# AM5530: ADVANCED FLUID MECHANICS

TERM PAPER PRESENTATION

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# CONTENTS

This presentation is mainly focussed on the presentation of analytical solutions for potential flow past multiple cylinders. Also presented is an independently developed code for flow past two cylinders at any configuration. The contents can be broadly classified into:


1. Flow past two cylinders

- Problem Introduction and Background
- Problem Definition and Assumptions
- Solution Algorithm
- Results and Discussions

2. Uniform Potential Flow past multiple cylinders

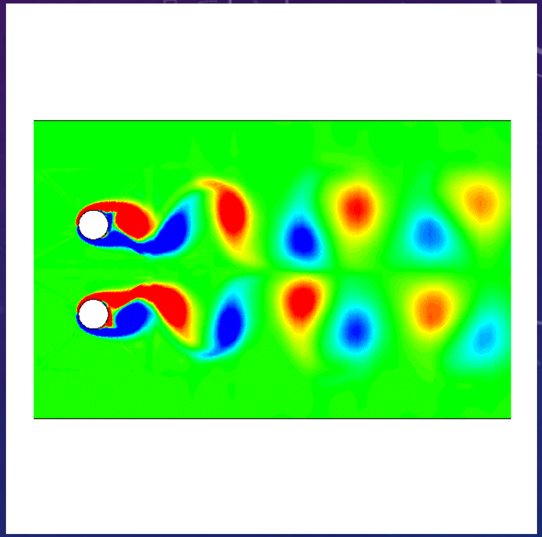
- Problem Definition
- Approach in case of Single Cylinder and validation
- Generalisation for multi cylinder
- Discussions

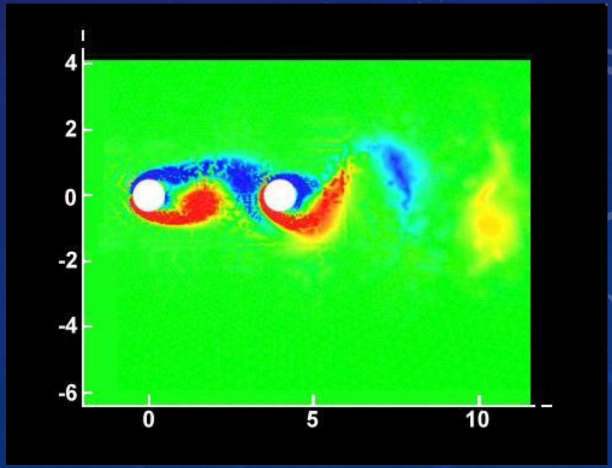
# FLOW PAST TWO CYLINDERS

The simpler of the two problems and is necessary to understand the more complex case of flow past multiple cylinders

# INTRODUCTION AND BACKGROUND

- A commonly studied configuration
- A good approximation for High-Re flows
- A large number of related scientific applications
- Extensive work done on single body problems, while work on multiple body problems is upcoming – this work provided an important base for the same
- Advantages of having analytical solutions include behaviour prediction, validation and the ability to interpret the effect of change in parameters on the solution.





# THE INCOMPRESSIBLE NAVIER-STOKES EQUATION

The incompressible Navier - Stokes Equation, obtained from the Reynolds Transport Theorem is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

With the transient term being zero (for steady flows) and $\boldsymbol{\mu}$ being zero for inviscid flow, the problem reduces to solving the stream function formulation of the same equation

The vorticity vector is as defined below (equals zero in this case):

$$\zeta = 2\omega = \nabla \times V$$

# STREAM FUNCTION-VORTICITY FORMULATION

- Cross differentiating the momentum equations and subtracting, we get the $\zeta$- $\psi$ formulation as shown:

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = \frac{1}{\mathrm{Re}} \left( \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)$$

- By definition of stream function, to elim

$$u = \frac{\partial \psi}{\partial y}; \; v = -\frac{\partial \psi}{\partial x}$$

This can hence be reduced to an equation nly! - which is solved analytically in the cases presented

$$\zeta = -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)$$

- Hence

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y} = \frac{1}{\mathrm{Re}} \left( \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)$$

# PROBLEM DEFINITION

The geometric and flow configuration for the problem is defined here

# FLOW CONFIGURATION

- The solution that is developed works for any configuration of two cylinders ( after reduction ) – different radii and given center to center distance and velocity components of the flow
- The solution is presented without non-dimensionalising, as coordinate (domain) transformations are involved

# BOUNDARY CONDITIONS

- Only boundary conditions associated with the cylinder walls and freestream velocity
- No slip boundary condition is not applicable!
- The velocity normal to the wall is zero, for both the cylinders

# SOLUTION ALGORITHM

Involves coordinate transformation to the bipolar domain, and subjecting the solution to satisfy the boundary conditions in the transformed domain. Complications arise, but are solved using a series solution

# BIPOLAR COORDINATE TRANSFORMATION

What is Bipolar Domain?

The coordinate transformation equation (along with scale factors) and inverse equation are given below :

$$x = \frac{a \sinh \xi}{\cosh \xi - \cos \theta}, \quad y = \frac{a \sin \theta}{\cosh \xi - \cos \theta}, \quad \text{and} \quad z = z$$

$$h_\theta = \frac{a}{\cosh \xi - \cos \theta}, \quad h_\xi = \frac{a}{\cosh \xi - \cos \theta}, \quad \text{and} \quad h_z = 1$$

$$\tanh \tau = \frac{2ax}{x^2 + y^2 + a^2} \qquad \tan \sigma = \frac{2ay}{x^2 + y^2 - a^2}.$$

In literature, $\xi$ and $\tau$ are used interchangeably and so are $\sigma$ and $\theta$. $\xi$ can vary from [-∞,∞] whereas $\theta$ varies from [0,2$\pi$]

# BIPOLAR COORDINATE TRANSFORMATION

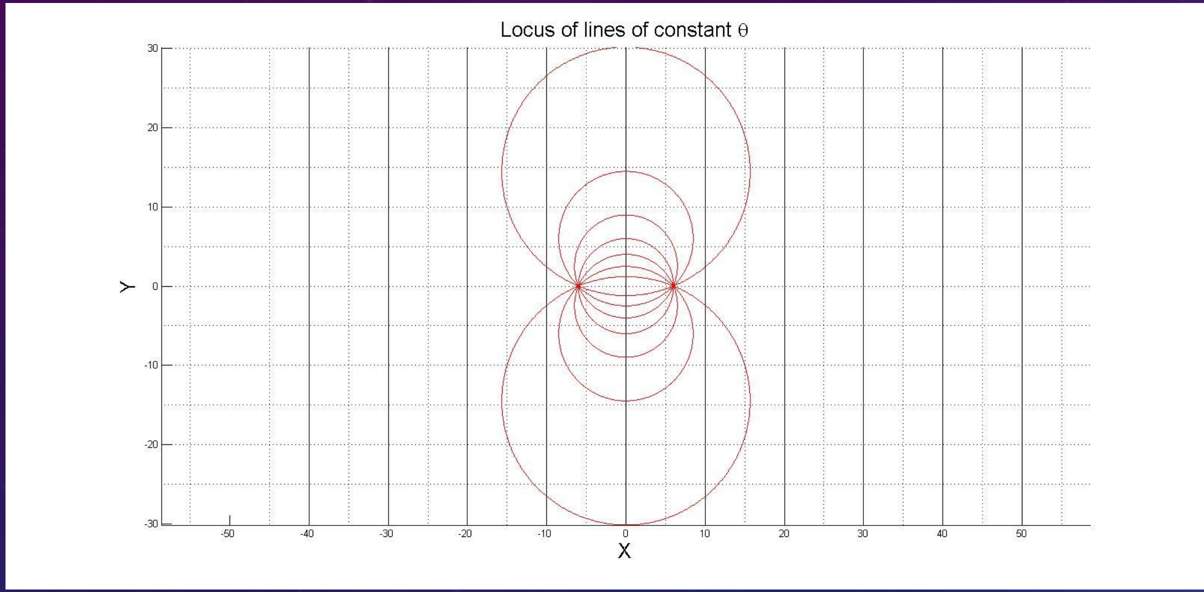
- Eliminating $\tau$ or $\sigma$ from either of the equations, we can be typecast it to get a relation between x, y and one of the transformed variable for better understanding. This leads to the following equations:

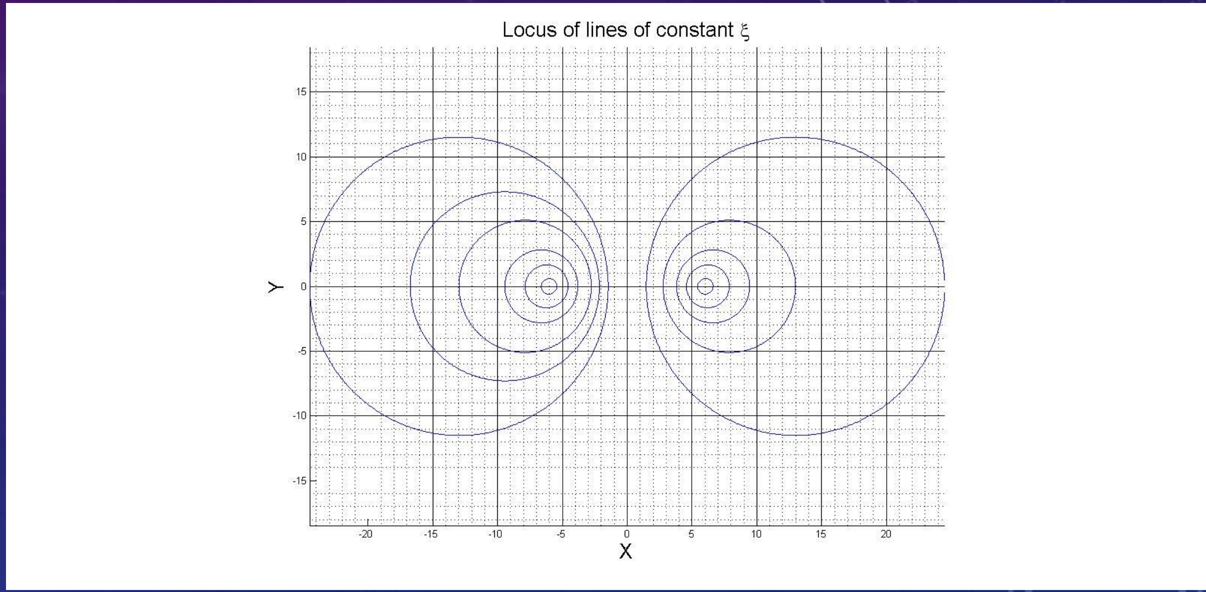
$$x^2 + (y - a\cot\sigma)^2 = \frac{a^2}{\sin^2\sigma}$$

$$y^2 + (x - a\coth\tau)^2 = \frac{a^2}{\sinh^2\tau}$$

- On plotting these in the x-y coordinates, these are circles with varying centres and radii plotted with their centres on one of the axis
- As $\sigma$ tends to extreme values (0 or $2\pi$), the first equation corresponds to the domain reduced to the entire of X axis
- Furthermore as $\tau$ tends to extreme values (-∞ or ∞), the second equation corresponds to a circle reduced to the point (-a,0) and (a,0) respectively

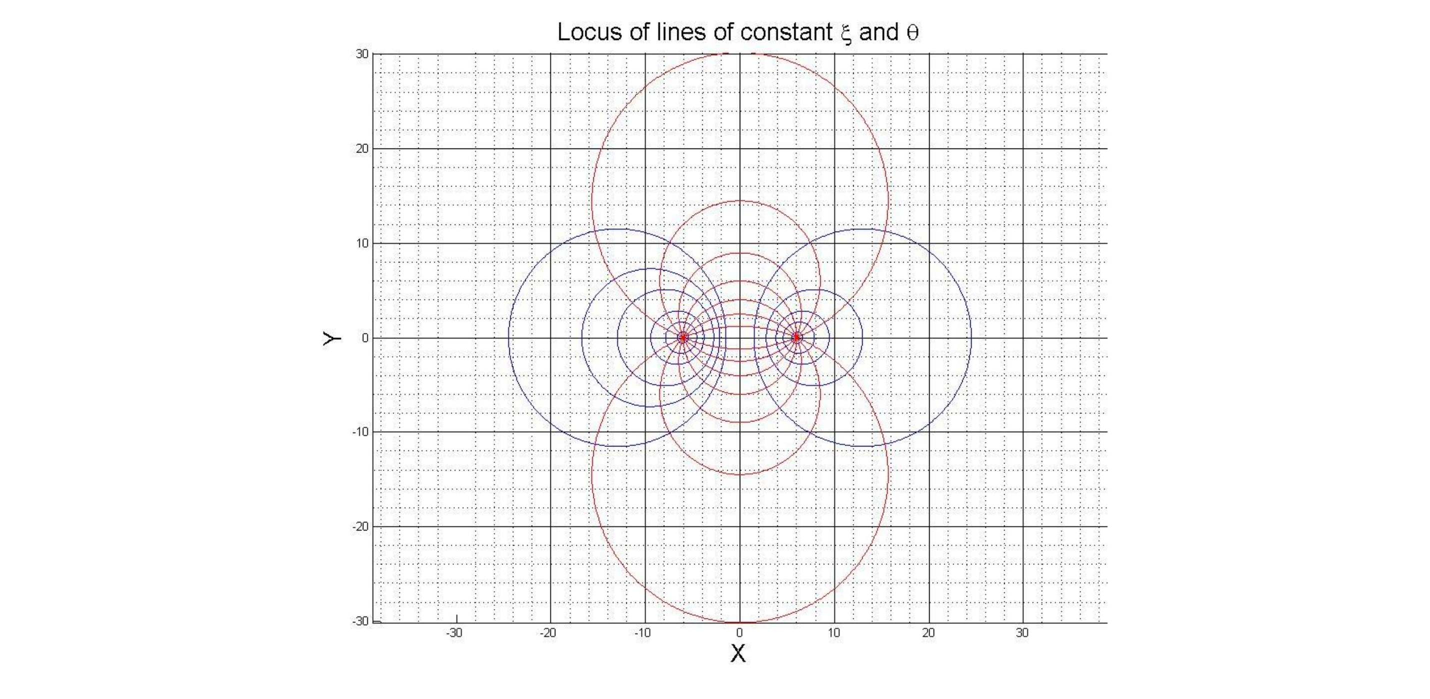
# BIPOLAR COORDINATE TRANSFORMATION




- They form a pair of infinite Apollonius circles – they intersect at right angles, and are valid as a set of orthogonal coordinates!
- We can see why they are a natural choice to represent circles – each value of $\xi$ or $\theta$ corresponds to a circle in the XY domain
- Furthermore as $\tau$ tends to extreme values (-∞ or ∞), the second equation corresponds to a circle reduced to the point (-a,0) and (a,0) respectively

# BIPOLAR COORDINATE TRANSFORMATION




- Detailed properties are avoided here – but the scale factors and coordinate transform as such is important
- By specifying the radius of the circles and the centre to centre distance, unique values of a, $\xi_1$ and $\xi_2$ can be generated ($\xi_1$ and $\xi_2$ corresponds to locus of the two cylinders being investigated)

# QUANTITIES IN BIPOLAR COORDINATES*

- The stream function in the bipolar coordinates can be written as:

$$h_\xi h_z w_\theta = +\frac{\partial \psi}{\partial \xi}, \quad h_\theta h_z w_\xi = -\frac{\partial \psi}{\partial \theta}$$

- The condition of zero vorticity, therefore becomes:

$$\frac{\partial}{\partial \theta}\left(\frac{h_\xi}{h_\theta h_z}\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \xi}\left(\frac{h_\theta}{h_\xi h_z}\frac{\partial \psi}{\partial \xi}\right) = 0$$

- To satisfy free stream conditions , the stream function in the transformed coordinates have to satisfy, for side by side flows and tandem flows respectively:

$$\frac{\partial \psi}{\partial \xi} \to \frac{aU_y(\cosh \xi \cos \theta - 1)}{(\cosh \xi - \cos \theta)^2} \quad \text{and} \quad \frac{\partial \psi}{\partial \theta} \to \frac{aU_y \sinh \xi \sin \theta}{(\cosh \xi - \cos \theta)^2}$$

$$\frac{\partial \psi}{\partial \xi} \to -\frac{aU_x \sinh \xi \sin \theta}{(\cosh \xi - \cos \theta)^2} \quad \text{and} \quad \frac{\partial \psi}{\partial \theta} \to -\frac{aU_x(1 - \cosh \xi \cos \theta)}{(\cosh \xi - \cos \theta)^2}$$

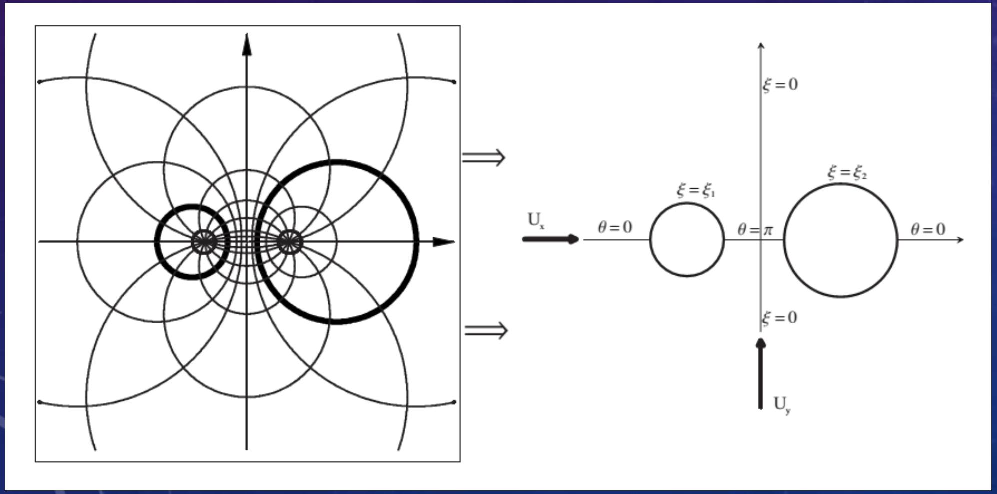
# QUANTITIES IN BIPOLAR COORDINATES*

- On integrating with respect to one of the variables, this results in :

$$\psi \rightarrow -\frac{aU_y \sinh \xi}{\cosh \xi - \cos \theta} \quad \text{as} \quad (\theta,\xi) \rightarrow (0,0)$$

$$\psi \rightarrow +\frac{aU_x \sin \theta}{\cosh \xi - \cos \theta} \quad \text{as} \quad (\theta,\xi) \rightarrow (0,0)$$

- (0,0) in the transformed coordinates represents a circles that engulfs the entire x-y domain! (This is basically the free stream condition)




- The configuration in given in picture besides, representing both the physical and transformed domain – note the variation in the values of $\xi$ and $\theta$
- The value of $\xi_1$ corresponding to the cylinder on the left is strictly negative and the value of $\xi_2$ corresponding to the cylinder on the right is strictly positive

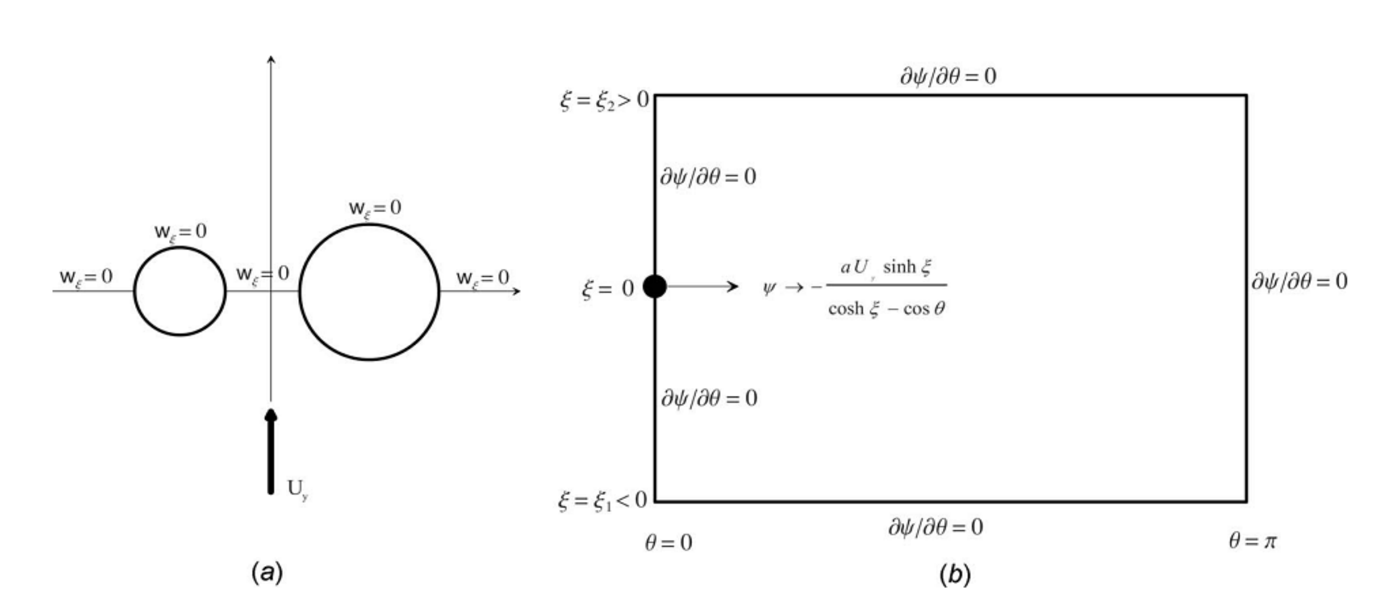
# SOLUTION METHODOLOGY

- The solution is broken down into two components in the transformed domain, one corresponding to flow in the x-direction in x-y domain (tandem flow) and the other corresponding to a flow in the y-direction in the x-y domain (side by side flow)
- Side by side flow is covered first, followed by the tandem configuration. Stream function is then super imposed in transformed domain, and converted back to the physical domain through inverse functions
- The problem thus formulated is not separable, because of the boundary conditions (explained in the next slide)
- Use of series solution to solve a perturbed problem ( Relaxation in BCs for solution) and mend it to the unperturbed (original) problem
- That is, a solution like :

$$\psi_y = -\frac{aU_y \sinh(\xi)}{\cosh(\xi) - \cos(\theta)} - \Phi - \Omega$$

# SIDE BY SIDE FLOW : BOUNDARY CONDITONS

- The boundary conditions are given below. BoCo’s on the centreline are a result of flow symmetry : (a) Physical domain (b) Transformed domain




$$\frac{\partial\psi}{\partial\xi} \to \frac{aU_y(\cosh\xi\cos\theta - 1)}{(\cosh\xi - \cos\theta)^2} \quad \text{and} \quad \frac{\partial\psi}{\partial\theta} \to \frac{aU_y \sinh\xi \sin\theta}{(\cosh\xi - \cos\theta)^2} \quad \longrightarrow \quad \psi \to -\frac{aU_y \sinh\xi}{\cosh\xi - \cos\theta} \quad \text{as} \quad (\theta,\xi) \to (0,0)$$

Free Stream BoCo

# SIDE BY SIDE FLOW : SOLUTION

- Define two harmonic functions independently satisfying BoCos on walls:

$$\Phi(\theta,\xi) = \sum_{n=1}^{\infty} c_n \cos n\theta \sinh n(\xi - \xi_1)$$

$$\Omega(\theta,\xi) = \sum_{n=1}^{\infty} d_n \cos n\theta \sinh n(\xi - \xi_2)$$

- They satisfy symmetry BoCos as well, but not the freestream BoCo
- $\Phi$ satisfies the top wall ($\xi_2$) and $\Omega$ satisfies the bottom wall ($\xi_1$)
- We can therefore use the walls to get the associated equations

$$\left.\frac{\partial \Phi}{\partial \theta}\right|_{\xi=\xi_2} = \sum_{n=1}^{\infty} -nc_n \sin n\theta \sinh n(\xi_2 - \xi_1) = \frac{aU_y \sinh \xi_2 \sin \theta}{(\cosh \xi_2 - \cos \theta)^2}$$

$$\left.\frac{\partial \Omega}{\partial \theta}\right|_{\xi=\xi_1} = \sum_{n=1}^{\infty} -nd_n \sin n\theta \sinh n(\xi_1 - \xi_2) = \frac{aU_y \sinh \xi_1 \sin \theta}{(\cosh \xi_1 - \cos \theta)^2}$$

# SIDE BY SIDE FLOW : SOLUTION

- The harmonics do not satisfy the condition of *Φ* and *Φ* vanishing at (0,0) (due to the nature of the solution). Therefore, other harmonics are added, so that they can vanish – the easiest way is to subtract the residual, while at the same time maintaining the wall BoCo. This is done through:

$$\Phi(\theta,\xi)=\sum_{n=1}^{\infty} c_n \cos n\theta \sinh n(\xi-\xi_1)+\sum_{n=1}^{\infty} c_n \sinh n(\xi_1)$$

$$\Omega(\theta,\xi)=\sum_{n=1}^{\infty} d_n \cos n\theta \sinh n(\xi-\xi_2)+\sum_{n=1}^{\infty} d_n \sinh n(\xi_2)$$

- The coefficients $c_n$ and $d_n$ can be determined through the orthogonality property of the trigonometric functions:

$$c_n=-\frac{aU_y \sinh \xi_2}{n\pi \sinh n(\xi_2-\xi_1)}\int_0^{2\pi}\frac{\sin\theta \sin n\theta}{(\cosh \xi_2-\cos\theta)^2}d\theta \\ =-\frac{2aU_y\ e^{-n\xi_2}}{\sinh n(\xi_2-\xi_1)}$$

$$d_n=-\frac{aU_y \sinh \xi_1}{n\pi \sinh n(\xi_1-\xi_2)}\int_0^{2\pi}\frac{\sin\theta \sin n\theta}{(\cosh \xi_1-\cos\theta)^2}d\theta \\ =+\frac{2aU_y\ e^{+n\xi_1}}{\sinh n(\xi_1-\xi_2)}$$

# SIDE BY SIDE FLOW : SOLUTION

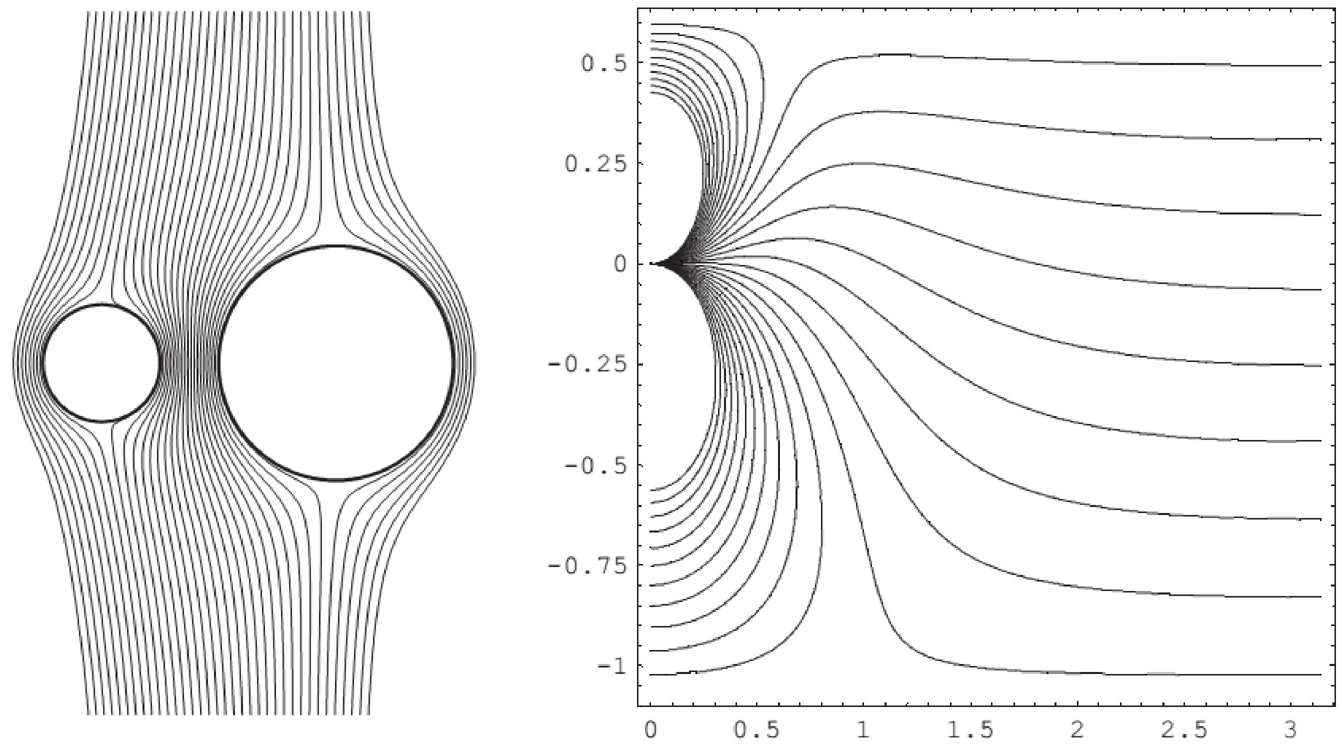
- This leads to the value of the stream function in the transformed variables, written after simplification as :

$$\frac{\psi_y(\theta,\xi)}{2aU_y} = -\frac{\sinh \xi}{2(\cosh \xi - \cos \theta)} + \sum_{n=1}^{\infty} \frac{\cos n\theta[e^{-n\xi_2} \sinh n(\xi - \xi_1) + e^{+n\xi_1} \sinh n(\xi - \xi_2)]}{\sinh n(\xi_2 - \xi_1)} + \sum_{n=1}^{\infty} \frac{\sinh n(\xi_2 + \xi_1)}{\sinh n(\xi_2 - \xi_1)}$$

- We can derive the values of velocity and pressure at any point once the values of the stream function in the physical domain are known, obtained through inverse transforms

# SIDE BY SIDE FLOW : SOLUTION

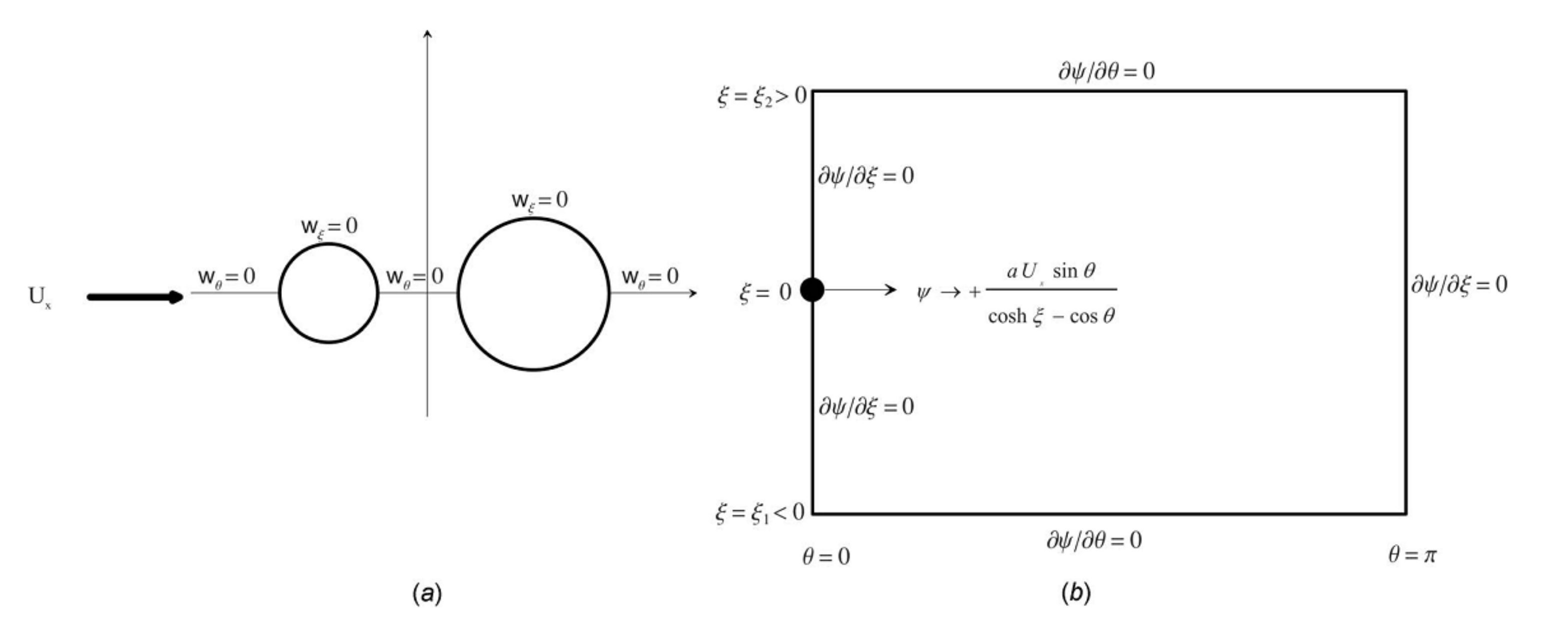
- The simulated value and results as given in the reference is given below:




**Streamline patterns for the case $r_1=1$, $r_2=2$, and $H=4$**

# TANDEM FLOW : SOLUTION

- A similar process is carried out for tandem flows (flow along x direction in the physical domain). Without going into the same details, the BoCos and solution is presented here.




$$\frac{\partial\psi}{\partial\xi} \to -\frac{aU_x \sinh\xi \sin\theta}{(\cosh\xi - \cos\theta)^2} \quad \text{and} \quad \frac{\partial\psi}{\partial\theta} \to -\frac{aU_x(1-\cosh\xi\cos\theta)}{(\cosh\xi - \cos\theta)^2} \quad \rightarrow \quad \psi \to +\frac{aU_x \sin\theta}{\cosh\xi - \cos\theta} \quad \text{as} \quad (\theta,\xi) \to (0,0)$$

Free Stream BoCo

# TANDEM FLOW : SOLUTION

We do not need any adjustment terms, as the solution taken automatically satisfies the free stream boundary condition

$$\psi_x(\theta,\xi) = +\frac{aU_x \sin\theta}{\cosh\xi - \cos\theta} - \sum_{n=1}^{\infty} f_n \sin n\theta \sinh n(\xi-\xi_1) - \sum_{n=1}^{\infty} g_n \sin n\theta \sinh n(\xi-\xi_2)$$

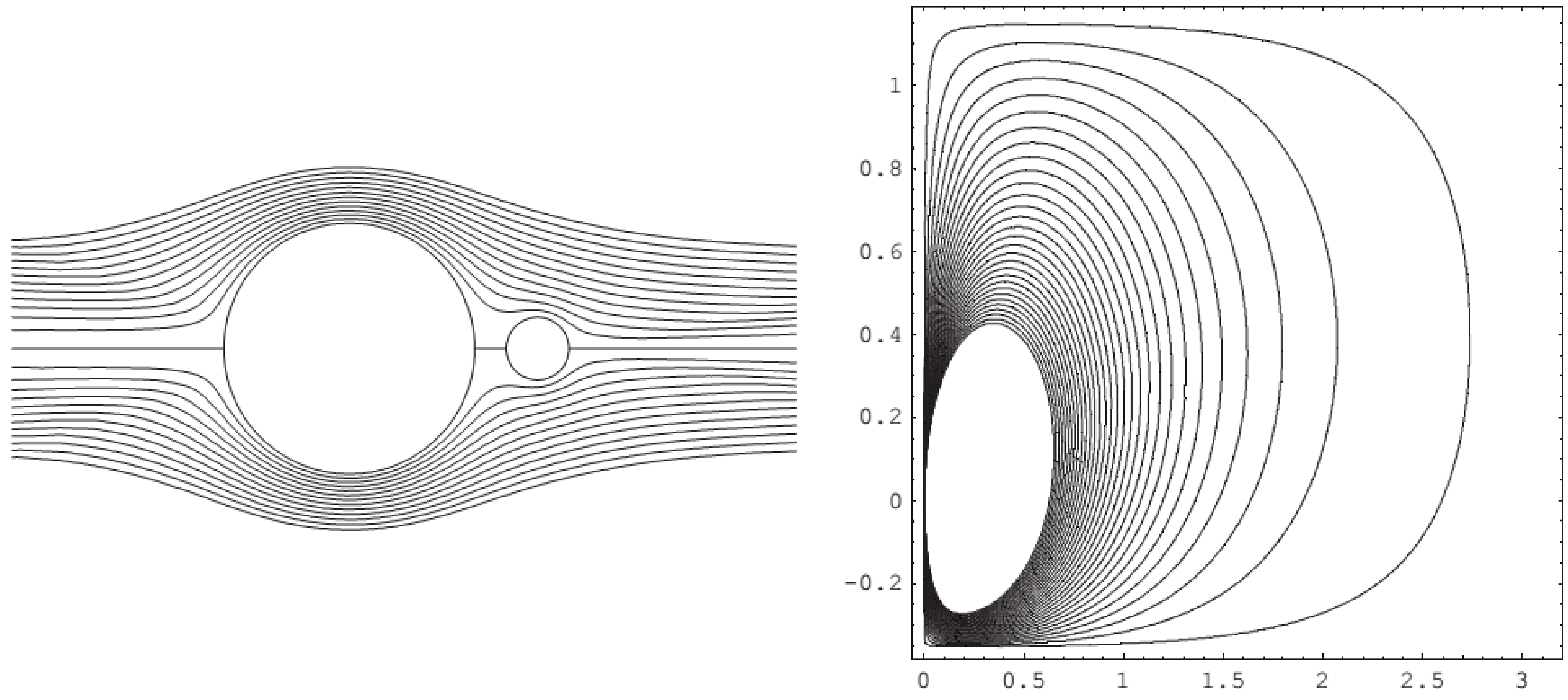
$$f_n = -\frac{aU_x}{n\pi \sinh n(\xi_2-\xi_1)} \int_0^{2\pi} \frac{\cos n\theta(1-\cos\theta\cosh\xi_2)}{(\cosh\xi_2-\cos\theta)^2} d\theta = +\frac{2aU_x e^{-n\xi_2}}{\sinh n(\xi_2-\xi_1)}$$

$$g_n = -\frac{aU_x}{n\pi \sinh n(\xi_1-\xi_2)} \int_0^{2\pi} \frac{\cos n\theta(1-\cos\theta\cosh\xi_1)}{(\cosh\xi_1-\cos\theta)^2} d\theta = +\frac{2aU_x e^{+n\xi_1}}{\sinh n(\xi_1-\xi_2)}$$

$$\frac{\psi_x(\theta,\xi)}{2aU_x} = +\frac{\sin\theta}{2(\cosh\xi-\cos\theta)} - \sum_{n=1}^{\infty} \frac{\sin n\theta[e^{-n\xi_2}\sinh n(\xi-\xi_1) - e^{+n\xi_1}\sinh n(\xi-\xi_2)]}{\sinh n(\xi_2-\xi_1)}$$

# TANDEM FLOW : SOLUTION

- The simulated value and results as given in the reference is given below:




**Streamline patterns for the case $r_1=4$, $r_2=1$, and $H=6$**

# FLOW IN AN ARBITRARY DIRECTION

- Consider the free stream flow , with velocity $U_o$ ,making an angle of $\delta$ with the x axis. The free-stream conditions will then be

$$\frac{\partial\psi}{\partial\theta} \to -\frac{aU_o\cos\delta(1-\cosh\xi\cos\theta)}{(\cosh\xi-\cos\theta)^2} + \frac{aU_o\sin\delta\sinh\xi\sin\theta}{(\cosh\xi-\cos\theta)^2}$$

$$\frac{\partial\psi}{\partial\xi} \to -\frac{aU_o\cos\delta\sinh\xi\sin\theta}{(\cosh\xi-\cos\theta)^2} + \frac{aU_o\sin\delta(\cosh\xi\cos\theta-1)}{(\cosh\xi-\cos\theta)^2}$$

That is,

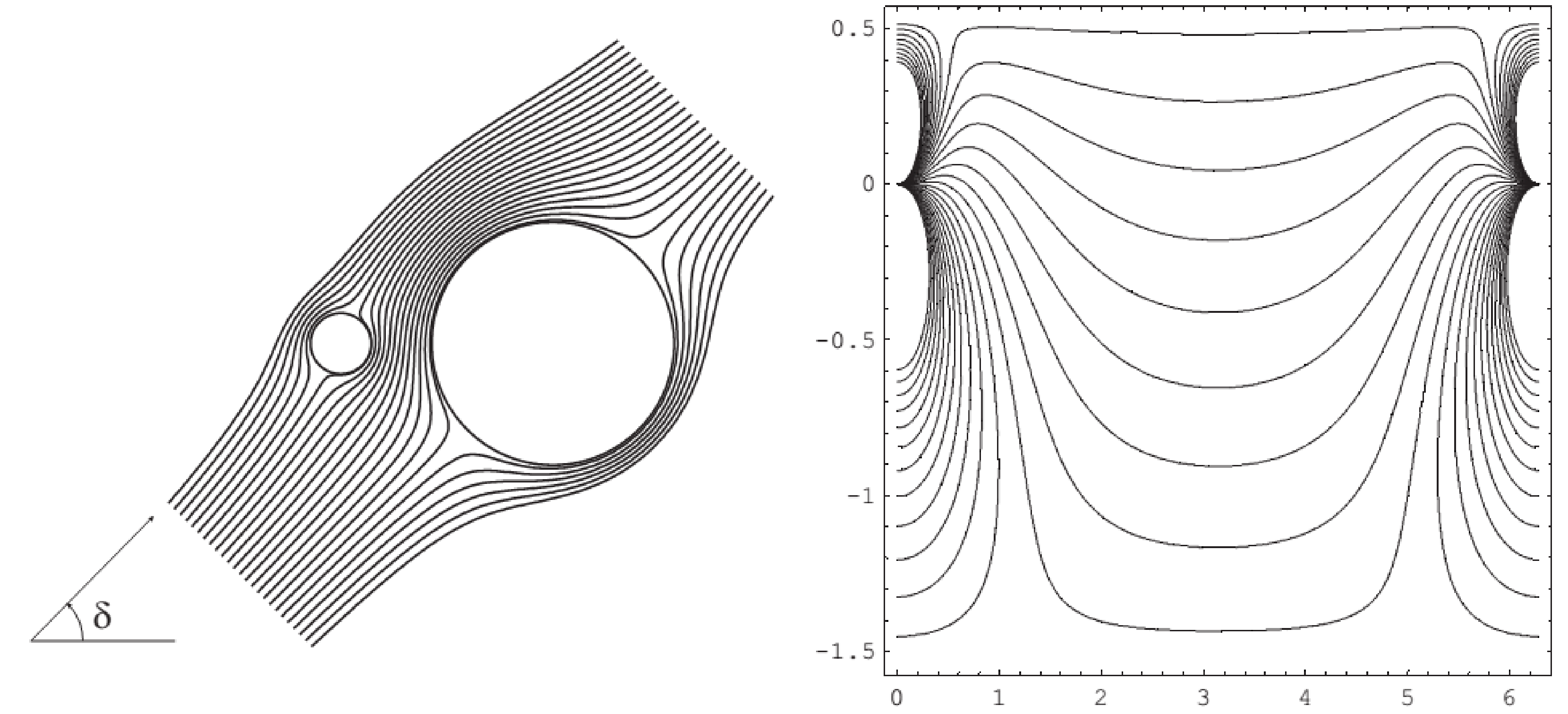
$$\psi \to +\frac{aU_o\cos\delta\sin\theta}{\cosh\xi-\cos\theta} - \frac{aU_o\sin\delta\sinh\xi}{\cosh\xi-\cos\theta} \quad \text{as} \quad (\theta,\xi)\to(0,0)$$

And this is satisfied by the following condition, with Ux and Uy replaced appropriately with $U_o$ cos($\delta$) and $U_o$ sin($\delta$) respectively

$$\psi(\theta,\xi) = \psi_y(\theta,\xi) + \psi_x(\theta,\xi)$$

# FLOW IN AN ARBITRARY DIRECTION

- The simulated value and results as given in the reference is given below:



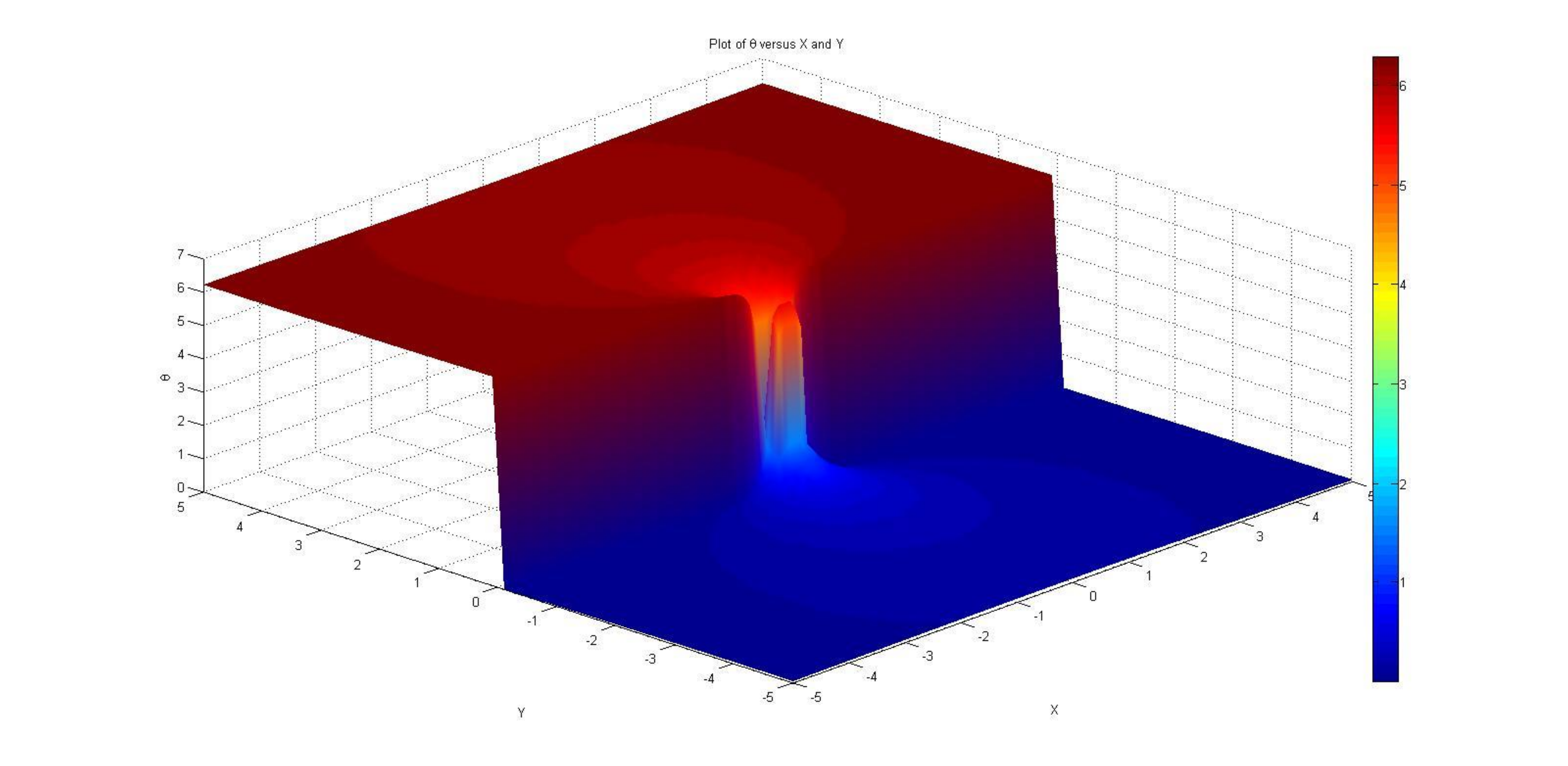

**Streamline patterns for the case $r_1$=1, $r_2$=4, $H$=7, and $\delta=\pi/4$**

# RESULTS

Discussions on the independently developed code and corresponding results are presented in this section , Codes in MATLAB and Mathematica

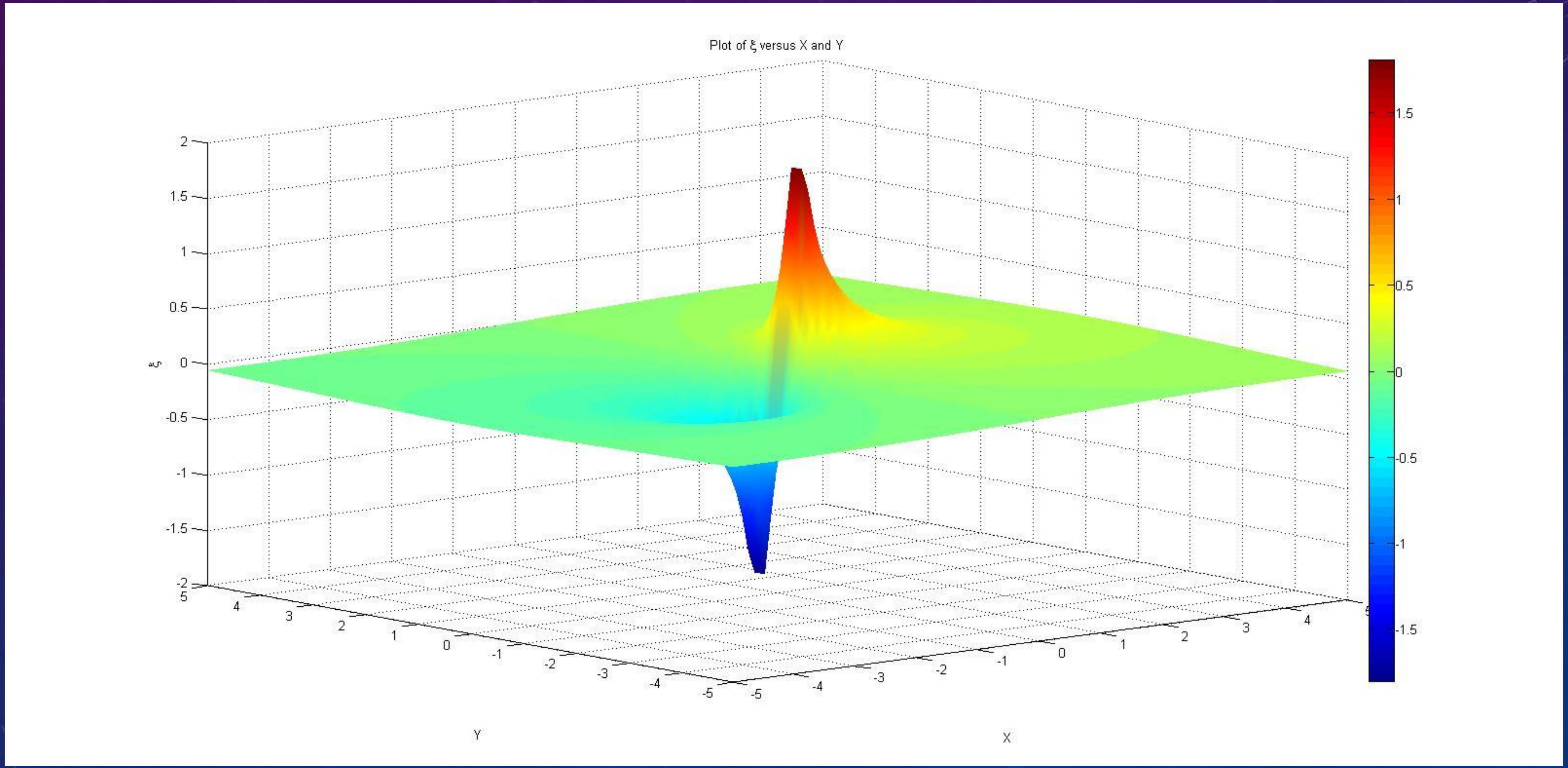
# PROFILE OF TRANSFORMED COORDINATE

- The transformed coordinate ($\theta$) is plotted as a function of the x and y coordinates.



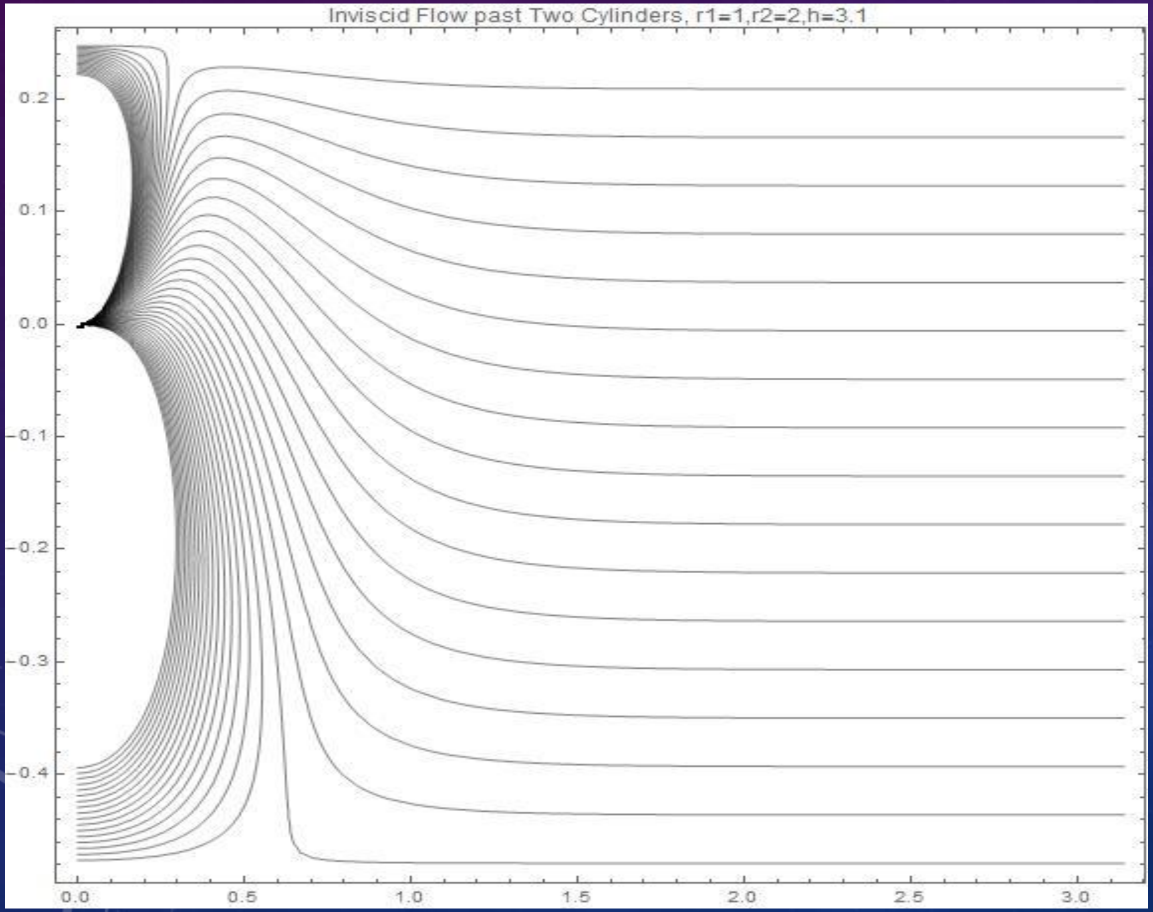
# PROFILE OF TRANSFORMED COORDINATE

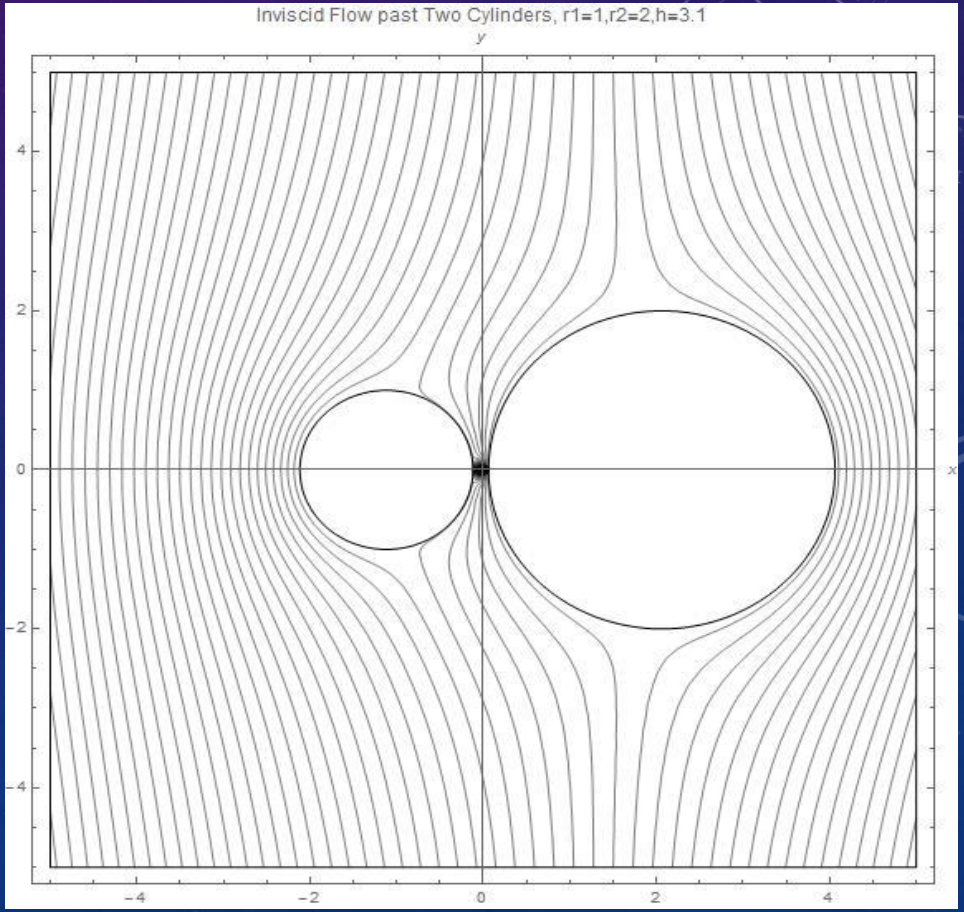
- The transformed coordinate ($\xi$) is plotted as a function of the x and y coordinates.



# STREAM FUNCTION IN $\theta$- $\xi$ AND X-Y COORDINATES, SIDE BY SIDE FLOW

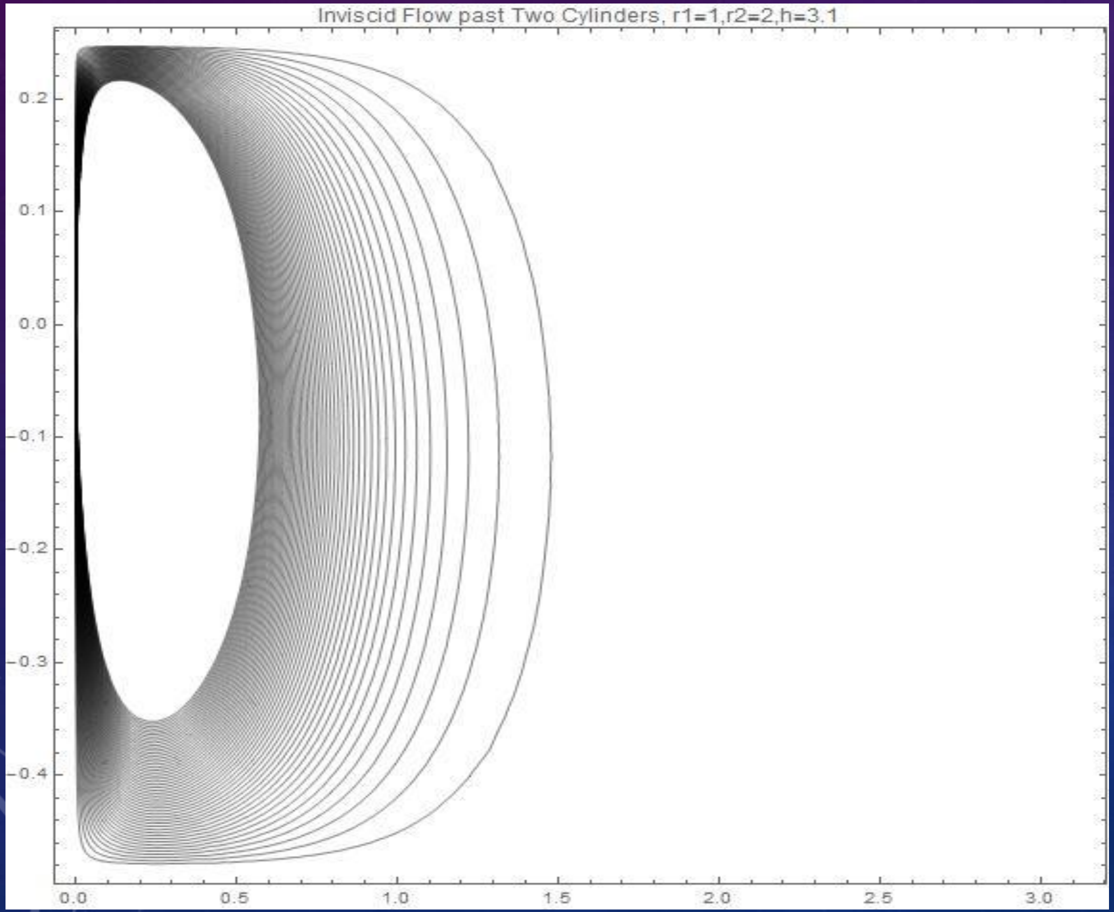
- Radius of Smaller Circle = 1m, Radius of Larger Circle =2m and Centre to Centre Distance =3.19m.

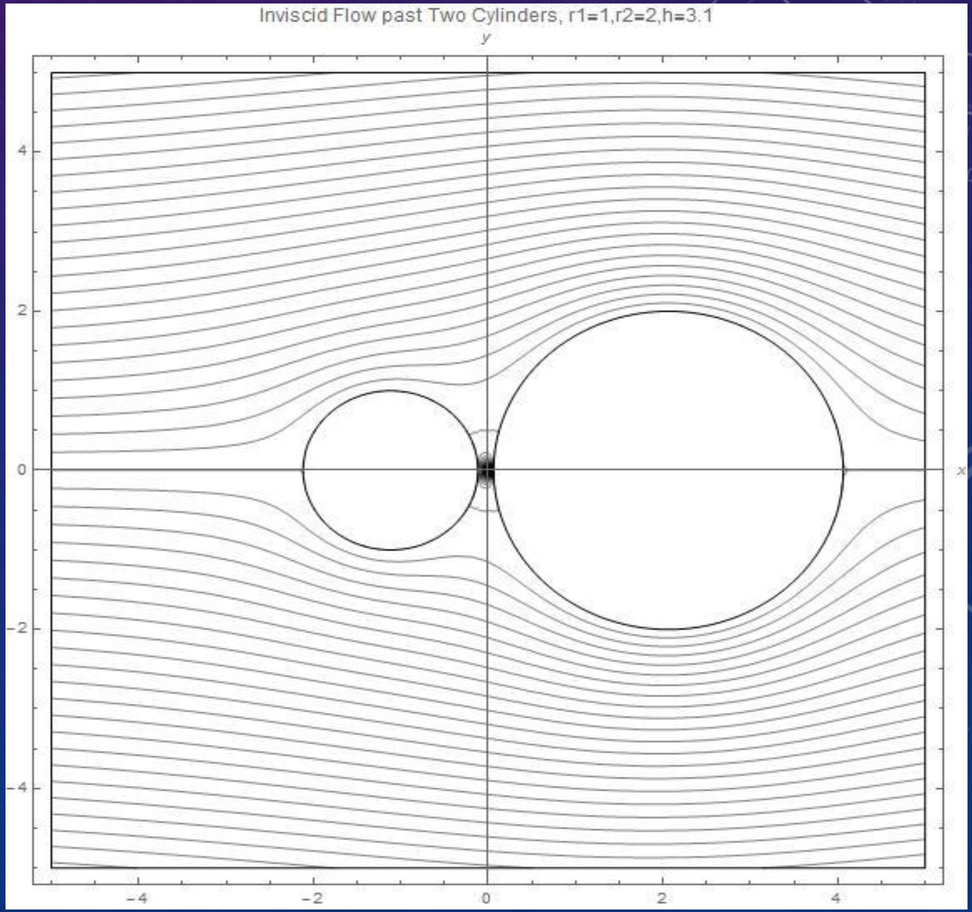


# STREAM FUNCTION IN $\theta$- $\xi$ AND X-Y COORDINATES, TANDEM FLOW

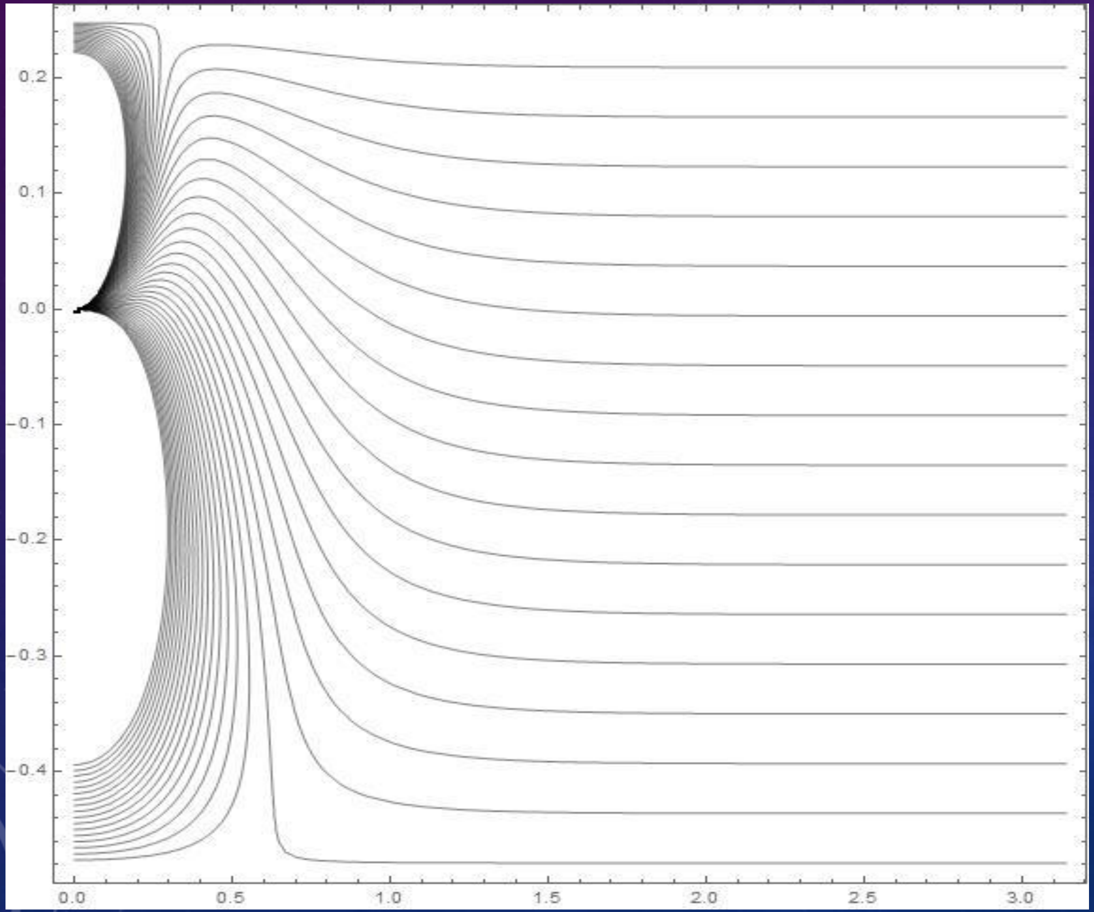
- Radius of Smaller Circle = 1m, Radius of Larger Circle =2m and Centre to Centre Distance =3.19m.

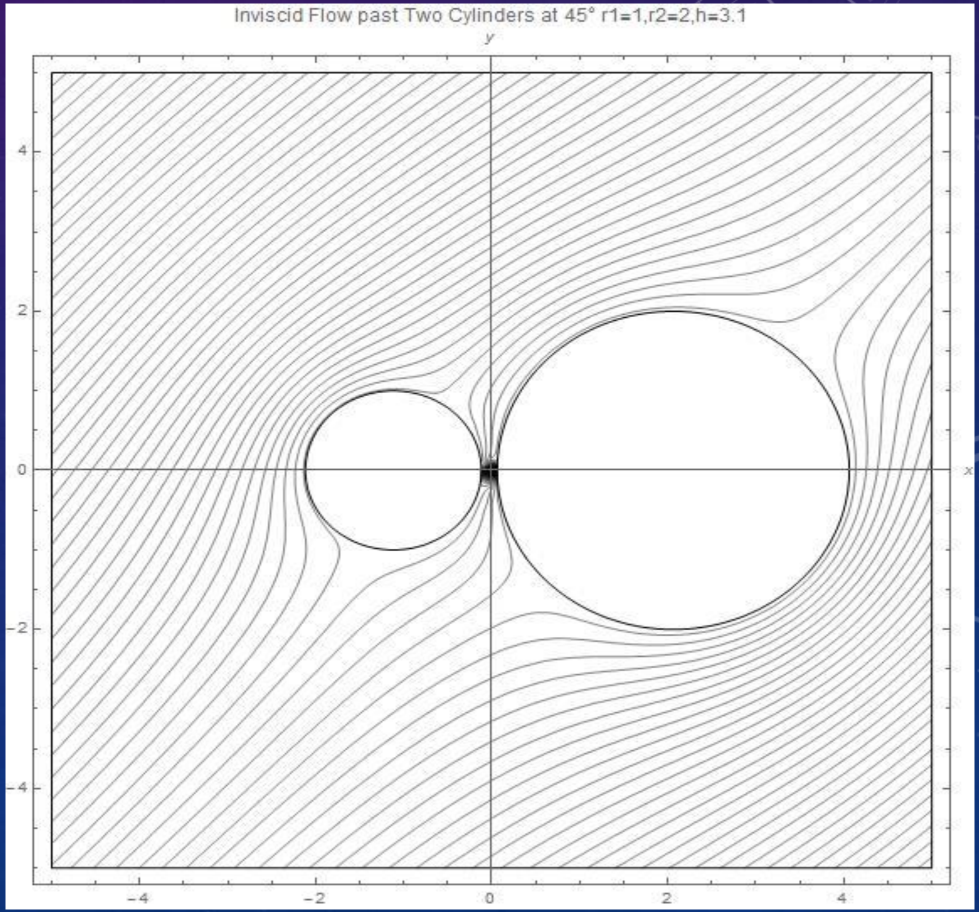


# STREAM FUNCTION IN $\theta$- $\xi$ AND X-Y COORDINATES, COMBINED FLOW

- Radius of Smaller Circle = 1m, Radius of Larger Circle =2m and Centre to Centre Distance =3.19m, Angle =$45^{o}$



# DISCUSSIONS

- (0,0) in the X-Y coordinate is a point of singularity in the $\theta$- $\xi$ coordinates
- Streamlines plotted in the $\theta$- $\xi$ coordinates appear to be sucked into the origin
- This aberration is reflected in the final streamlines plotted in the X-Y coordinate as well
- Quantities such as the velocity, pressure can also be independently plotted as the solution for the equation depends on the stream function alone
- Analytical solutions for inviscid flow over any two cylinder configuration can be derived using this method, and plotted

# ANALYTICAL SOLUTION FOR UNIFORM POTENTIAL FLOW PAST MULTIPLE CYLINDERS

# PROBLEM DEFINITION

Using Circular Domain transformation approach to get analytical Solution of a irrotational flow past a multi cylinder array.

# APPROACH IN CASE OF SINGLE CYLINDER

- The analysis is performed in a *circular domain* $D\zeta$ of a parametric $\zeta$ -plane consisting of the unit $\zeta$ -disc with some smaller interior circular discs excised
- Such circular domains are a canonical class of multiply connected domains that is, it is known that some choice of bounded multiply connected circular domain $D\zeta$ can be conformally mapped (by some function $z(\zeta)$) to the unbounded fluid region, $Dz$ say, exterior to any given collection of obstacles
- If $w1(z)$ is the complex potential in $Dz$ then we define $W1(\zeta) = w1(z(\zeta))$ and find explicit expressions for $W1(\zeta)$ in the domain $D\zeta$
- This implies that, up to knowledge of the appropriate conformal mapping $z(\zeta)$ (the existence of which is guaranteed by the Riemann mapping theorem), the solution for the complex potential associated with the flow is known.

# UNIFORM FLOW PAST CYLINDER

Validation Conditions

- In far field, the complex potential function must be reduced to uniform flow
- Also the surface of the cylinder is stream line flow i.e., basically $v_n = 0$

Flow past cylinder can be taken as uniform flow plus doublet

- Complex potential function for Uniform flow is given by

$$W(z) = Uz$$

Where is U is uniform velocity,

Z = x+iy is a complex number

- Complex potential function for Doublet is given by $W(z) = \mu / \pi z$

Where $\mu = me/2$ , m is flow rate per second and e is distance between the source and sink

- For cylinder $me/2\pi U = R^2$

- Therefore, Complex potential function for uniform flow past cylinder of radius R is given by

$$W(z) = U(z+R^2/z)$$ , R is radius of cylinder Eq. 1

- This is the result to be verified

# SOLUTION

- *z(ζ )* be a conformal mapping from the interior of the unit *ζ* -disc to the exterior of the unit *z*-disc mapping $\zeta = \beta$ to $z=\infty$

$$z \sim \frac{a}{\zeta - \beta}, \quad \text{as } \zeta \to \beta$$

a and $\beta$ are constants chose such that it shouldn't violate the Riemann mapping theorem

- Complex Potential in a complex $\zeta$ plane at a position due to a single unit-circulation point vortex at position $\beta$ *inside* a unit-disc $|\zeta| < 1$ is given by $-(i/2\pi)\, W_0(\zeta, \beta)$

Where

$$W_0(\zeta, \beta) \equiv \log\left(\frac{(\zeta - \beta)}{|\beta|(\zeta - \bar{\beta}^{-1})}\right).$$

- On $|\zeta| = 1$, the $\mathrm{Re}[W_0(\zeta, \beta)] = C$ (derivation)
- Complex potential is

$$W(\zeta, \beta) = Ua\left(\frac{\partial}{\partial\bar{\beta}} - \frac{\partial}{\partial\beta}\right)W_0(\zeta, \beta) = Ua\left(\frac{1}{\zeta - \beta} + \frac{1}{2\beta} - \frac{1}{\bar{\beta}^2(\zeta - \bar{\beta}^{-1})} - \frac{1}{2\bar{\beta}}\right).$$

The complex potential will reduce to Eq. 1 as $\beta$ tends to zero and $a = 1$

# ANOTHER APPROACH

- Consider the unit-radius disc in a $\zeta$ -plane and let $z(\zeta)$ be the conformal map taking the interior of the unit $\zeta$ -disc to the exterior of the unit $z$-disc

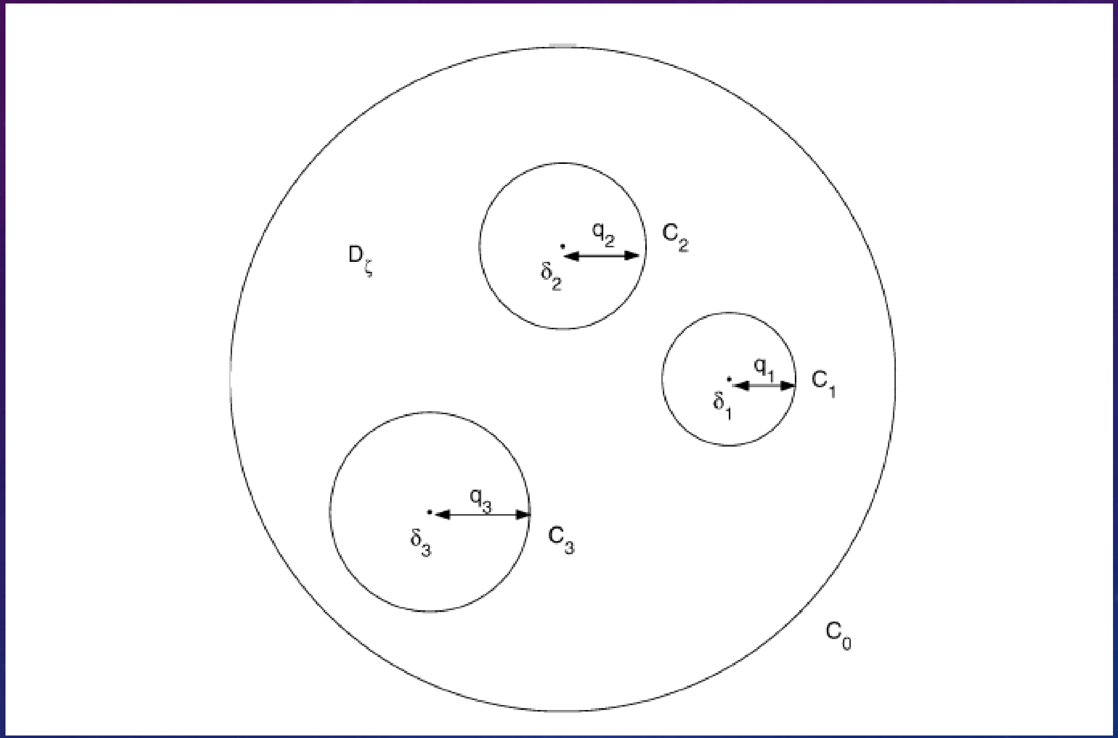
$$z(\zeta) = \zeta^{-1}$$

- Consider the conformal mapping to a complex $\eta$-plane given by

$$\eta(\zeta) = \frac{1}{\zeta} + \zeta.$$

$$W(\eta(\zeta(z))) = U\eta(\zeta(z)) = U\left(\frac{1}{\zeta(z)} + \zeta(z)\right) = U\left(z + \frac{1}{z}\right)$$

# GENERALIZATION




Schematic of a typical multiply connected circular region *Dζ*

$C_1$,$C_2$,$C_3$ are different circular discs enclosed in $C_0$ whose $|\zeta|$ = 1 and center of circular discs given by $\zeta = \delta j \in C$ with radius $q_j$

- Solving the problem of uniform flow in the unbounded fluid region *Dz* exterior to a finite collection of obstacles is equivalent to finding a function *w1(z)*, a complex potential that is analytic everywhere in the fluid region *Dz* except at infinity
- Analytic Complex potential function in the fluid region *Dz* except at infinity is given by

$$w_1(z) \sim U\,\mathrm{e}^{-\mathrm{i}\chi} z + \mathcal{O}(1), \quad \text{as } z \to \infty$$

Where *U* and *χ(direction of U with positive real axis)* are real constants

- This ensures that the flow speed at infinity is *U* while its direction makes an angle *χ* with the positive real axis
- *w1(z)* must also satisfy the boundary conditions that Im[*w1(z)*] is constant on the boundaries of all the obstacles. This ensures that all the obstacle boundaries are streamlines

- Let $z(\zeta)$ is a conformal mapping from the circular domain $D\zeta$ to the fluid region $Dz$ exterior to the collection of obstacles
- Suppose $\zeta = \beta$ is the point in $D\zeta$ mapping to $Z = \infty$ and that, as $\zeta \to \beta$,

$$z(\zeta) = \frac{a}{\zeta - \beta} + \mathcal{O}(1)$$

# FINDING COMPLEX POTENTIAL AS A FUNCTION OF ZETA

Conditions to be satisfied by the complex potential function $W_0(\zeta,\beta)$

- The circulations around the *M* excised circular discs are zero
- The function must be analytic (but not necessarily single-valued) everywhere in $D\zeta$ except for a logarithmic singularity at $\zeta = \beta$ corresponding to the point vortex and also it should satisfy

$$\mathrm{Re}\left[W_0(\zeta,\beta)\right] = 0, \quad \text{on } |\zeta| = 1,$$

and

$$\mathrm{Re}\left[W_0(\zeta,\beta)\right] = c_j, \quad \text{on } C_j, \; j = 1, \ldots, M,$$

$C_j$ are real constants

# FUNCTION

- All the above conditions satisfied by the function given in Crowdy and Marshall

$$W_0(\zeta,\beta) = \log\left(\frac{\omega(\zeta,\beta)}{|\beta|\omega(\zeta,\bar{\beta}^{-1})}\right).$$

- The function *ω(ζ, ·)* is defined as follows. For each interior circle $\{C_j \mid j = 1, \ldots, M\}$ of the domain *Dζ*, define the conformal map

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma)$$

where

$$\omega'(\zeta,\gamma) = \prod_{\theta_k} \frac{(\theta_k(\zeta)-\gamma)(\theta_k(\gamma)-\zeta)}{(\theta_k(\zeta)-\zeta)(\theta_k(\gamma)-\gamma)}$$

- where the product is over all compositions of the basic maps $\{\Theta_j, \Theta^{-1}_j \mid j = 1, \ldots, M\}$ *excluding* the identity map and all inverse maps

$$\theta_j(\zeta) = \frac{a_j\zeta + b_j}{c_j\zeta + d_j}, \quad j = 1, \ldots, M,$$

where

$$a_j = q_j - \frac{|\delta_j|^2}{q_j}, \quad b_j = \frac{\delta_j}{q_j}, \quad c_j = -\frac{\overline{\delta_j}}{q_j}, \quad d_j = \frac{1}{q_j}.$$

- Conformal maps of the linear-fractional form $\Theta_j$ are known as Möbius maps

- Potential function is given by

$$W_1(\zeta,\beta) = Ua\left[e^{i\chi}\frac{\partial}{\partial\bar{\beta}} - e^{-i\chi}\frac{\partial}{\partial\beta}\right]W_0(\zeta,\beta).$$

Eq. 2

# IMPLEMENTATION TO SIMPLE CONNECTED CASE

Conditions:

- M = 0
- The fluid region is then simply connected and there are no enclosed circles to generate any Möbius maps $\omega(\zeta,\gamma) = (\zeta - \gamma)$
- $\beta = 0$ with $\chi = 0$ gives single cylinder case which is obtained earlier (derivation)

# DOUBLE CONNECTED CASE

- Uniform flow past two cylinders
- In this case, $\delta_1 = 0$ and $q_1 = q$, so that the single Möbius map given by is $\Theta_1(\zeta) = q^2\zeta$

$$\omega(\zeta,\gamma) = -\frac{\gamma}{C^2} P\left(\frac{\zeta}{\gamma}, q\right)$$

where

$$P(\zeta, q) \equiv (1-\zeta) P'(\zeta, q)$$

and

$$P'(\zeta, q) \equiv \prod_{k=1}^{\infty} (1 - q^{2k}\zeta)(1 - q^{2k}\zeta^{-1}), \qquad C \equiv \prod_{k=1}^{\infty} (1 - q^{2k}).$$

# SOLUTION

$$W_1(\zeta,\beta) = \frac{Uai}{\beta}\sin\chi + \frac{Ua}{\beta}\left(\mathrm{e}^{-\mathrm{i}\chi}K\left(\zeta\beta^{-1},q\right) - \mathrm{e}^{\mathrm{i}\chi}K(\zeta\beta,q)\right)$$

where we have taken $\beta$ to be real and where we define

$$K(\zeta,q) \equiv \frac{\zeta P_\zeta(\zeta,q)}{P(\zeta,q)}.$$

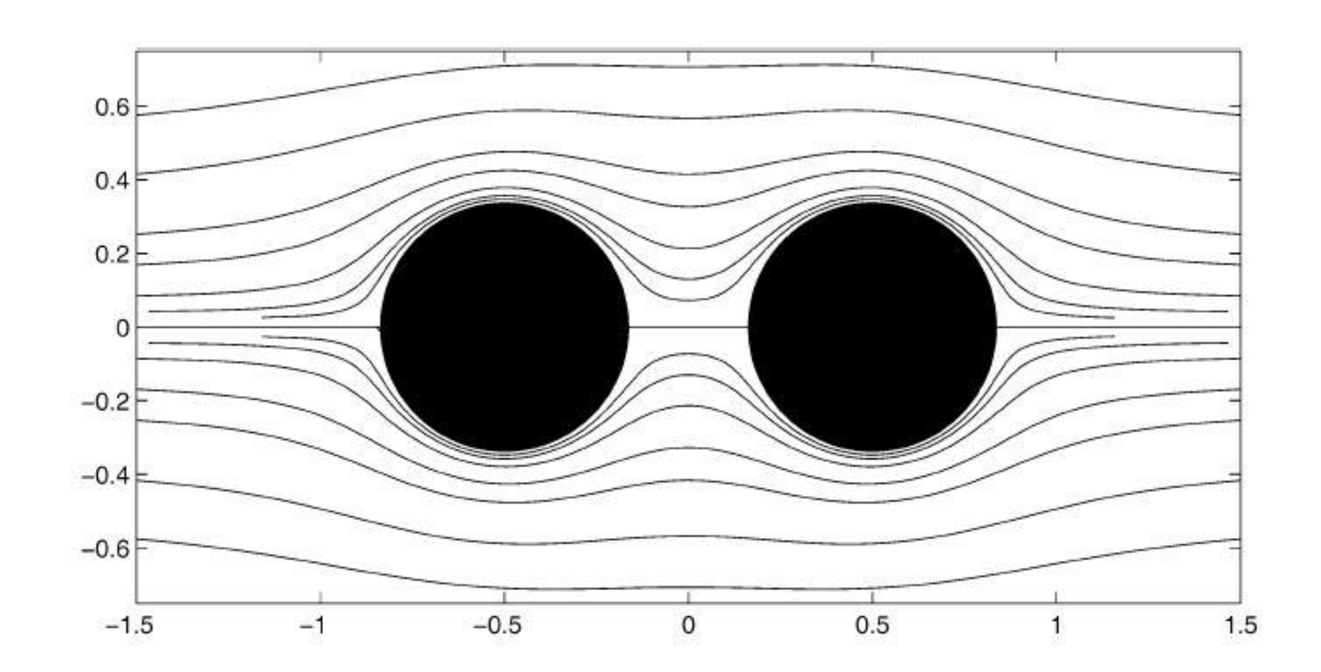
*Pζ (ζ, q)* denotes the derivative of *P(ζ,q)* with respect to its first argument.

Streamlines for uniform potential flow, with $\chi = 0$, past two equal cylinders aligned horizontally

- In this case, the conformal mapping from the annulus $q < |\zeta| < 1$ is given by the Möbius mapping, $\beta = \sqrt{q}$ and $\chi = 0$

$$z(\zeta) = R\left(\frac{\zeta + \sqrt{q}}{\zeta - \sqrt{q}}\right)$$

- Where R is real constant



# UNIFORM FLOW MULTIPLE CYLINDERS

- The conformal map to be used is

$$z(\zeta) = \frac{a}{\zeta} + b$$

- In this case $\beta = 0$ Eq. 3
- Let $Q_j$ be the radius and $D_j$ the position of the centre of the $j$ -th cylindrical obstacle ($j = 1, \ldots, M$) in the physical plane

$$W_0(\zeta, 0) = \log\left(\zeta \frac{\omega'(\zeta, 0)}{\omega'(\zeta, \infty)}\right)$$

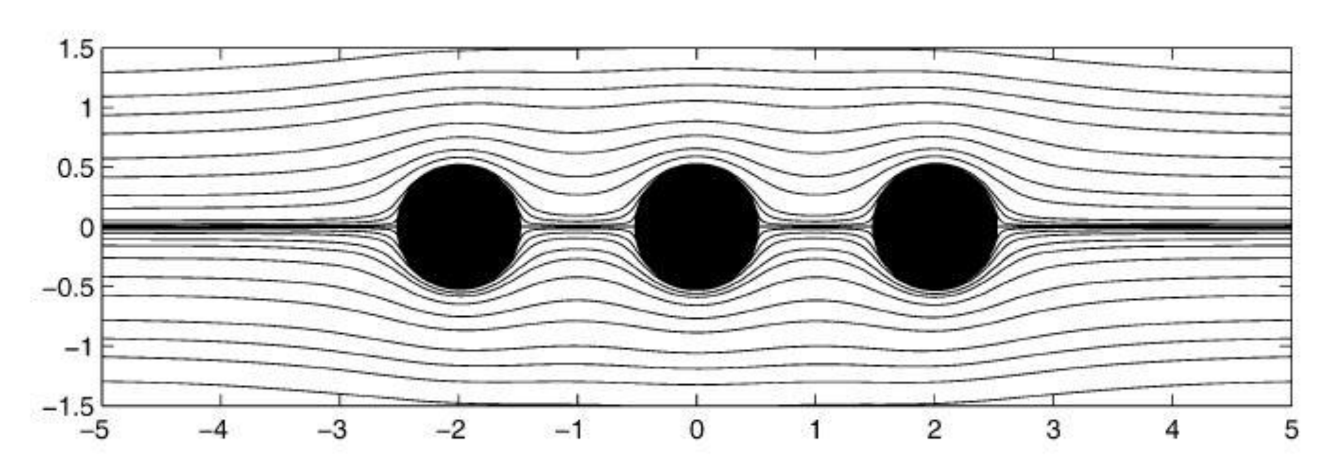
$$D_j = b + \frac{a\bar{\delta}_j}{|\delta_j|^2 - q_j^2}$$

and

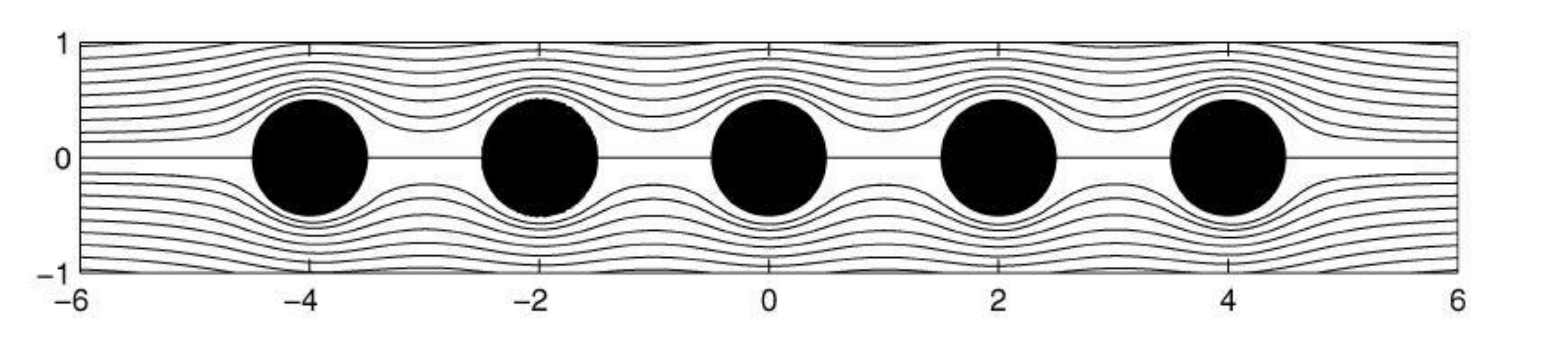
$$Q_j = \frac{q_j |a|}{|\delta_j|^2 - q_j^2}.$$

- It remains to determine the parameters $\{q_j, \delta_j \mid j = 1, \ldots, M\}$ from the known parameters $a$, $b$, $\{Q_j, D_j \mid j = 1, \ldots, M\}$.

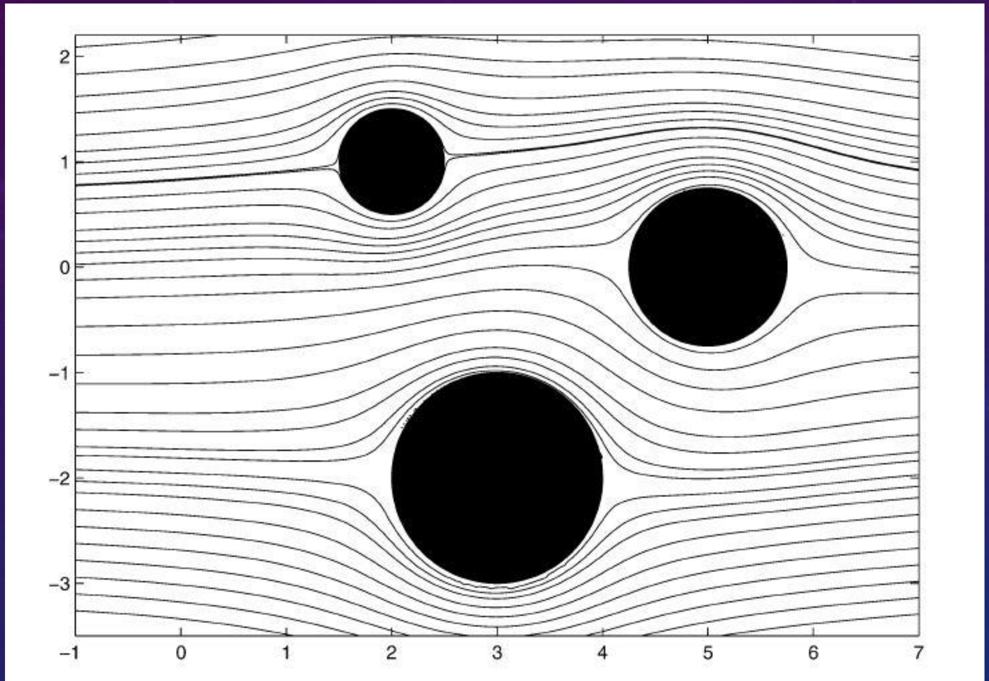
Streamlines for uniform flow, with $\chi = 0$, past a line of three equal cylinders




Streamlines for uniform flow, with $\chi = 0$, past a line of five equal cylinders



Streamlines for uniform flow, with $\chi = 0$, around an arbitrary non-aligned distribution of three circular cylinders of varying radii




All plots are plotted using $\mathrm{Im}[W_1(\zeta,\beta)]$ and equations 2 and 3

# DISCUSSIONS

- To compute the flow past a more general configuration of obstacles, it is necessary to find the conformal mapping $z(\zeta)$ to the flow region from *some* canonical circular domain $D\zeta$ .
- The parameters $\{q_j, \delta_j \mid j = 1, \ldots, M\}$ determining $D\zeta$ must be found from the given domain $Dz$ (they are known as the *conformal moduli* of the domain $Dz$).
- Then, the relevant complex potential for the flow is given, as a function of $\zeta$ , by Eq. 2
- Complex velocity field is given by

$$u - \mathrm{i}v = \frac{W_{1\zeta}(\zeta, \beta)}{z_\zeta(\zeta)}.$$

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# THANK YOU