Hints to practice problems : Logic

MA221: Discrete Mathematics

Q.5 $\sim P$ can be proved from $P \to Q$ and $\sim Q$. So $\sim P$ is valid.
If P is also valid, then ~~$P \wedge Q$~~ $(P \to Q) \wedge \sim Q$ will be a contradiction. However, $(P \to Q) \wedge \sim Q$ takes T whenever P, Q both takes F. Hence P is not a valid conclusion from $\{P \to Q, \sim Q\}$

Q.8 To check if $\{P \to Q, Q\} \Rightarrow P$ is valid.

Q.9. (a) To check if $\{P \to Q, \sim P \to R, R \to S\} \Rightarrow \sim Q \to S$

(b) To check if $\{(P \wedge Q) \to R, S \to Q, A \to P\} \Rightarrow \sim R \to (\sim S \vee \sim A)$

(c) To check if $\{P \to (Q \vee R), \sim R\} \Rightarrow \sim Q \to \sim P$.

(d) To check if $\{\sim A \to B, \sim B \to C, \sim C \to A\} \Rightarrow (A \wedge B) \vee (B \wedge C) \vee (C \wedge A)$.

(e) To check if $\{P \to Q, R \vee S, A \to (\sim P \vee R), P \vee S\} \Rightarrow \sim Q \to \sim A$.

(f) To check if $\{P \to (Q \vee R \vee S), Q \to \sim A, R \to \sim B, S \to C, P, B, A\} \Rightarrow C$.

Q.11 $(\forall x) \sim P(x) \vee (\forall x) P(x) \equiv \left(\sim P(a) \wedge \sim P(b) \wedge \sim P(c)\right) \vee \left(P(a) \wedge P(b) \wedge P(c)\right)$

$$\equiv \sim\left(P(a) \vee P(b) \vee P(c)\right) \vee \left(P(a) \wedge P(b) \wedge P(c)\right)$$

Q.12 (a) T (b) F (c) T.

Q.13 Take $UD = \{2, 3\}$.

Q.14 $(\exists z)\left(P(z) \wedge F(x, z) \wedge M(z, y)\right)$

Q.15 $(\forall x)\left(P(x) \to (\forall y)\left(\left(P(y) \wedge L(y)\right) \to R(x, y)\right)\right)$

Q.16

(a) $(\forall x)\left(P(x) \to G(x)\right)$

(b) $(\forall x)\left(P(x) \to (\exists y)\left(P(y) \wedge G(x, y)\right)\right)$

(c) $(\forall x)\left(P(x) \to \left(B(x) \vee M(x)\right)\right)$

(d) $(\exists x)\left(P(x) \wedge \left(H(x) \vee E(x)\right)\right)$

(e) $(\forall x)\left(N(x) \to (\exists y)\left(N(y) \wedge Q(x, y)\right) \vee R(x)\right)$

(f) $(\forall x)\left(R(x) \to (\exists y)\left(R(y) \wedge G(x, y)\right)\right)$

(a) UD = all human being in the world.
$(\forall x)\, G(x)$.

(b) $UD = \mathbb{N}$, $(\forall x)(\exists y)\, G(x, y)$.

(c) UD = all persons in the room
$(\forall x)\left(B(x) \vee M(x)\right)$

(d) UD = all students in the room
$(\exists x)\left(H(x) \vee E(x)\right)$

(e) $UD = \mathbb{N}$, $(\forall x)\left((\exists y) Q(x, y) \vee R(x)\right)$

(f) $UD = \mathbb{R}$, $(\forall x)(\exists y)\, G(x, y)$.

Q.17 $(\forall x)(S(x)\to H(x))$, $(\forall x)(\exists y)(S(y)\wedge C(x,y)) \to (\exists z)(H(z)\wedge C(x,z)))$

OR $(\forall x)(S(x)\to H(x))$, $(\forall x)(S(x)\to(\forall y)(C(y,x)\to(\exists z)(H(z)\wedge C(y,z))))$

Q.18. UD $=\mathbb{R}$.

$(\forall x)(\forall \varepsilon)(R(\varepsilon)\to((\exists \delta)\, R(\delta)\wedge(P(x,\delta)\to Q(x,\varepsilon))))$.

$(\exists \varepsilon)(R(\varepsilon)\wedge(\forall \delta)(R(\delta)\to(\exists x)(P(x,\delta)\wedge \sim Q(x,\varepsilon))))$.

Q.19. $(\forall x)(G(x)\to(\exists y)(G(y)\wedge C(x,y)))$

Q.20. (b) If every rational number is bigger than 2 then every real number

Q.21. Consider an arbitrary interpretation, and let, $(\exists x)(p(x)\wedge q(x))$ be true. Then there is an x_0 such that $p(x_0)\wedge q(x_0)$ is true. So, both $p(x_0)$ ~~and q(~~ and $q(x_0)$ are true. Thus $(\exists x)p(x)$ and $(\exists x)q(x)$ are true. ~~Hence~~ Therefore $(\exists x)p(x)\wedge(\exists x)q(x)$ is true.

Hence the given formula is valid.

For the converse, consider UD $=\mathbb{N}\setminus\{2\}$,
$p(x)$: x is a prime number, $q(x)$ = x is an even number.

Q.22 Take UD $\equiv \mathbb{R}$, $r(x) : x>0$, $p(x,y) : x+y=0$

Q.24. UD $=\{2^n : n\in\mathbb{N}\}$, $p(x)$: The power of x in base 2 is even.
$q(x)$: $\sqrt{x}$ is a natural number.

Q.23 (d) Let $(\forall x)p(x) \vee (\forall x)q(x)$ be true (T).
i.e. $(\forall x)p(x)$ is T or $(\forall x)q(x)$ is T
i.e. for all x, $p(x)$ is T or for all x, $q(x)$ is T
i.e. for all x, $p(x)\vee q(x)$ is T
i.e. ~~for all x~~, $(\forall x)(p(x)\vee q(x))$ is T

Thus $((\forall x)p(x) \vee (\forall x)q(x)) \Rightarrow (\forall x)(p(x)\vee q(x))$

However,
$(\forall x)(p(x)\vee q(x)) \Rightarrow ((\forall x)p(x) \vee (\forall x)q(x))$
is not valid.
Take UD $\equiv \mathbb{N}$, $p(x)$: x is even
$q(x)$: x is odd

Formula's ~~Rules~~ used for Inference

R1) $(P \to Q) \Leftrightarrow (\sim P \vee Q)$

R2) $(P \to Q) \Leftrightarrow (\sim Q \to \sim P)$

R3) $(P \rightleftarrows Q) \Leftrightarrow (P \to Q) \wedge (Q \to P)$

R4) $(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q)$

R5) ~~Idem~~ $P \vee P \Leftrightarrow P$, Idempotent

R6) Associative

R7) Commutative

R8) Distributive

R9) $P \vee F \Leftrightarrow P$, $P \wedge T \Leftrightarrow P$.

R10) $P \vee \sim P \Leftrightarrow T$, $P \wedge \sim P \Leftrightarrow F$

R11) Absorption.

R12) De-Morgan's.

R13) $\sim\sim P \Leftrightarrow P$

R14) $P \wedge Q \Rightarrow P$, $P \wedge Q \Rightarrow Q$

R15) $P \Rightarrow P \vee Q$, $Q \Rightarrow P \vee Q$

R16) $\sim P \Rightarrow P \to Q$, $Q \Rightarrow P \to Q$

R17) $P \wedge (P \to Q) \Rightarrow Q$. (Modus ponens)

R18) $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$ (Hypothetical syllogism)

R19) $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$ (dilemma)

R20) $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$

R21) $\sim (P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \sim Q$

R22) $\sim (P \to Q) \Rightarrow P$, $\sim (P \to Q) \Rightarrow \sim Q$

R21) $\{P, Q\} \Rightarrow P \wedge Q$

R22) $\{\sim P, P \vee Q\} \Rightarrow Q$. (disjunctive syllog

R23) $\{\sim Q, P \to Q\} \Rightarrow \sim P$ (Modus Tollens)

Hints / solutions to practice problems: Logic.

MA 221 : Discrete Mathematics

Logic

Q2. a

1. $P \land Q$
2. Q
3. $\sim P \lor Q$
4. $P \to Q$

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1. $P \to Q$
2. P (assumed premises)
3. Q (MP)
4. $P \land Q$
5. $P \to (P \land Q)$ (CP)

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1. $(P \to Q) \to Q$
2. $\sim(P \to Q) \lor Q$
3. $\sim(\sim P \lor Q) \lor Q$
4. $(\sim\sim P \land \sim Q) \lor Q$
5. $(P \land \sim Q) \lor Q$
6. $(P \lor Q) \land (\sim Q \lor Q)$
7. $P \lor Q$

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1. $P \to (Q \to R)$
2. $P \to Q$ (assumed premise)
3. $\sim P \lor Q$
4. $\sim P \lor (Q \to R)$
5. $\sim P \lor (\sim Q \lor R)$
6. $(\sim P \lor Q) \lor (\sim P \lor (\sim Q \lor R))$
7. $(\sim P \lor \sim P) \lor (Q \lor \sim Q) \lor R$
8. $\sim P \lor R$
9. $P \to R$
10. $(P \to Q) \to (P \to R)$ Rule CP.

Q.3 **First**

$\cancel{1.\ P \to (Q \to P)}$

$\cancel{2.\ \sim P}$ (assumed)

$\cancel{3.\ \sim P \lor (\sim Q \lor P)}$

$\cancel{4.\ P \lor (\sim P \lor \sim Q)}$

1. $P \lor \sim P$
2. $(\sim\sim P) \lor \sim P$
3. $(\sim\sim P) \lor \sim P \lor Q$
4. $(\sim\sim P) \lor (P \to Q)$
5. $\sim P \to (P \to Q)$.

Again.

1. $\sim P \lor P$
2. $\sim P \lor P \lor \sim Q$
3. $P \to (P \lor \sim Q)$
4. $P \to (\sim Q \lor P)$
5. $P \to (Q \to P)$.

Hence $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$

2nd

1. $P \to (Q \lor R)$
2. $\sim P \lor (Q \lor R)$ 1m)
3. $(\sim P \lor \sim P) \lor (Q \lor R)$
4. $(\sim P \lor Q) \lor (\sim P \lor R)$
5. $(P \to Q) \lor (P \to R)$.

Again

1. $(P \to Q) \lor (P \to R)$
2. $(\sim P \lor Q) \lor (\sim P \lor R)$
3. $(\sim P \lor \sim P) \lor (Q \lor R)$
4. $\sim P \lor (Q \lor R)$
5. $P \to (Q \lor R)$

So, $P \to (Q \lor R) \Leftrightarrow (P \to Q) \lor (P \to R)$.

~~3rd~~ 3rd

1. $\sim(P \rightleftarrows Q)$
2. $\sim((P \to Q) \land (Q \to P))$
3. $\sim((\sim P \lor Q) \land (\sim Q \lor P))$
4. $\sim(\sim P \lor Q) \lor \sim(\sim Q \lor P)$
5. $(\sim\sim P \land \sim Q) \lor (\sim\sim Q \land \sim P)$
6. $(P \land \sim Q) \lor (Q \land \sim P)$
7. $(P \land \sim Q) \lor (\sim P \land Q)$

These steps can easily be reversed, and hence the equivalence.

4th ~~Also~~

1. $\sim(P \to Q)$
2. $\sim(\sim P \lor Q)$
3. $(\sim\sim P \land \sim Q)$
4. $P \land \sim Q$

These steps can easily be reversed, and so the inference for the reverse implication follows similarly, ~~&~~ just by reversing these four steps.

Q.3

5th This is R4, but we want to prove it using other formulas.

1. $P \rightleftarrows Q$.
2. $(P \to Q) \wedge (Q \to P)$
3. $(\sim P \vee Q) \wedge (\sim Q \vee P)$
4. $((\sim P \vee Q) \wedge \sim Q) \vee ((\sim P \vee Q) \wedge P)$
5. $(\sim P \wedge \sim Q) \vee (Q \wedge \sim Q) \vee (\sim P \wedge P) \vee (Q \wedge P)$
6. $(\sim P \wedge \sim Q) \vee (P \wedge Q)$
7. $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.

Reverse part is similar.

6th

1. P.
2. $P \vee Q$
3. $P \vee \sim Q$.
4. $(P \vee Q) \wedge (P \vee \sim Q)$

Also,
1. $(P \vee Q) \wedge (P \vee \sim Q)$
2. $P \vee (Q \wedge \sim Q)$
3. $P \vee F$
4. P.

So $P \Leftrightarrow (P \vee Q) \wedge (P \vee \sim Q)$

Q.4

First

1. $\sim (P \wedge \sim Q)$
2. $\sim Q \vee R$
3. $\sim R$
4. $\sim Q$ R22 (Dis. syll.)
5. $\sim P \vee \sim\sim Q$
6. $\sim P \vee Q$
7. $\sim P$.   R22

2nd

1. $(P \to Q) \to R$
2. ~~$P \wedge S$~~
3. $Q \wedge T$.
4. ~~P~~.
5. Q.
6. $\sim P \vee Q$
7. $P \to Q$
8. R 1, ~~7~~, 8 MP. (R.17)

3rd

1. $P \to Q$
2. P (assumed)
3. Q.   R17
4. $P \wedge Q$
5. $P \to (P \wedge Q)$ (Rule CP)

4th

1. $\sim P \vee Q$
2. $\sim Q \vee R$
3. $R \to S$
4. $P \to Q$
5. $Q \to R$
6. $P \to R$ R18
7. $P \to S$ R18.

5th

1. P
2. $P \to (Q \to (R \wedge S))$
3. $Q \to (R \wedge S)$
4. Q (assumed)
5. $R \wedge S$
6. S
7. $Q \to S$ Rule CP

6th

1. $(P \vee Q) \to R$
2. $P \wedge Q$ (assumed)
3. P
4. $P \vee Q$
5. R
6. $(P \wedge Q) \to R$ (Rule CP)

8.5 If $\{P\to Q, \sim Q\} \Rightarrow P$, then $((P\to Q)\wedge \sim Q)\to P$ is a tautology.

However, $((P\to Q)\wedge\sim Q)\to P \Leftrightarrow (\sim((P\to Q)\wedge\sim Q))\vee P$

$\Leftrightarrow ((\sim P\vee Q)\wedge\sim Q)\vee P$

$\Leftrightarrow \cancel{((\sim P\wedge\sim Q)\vee(Q\wedge\sim Q))\vee P}$

$\Leftrightarrow \cancel{(\sim P\wedge\sim Q)\vee P}$

$\Leftrightarrow ((P\wedge\sim Q)\vee Q)\vee P$

8.5 We have

1. $P\to Q$
2. $\sim Q$
3. $\sim Q\to\sim P$
4. $\sim P$.

So $\sim P$ is a valid conclusion from $\{P\to Q,\sim Q\}$. If P is also a valid conclusion from $\{P\to Q,\sim Q\}$, then $(P\to Q)\wedge\sim Q$ would be a contradiction. However, $(P\to Q)\wedge\sim Q$ is not a contradiction, as its takes T whenever P, & Q both takes F.

8.6

First

1. $P\to Q$
2. $P\to R$
3. $Q\to\sim R$
4. P
5. Q MP
6. $\sim R$ MP
7. $\sim P$ MT
8. $P\wedge\sim P$

So, Not consistent

2nd

1. $A\to(B\to C)$
2. $D\to(B\wedge\sim C)$
3. $A\wedge D$
4. A.
5. D
6. $B\to C$
7. $B\wedge\sim C$
8. B
9. $\sim C$
10. C
11. $C\wedge\sim C$.

Not consistent

3rd

1. $A\to B$
2. $B\to C$
3. $D\to\sim C$
4. $A\wedge D$
5. A
6. D
7. B
8. C
9. $\sim C$
10. $C\wedge\sim C$.

7.

(a)
1. $(A \vee B) \wedge \sim A \wedge (B \to C)$ ~~(assumed)~~
2. $A \vee B$
3. $\sim A$
4. $B \to C$
5. B (R22)
6. C
7. $B \wedge C$

~~8. $(A \vee B) \wedge \sim A \wedge (B \to C) \to (B \wedge C)$. Rule CP~~

(c)
1. $\sim A \vee A$
2. $\sim A \vee (A \vee \sim B)$
3. $A \to (A \vee \sim B)$
4. $A \to (\sim B \vee A)$
5. $A \to (B \to A)$

(f)
1. $A \vee B$
2. $\sim (B \vee A)$ (assumed)
3. $\sim B \wedge \sim A$
4. $\sim B$
5. $\sim A$
6. $\sim A \wedge \sim B$
7. $\sim (A \vee B)$
8. $(A \vee B) \wedge \sim (A \vee B)$.

So, by method of contradiction, $(A \vee B) \to (B \vee A)$.
Similarly, $(B \vee A) \to (A \vee B)$.

(h)
1. A
2. $\sim (B \to (A \wedge B))$ (assumed)
3. $\sim (\sim B \vee (A \wedge B))$
4. $\sim\sim B \wedge \sim (A \wedge B)$
5. ~~$B \wedge \sim(\sim A \wedge \sim B)$~~ $B \wedge \sim (A \wedge B)$
6. B

~~7. $\sim A \vee \sim B$~~
~~8. $A \vee (\sim A \vee \sim B)$~~
~~9. $(A \vee \sim A) \vee \sim B$~~
~~10. $\sim B$~~

7. $\sim (A \wedge B)$
8. $A \wedge B$ from 1, 6.
9. $(A \wedge B) \wedge \sim (A \wedge B)$

So follows from contradiction.

(b)
1. $(A \vee B) \to (B \wedge C)$ Rule P
2. B (assumed)
3. $A \vee B$
4. $B \wedge C$
5. C
6. $B \to C$ Rule CP

(d)
1. $A \vee \sim A$
2. $A \vee \sim A \vee B$
3. $A \vee (\sim A \vee B)$
4. $A \vee (A \to B)$
5. $\sim\sim A \vee (A \to B)$
6. $\sim A \to (A \to B)$.

(g)
1. $A \to B$
2. $\sim (\sim A \vee B)$ (assumed)
3. $\sim\sim A \wedge \sim B$
4. $A \wedge \sim B$
5. A
6. $\sim B$
7. B
8. $B \wedge \sim B$

So, follows from contradiction.

(i)
1. A
2. $\sim (\sim B \to (A \wedge \sim B))$
3. $\sim (\sim\sim B \vee (A \wedge \sim B))$
4. $\sim (B \vee (A \wedge \sim B))$
5. $\sim B \wedge \sim (A \wedge \sim B)$
6. $\sim B$
7. $\sim (A \wedge \sim B)$
8. $A \wedge \sim B$
9. $(A \wedge \sim B) \wedge \sim (A \wedge \sim B)$.

So, follows from contradiction.

(i) is a substitution instance of th (h).

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7. (j)

1. $((A \vee B) \to C) \wedge A$
2. A
3. $(A \vee B) \to C$ $\Big\}$ $P \wedge Q \Rightarrow P, Q$.
4. $A \vee B$ $A \Rightarrow A \vee B$
5. C MP

(k) 1. $A \to C$
2. $\sim((A \wedge B) \to C)$ (assumed)
3. $A \wedge B$ $\Big\}$ $\sim(P \to Q) \Rightarrow P, \sim Q$
4. $\sim C$
5. A.
6. C
7. $C \wedge \sim C$

(l) 1. $A \to C$
2. $\sim(A \to (B \vee C))$ (assumed)
3. $\sim(\sim A \vee (B \vee C))$
4. $\sim\sim A \wedge \sim(B \vee C)$
5. $A \wedge \cancel{(\sim B \wedge \sim C)}\ \sim(B \vee C)$
6. A
7. $\sim(B \vee C)$
8. C
9. $B \vee C$
10. $(B \vee C) \wedge \sim(B \vee C)$

(m) 1. $A \to (B \to C)$
2. $\sim(B \to (A \to C))$ (assumed)
3. $\sim(\sim B \vee (\sim A \vee C))$
4. $\sim\sim B \wedge (\sim\sim A \wedge \sim C)$
5. $B \wedge (A \wedge \sim C)$
6. A
7. B
8. $\sim C$
9. $B \to C$
10. C
11. $C \wedge \sim C$

(n) 1. $B \to C$
2. $\sim((A \wedge B) \to (A \wedge C))$
3. $\sim(\sim(A \wedge B) \vee (A \wedge C))$
4. $\sim\sim(A \wedge B) \wedge \sim(A \wedge C)$
5. $(A \wedge B) \wedge \sim(A \wedge C)$
6. $A \wedge B$
7. $\sim(A \wedge C)$
8. A
9. B
10. C
11. $A \wedge C$
12. $(A \wedge C) \wedge \sim(A \wedge C)$

(o) 1. $A \to B$
2. $\sim((C \vee A) \to (C \vee B))$
3. $\sim(\sim(C \vee A) \vee (C \vee B))$
4. $\sim\sim(C \vee A) \wedge \sim(C \vee B)$
5. $(C \vee A) \wedge \sim(C \vee B)$
6. $C \vee A$
7. $\sim(C \vee B)$
8. $\sim C \wedge \sim B$
9. $\sim C$
10. $\sim B$
11. $\sim A$
12. $\sim C \wedge \sim A$
13. $\sim(C \vee A)$
14. $(C \vee A) \wedge \sim(C \vee A)$

8. To check if $\{P \to Q, Q\} \Rightarrow P$ is valid.

~~Method 1~~ If P takes F, Q takes T, then both $P \to Q$, Q takes T.

However P takes F and so $((P \to Q) \wedge Q) \to P$ is not a tautology

~~Method 2~~.

10. (a) To check $\{P \to Q, \sim P \to R, R \to S\} \Rightarrow \sim Q \to S$.

Now

1. $P \to Q$
2. $\sim P \to R$
3. $R \to S$
4. $\sim Q$ (assumed)
5. $\sim P$ (MT)
6. R MP
7. S MP
8. $\sim Q \to S$ CP

(b) To check

$\{(P \wedge Q) \to R, S \to Q, A \to P\} \Rightarrow \sim R \to (\sim S \vee \sim A)$

1. $(P \wedge Q) \to R$
2. $S \to Q$
3. $A \to P$
4. $S \wedge A$ (assumed)

~~5. $\sim(P \wedge Q)$~~

~~6. $\sim P \vee \sim Q$~~

~~7. $\sim P \to \sim A$~~

~~8. $\sim Q \to \sim S$~~

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5. S
6. A
7. Q
8. P
9. $P \wedge Q$
10. R
11. $S \wedge A \to R$ Rule CP
12. $\sim R \to \sim(S \wedge A)$
13. $\sim R \to \sim S \vee \sim A$

(c) To check

$\{P \to (Q \vee R), \sim R\} \Rightarrow \sim Q \to \sim P$

1. $P \to Q \vee R$
2. $\sim R$
3. $\sim Q$ (assumed)
4. $\sim Q \wedge \sim R$
5. $\sim(Q \vee R)$
6. $\sim(Q \vee R) \to \sim P$
7. $\sim P$
8. $\sim Q \to \sim P$ Rule CP

(d) To check if

$\{\sim A \to B, \sim B \to C, \sim C \to A\} \Rightarrow (A \wedge B) \vee (B \wedge C) \vee (C \wedge A)$

1. $\sim A \to B$
2. $\sim B \to C$
3. $\sim C \to A$

~~4. $\sim((A \wedge B) \vee (B \wedge C) \vee (C \wedge A))$ (assumed~~

4. $\sim\sim A \vee B$
5. $\sim\sim B \vee C$
6. $\sim\sim C \vee A$
7. $A \vee B$
8. $B \vee C$
9. $C \vee A$

~~11. $\sim(A \wedge B) \wedge \sim(B \wedge C) \wedge \sim(C \wedge A)$~~

10. $(B \vee C) \wedge (C \vee A)$

11. $C \vee (A \wedge B)$ (distributive, commutative

12. $(A \vee B) \wedge [C \vee (A \wedge B)]$

13. $[(A \vee B) \wedge C] \vee [(A \vee B) \wedge (A \wedge B)]$ (distributive)

13. $[(A \wedge C) \vee (B \wedge C)] \vee [((A \vee B) \wedge A) \wedge B]$

14. $(C \wedge A) \vee (B \wedge C) \vee (A \wedge B)$ absorption.

15. $(A \wedge B) \vee (B \wedge C) \vee (C \wedge A)$, commutative, associative.

10.(e)

To check

$\{P \to Q,\ R \vee S,\ A \to (\sim P \vee R),\ P \vee S\} \Rightarrow \sim Q \to \sim A$

Let us take the truth values of P, Q, R, S and A be F, F, T, T, and T.

Then $P \to Q$ is T

$R \vee S$ is T

$A \to (\sim P \vee R)$ is T

$P \vee S$ is T.

But $\sim Q \to \sim A$ is F.

Hence the argument is not valid.

(f) To check

$\{P \to (Q \vee R \vee S),\ Q \to \sim A,\ R \to \sim B,\ S \to C,\ P, B, A\} \Rightarrow C$

1. $P \to (Q \vee R \vee S)$
2. $Q \to \sim A$
3. $R \to \sim B$
4. $S \to C$
5. P
6. B
7. A
8. $Q \vee R \vee S$
9. $\sim Q$, 2, 7, MT
10. $\sim R$ 3, 6, MT
11. $\sim Q \wedge \sim R$
12. $\sim (Q \vee R)$
13. $\sim (Q \vee R) \vee ((Q \vee R) \vee S)$
14. $(\sim (Q \vee R) \vee (Q \vee R)) \vee S$
15. S
16. C 4, 15, MP

11. (a) The scope of first $(\forall x)$ is $P(x) \wedge R(x)$ and all the first three ~~two~~ occurances are bound to the first quantifier.

The scope of the 2nd $(\forall x)$ is $P(x)$, and the 4th, 5th occurances of x are bound to the 2nd quantifier.

The last occurance of x is free.

The last occurance of x is free.

(b) The scope of first $(\forall x)$ is $P(x) \wedge (\exists x)(Q(x)) \vee ((\forall x) P(x) \to Q(x))$.

The scope of $(\exists x)$ is $Q(x)$.

The scope of 2nd $(\forall x)$ is $P(x)$.

w.r.t the first $(\forall x)$, the first two and the last x are bound, others are free.

w.r.t the $(\exists x)$, the 3rd, 4th x are bound, others are free.

w.r.t the last $(\forall x)$, the 5th, 6th x are bound, others are free.

(c) The scope of $(\forall x)$ is $((P(x) \rightleftarrows Q(x)) \wedge (\exists x) R(x))$.

The scope of $(\exists x)$ is $R(x)$.

The last occurance of x is free, others are bound.

12. $(\forall x) R(x) \wedge (\forall x) S(x) \equiv (R(a) \wedge R(b) \wedge R(c)) \wedge (S(a) \wedge S(b) \wedge S(c))$

$(\forall x) R(x) \wedge (\exists x) S(x) \equiv (R(a) \wedge R(b) \wedge R(c)) \wedge (S(a) \vee S(b) \vee S(c))$

$(\forall x)(P(x) \to Q(x)) \equiv (P(a) \to Q(a)) \wedge (P(b) \to Q(b)) \wedge (P(c) \to Q(c))$

$(\forall x) \sim P(x) \vee (\forall x) P(x) \equiv (\sim P(a) \wedge \sim P(b) \wedge \sim P(c)) \vee (P(a) \wedge P(b) \wedge P(c))$

$\equiv \sim (P(a) \vee P(b) \vee P(c)) \vee (P(a) \wedge P(b) \wedge P(c))$.

13. (a) $(\forall x)\big(P(x) \vee Q(x)\big) \equiv \big(P(1) \vee Q(1)\big) \wedge \big(P(2) \vee Q(2)\big)$

As $P(1)$ is true & $Q(2)$ is true, the required truth value is T.

(b) Here P is true, but $Q(6)$ is false. So $(\forall x)(P \to Q(x))$ is false.

Also $R(a)$ is false. So, the required truth value is F.

(c) Here both $P(1)$, $Q(1)$ are false, and so $(\exists x)(P(x) \to Q(x))$ is true. So, the required truth value is T.

---

14. Take $UD = \{2, 3\}$, $R(x)$: x is even, $Q(x)$: x is odd, $P(x)$: x is a prime.

Then both $(\exists x)(P(x) \wedge Q(x))$ and $(\exists y)(P(y) \wedge R(y))$ are true, while

$(\exists z)(Q(z) \wedge R(z))$ is false.

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15. $(\exists z)\big(P(z) \wedge F(x,z) \wedge M(z,y)\big)$.

16. Let $P(x)$: x is a person, $L(x)$: x is a lover., $R(x,y)$: x loves y.

Then $(\forall x)\Big(P(x) \to (\forall y)\big((P(y) \wedge L(y)) \to R(x,y)\big)\Big)$.

---

17. (a) $P(x)$: x is a man $G(x)$: x is a giant.

Then $(\forall x)(P(x) \to G(x))$. If UD = all human in this world, then $(\forall x)\, G(x)$.

(b) $P(x)$: x is a +ve integer

$G(x,y)$: $y > x$

$(\forall x)\Big(P(x) \to (\exists y)\big(P(y) \wedge G(x,y)\big)\Big)$.

(b) If $UD = \mathbb{N}$. Then

$(\forall x)(\exists y)\, G(x,y)$.

(c) $P(x)$: x is a person in this room

$B(x)$: x is a B.Tech student

$M(x)$: x is an M.Sc student

So $(\forall x)\big(P(x) \to (B(x) \vee M(x))\big)$.

(c) If UD: all the persons in this room

$(\forall x)(B(x) \vee M(x))$.

(d) $P(x)$: x is a student in this room.

$H(x)$: x speaks Hindi

$E(x)$: x speaks English.

$(\exists x)\big(P(x) \wedge (H(x) \vee E(x))\big)$.

UD: all the students in this room

$(\exists x)\big(H(x) \vee E(x)\big)$.

(e) $N(x)$: x is a natural number.

$Q(x,y)$: $x = y^2$

~~$R(x,y)$: $\sqrt{x}$~~ $R(x)$: $\sqrt{x}$ is irrational)

$(\forall x)\Big(N(x) \to (\exists y)(N(y) \wedge Q(x,y)) \vee R(x)\Big)$

UD: $\mathbb{N}$. Then

$(\forall x)\Big((\exists y)\, Q(x,y) \vee R(x)\Big)$

(f) $R(x)$: x is a real number

$G(x,y)$: $x + y = 0$

$(\forall x)\Big(R(x) \to (\exists y)\big(R(y) \wedge G(x,y)\big)\Big)$

UD: $\mathbb{R}$.

$(\forall x)(\exists y)\, G(x,y)$.

7. Translate the following into formal statements.

8.18

'All scientists are human beings. Therefore, all children of scientists are children of human beings.'

A: Let Sx : 'x is a scientist'; Hx : 'x is a human being' and Cxy : x is a child of y.

Let the hypothesis be $\forall x(Sx \to Hx)$. Then, the possible translations of the conclusion are the following.

(a) $\forall x(\exists y(Sy \wedge Cxy) \to \exists z(Hz \wedge Cxz))$. It means 'for each x, if x has a scientist father, then x has a human father'.

Wrong (b) $\forall x[\forall y(Sy \wedge Cxy) \to \forall z(Hz \wedge Cxz)]$. This is wrong, as the statement means 'for all x, if x is a common child of all scientists, then x is a common child of all human beings'.

(c) $\forall x(Sx \to \forall y(Cyx \to \exists z(Hz \wedge Cyz)))$. This means 'for each x, if x is a scientist, then each child of x has a human father'.

(d) $\forall x \forall y(Sx \wedge Cyx) \to \forall x \forall y(Hx \wedge Cxy)$. What? This means 'if each x is a scientist and Wrong each y is a child of x (including x it self!), then each x is a human being and each y is a child of x'.

Exercise **4.2.14.** *1. Write a formal definition of* $\lim_{x \to a} f(x) \neq l$.

2. *Is* $\exists x\,[p(x) \wedge q(x)] \to \exists x\, p(x) \wedge \exists x\, q(x)$ *valid? Is it's converse valid?*

3. [**common ones**] *If* r *does not involve* x*, then establish the following assertions.*

   (a) $\neg \forall x\, p(x) \equiv \exists x\, \neg p(x)$; $\quad \neg \exists x\, p(x) \equiv \forall x\, \neg p(x)$

   (b) $\exists x\, \big(p(x) \vee q(x)\big) \equiv \exists x\, p(x) \vee \exists x\, q(x)$; $\quad \exists x\, \big(p(x) \wedge q(x)\big) \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$.

   (c) $\forall x\, \big(p(x) \wedge q(x)\big) \equiv \forall x\, p(x) \wedge \forall x\, q(x)$; $\quad \forall x\, \big(p(x) \vee q(x)\big) \Leftarrow \forall x\, p(x) \vee \forall x\, q(x)$.

   (d) $\forall x\, \big(r \vee q(x)\big) \equiv r \vee \forall x\, q(x)$; $\quad \forall x\, \big(r \to q(x)\big) \equiv r \to \forall x\, q(x)$

   (e) $\exists x\, \big(r \wedge q(x)\big) \equiv r \wedge \exists x\, q(x)$; $\quad \exists x\, \big(r \to q(x)\big) \equiv r \to \exists x\, q(x)$.

   (f) $\forall x\, p(x) \to r \equiv \exists x\, \big(p(x) \to r\big)$; $\quad \exists x\, p(x) \to r \equiv \forall x\, \big(p(x) \to r\big)$.

4. *Translate and check for validity of the following arguments.*

   (a) *Recall that the decimal representation of a rational number either terminates or begins to repeat the same finite sequence of digits, whereas that of an irrational number neither terminates nor repeats. The square root of a natural number either has a decimal representation which is terminating or has a decimal representation which is non-terminating and non-repeating. The square root of all natural numbers which are squares have terminating decimal representation. Therefore, the square root of a natural number which is not a square is an irrational number.*

   (b) *For any two algebraic numbers* a *and* b*,* $a \neq 0, 1$ *and* b *irrational, we have that* a^b *is transcendental. The number* i *(imaginary unit) is irrational and algebraic. The number* i *is not equal to* 0 *or* 1*. Therefore, the number* i^i *is transcendental.*

5. (a) *Give an interpretation to show that* $\forall x\, \Big(r(x) \to \exists y\, \big(r(y) \wedge p(x, y)\big)\Big)$ *is not valid.*

18. $S(x)$: x is a scientist, $C(x)$: x is a children of a scientist
$H(x)$: x is a human $D(x)$: x is a children of a human.

So $\big((\forall x)(S(x) \to H(x))\big) \to \big((\forall y)(C(y) \to D(y))\big)$.

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19. $\underset{x \to a}{Lt} f(x) = l$ i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$.

So, let UD : $\mathbb{R}$, $R(x)$: $x > 0$ $P(x,y)$: $0 < |x-a| < y$, $Q(x,y)$: $|f(x) - l| < y$.

So $(\forall x)(\forall \varepsilon)\Big(R(\varepsilon) \to \big((\exists \delta) R(\delta) \wedge (P(x,\delta) \to Q(x,\varepsilon))\big)\Big)$

$\lim_{x \to a} f(x) \neq l$ i.e. $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x_0$ satisfying $0 < |x-a| < \delta$ and $|f(x_0) - l| \geq \varepsilon$.

~~$(\exists \varepsilon)\big(R(\varepsilon) \wedge (\forall \delta)(R(\delta) \to (\exists x_0)(P(x_0,\delta) \wedge \sim Q(x,\varepsilon)$~~

$(\exists \varepsilon)\Big(R(\varepsilon) \wedge (\forall \delta)\big(R(\delta) \to (\exists x)(P(x,\delta) \wedge \sim Q(x,\varepsilon))\big)\Big)$

---

20. $G(x)$: x is a student in IIT G. $C(x,y)$: y has CPI more than x.

$(\forall x)\big(G(x) \to (\exists y)(G(y) \wedge C(x,y))\big)$

---

21. $p(x)$: x is a rational number, $q(x)$: x is a real number.

(a) $(\forall x)(p(x) \to q(x))$: every rational number is a real number.

$(\exists x)(\sim p(x) \wedge q(x))$: ~~there is a number which is not rati~~
there is a real number which is not rational.

(b) $\big((\forall x)(p(x) \wedge (x > 2))\big) \to \big((\forall x)(q(x) \wedge (x < 2))\big)$:

~~If x is a rational number bigger than two then x is a real number lesser than two~~

---

22. If every rational number is bigger than two then every real number is lesser than two.

---

23. Let UD : $\mathbb{R}$, $r(x)$: $x > 0$, $p(x,y)$: $x + y = 0$

Then $r(1)$ is true, while, $\forall y$, $p(1,y)$ is false.

So $(\forall x)\big(r(x) \to (\exists y)(r(y) \wedge p(x,y))\big)$.

---

25. UD : $\mathbb{Q}$ $p(x)$: x is an integer. $q(x)$: x is rational.

Then $(\forall x)(p(x) \to q(x))$ is true. But $((\exists x)(\sim p(x) \to \sim q(x)))$ is false

because, $\sim((\exists x)(\sim p(x) \to \sim q(x)))$

$\equiv (\forall x) \sim(\sim p(x) \to \sim q(x))$ and $\sim p(\frac{1}{2}) \to \sim q(\frac{1}{2})$ is false

$\equiv (\forall x)(q(x) \to p(x))$

and so $\sim(\sim p(\frac{1}{2}) \to \sim q(\frac{1}{2}))$ is true

Solution: We have the following sequence of arguments:

$$\begin{array}{rll}
(1) & A \to B, & \text{Rule } \mathbf{P} \\
(2) & B \to C, & \text{Rule } \mathbf{P} \\
(3) & A \to C, & \text{Rule } \mathbf{T}\ ;\ (1),(2) \text{ and hypothetical syllogism} \\
(4) & D \to \neg C, & \text{Rule } \mathbf{P} \\
(5) & \neg\neg C \to \neg D, & \text{Rule } \mathbf{T}\ ;\ (4) \text{ and } P \to Q \Leftrightarrow \neg Q \to \neg P \\
(6) & C \to \neg D, & \text{Rule } \mathbf{T}\ ;\ (5) \text{ and } \neg\neg P \Leftrightarrow P \\
(7) & A \to \neg D, & \text{Rule } \mathbf{T}\ ;\ (3),(6) \text{ and hypothetical syllogism} \\
(8) & \neg A \vee \neg D, & \text{Rule } \mathbf{T}\ ;\ (7) \text{ and } P \to Q \Leftrightarrow \neg P \vee Q \\
(9) & \neg(A \wedge D), & \text{Rule } \mathbf{T}\ ;\ (8) \text{ and } \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q \\
(10) & A \wedge D, & \text{Rule } \mathbf{P} \\
(11) & \neg(A \wedge D) \wedge (A \wedge D), & \text{Rule } \mathbf{T}\ ;\ (9),(10) \text{ and } P, Q \Rightarrow P \wedge Q.
\end{array}$$

Since $\neg(A \wedge D) \wedge (A \wedge D)$ is a contradiction, we conclude that the given set $\{A \to B, B \to C, D \to \neg C, A \wedge D\}$ of premises is not consistent.

11. Prove that $(\exists x\ (p(x) \wedge q(x))) \to (\exists x\ p(x) \wedge \exists x\ q(x))$ is a valid formula. Check whether its converse is also valid or not? **2+2**

Q22

Solution: Let us consider an arbitrary interpretation of the given formula $(\exists x\ (p(x) \wedge q(x))) \to (\exists x\ p(x) \wedge \exists x\ q(x))$. Let $\exists x\ (p(x) \wedge q(x))$ be true in this interpretation. Then we find an x_0 in the associated universe of discourse such that the statement $p(x_0) \wedge q(x_0)$ is true. Therefore both the statements $p(x_0)$ and $q(x_0)$ are true. Thus the formulas $\exists x\ p(x)$ and $\exists x\ q(x)$ are true in this interpretation. And therefore the formula $\exists x\ p(x) \wedge \exists x\ q(x)$ is true in this interpretation. Thus we have seen that the formula $\exists x\ p(x) \wedge \exists x\ q(x)$ is true whenever the formula $\exists x\ (p(x) \wedge q(x))$ is true in any arbitrary interpretation. Hence $(\exists x\ (p(x) \wedge q(x))) \to (\exists x\ p(x) \wedge \exists x\ q(x))$ is a valid formula.

The converse of the given formula is $(\exists x\ p(x) \wedge \exists x\ q(x)) \to (\exists x\ (p(x) \wedge q(x)))$. Let us consider the following interpretation of this formula, where

Universe of Discourse $= \mathbb{N} \setminus \{2\}$,

$p(x)$: x is a prime number, and

25. Take UD: $\{2^n : n \in \mathbb{N}\}$. $p(x)$ = the power of x in base 2 is even.
$q(x) = \sqrt{x}$ is a natural number.

It is clear that for all x in UD, $p(x)$ is true iff $q(x)$ is true.

So, $(\forall x)(p(x) \to q(x))$ has truth value T.

Also $\sim((\exists x)(\sim p(x) \to \sim q(x)) \equiv (\forall x) \sim(\sim p(x) \to \sim q(x))$
$\equiv \forall x(\sim\sim q(x) \to \sim\sim p(x)) \equiv (\forall x)(q(x) \to p(x))$ has truth value T.

So, $(\exists x)(\sim p(x) \to \sim q(x))$ has truth value F.

---

24. For all the problems L means left part & R means right part of the given equivalence/implication to be proved.

(a) Let L be T, i.e. for all x & for all y, $p(x,y)$ is T
i.e. for all y & for all x, $p(x,y)$ is T
i.e. $(\forall y)(\forall x)\, p(x,y)$ is T.

~~Again let L be F i.e.~~ Similarly if R is T, then L is T.

2nd part is in Lecture note

(b) First part is in Lecture note

To show $\sim(\exists x)\, p(x) \equiv (\forall x) \sim p(x)$

Let $\sim(\exists x)\, p(x)$ be T, i.e. $(\exists x) p(x)$ is F i.e. for all x, $p(x)$ is F
i.e. for all x, $\sim p(x)$ is T
i.e. $(\forall x) \sim p(x)$ is T.

Again, let $\sim(\exists x)\, p(x)$ be F i.e. $(\exists x)\, p(x)$ is T i.e. there is an x_0, s.t. $p(x_0)$ is T
ie. $(\exists x) \sim p(x)$ is F.

Hence $\sim(\exists x)\, p(x) \equiv (\forall x) \sim p(x)$

(c) 2nd part is Q.22

First part. Let L be T. Then
there is an x_0 s.t. $p(x_0) \vee q(x_0)$ is T
ie. there is an x_0 s.t. $p(x_0)$ is T or there is an x_0 s.t. $q(x_0)$ is T
ie. $(\exists x)\, p(x) \vee (\exists x) q(x)$ is T
ie. R is T

Again, let L be F, Then
for all x, $p(x) \vee q(x)$ is F
i.e. for all x, $p(x)$ and $q(x)$ both are F
ie. for all x, $p(x)$ is F and for all x, $q(x)$ is F
ie. $(\exists x) p(x)$ is F and $(\exists x) q(x)$ is F
ie. $(\exists x) p(x) \vee (\exists x) q(x)$ is F

Aliter R is T,
i.e. $(\exists x)\, p(x)$ is T or $(\exists x)\, q(x)$ is T
ie. there is x_0 s.t. $p(x_0)$ is T or there is y_0 s.t. $q(y_0)$ is T
ie. $p(x_0) \vee q(x_0)$ is T or $p(y_0) \vee q(y_0)$ is T
ie. $(\exists x)(p(x) \vee q(x))$ is T.

24. (d). Let $(\forall x)\, p(x) \vee (\forall x)\, q(x)$ be T

i.e. $(\forall x)\, p(x)$ is T or $(\forall x)\, q(x)$ is T

i.e. for all x, $p(x)$ is T or for all x, $q(x)$ is T

i.e. for all x, $p(x) \vee q(x)$ is T

i.e. $(\forall x)\,(p(x) \vee q(x))$ is T. So $(\forall x)\, p(x) \vee (\forall x)\, q(x) \Rightarrow (\forall x)\,(p(x) \vee q(x))$.

However $(\forall x)(p(x) \vee q(x)) \Rightarrow (\forall x)\, p(x) \vee (\forall x)\, q(x)$ is invalid, as can be seen by the following interpretation:

Take UD: $\mathbb{N}$, $p(x)$: x is even, $q(x)$: x is odd.

2nd part Let $(\forall x)\,(p(x) \wedge q(x))$ be T

i.e. for all x, $p(x) \wedge q(x)$ is T

i.e. for all x, $p(x)$ is T and $q(x)$ is T

i.e. for all x, $p(x)$ is T and for all x, $q(x)$ is T

i.e. $(\forall x)\, p(x)$ is T and $(\forall x)\, q(x)$ is T

i.e. $(\forall x)\, p(x) \wedge (\forall x)\, q(x)$ is T.

Let $(\forall x)(p(x) \wedge q(x))$ be F

i.e. there is x_0 s.t. $p(x_0) \wedge q(x_0)$ is F

i.e. there is x_0 s.t. $p(x_0)$ is F or $q(x_0)$ is F

i.e. there is x_0 s.t. $p(x_0)$ is F or there is x_0 s.t. $q(x)$ is F

i.e. $(\forall x)\, p(x)$ is F or $(\forall x)\, q(x)$ is F

i.e. $(\forall x)\, p(x) \wedge (\forall x)\, q(x)$ is F

(e) Let $(\forall x)\,(r \vee p(x))$ be T

i.e. for all x, $r \vee p(x)$ is T

i.e. for all x, r is T or $p(x)$ is T

i.e. r is T or for all x, $p(x)$ is T

i.e. $r \vee (\forall x)\, p(x)$ is T

Again, $(\forall x)\,(r \vee p(x))$ is F

i.e. there is x_0 s.t. $r \vee p(x_0)$ is F

i.e. there is x_0 s.t. r is F and $p(x_0)$ is F

i.e. r is F and there is x_0 s.t. $p(x_0)$ is F

i.e. r is F and $(\forall x)\, p(x)$ is F

i.e. $r \vee (\forall x)\, p(x)$ is F.

2nd part Let $(\forall x)\,(r \to p(x))$ be T

i.e. for all x, if r is T then $p(x)$ is T

i.e. if r is T then for all x, $p(x)$ is T

i.e. $r \to (\forall x)\, p(x)$ is T

Again let $(\forall x)\,(r \to p(x))$ be F

i.e. there is x_0 s.t. r is T but $p(x_0)$ is F

i.e. r is T and there is x_0 s.t. $p(x_0)$ is F

i.e. r is T and $(\forall x)\, p(x)$ is F

i.e. $r \to (\forall x)\, p(x)$ is F

Aliter If r is F, then both sides have truth value T. Now let r be T.

Then $(\forall x)\,(r \to p(x))$ is T

i.e. for all x, $r \to p(x)$ is T

i.e. for all x, $p(x)$ is T (as r is T)

i.e. $(\forall x)\, p(x)$ is T

i.e. $r \to (\forall x)\, p(x)$ is T.

Similarly, $(\forall x)\,(r \to p(x))$ is F

i.e. there x_0 s.t. $r \to p(x_0)$ is F

i.e. there is x_0 s.t. $p(x_0)$ is F (as r is T)

i.e. $(\forall x)\, p(x)$ is F

i.e. $r \to (\forall x)\, p(x)$ is F.

24. (f) If r is F then both sides have ~~th~~ same truth value F.
So, let r be T. Then

$(\exists x)(r \wedge p(x))$ is T

ie. there is x_0 s.t. $r \wedge p(x_0)$ is T
ie. there is x_0 s.t. $p(x_0)$ is T
ie. $(\exists x)\, p(x)$ is T
ie. $r \wedge (\exists x)\, p(x)$ is T (as r is T)

Again, ~~$r \wedge (\exists x)\, p(x)$ is F~~
$(\exists x)(r \wedge p(x))$ is F

ie. for all x, $r \wedge p(x)$ is F
ie. for all x, $p(x)$ is F (as r is T)
ie. $(\exists x)\, p(x)$ is F
ie. $r \wedge (\exists x)\, p(x)$ is F.

2nd part If r is F, then both sides have same truth value T.
So, let r be T. Then

$(\exists x)(r \to p(x))$ is T

ie. there is x_0 s.t. $r \to p(x_0)$ is T
ie. there is x_0 s.t. $p(x_0)$ is T, as r is T
ie. $(\exists x)\, p(x)$ is T
ie. $r \to (\exists x)\, p(x)$ is T

Again, $r \to (\exists x)\, p(x)$ is T
ie. $(\exists x)\, p(x)$ is T (as r is T)
ie. there is an x_0 s.t. $p(x_0)$ is T
ie. there is an x_0 s.t. $r \to p(x_0)$ is T (as r is T)
ie. ~~there is x_0 s.t.~~ $(\exists x)(r \to p(x))$ is T

(g) If r is T then both sides have truth value T. So, let r be F. Then

If $(\forall x)\, p(x) \to r$ is T
ie. $(\forall x)\, p(x)$ is F (as r is F)
ie. there is x_0 s.t. $p(x_0)$ is F
ie. there is x_0 s.t. $p(x_0) \to r$ is T
ie. $(\exists x)(p(x) \to r)$ is T.

Again, $(\forall x)\, p(x) \to r$ is F
ie. $(\forall x)\, p(x)$ is T (as r is F)
ie. for all x, $p(x)$ is T
ie. for all x, $p(x) \to r$ is F
ie. there is no x s.t. $p(x) \to r$ is T
ie. $(\exists x)(p(x) \to r)$ is F

2nd part If r is T, then both sides have truth value T. So, let r be F. Then

$(\exists x)\, p(x) \to r$ is T

ie. $(\exists x)\, p(x)$ is F (as r is F)
ie. for all x, $p(x)$ is F
ie. for all x, $p(x) \to r$ is T
ie. $(\forall x)(p(x) \to r)$ is T.

Again, $(\exists x)\, p(x) \to r$ is F
ie. $(\exists x)\, p(x)$ is T (as r is F)
ie. there is x_0 s.t. $p(x_0)$ is T
ie. there is x_0 s.t. $p(x_0) \to r$ is F
ie. $(\forall x)(p(x) \to r)$ is F