DEPARTMENT OF MATHEMATICS, IIT Guwahati

MA221: Discrete Mathematics, July - November 2019

Practice Problems: Logic

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1. Construct the truth tables of the following well formed formulas. Check if some of them are tautologies.
   $(P \leftrightarrows Q) \leftrightarrows (P \wedge Q) \vee (\neg P \wedge \neg Q)$, $\quad \neg(P \vee (Q \wedge R)) \leftrightarrows ((P \vee Q) \wedge (P \vee R))$,
   $((P \to Q) \wedge (Q \to R)) \to (P \to R)$, $\quad (P \to \neg P) \to \neg P$.

2. Prove the following implications (once using truth table and once using inference theory):
   $P \wedge Q \Rightarrow P \to Q$, $P \to (Q \to R) \Rightarrow (P \to Q) \to (P \to R)$,
   $P \to Q \Rightarrow P \to (P \wedge Q)$, $\quad (P \to Q) \to Q \Rightarrow P \vee Q$.

3. Prove the following equivalances (once using truth table and once using inference theory):
   $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$, $\quad P \to (Q \vee R) \Leftrightarrow (P \to Q) \vee (P \to R)$,
   $\neg(P \leftrightarrows Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$, $\quad \neg(P \to Q) \Leftrightarrow P \wedge \neg Q$,
   $P \rightleftarrows Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$, $\quad P \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q)$.

4. Using the rules of inference, check the following implications:
   $\{\neg(P \wedge \neg Q), \neg Q \vee R, \neg R\} \Rightarrow \neg P$, $\quad \{(P \to Q) \to R, P \wedge S, Q \wedge \mathbf{T}\} \Rightarrow R$, where $\mathbf{T}$ is a tautology, $P \to Q \Rightarrow P \to (P \wedge Q)$, $\quad \{\neg P \vee Q, \neg Q \vee R, R \to S\} \Rightarrow P \to S$,
   $\{P, P \to (Q \to (R \wedge S))\} \Rightarrow Q \to S$, $\quad \{(P \vee Q) \to R\} \Rightarrow (P \wedge Q) \to R$.

5. Determine whether P is a valid conclusion of the set of premises $\{P \to Q, \neg Q\}$.

6. Using the rules of inference, check the consistency of the following set of premises: $\{P \to Q, P \to R, Q \to \neg R, P\}$, $\quad \{A \to (B \to C), D \to (B \wedge \neg C), A \wedge D\}, \{A \to B, B \to C, D \to \neg C, A \wedge D\}$.

7. Use rules of inference to show that the following wffs are tautologies.
   (a) $((A \vee B) \wedge \neg A \wedge (B \to C)) \to (B \wedge C)$.
   (b) $((A \vee B) \to (B \wedge C)) \to (B \to C)$.
   (c) $A \to (B \to A)$.
   (d) $\neg A \to (A \to B)$.
   (e) $(A \vee B) \to (B \vee A)$, $\quad (B \vee A) \to (A \vee B)$, that is, $(A \vee B) \rightleftharpoons (B \vee A)$.
   (f) $(A \to B) \rightleftharpoons (\neg A \vee B)$.
   (g) $A \to (B \to (A \wedge B))$.
   (h) $A \to (\neg B \to (A \wedge \neg B))$.
   (i) $(((A \vee B) \to C) \wedge A) \to C$.
   (j) $(A \to C) \to ((A \wedge B) \to C)$.
   (k) $(A \to C) \to (A \to (B \vee C)$.
   (l) $(A \to (B \to C)) \to (B \to (A \to C))$.

(m) $(B \to C) \to ((A \wedge B) \to (A \wedge C))$.

(n) $(A \to B) \to ((C \vee A) \to (C \vee B))$.

Note: Some of the formulas in Q.2,3,4 and Q7 are from the list of tautologies/implications mentioned in the lecture note that were intended to be used in inference as Rule T. Those formulas in these questions are supposed to be proved using other formulas in the lecture note (from the approved list.) This shows that many in the list of 22 formulas of the lecture note are redundant.

8. Consider the following three statements: A: 'if $x = 4$, then discrete math is bad'; B: 'discrete math is bad'; C: '$x = 4$'. Does C logically follow from A and B?

9. Determine validity of the following arguments:

   (a) If you send me an e-mail message, then I will finish writing the program", "If you do not send me an e-mail message, then I will go to sleep early", and "If I go to sleep early, then I will wake up feeling refreshed". Symbolize these hypotheses and use rules of inference to arrive at the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed.

   (b) The meeting can take place if all the members are informed in advance and it is quorum. It is quorum if at least 15 members are present. Members would have been informed in advance if there was not a postal strike. Therefore, if the meeting was canceled, then either there were fewer than 15 members present or there was a postal strike.

   (c) If the Big-Bang theory is correct, then either there was a time before anything existed, or the world will come to an end. The world will not come to an end. Therefore if there was no time before anything existed, the Big-Bang theory is incorrect.

   (d) There are three persons Mr X, Mr Y, and Mr Z making statements. If Mr X is wrong then Mr Y is right. If Mr Y is wrong then Mr Z is right. If Mr Z is wrong then Mr X is right. Therefore some two of them are always right.

   (e) If the colonel was out of the room when the murder was committed, then he could not have been right about the weapon used. Either the butler is lying or he knows who the murderer was. If lady Barntree was not the murderer, then either the colonel was in the room at the time of the murder or the butler was lying. Either the butler knows who the murderer is or the colonel was out of the room at the time of murder. The policeman deduced that if the colonel was right about the weapon used, then the lady Barntree is the murderer.

   (f) If the lecture proceeds then either blackboard is used or the slides are shown or the tablet PC is used. If the blackboard is used, then students at the back bench are not comfortable in reading the blackboard. If the slides are shown then the students are not comfortable with the speed. If the tablet PC is used then it causes lots of small irritating disturbances to the instructor. The lecture proceeds. The students are comfortable. So, it is deduced that the instructor faces the disturbances.

10. Determine the scope of the quantifiers, and indicate the free and bound occurances of the variable in the following:

    (a) $(\forall x)(P(x) \wedge R(x)) \rightarrow (\forall x)P(x) \wedge Q(x)$.

    (b) $(\forall x)(P(x) \wedge (\exists x)(Q(x)) \vee ((\forall x)P(x) \rightarrow Q(x))$.

    (c) $(\forall x)((P(x) \rightleftharpoons Q(x)) \wedge (\exists x)R(x)) \wedge S(x)$.

11. If the UD is $\{a, b, c\}$, write the following formula by eliminating the quantifiers: $(\forall x)P(x)$, $(\forall x)R(x) \wedge (\forall x)S(x)$, $(\forall x)R(x) \wedge (\exists x)S(x)$, $(\forall x)(P(x) \rightarrow Q(x))$, $(\forall x)\neg P(x) \vee (\forall x)P(x)$.

12. Find the truth values of

    (a) $(\forall x)(P(x) \vee Q(x))$, where $P(x) : x = 1$, $Q(x) : x = 2$ and the UD is $\{1, 2\}$.

    (b) $(\forall x)(P \rightarrow Q(x)) \vee R(a)$, where $P : 2 > 1$, $Q(x) : x \leq 3$, $R(x) : x > 5$, $a : 5$ and the UD is $\{-2, 3, 6\}$.

    (c) $(\exists x)(P(x) \rightarrow Q(x)) \wedge T$, where $P(x) : x > 2$, $Q(x) : x = 0$, T is any tautology and the UD is $\{1\}$.

13. Show that $(\exists z)(Q(z) \wedge R(z))$ is not implied by the formulas $(\exists x)(P(x) \wedge Q(x))$ and $(\exists y)(P(y) \wedge R(y))$, by assuming a UD which has two elements.

14. Let $P(x) :$ x is a person, $F(x, y) :$ x is the father of y, and $M(x, y) :$ x is the mother of y. Write the predicate "x is the father of the mother of y".

15. Symbolize the expression "All the world loves a lover".

16. Write the following as predicate formula, once using a UD and once without using any UD.

    (a) All men are giants.

    (b) Given any positive integer, there is a bigger positive integer.

    (c) Each person in this class room is either a B.Tech student or an M.Sc student.

    (d) There is a student in this class room who speaks Hindi or English.

    (e) Every natural number is either square of a natural number or its square root is irrational.

    (f) For every real number x, there is a real number y such that $x + y = 0$.

17. Translate the following into formal wff of predicate logic.

    (a) All scientists are human beings. Therefore, all children of scientists are children of human beings.

18. Write a formal definition of $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) \neq l$.

19. Take UD:= all students in all IIT's in India to write a formal statement of "for each students in IITG there is a student in IITG with more CPI".

20. Let $p(x)$: x is a rational number and $q(x)$: x is a real number. Translate the following wff into English.

    (a) $(\forall x)(p(x) \to q(x))$, $\quad (\exists x)(\neg p(x) \wedge q(x))$.

    (b) $((\forall x)(p(x) \wedge (x > 2))) \to ((\forall x)(q(x) \wedge (x < 2)))$.

21. Check if $(\exists x)(p(x) \wedge q(x)) \to (\exists x)p(x) \wedge (\exists x)q(x)$ is valid. Is it's converse valid?

22. Give an interpretation to show that $(\forall x)(r(x) \to (\exists y)(r(y) \wedge p(x, y)))$ is not valid.

23. Assuming that the formula r does not contain x, prove the following:

    (a) $(\forall x)(\forall y)p(x, y) \equiv (\forall y)(\forall x)p(x, y)$, $\quad (\exists x)(\exists y)p(x, y) \equiv (\exists y)(\exists x)p(x, y)$.

    (b) $\neg(\forall x)p(x) \equiv (\exists x)\neg p(x)$, $\quad \neg(\exists x)p(x) \equiv (\forall x)\neg p(x)$;

    (c) $(\exists x)(p(x) \vee q(x)) \equiv (\exists x)p(x) \vee (\exists x)q(x)$, $\quad (\exists x)(p(x) \wedge q(x)) \Rightarrow (\exists x)p(x) \wedge (\exists x)q(x)$;

    (d) $(\forall x)(p(x) \vee q(x)) \Leftarrow (\forall x)p(x) \vee (\forall x)q(x)$, $\quad (\forall x)(p(x) \wedge q(x)) \equiv (\forall x)p(x) \wedge (\forall x)q(x)$;

    (e) $(\forall x)(r \vee p(x)) \equiv r \vee (\forall x)p(x)$, $\quad (\forall x)(r \to p(x)) \equiv r \to (\forall x)p(x)$;

    (f) $(\exists x)(r \wedge p(x)) \equiv r \wedge (\exists x)p(x)$, $\quad (\exists x)(r \to p(x)) \equiv r \to (\exists x)p(x)$;

    (g) $(\forall x)p(x) \to r \equiv (\exists x)(p(x) \to r)$, $\quad (\exists x)p(x) \to r \equiv (\forall x)(p(x) \to r)$.

24. Find an appropriate UD and appropriate predicates to show that the formula $((\forall x)(p(x) \to q(x))) \to ((\exists x)(\neg p(x) \to \neg q(x)))$ is invalid.

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