Hints to practice problems: Graph Theory

Q.1 ~~Take~~ Any two vertices u, v of the induced subgraph are adjacent.

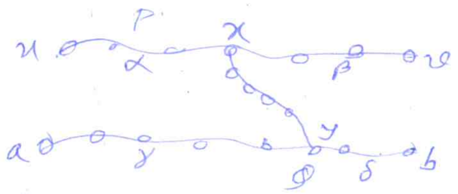
Q.2 If $V(G) = X \cup Y$, then $V(H) = (V(H) \cap X) \cup (V(H) \cap Y)$.

Q.4. $\delta \leq d(v) \leq \Delta$ and $\sum d(v) = 2m$.

Q.3 Let, two longest path P and Q have length l, and have no common ver

Consider lengths $\alpha, \beta, \gamma, \delta$.

α = length of (u,x)-section of P.



If $\alpha \geq \beta$, $\gamma \geq \delta$, then $(u-x-y-a)$-path have length greater than l.

Q.5 X = all k-tuples with even no. of 1's

Y = all k-tuples with odd no. of 1's.

Q.6 Consider corresponding graph ~~on~~ G on 6 vertices, and ~~its~~ G^c.

For $v \in V(G)$, without loss of generality, $d(v) \geq 3$, say $x, y, z \in N(v)$.

Either none of x, y, z are adjacent to each other or at leas ~~on~~ two of them are ~~adja~~ adjacent.

Q.7 Consider a longest path P, and its one end vertex u. $N(u) \subseteq V(P)$.

Q.8 $\Rightarrow$ For $e = uv$, take a ~~path~~ (u,v)-path in $G - e$.

$\Leftarrow$ If any (x,y)-path includes e, then traverse through the other portion of the cycle that contains e.

Q.9 ~~Take~~ Give order to the vertices of P and Q.

Take a vertex u common to P & Q s.t. its next vertex is not common.

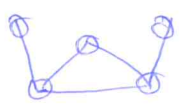
~~Take common vertex v next to u.~~

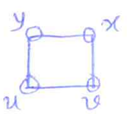
Q.10 If u & v are in the same component, take w in a another component.

Q.11 ~~For $e = uv \in E(G)$, $d_{L(G)}(e) =$~~

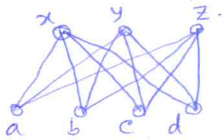
For each v, there are $\binom{d(v)}{2}$ edges in $L(G)$

Q.12 For an x_i, consider the unit circle centered at x_i. At three other x_j's can be on this circle. So, in the corresponding graph, $d(x_i) \leq 6 \Rightarrow \sum d(x_i) \leq 6n$.

Q.13  , $|E(G)| + |E(G^c)| = \frac{n(n-1)}{2}$ and $|E(G)| = |E(G^c)|$

Q.14 For 4-cycle 

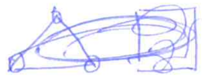
, u and x are not adjacent, v and y are not adjacew and $d(u) = d(v) = 2k$.

Q.15 

. Take, e.g., $a \mapsto b$, $b \mapsto a$ and others are fixed.

Q.16. ~~Consider Consider a path P of longest le~~ Consider a longest path and its end vertex u. Now $N(u) \subseteq V(P)$ and $\delta \geq$

Q.17 Pigeon-Hole principle on the vertex degrees. Take 

Q.18.  Take $G = C_3 \cup C_4$.

Q.19. (b) If $n_1, n_2, \cdots, n_\omega$ are the no. of vertices of the components, then $n_1 - 1, n_2 - 1, \cdots, n_\omega - 1$ are the no. of edges.

Q.20 Take a path P_1 joining two odd degree vertices in G.
Take P_2 similarly in $G - E(P_1)$, and so on.
Then $G - (E(P_1) \cup E(P_2) \cup \cdots \cup E(P_k))$ is empty graph.

Q.21. If $d(u) = \Delta$ and $N(u) = \{v_1, \cdots v_\Delta\}$, then no two v_i are connected via a path in $T - u$, so each of them belong to different trees in $T - u$.

Q.22. The corresponding graph ~~has~~ is a tree by its ~~def~~ definition.

Q.23 If maximal acyclic subgraph T is not spanning, say $u \notin V(T)$, then $T \cup \{u\}$ is also an acyclic subgraph.

Q.24 Take a spanning tree T. For each $e \in E(G) \setminus E(T)$, $T + e$ have a unique cycle.

Q.25. Tree cannot have odd cycle.
Fix vertex u and take $X = \{x \mid d(u,x) \text{ is even}\}$, $Y = V(T) \setminus X$
If $|X| > |Y|$ and X does not have a pendent vertex, then

$$\sum_{x \in X} d(x) > \sum_{y \in Y} d(y),$$ a contradiction.

Q.26 Extend P as far as possible. The last vertex must be pendent.

Q.27 True: There must be a component with fewer edges than vertices.

Q.28. only K_2. A tree must have a pendent vertex.

Q.29. If degree of end vertex ≥ 2, then either a cycle or a longer path produces.

Q.31. $(n-1) + (n-1) = \frac{n(n-1)}{2} \Rightarrow n = 1, 4.$

Q.30 $e = uv$ is the only (u,v)-path. So $T-e$ is disconnected. Further, an edge can make connected between two components only.

Q.32. The unique (x,y)-path P in T, along with xy, makes a cycle.

Q.33. $V(K_{2n}) = \{v_1, \dots v_{2n}\}$. For $i = 1, 2, \dots, 2n-1$, define,

$M_i = \{v_i v_{2n}\} \cup \{v_{i-j} v_{i+j} \mid j = 1, 2, \dots, n-1\}$, $i-j$, $i+j$ are read modulo $2n$

Take, $V(K_{n,n}) = \{u_1, \dots u_n\} \cup \{v_1, \dots v_n\}$. For $i = 0, 1, \dots, n-1$,

$M_i = \{u_j v_{j+i} : j = 1, \dots n\}$, $i+j$ is read modulo n.

Q.34 Symmetric difference of two perfect matching is empty.

Q.35 Color all the odd-cycles in 3-colors. Now delete the odd cycles and color the remaining in 2-colors.

Q.36 A $(\chi(G) + \chi(H) - 1)$-coloring of $G+H$ is not proper.

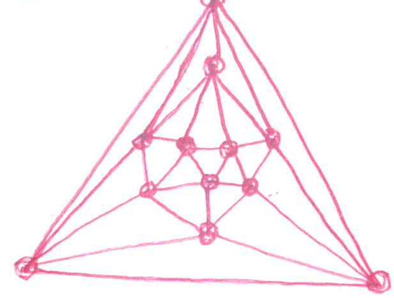
Q.37 For a proper-coloring, for each pair ~~of colo~~ i, j of colors, $\exists$ an edge $e = uv$, where u has color i and v has color j. Else, re-color the vertices of color i with color j.

Q.38. Let $n = \chi(G)$, $k_1, k_2 \dots, k_n$ be the size of ~~proper~~ color-classes of a proper n-coloring. Then

$2m \le k_1(n-k_1) + \dots + k_n(n-k_n) = n^2 - \sum k_i^2 \longrightarrow (1)$

The minimum of $g = \sum_{i=1}^{n} x_i^2$ subject to $\sum_{i=1}^{n} x_i = n$ exists at $x_i = \frac{n}{n}$. So, $g \ge \frac{n^2}{n}$. Then

$(1) \Rightarrow 2m \le n^2 - \frac{n^2}{n}$.



Icosahedron

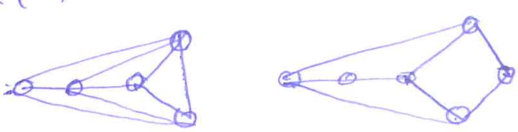
Q.39 Ref. Q.37.

Q.40. Ref. Q. Q 38.

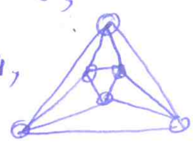
Q.41. $K_{1,n}$.

Q.42 $\varepsilon(G) \le 3n-6 = 27. \Rightarrow \varepsilon(G^c) \ge 55-27 = 28$

Q.43.



Q.44 For $n \ge 6$, $m \ge 3n$ and so $3n \le m \le 3n-6$, a contradiction.

For $n = 0, 1, 2, 3, 4, 5$, the graphs are K_1, K_2, K_3, K_4,  and the Icosahedron graph.

Q.45 $n-m+f=2$, $2m=3n$, $5f \le 2m$.

Q.46 Adding edges, if needed, convert it to maximal planar. $\Rightarrow \delta \geq 3$.
Let n_i = no. of vertices of degree i. Then

$$2(3n-6) = 2m = 3n_3+4n_4+5n_5+6n_6+\cdots \geq 3(n_3+n_4+n_5)+6\sum_{i\geq 6} n_i$$

Add $3(n_3+n_4+n_5)$ and use $n=\sum_{i\geq 3} n_i$. ~~in the left side~~.

Q.47 Easy.

Q.48. $n \leq 11$ and $m \leq 3n-6 \Rightarrow \sum d(v) \leq 6n-12 < 5n$, as $n \leq 11$. Now use pigeon-hole principle.

Q.49. For bi-partite planar graph $2m \leq 4n-8$, not possible if $\delta \geq 4$

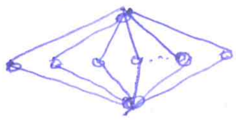
Q.50. If two unit-distance cross, ~~the corresponding four points form a rhombus,~~ ~~and so~~ then one of the four distances among these four points have distance less than 1. Thus, the corresponding graph is planar.

Q.51 Ref. Q.46.

Q.52. Use Q.51 and induction on n.

Q.53 only $K_{1,s}$ and $K_{2,s}$. Else, $K_{r,s}$ will contain $K_{3,3}$.

Q.54.



Q.55. Introduce a new vertex v and join it to ~~all old~~ odd degree vertices. The new graph is Eulerian. Remove v from the Euler tour.

Q.56. Introduce new vertices $v_1, v_2, \dots, v_k$, add each to two odd degree vertices

Q.57. Introduce a new vertex x, add it to all old vertices; and use Ore' Thm.

Q.58. Since each degree 3, degree of each vertex in the closed trail is 2.

Q.59. Use Ore' Thm.

Q.60. If $u_1 u_2 u_3 \dots u_n u_1$ is a Hamitonian cycle and $u_i \in X$, then $u_{i+1} \in Y$.