has a minimum for β_0 which is more pronounced for the heavy top due to the additional term in $\cos \beta$. The angle between the symmetry axis and the vertical will remain constant in time for this value of β . This angle of inclination of the symmetry axis with respect to the vertical (β_0) is determined by

$$\left. \frac{dU_{\text{eff, heavy}}}{d\beta} \right|_{\beta_0} = 0.$$

Direct evaluation (● D.tail 6.11) leads to the condition

$$(C_1 - C_2 \cos \beta_0) = \frac{C_2 \sin^2 \beta_0}{2 \cos \beta_0} \left[1 \pm \sqrt{1 - \frac{4M_K g s I \cos \beta_0}{C_2^2}} \right]. \quad (6.176)$$

The radicand in (6.176) can only be positive for $\beta_0 \leq \pi/2$, if the condition

$$C_2^2 \geq 4M_K g s I \cos \beta_0$$

or using (6.163)

$$\omega_3 \geq \frac{2}{I_3} \sqrt{M_K g s I \cos \beta_0}$$

¹⁷ For a comparison with the elliptic integral in Chap. 4.2.1 see ● Math.Chap. 4.3.4.

is satisfied. If the initial conditions lead to the angle β_0 , the following motion can be observed.

- A quasi-regular precession of the symmetry axis of the heavy top occurs for angles with $\beta_0 \leq \pi/2$ provided the inequality for the angular momentum component ω_3 is satisfied as well. This precession is described by the angle $\alpha(t)$. The symmetry axis rotates about the vertical with a constant angular velocity (see (6.164))

$$\dot{\alpha} = \frac{C_1 - C_2 \cos \beta_0}{I \sin^2 \beta_0}. \quad (6.177)$$

There exists a slower or a faster version of this precession as the condition (6.176) admits two different positive values for $C_1 - C_2 \cos \beta_0$.

- The case with $\beta_0 \geq \pi/2$ corresponds to a situation in which the point of support (a point of suspension actually) is above the centre of mass, a situation which can only be realised with a gyroscopic suspension. The condition (6.176) leads to two values for $(C_1 - C_2 \cos \beta_0)$ with a different sign as $\cos \beta_0$ is negative for this range of angles. The relation (6.177) then shows that the top rotates in the same sense as for the case $\beta_0 < \pi/2$ for the faster (positive) component but in the opposite sense for the slower (negative) one.

The motion of the symmetry axis is more complicated, if the top is not in the minimum of the effective potential. The sign of $\dot{\alpha}$ changes also according to (see (6.164) again)

$$\dot{\alpha} = \frac{C_1 - C_2 \cos \beta}{I \sin^2 \beta},$$

while the angle β varies between the two limiting values. Details depend on the integration constants C_1 and C_2 . The top precesses monotonously if there is no change of sign. The projection of the symmetry axis onto a unit sphere describes a sort of oscillation within the band between the limiting angles β_1 and β_2 (Fig. 6.44a). Such a motion is called a **nutation**. The oscillation of the symmetry axis is replaced by a looping motion (Fig. 6.44b), if the sign of $\dot{\alpha}$ changes with a change of the values of β , because the precession for the upper and the lower values of β has a different sign. A special case may occur for

$$(C_1 - C_2 \cos \beta_i) = 0$$

which gives

$$\dot{\alpha}(\beta_i) = 0 \quad \dot{\beta}(\beta_i) = 0 \quad (\text{for } i = 1 \text{ or } 2).$$

This leads to a projection of the motion of the axis of symmetry on the unit sphere with spikes and arcs (Fig. 6.44c).

This discussion could also be based on the radicand in (6.174). The qualitative behaviour of the function

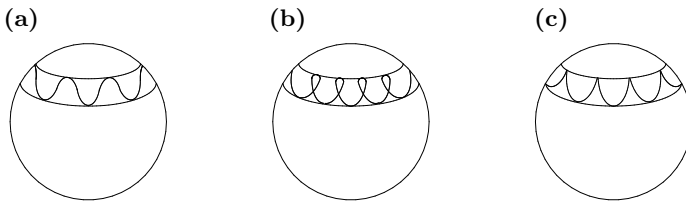


Fig. 6.44. Motion of a heavy top: Projection of the symmetry axis on a unit sphere for different ratios of the integration constants C_1/C_2

$$f(q) = (a - bq)(1 - q^2) - (a_1 - a_2q)^2$$


is similar to that of the function (5.97) found for the discussion of the spherical pendulum. The function $f(q)$ can, in general, have three zeros. Two of the zeros limit the range of positive values of the function f . They must lie in the range of physical interest, the interval $-1 \leq q \leq 1$. These zeros correspond to the two limiting angles β_1 and β_2 .

The treatment of the asymmetric top (free or heavy) is even more involved. In the calculation of the motion of the free asymmetric top elliptic integrals occur already during the solution of the simpler Euler equations of motion (6.152). The various cone constructions can be used, but the guiding curves are not circles as in the case of the symmetric top. They are transcendental curves, which do not close in the general case.¹⁸

¹⁸ The standard text on the theory of tops (list of references [8]) has been published more than 100 years ago.

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This list of references contains

- the explicit citations in the text
- text books on Theoretical Mechanics
- text books on mathematical topics. A more extensive documentation of the mathematical literature can be found in the 'Mathematical Complements' on the accompanying virtual  CD-ROM.

The text books are listed alphabetically. The sequence is no indication of the level of presentation or the difficulty. Books, which are not available for sale any more (as far as can be ascertained from the internet pages of the publishers), are marked by an asterisk (*).

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- see also the relevant sections of the general handbooks

Appendix

The following data are collected in this Appendix:

- The brief list of biographical data of scientists is intended to put the topics associated with these names into the proper historical context.
- The Greek alphabet which is used extensively in physics and mathematics.
- A short list of other mathematical symbols found in the text.
- The physical quantities relevant in mechanics are listed together with their units in the CGS- and SI-systems.
- Conversion factors for common physical quantities.
- A selection of astronomical data (for planetary motion).
- A rudimentary collection of formulae.
- A listing and a brief outline of the problems sorted by chapter which can be found on the virtual  CD-ROM.

A Biographical data

The following data are extracted from a number of sources (encyclopaediae, internet) and from D. Hoffmann (ed.) et al.: 'Lexikon der bedeutenden Naturwissenschaftler' (Spektrum Akademischer Verlag, Heidelberg, 2007). Uncertain dates are marked by (?).

d'Alembert, Jean le Rond	French philosopher * 16.11.1717 Paris (France) † 29.10.1783 Paris (France)
Archimedes	Greek mathematician and physicist * 287(?) v. Chr. Syracuse (Italy) † 212 v. Chr. Syrakus (Italy)
Atwood, George	English mathematician and physicist * 1745/46(?) London (England) † 11.07.1807 London (England)
Bernoulli, Jacob	Swiss mathematician * 27.12.1654 Basel (Switzerland) † 16.08.1705 Basel (Switzerland)
Bernoulli, Johann	Swiss mathematician * 27.07.1667(?) Basel (Switzerland) † 01.01.1748 Basel (Switzerland)
Cavendish, Henry	English physicist and chemist * 10.10.1731 Nice (France) † 24.02.1810 London (England)
Chadwick, Sir James	English physicist Nobel price 1935 * 20.10.1891 Manchester (England) † 24.07.1974 Cambridge (England)
Chandler, Seth Carlo	American astronomer * 16.09.1846 Boston (USA) † 13.12.1913 Wellesley Hills (USA)
Chasles, Michel	French mathematician * 15.11.1793 Epervon (France) † 18.12.1880 Paris (France)

de Coriolis, Gustave Gaspard	French physicist * 21.05.1792 Paris (France) † 19.09.1843 Paris (France)
Cockroft, Sir John Douglas	English physicist Nobel price 1951 * 27.05.1887 Todmorden (England) † 18.09.1967 Cambridge (England)
de Coulomb, Charles Augustin	French physicist * 14.06.1736 Angoulême (France) † 23.08.1806 Paris (France)
Descartes, René, (Renatus Cartesius)	French philosopher and mathematician * 31.03.1596 La Haye (France) † 11.02.1650 Stockholm (Sweden)
Einstein, Albert	German physicist Nobel price 1921 * 14.03.1897 Ulm (Germany) † 18.04.1955 Princeton (USA)
Euler, Leonhard	Swiss mathematician and astronomer * 15.04.1707 Basel (Switzerland) † 18.09.1783 St Petersburg (Russia)
Feynman, Richard Phillips	American physicist Nobel price 1965 * 11.05.1918 New York (USA) † 15.02.1988 Los Angeles (USA)
Foucault, Jean Bernard Léon	French physicist * 18.09.1819 Paris (France) † 11.02.1868 Paris (France)
Fourier, Joseph Baron	French mathematician * 21.03.1768 Auxerre (France) † 16.05.1830 Paris (France)
Galilei, Galileo	Italian physicist and astronomer * 15.02.1564 Pisa (Italy) † 08.01.1642 Arcetri (Italy)
Geiger, Hans Wilhelm	German physicist * 30.09.1882 Neustadt (Germany) † 24.09.1945 Potsdam (Germany)

Halley, Edmond	English astronomer * 08.11.1656 London (England) † 14.01.1742 Greenwich (England)
Hamilton, Sir William Rowan	Irish mathematician * 03/04.08.1805 Dublin (Ireland) † 02.09.1865 Dunsink (Ireland)
Heisenberg, Werner Karl	German physicist Nobel price 1932 * 05.12.1901 Würzburg (Germany) † 01.02.1976 München (Germany)
Hooke, Robert	English physicist * 18.07.1635 Freshwater (England) † 03.03.1703 London (England)
Hubble, Edwin Powell	American astronomer * 20.11.1889 Marshfield (USA) † 28.09.1953 San Marino (USA)
Huygens, Christiaan	Dutch physicist and astronomer * 14.04.1629 Den Haag (Holland) † 08.07.1695 Den Haag (Holland)
Jacobi, Carl Gustav Jacob	German mathematician * 10.12.1804 Potsdam (Germany) † 18.02.1851 Berlin (Germany)
Joule, James Prescott	English physicist * 24.12.1818 Salford (England) † 11.10.1889 Sale (England)
Joyce, James Augustine	Irish author * 02.02.1882 Dublin (Ireland) † 13.01.1941 Zürich (Switzerland)
Kepler, Johannes	German astronomer * 27.12.1571 Weil der Stadt (Germany) † 15.11.1630 Regensburg (Germany)
de Lagrange, Joseph-Louis comte	Italian-French mathematician * 25.01.1736 Turin (Italy) † 10.04.1813 Paris (France)

Legendre, Adrien Marie	French mathematician * 18.09.1752 Paris (France) † 10.01.1833 Paris (France)
Lissajous, Jules Antoine	French physicist * 04.03.1822 Versailles (France) † 24.06.1880 Plombières-les-Bains (France)
Marsden, Sir Ernest	English physicist * 19.02.1889 Rishton (England) † 14.12.1970 Wellington (New Zealand)
Michelson, Albert Abraham	American physicist Nobel price 1907 * 19.12.1852 Strzelno (Poland) † 09.05.1931 Pasadena (USA)
Morley, Edward Williams	American chemist * 29.01.1838 Newark (USA) † 24.02.1923 West Hartford (USA)
Newton, Sir Isaac	English physicist and mathematician * 04.01.1643 Woolsthorpe (England) † 31.03.1727 Kensington (England)
Poincaré, Jules Henri	French mathematician and philosopher * 29.04.1854 Nancy (France) † 17.07.1912 Paris (France)
Poisson, Siméon Denis	French mathematician and physicist * 21.06.1781 Pithiviers (France) † 25.04.1840 Sceaux (France)
Poinsot, Louis	French mathematician * 03.01.1777 Paris (France) † 05.12.1859 Paris (France)
Lord Rayleigh, John William	English physicist Nobel price 1904 * 12.11.1842 Langford Grove (England) † 30.06.1919 Terling Place (England)
Riemann, Bernhard Georg Friedrich	German mathematician * 17.09.1826 Breselenz (Germany) † 20.07.1866 Selasca (Italy)

Rutherford, Ernest	English physicist Nobel price 1908 * 30.08.1871 Brightwater (New Zealand) † 19.10.1937 Cambridge (England)
Steiner, Jakob	Swiss mathematician * 18.03.1796 Utzenstorf (Switzerland) † 01.04.1863 Bern (Switzerland)
Stokes, Sir George Gabriel	English mathematician and physicist * 13.08.1819 Skreen (Ireland) † 01.02.1903 Cambridge (England)
Taylor, Brook	English mathematician * 18.08.1685 Edmonton (England) † 29.12.1731 London (England)
Walton, Ernest Thomas	Irish physicist Nobel price 1951 * 06.10.1903 Dungarvan (Ireland) † 25.06.1995 Belfast (Ireland)

B The Greek Alphabet

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ, ε	E	epsilon
ζ	Z	zeta
η	H	eta
θ, ϑ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ, ϱ	R	rho
σ, ς	Σ	sigma
τ	T	tau
ϕ, φ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega
v	Υ	upsilon

C Nomenclature

Symbols

\equiv	equivalent
\approx	approximately equal
\propto	proportional to
\boldsymbol{v}	vector v
\hat{A}	matrix A
mod	modulo
$\boldsymbol{x} \cdot \boldsymbol{y}$	scalar product of the vectors \boldsymbol{x} and \boldsymbol{y}
$\boldsymbol{x} \times \boldsymbol{y}$	vector (cross) product of \boldsymbol{x} and \boldsymbol{y}
$\text{grad } \phi = \nabla \phi$	gradient of ϕ
$\text{div } \boldsymbol{f} = \nabla \cdot \boldsymbol{f}$	divergence of \boldsymbol{f}
$\text{rot } \boldsymbol{f} = \nabla \times \boldsymbol{f}$	rotation (curl) of \boldsymbol{f}
$[\]$	unit/dimension
O	order of an expansion

D Physical Quantities

Table D.1. Systems of Basic Units

Notation	Explanation	Units
CGS System	Centimetre-Gram-Second System	cm, g, s
SI System	Système Internationale (Metre-Kilogram-Second System)	m, kg, s

Table D.2. Prefixes for Powers of Ten

nomenclature	symbol	powers of ten
tera	T	10^{12}
giga	G	10^9
mega	M	10^6
kilo	k	10^3
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

Table D.3. Physical units in the CGS- and SI-Systems

Physical Quantity	Symbol	Units in	
		CGS-System	SI-System
length	L	cm	m
mass	M	g	kg
time	T	s	s
velocity	L/T	cm/s	m/s
acceleration	L/T ²	cm/s ²	m/s ²
force	ML/T ²	g cm/s ² = dyn	kg m /s ² = N
momentum	ML/T	g cm/s = dyn s	kg m /s = N s
energy, work	ML ² /T ²	g cm ² /s ² = dyn cm = erg	kg m ² /s ² = N m = J
power	ML ² /T ³	g cm ² / s ³ = dyn cm/s	kg m ² /s ³ = J/s=W
volume	L ³	cm ³	m ³
density	M/L ³	g/cm ³	kg/m ³
angle	–	rad	rad
angular velocity	1/T	rad/s	rad/s
angular acceleration	1/T ²	rad/s ²	rad/s ²
torque	ML ² /T ²	g cm ² /s ²	kg m ² /s ²
angular momentum	ML ² /T	g cm ² /s	kg m ² /s
moment of inertia	ML ²	g cm ²	kg m ²
pressure	M/(LT ²)	g/(cm s ²)= dyn/cm ²	kg/(m s ²)= N/m ²

Table D.4. Some Conversion factors I

Physic. Quantity	Unit	Symbol Unit	Value in other Unit
length	1 kilometre	km	1000 m
	1 metre	m	100 cm
	1 centimetre	cm	10^{-2} m
	1 millimetre	mm	10^{-3} m
	1 micrometre	μm	10^{-6} m
	1 nanometre	nm	10^{-9} m
	1 Ångström	Å	10^{-10} m
area	1 square kilometre	km^2	10^6 m^2
	1 square metre	m^2	10^4 cm^2
	1 ar	a	10^2 m^2
volume	1 litre	l	1000 cm^3
	1 cubic metre	m^3	1000 l
mass	1 kilogram	kg	1000 g
	1 ton	t	1000 kg
velocity	1 kilometre per hour	km/h	$\approx 0.2778 \text{ m/s}$
density	1 gram per	g/cm^3	10^3 kg/m^3
	cubic centimetre		
force	1 dyne	dyn	1 g cm/s^2
	1 kilopond	kp	9.807 N
	1 Newton	N	$10^5 \text{ dyn} =$ 0.10197 kp

Table D.5. Some Conversion Factors II

Physic. Quantity	Unit	Symbol Unit	Value in other Unit
energy	1 Joule	J	1 Nm= 10^7 erg =0.2389 cal
	1 kilo calorie	kcal	1000 cal = 4184 J
	1 kilowatt hour	kWh	$3.6 \cdot 10^6$ J = 859.8 kcal
power	1 electron volt	eV	$1.602 \cdot 10^{-19}$ J
	1 watt	W	1 J/s= 10^7 erg/s = 0.2389 cal/s
	1 horse power	PS	75 kp m/s = 735.5 W
pressure	1 kilowatt	kW	1.360 PS
			1 N/m ² = 10 dyn/cm ²
	1 bar	bar	10^5 N/m ²
	1 phys. atmosphere	atm	1.013 bar
	1 torr		1/760 atm
angle	1 techn. atmosphere	at	1 kp/cm ² = 0.9807 bar
	1 hectopascal	hPa	100 N/m ² = 10^3 bar
	1 radian	rad	$\approx 57.2958^\circ$
	1°	0.017453 rad	

E Some Constants and Astronomical Data

Table E.1. Constants

Name	Symbol	Value
universal gravitational constant	γ	$6.674 \cdot 10^{-8} \text{ cm}^3/(\text{g s}^2)$
mean gravitational acceleration at the surface of the earth	g	9.81 m/s^2
velocity of light	c	$2.997925 \cdot 10^8 \text{ m/s}$

The numbers are adapted from R.Wielen (ed.): 'Planeten und ihre Monde'
(Spektrum Akademischer Verlag, Heidelberg, 1997)

Table E.2. Astronomical Data I

Mass	kg
Sun	$1.99 \cdot 10^{30}$
Earth	$5.98 \cdot 10^{24}$
Moon	$7.35 \cdot 10^{22}$
Jupiter	$1.90 \cdot 10^{27}$
Saturn	$5.68 \cdot 10^{26}$
Venus	$4.87 \cdot 10^{24}$
Mars	$6.42 \cdot 10^{23}$
Radius (mean)	km
Sun	$6.96 \cdot 10^5$
Earth	$6.37 \cdot 10^3$
Moon	$1.74 \cdot 10^3$
Jupiter	$7.15 \cdot 10^4$
Saturn	$6.03 \cdot 10^4$
Venus	$6.05 \cdot 10^3$
Mars	$3.40 \cdot 10^3$
Radius of Orbit (mean)	km
Earth	$1.50 \cdot 10^8$
Moon	$3.84 \cdot 10^5$
Jupiter	$7.78 \cdot 10^8$
Saturn	$1.43 \cdot 10^9$
Venus	$1.08 \cdot 10^8$
Mars	$2.28 \cdot 10^8$
Period of Rotation (axis, mean)	s
Earth	$8.62 \cdot 10^4$
Moon	$2.36 \cdot 10^6$
Jupiter	$3.53 \cdot 10^4$
Saturn	$3.88 \cdot 10^4$
Venus	$\approx 2.1 \cdot 10^7$
Mars	$8.83 \cdot 10^4$

Table E.3. Astronomical Data II

Gravitational Acceleration (mean)	m/s²
Earth	9.81
Moon	1.63
Jupiter	24.82
Saturn	10.45
Venus	8.87
Mars	3.70
Period (orbital, mean)	s
Earth	$3.16 \cdot 10^7$
Moon	$2.36 \cdot 10^6$
Jupiter	$3.76 \cdot 10^8$
Saturn	$9.31 \cdot 10^8$
Venus	$1.96 \cdot 10^7$
Mars	$5.93 \cdot 10^7$

F Formulae

F.1 Plane Polar Coordinates

Definition

$$x = r \cos \varphi \quad y = r \sin \varphi$$

Length of a vector \mathbf{r}

$$r(t) = [x^2(t) + y^2(t)]^{1/2}$$

Angle between \mathbf{r} and x -axis

$$\varphi(t) = \arctan \frac{y(t)}{x(t)}$$

$$\mathbf{r}(t) = r(t) \mathbf{e}_r(t)$$

$$\mathbf{v}(t) = \dot{r} \mathbf{e}_r + r \dot{\varphi} \mathbf{e}_\varphi = v_r \mathbf{e}_r + v_\varphi \mathbf{e}_\varphi$$

v_r : radial velocity

v_φ : azimuthal velocity

$$\mathbf{a}(t) = (\ddot{r} - r \dot{\varphi}^2) \mathbf{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi}) \mathbf{e}_\varphi = a_r \mathbf{e}_r + a_\varphi \mathbf{e}_\varphi$$

a_r : radial acceleration

a_φ : azimuthal acceleration

$$\mathbf{e}_r(t) = \cos \varphi(t) \mathbf{e}_x + \sin \varphi(t) \mathbf{e}_y$$

$$\mathbf{e}_\varphi(t) = -\sin \varphi(t) \mathbf{e}_x + \cos \varphi(t) \mathbf{e}_y$$

$$\mathbf{e}_x = \cos \varphi(t) \mathbf{e}_r(t) - \sin \varphi(t) \mathbf{e}_\varphi(t)$$

$$\mathbf{e}_y = \sin \varphi(t) \mathbf{e}_r(t) + \cos \varphi(t) \mathbf{e}_\varphi(t)$$

$$dx dy = r dr d\varphi$$

F.2 Cylinder Coordinates

$$\begin{aligned}x &= \rho \cos \varphi & y &= \rho \sin \varphi & z &= z \\ \rho &= \sqrt{x^2 + y^2} & \varphi &= \arctan \frac{y}{x} & z &= z\end{aligned}$$

$$\begin{aligned}\mathbf{r}(t) &= \rho \mathbf{e}_\rho(t) + z \mathbf{e}_z \\ \mathbf{v}(t) &= \dot{\rho} \mathbf{e}_\rho(t) + \rho \dot{\varphi} \mathbf{e}_\varphi(t) + \dot{z} \mathbf{e}_z \\ \mathbf{a}(t) &= (\ddot{\rho} - \rho \dot{\varphi}^2) \mathbf{e}_\rho(t) + (\rho \ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) \mathbf{e}_\varphi(t) + \ddot{z} \mathbf{e}_z\end{aligned}$$

$$\begin{aligned}r(t) &= \sqrt{\rho^2 + z^2} \\ v(t) &= \sqrt{\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2} \\ a(t) &= \sqrt{(\ddot{\rho} - \rho \dot{\varphi}^2)^2 + (\rho \ddot{\varphi} + 2\dot{\rho}\dot{\varphi})^2 + \ddot{z}^2}\end{aligned}$$

$$\begin{aligned}\mathbf{e}_\rho(t) &= \cos \varphi(t) \mathbf{e}_x + \sin \varphi(t) \mathbf{e}_y \\ \mathbf{e}_\varphi(t) &= -\sin \varphi(t) \mathbf{e}_x + \cos \varphi(t) \mathbf{e}_y \\ \mathbf{e}_z(t) &= \mathbf{e}_z\end{aligned}$$

$$\begin{aligned}\mathbf{e}_x &= \cos \varphi(t) \mathbf{e}_\rho(t) - \sin \varphi(t) \mathbf{e}_\varphi(t) \\ \mathbf{e}_y &= \sin \varphi(t) \mathbf{e}_\rho(t) + \cos \varphi(t) \mathbf{e}_\varphi(t) \\ \mathbf{e}_z &= \mathbf{e}_z(t)\end{aligned}$$

$$dx \, dy \, dz = r \, d\rho \, d\varphi \, dz$$

F.3 Spherical Coordinates

$$\begin{aligned}x &= r \cos \varphi \sin \theta \\ y &= r \sin \varphi \sin \theta \\ z &= r \cos \theta\end{aligned}$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad \varphi = \arctan \frac{y}{x} \quad \theta = \arctan \frac{\sqrt{x^2 + y^2}}{z}$$

$$\mathbf{v}(t) = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\varphi} \sin \theta \mathbf{e}_\varphi$$

$$\begin{aligned}
\mathbf{a}(t) = & \mathbf{e}_r \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2 \theta \right) \\
& + \mathbf{e}_\theta \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin \theta \cos \theta \right) \\
& + \mathbf{e}_\varphi \left(r\ddot{\varphi} \sin \theta + 2\dot{r}\dot{\varphi} \sin \theta + 2r\dot{\theta}\dot{\varphi} \cos \theta \right)
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_r &= (\sin \theta \cos \varphi) \mathbf{e}_x + (\sin \theta \sin \varphi) \mathbf{e}_y + (\cos \theta) \mathbf{e}_z \\
\mathbf{e}_\theta &= (\cos \theta \cos \varphi) \mathbf{e}_x + (\cos \theta \sin \varphi) \mathbf{e}_y + (-\sin \theta) \mathbf{e}_z \\
\mathbf{e}_\varphi &= -\sin \varphi \mathbf{e}_x + \cos \varphi \mathbf{e}_y
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_x &= (\sin \theta \cos \varphi) \mathbf{e}_r - \sin \varphi \mathbf{e}_\varphi + (\cos \theta \cos \varphi) \mathbf{e}_\theta \\
\mathbf{e}_y &= (\sin \theta \sin \varphi) \mathbf{e}_r + \cos \varphi \mathbf{e}_\varphi + (\cos \theta \sin \varphi) \mathbf{e}_\theta \\
\mathbf{e}_z &= \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta
\end{aligned}$$

$$dx \, dy \, dz = r^2 \, dr \, \sin \theta \, d\theta \, d\varphi$$

F.4 Sum Formulae / Moivre Formula

x, y real

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\sin 2x = 2 \cos x \sin x = \frac{2 \tan x}{1 + \tan^2 x}$$

$$\cos 2x = \cos^2 x - \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$a = x + iy = |a|(\cos \varphi + i \sin \varphi)$$

$$a^n = (|a|(\cos \varphi + i \sin \varphi))^n = |a|^n (\cos n\varphi + i \sin n\varphi) = |a|^n e^{in\varphi}$$

$$\cos \varphi = \frac{1}{2} (e^{i\varphi} + e^{-i\varphi})$$

$$\sin \varphi = \frac{1}{2i} (e^{i\varphi} - e^{-i\varphi})$$

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

F.5 Hyperbolic Functions

$$\sinh x = \frac{1}{2} (e^x - e^{-x})$$

$$\cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$\tanh x = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

F.6 Series Expansions

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$(1 \pm x)^n = 1 \pm \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \dots \quad |x| \leq 1$$

$$\binom{n}{m} = \frac{n(n-1)\dots(n-m+1)}{m!}$$

F.7 Approximations (δ small)

$$(1 \pm \delta)^\alpha \approx 1 + \alpha \delta$$

$$e^\delta \approx 1 + \delta$$

$$\ln(1 + \delta) \approx \delta$$

$$\sin \delta \approx \delta$$

$$\cos \delta \approx 1 - \frac{1}{2} \delta^2$$

$$\tan \delta \approx \delta$$

G Problems on the virtual CD-ROM

2.1 Vertical projectile motion

The discussion of the one dimensional motion of a mass point under the influence of a constant gravitational acceleration is simple. The discussion is more involved, if the simultaneous motion of two mass points is considered. They start upwards at different times with different velocities. The task is the calculation of the time of ascent, the maximal height, the time and the position at which the mass points meet (if they do) and some velocities.

2.2 A comparison of the motion of two objects

The velocity and the acceleration of a mass point can be obtained if the time development of its position is given. In the present exercise, all the details of the motion of two objects in a one dimensional world are to be analysed and compared.

2.3 Projectile motion

One of the classical problems of mechanics is the discussion of projectile motion on the flat earth. Questions concerning the range of the projectile, the maximum height of the trajectory, the velocity at impact etc. are to be answered in the present exercise.

2.4 Variation of the problem of projectile motion

The discussion of projectile motion on the flat earth is more complicated if the target point is above the ground. The structure of the present problem differs for this reason from the structure of the previous problem (Probl. 2.3). Nonetheless, the same kind of information concerning the maximum height, the velocities at different point of the trajectory etc. is obtained here. It is suggested that you also approach the present problem in the same manner as the previous one (even if no guidance is offered).

2.5 Problems of Motion: Initial Conditions

The specification of the acceleration and of the initial conditions allows the calculation of the actual trajectory of a point particle. The variation of the pattern of the motion with different initial conditions is investigated in the present exercise. It only requires the discussion of simple curves in three space dimensions (\mathcal{R}_3).

2.6 Motion on a planar spiral

Two coupled differential equations in Cartesian coordinates define the present problem of motion in a plane. It can be demonstrated, that the calculation of the trajectory is much simpler if polar coordinates are used. The actual task is the characterisation of the trajectory and a detailed discussion of the time development of the motion.

2.7 Discussion of the cardioid

The calculation of the lengths of arcs and of areas enclosed by planar curves is the task in this exercise. Given is the parametric representation of a heart-shaped curve, the cardioid, in terms of Cartesian as well as polar coordinates. Next to the calculation of the area and the circumference, there is a message in the comparison of the parametric representations of the cardioid with the representation by an implicit function.

2.8 Calculation of the area of a Lissajous figure

Areas, which are fully enclosed by a planar curve, can be determined by tracing their boundary. This technique is used here to determine the area enclosed by a Lissajous figure. The accompanying applet allows the production and investigation of a much larger variety of these fascinating figures.

2.9 Kinematics of a Lissajous Ellipse

Even a set of simple differential equations can lead to a relatively complicated time development of the motion of a mass point. In this exercise the motion on a Lissajous ellipse is investigated in some detail using Cartesian as well as polar coordinates.

2.10 Cartesian and spherical coordinates

The relation between the basis vectors of a Cartesian coordinate system and the local basis vectors of spherical coordinates has been discussed in Chap. 2.4.2. Two alternative approaches for the determination of these relations are studied in this problem.

2.11 Elliptical coordinates

Local coordinate systems can be found for every family of orthogonal curves (2 dimensions) or orthogonal surfaces (3 dimensions). A useful system in two dimensions, the elliptical coordinates, is generated by sets of orthogonal hyperbolae and ellipses. These coordinates are introduced in the present problem.

3.1 Analysis of a parallelogram of forces

The geometry of a parallelogram of forces is analysed in this problem. It is created by a lantern, standing in for a mass point, which hangs from a slack suspension wire between two posts. Only basic knowledge of trigonometry and vector calculus is required for the solution of this exercise.

3.2 The inverse force problem: frictional forces

A force, which acts on a mass point, can be determined if the trajectory is known. The task set in this problem is the analysis of all the details of the motion and of the acting force for a given planar trajectory.

3.3 The motion of a rocket

A rocket is a prime example of a system characterised by a time dependent mass. In establishing the equations of motion of a rocket one can encounter difficulties. The situation has to be analysed with sufficient care. The derivation of the equation of motion of a rocket, with simplifying assumptions for the ejection of the combustion products, is the task in this problem.

3.4 The 'force-free' rocket

The simplest example for the motion of a rocket is the so-called 'force-free' motion. The rocket is only subjected to the thrust of the combustion products. The solution of the corresponding equation of motion allows the calculation of the time dependence of the mass, of the velocity and of the kinematic energy of the rocket. All these quantities are calculated and discussed in the present exercise.

3.5 A rocket moving in a uniform gravitational field

The solution of the equation of motion of a rocket is still rather simple if the rocket moves upwards in a uniform gravitational field. The task in this exercise is the calculation of the height above ground and the velocity as a function of time.

3.6 Pirouettes

Angular momentum conservation explains the change of the angular velocity during a pirouette. The present problem is slightly more involved. A rotating mass is pulled towards the centre of a circle by a central force. The dynamical aspects (trajectory, forces, work) which govern this motion are to be calculated.

3.7 Angular momentum and projectile motion

Projectile motion of a mass point can be viewed as a part-rotation with respect to the initial position, the origin of the coordinate system. The concepts involved in the description of rotational motion are to be recapitulated here via the discussion of this simple planar problem.

3.8 Questions concerning a ballistic pendulum

The ballistic pendulum constitutes a simple example for the application of conservation laws. This device is introduced here with 10 explicit questions concerning its possible use. There is a pattern for finding the correct answers.

3.9 Forces acting in a linear chain of barges

A chain of barges (objects), each with a different mass, is pulled by a tug boat (the agent of the motion). The forces acting on each of the barges in the chain can be determined with the aid of the individual equations of motion. Frictional effects can *not* be neglected. Given the forces, the work situation can be analysed.

3.10 A ball and a point particle on a track with a loop

The principle of energy conservation allows a partial analysis of simpler problems of motion. The example of Chap. 3.2.3.3 – a point particle moves under the influence of gravity without friction on a track with a circular loop – can be discussed in this fashion. A comparison of the motion of the point particle with the motion of a rolling ball offers additional insights.

3.11 Work done along a trajectory in \mathcal{R}_3

The task of this exercise is the calculation of the trajectory of a point particle in a time dependent field of force. Once the trajectory is known, the energy transferred to the particle during a specified interval of time can be obtained.

3.12 Work through gravitation

Some machines work with pulleys and weights. In this machine a mass m is moved along the horizontal by the action of the gravitation on a mass M , which is connected to m by a pulley. The situation of the work supplied by gravity is to be analysed.

3.13 Forces induced by a falling chain

A moving mass, which hits a wall, exerts a force on the wall by transfer of momentum. In the present example a section of a chain has fallen on some scales under the action of gravity, while the remainder is still underway. The question is: What is the apparent weight of the section of the chain, including the effects of the dynamical force?

3.14 The inverse force problem: central forces

Newton has been able to determine the law of gravitation using Kepler's summary of planetary motion. The extraction of a force law can be carried out quite elegantly, if the equation of the trajectory is given explicitly. The task set in this problem, following in the footsteps of Newton, is the calculation of three forces on the basis of the specification of three trajectories.

3.15 The gravitational potential of a hollow sphere

The calculation of potential energies, or potentials, for a given mass or charge distribution, is a standard task of Theoretical Physics. The present exercise deals with the gravitational potential of a hollow spherical mass distribution. The solution of this problem allows, in addition, an explicit verification of the principle of superposition.

3.16 Gravitational field of a spherical cavity embedded in a sphere

The calculation of the (gravitational) field for a more complicated geometry can be tedious, if one relies on the evaluation of the integrals for the three field components. The present exercise addresses such an example: the field of a spherical cavity embedded in a sphere. This field can be obtained readily by application of the principle of superposition.

3.17 The action of gravitation in a tunnel through the earth

This somewhat futuristic problem can be found in many introductory texts of physics. A straight tunnel through the Earth connects two towns with a sufficiently large separation (here Frankfurt and San Francisco). The task is the discussion of the motion of a mass point along the tunnel. Even though it is not easy to determine the force that acts on the mass, the final answer is quite simple, as the tunnel constrains the motion. Who is able to guess the answer?

3.18 Motion in an exponential potential

The principle of energy conservation is the first integral of the equations of motion. It is, for this reason, possible to extract a differential equation of first

order from the specification of the potential energy, provided one deals with a problem in one space dimension. The equation of motion of a mass point in an exponential potential and its solution are to be discussed in the present problem.

3.19 Elastic collisions

The simplest (classical) two body collision problem is the central, elastic collision. The basic formulae, which allow the calculation of the final velocities of two colliding mass points for a given set of initial velocities and masses can be obtained from the laws of energy and momentum conservation. The equations, relating final and initial velocities, are derived here. They are then applied to a selection of situations.

4.1 Masses of celestial bodies with Kepler's third law

An application of Kepler's third law allows the determination of the masses of central bodies from the data of their moons. This task is to be carried out in the present example for the mass of Earth and of Saturn. By the processing of standard astronomical data a feeling for their accuracy can be obtained. The data provided are taken from

R. Wielen (ed.), 'Planeten und ihre Monde'
Spektrum Akademischer Verlag, Heidelberg (1997)

Warning: Variations of the values of astronomical data from different sources are often encountered.

4.2 Stable circular orbits of central forces

The question, whether a central force in the form of a power law can support stable *circular* orbits, can be answered by different methods. A simple and direct answer can be obtained with a cursory look at the expected trajectory and an analysis of the corresponding potential energy.

4.3 Geostationary and planetostationary orbits

Satellites stay above the same point of the equator of the uniformly rotating earth, if they are on a geostationary orbit. An equation, which connects the radius of this orbit with the frequency of the earth's rotation can be obtained either via conservation laws or via a consideration of the forces. Some representative numbers will be calculated.

4.4 A comet traverses the orbit of the earth

The sighting of a comet raises the question whether and for how long it will travel within the orbit of the earth. The question concerning the duration of a possible transit will be answered in this exercise within a simplified scenario. Techniques as application of conservation laws and integration of a radial equation of motion are the tools to be used in this modified Kepler problem.

4.5 A parametric representation of the Kepler hyperbolae

A parametric representation of the coordinates and of time for the motion on a Kepler ellipse has been presented in Chap. 4.1.2.6. A corresponding representation can be found for the motion of comets (parabola and hyperbolae). The determination and discussion of the parametric representation of Kepler hyperbolae is the aim of this exercise.

4.6 A Kepler-type problem: the $1/r^2$ -potential

An answer to the question, which trajectories could be expected, if the central potential obeys a power law different from that of the Kepler problem, is not easy. Instead of the general case, the trajectories of the $1/r^2$ -potential are investigated in this exercise in detail. The technical aspects do not differ from those of the Kepler problem.

4.7 Details of the classical collision problem

This exercise addresses once more the motion of comets by an explicit solution of the Kepler problem for positive initial energies. It involves the derivation of the formulae necessary for the discussion of the classical problem of a collision of a small by a large mass (Chap. 4.1.3.2). One example of such formulae is the derivation of the central relation between the impact parameter and the scattering angle.

4.8 Scattering by a potential step: the scattering angle

The scattering of a particle (mass point) from a spherical potential step is a central force problem. The same techniques that are used in the solution of this problem, can be applied to the discussion of the relative motion of two colliding particles with a corresponding interaction. The motion of the (effective) particle can be traced explicitly in this classical mechanics problem, so

that useful insights into a collision situation can be gained. The solution of the problem outlined will be treated in some detail. The first part addresses the dependence of the scattering angle on the impact parameter. The extraction of cross sections is the task set in the companion problem, Probl. 4.9. It is worth noting that the corresponding quantum problem can be found in most quantum mechanics texts (e.g. Vol. 3 of this series).

4.9 Scattering by a potential step: cross sections

The discussion of the scattering of one particle by a potential or the collision of two particles with a corresponding interaction is essentially identical. The results obtained in the previous problem, Probl. 4.8, for the relation between the scattering angle and the impact parameter will be used here to calculate and discuss the differential and total cross sections. Can you guess the result for the total cross section?

4.10 The collision problem in the laboratory and the centre of mass systems

A question that arises in the discussion of collision experiments is the connection of experiment (which is carried out in the laboratory system) and theory (which relies on the centre of mass system, as for example for the Rutherford formula of Chap. 4.1.3.2). This statement applies to the collision of two masses (in classical mechanics) as well as the collision of charges (in electrodynamics and quantum mechanics). In order to compare experimental data with theoretical differential cross section a relation that allows the conversion between the two reference systems, is needed. The derivation of this relation is a must for every student of physics.

4.11 Calculation of effective spring constants

Hooke's law is the basis for the discussion of oscillating mass-spring systems. Simpler models (see Chap. 6.1) are a chain with alternating springs and masses, which oscillate in the longitudinal direction. However, more complicated arrangements of springs between masses can be imagined. The effective spring constants of two basic arrangements of springs as well as of some combinations of the basic arrangements are investigated in the present exercise. The results of an electro-technical problem, the combination of electric resistors (instead of springs), are quite similar.

4.12 The fast mathematical pendulum

The mathematical pendulum is a favourite example for the illustration of a more complicated periodic motion which is obtained as the solution of a non-linear differential equation. In most cases the treatment is restricted to the case of a maximal displacement smaller than 180° . The full rotation of the pendulum is treated and discussed in this exercise.

4.13 The cycloid pendulum of Huygens

The period of a mathematical pendulum is a function of the maximum displacement. Frictional effects, which lead to a change of the maximum displacement, can not be avoided during its operation. This pendulum is therefore not an ideal instrument for the precise measurement of the time. Christian Huygens has already suggested an alternative in the 17th century, the cycloid pendulum. The discussion of this pendulum is the topic of the present exercise. A number of different cycloids are presented before the equation of motion characterising Huygens' pendulum is established. The practical implementation of this apparatus can then be checked with modern means.

4.14 Energy loss of the damped harmonic oscillator

The calculation of the energy loss of a damped harmonic oscillator involves basically the evaluation of integrals over the square of the velocity of the 'oscillating' mass. This (essentially technical) exercise is carried out for the three forms of motion of the damped oscillator. Comments on the results are required.

4.15 An initial value problem: the forced oscillator

The discussion of the differential equation of the forced damped oscillator is an initial value problem, in which the general solution of a homogeneous linear differential equation has to be matched to a special solution of an inhomogeneous differential equation. The classical damped oscillator is driven in this exercise by a harmonic force.

5.1 Problems with constraints: the upright wheel

The upright wheel is an example of a system which can not be characterised by holonomic constraints. The present problem deals with a strictly upright

wheel. The more realistic problem of a wheel that can tilt over is more involved, as an additional degree of freedom comes into play. The proof that the motion of the upright wheel is not compatible with holonomic constraints leads quite naturally to the formulation of the relevant nonholonomic constraints.

5.2 Constraints: the moving plane

The problem of a mass point moving under the influence of gravity on a moving plane has been indicated in Chapter 5.1.2.5. This problem should be discussed independently in this exercise, in particular, as the consideration of additional points will help to gain a better understanding of rheonomic problems.

5.3 Constraints: another free fall machine

The original form of Atwood's machine is varied, though only slightly, in this exercise. This problem with constraints can be treated with standard methods.

5.4 Constraining forces of a planar guide rail

The motion of objects along a given curve (e.g. a mass point in a plane) can be treated with the Lagrange equations of the first kind. The first step is the determination of the constraining forces. The next step, the solution of the resulting equations of motion has to be approached by numerical means in most cases. The task of the present problem is restricted to the analysis of the constraining forces of two guiding rails and answers to explicit questions concerning their effect on the motion.

5.5 Constraints: the crank mechanism

The crank mechanism, which is used in many machines for the transmission of power, is a system with constraints. By analysis of these constraints one is able to understand the interplay between the forces in different parts of this device. Basis of the discussion is the principle of d'Alembert.

5.6 Motion in a crank mechanism

The crank mechanism is a system with constraints. The motion of two points in such a device is analysed in this exercise. Of particular interest are the limiting cases of a long drive rod in comparison with the radius of the crank wheel as well as the case that the length of the drive rod equals the radius of the crank wheel. Can you guess the form of the motion in these limiting cases without an explicit calculation.

5.7 Motion on a 'breathing' cylinder

The motion of a mass point on the surface of a cylinder with a given radius under the influence of a harmonic restoring force has been treated in Chapter 5.3.1.4. A possible variant of this holonomic problem addresses rheonomic constraints: the mass moves on a cylinder with a time dependent radius. The equations of motion and their solution are discussed for the general case and applied to an explicit example, a pulsating cylinder.

5.8 Solution of the Kepler problem with Lagrange

The standard solution of the Kepler problem is based on Newton's equations of motion. The approach to this problem on the basis of the Lagrange formalism provides an interesting alternative. It actually leads to the discussion of an oscillator problem. In the end there appears, naturally, the same solution

5.9 A driven mathematical pendulum: Planar movement of the point of suspension

The discussion of the motion of a mathematical pendulum for which the point of suspension moves on an ellipse, is a problem with holonomic/rheonomic constraints. Problems of this kind can be treated in terms of the Lagrange equations of the second kind. The resulting equations of motion can, however, not be solved analytically. This is the reason why only the formulation of the equation of motion is required in this exercise. The structure of this differential equation is discussed for some special cases. In addition, it is of interest to observe that the action of gravitation can be simulated by a circular motion of the point of suspension.

5.10 A rotating mathematical pendulum

The discussion of the mathematical pendulum can be varied in many ways. For example, the point of suspension can rotate with a constant angular velocity. Experiments indicate that stable equilibrium positions of the angle between the vertical and the rod of the pendulum occur. The calculation and discussion of these angles will be undertaken here. Harmonic oscillations about the stable configurations can be expected for small initial displacements. The frequencies of these oscillations should be determined as well. The present problem has some similarity with the spherical pendulum. Does this help in finding a direct access?

6.1 The Short Asymmetric Oscillator Chain

A linear chain of masses and springs represents a basic model of an interacting (via next neighbours) many particle system. A general analytic solution can be obtained in the case of equal masses and equally strong springs, even for very large systems. The solution of the problem is more tedious, if the masses and the springs are different. The detailed discussion of the classical problem of longitudinal oscillations of a system with two different masses connected by three different springs is the task of this problem.

6.2 A pendulum in the form of a V

Oscillating systems with one mass and two springs need not be arranged in a linear manner. The somewhat more complicated geometry in this problem is produced by a symmetric arrangement of two identical springs in the form of a **V**. The discussion centres on small vertical oscillations of the mass. The derivation of the correct linear approximation is not a trivial task in this case.

6.3 The planar double pendulum, with and without springs

The double pendulum is a good example for the occurrence of chaotic motion through the non-linear coupling of the motion of two masses. The task set here is the derivation and the discussion of the corresponding equations of motion rather than their solution. The variants which can be discussed are, for example,

- a pendulum with the sequence of elements 'point of suspension-rigid rod-mass-rigid rod-mass'
- or the replacement of any or of all of the rods by a spring.

The technical steps involved in this derivation are basically elementary, but they do require a certain amount of 'mathematical stamina'. The intricate motion of the pendulum can be studied in the accompanying applet.

6.4 Projectile motion on the rotating earth

The description of the motion of objects from the point of view of the rotating earth has to include apparent forces, in particular the Coriolis force. Projectile motion can serve as a proof of the earth's rotation in the same way as the simpler free fall motion or the Foucault pendulum. The effects of the pseudo forces 'acting' during the projectile motion are especially noticeable for larger distances and velocities.

6.5 The rotating fountain

A circular fountain is equipped with four (or more) nozzles. Four jets of water emanate and meet at the centre of the circle. The question, which is discussed in this problem, is: how would the jets behave if the fountain is rotating uniformly. It is necessary to model the jets in a simple fashion in order to answer this question. Besides the correct approach, involving both the Coriolis and the centrifugal forces, an artificial problem with exclusion of centrifugal effects is discussed. This problem is considered in order to highlight the nature of apparent forces.

6.6 Inertia matrix: cuboid and ellipsoid

The knowledge of the inertial properties of some basic objects is necessary for a physicist. The evaluation of the elements of the matrix of inertia is an exercise in triple integration. It involves the question of suitable substitutions with optimal variables of integration. Some basic rigid bodies are the cuboid and ellipsoid with a homogeneous density distribution which are considered here.

6.7 Inertia matrix: miscellaneous spheres

It is not sufficient to calculate only the elements of the inertia matrix for homogeneous objects. The inertial behaviour of objects with a more complicated density distribution is also required in many instances. This topic is treated here with the calculation of the inertia matrix of three objects with a spherical shape. The density distribution varies with the angles in two of the cases, the third case deals with a hollow sphere.

6.8 Inertia matrix: a selection of objects

The effort involved in the calculation of moments of inertia can often be reduced with a little reflection. The three objects considered in this problem are often encountered in the application of the theory of rigid bodies.

6.9 Rotation of a disc: the load on the bearing

The discussion of the rotation of rigid bodies generally calls for a more extensive calculation. The following problem can, however, be discussed much more simply if the proper set of equations and the correct choice of the system of reference is used.

6.10 A cylinder on an inclined plane

A popular problem with constraints is the motion of a point particle on an inclined plane. This problem can be varied. In the present case the mass point is replaced by a rolling cylinder. The rolling cylinder has to be characterised by more constraints as additional degrees of freedom are involved (can you say how many?). A comparison of the forces and of the motion of the two 'objects' will naturally be included.

6.11 Stability of the rotation of a force-free, asymmetric top

The calculation and the discussion of the motion of an asymmetric top (whether force-free or not) is not simple. The time development of the vector of the rotational velocity and of the Euler angles can be represented in terms of elliptical integrals. It is nonetheless not easy to picture the actual motion of the top. One question can, however, be answered with modest means: The question of whether the rotation about any of the three different principal axes of the force-free, asymmetric top is stable or not.

6.12 The rolling circular cone

It is not easy to visualise the rotational motion of a rigid body. The following discussion of a rolling circular cone is meant to help the imagination. The key points for the solution of this problem are an optimal choice of a body fixed system and an appropriate sorting of the angles and axes of rotation which are used to describe the rotation.

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