DEPARTMENT OF MATHEMATICS, IIT Guwahati
MA221: Discrete Mathematics, July - November 2019
Practice Problems: Graph Theory

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1. Prove that every vertex induced subgraph of a complete graph is complete.

2. Prove that every subgraph of a bipartite graph is bipartite.

3. Show that any two longest paths in a connected graph have a vertex in common.

4. Let G be a simple graph on n vertices and m edges. Show that $\delta \leq \frac{2m}{n} \leq \Delta$.

5. The k-cube is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. Show that the k-cube has 2^k vertices, $k2^{k-1}$ edges and is bipartite.

6. Show that in any group of 6 people, there are either 3 mutual friends or 3 mutual strangers.

7. Show that if G is a simple graph and $\delta \geq 2$, then G has a cycle of length at least $\delta + 1$.

8. Prove that a connected graph G remains connected after removing an edge e from G if and only if e is in some cycle of G.

9. If the intersection of two paths in a graph is a disconnected subgraph, then show that their union will contain at least one cycle.

10. Show that a graph G is disconnected then G^c is connected.

    Definition: Let G be a graph. The line graph of G, denoted $L(G)$, is the graph whose vertex set is $E(G)$ and for $e, f \in E(G)$, $ef \in E(L(G))$ iff e and f are adjacent in G.

11. Show that the number of vertices and edges of the line graph $L(G)$ of a simple graph G are $|E(G)|$ and $\sum\limits_{v \in V(G)} \binom{d_G(v)}{2}$, respectively.

12. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of points in the plane such that the Euclidean distance between any two points is at least one. Show that there are at most $3n$ pairs of points at distance exactly one.

13. A graph G is called *self-complementary* if G and G^C are isomorphic. Check if the graphs P_4, P_5 and C_5 are self-complementary. Find a self-complementary graph on 5 vertices (other than C_5). Show that if a graph G on n vertices is self-complementary then $n \equiv 0, 1 (\bmod 4)$.

14. Let G be a k-regular graph in which the length of a shortest cycle is 4. Prove that G has at least $2k$ vertices.

15. Exhibit, with verification, a non-trivial automorphism of the graph $K_{3,4}$.

16. Show that if a graph G is simple and $\delta \geq k$, then G has a path of length k.

17. Prove that every simple graph with at least two vertices has two vertices of equal degree. Is the conclusion true for non-simple graphs?

18. Prove or disprove: If every vertex of a simple graph G has degree 2 then G is a cycle.

19. An acyclic graph is called a forest. Show that

    (a) each component of a forest is a tree;

    (b) G is a forest with n vertices, m edges and ω components iff $m = n - \omega$.

20. Show that if G is a forest with exactly $2k$ vertices of odd degree then there are k edge-disjoint paths $P_1, P_2 \ldots, P_k$ in G such that $E(G) = E(P_1) \cup E(P_2) \cup \ldots \cup E(P_k)$.

21. Show that if G is a tree with $\Delta \geq k$, then G has at least k vertices of degree one.

22. A saturated hydrocarbon is a molecule C_mH_n in which every carbon atom has four bonds, every hydrogen atom has one bond, and no sequence of bonds forms a cycle. Show that, for every positive integer m, C_mH_n can exist only if $n = 2m + 2$.

23. Prove that a maximal acyclic subgraph of a connected graph G is a spanning tree of G.

24. Prove that every n vertex graph with m edges has at least $m - n + 1$ cycles.

25. Let T be a tree. Show that T is bipartite and find a bipartition of T. Further, prove that T has a pendent vertex in its larger partite set.

26. Let T be a tree. Prove that for every path P starting at a vertex v, there exists a pendent vertex w such that P is a section of the (v, w)-path.

27. Prove or disprove: Every graph with fewer edges than vertices has a tree component.

28. Find all regular trees.

29. Show that the end vertices of a longest path in a non-trivial tree have degree one.

30. Let e be an edge of a tree T. Show that $T - e$ has exactly two components.

31. Find all trees that are self-complementary graphs.

32. Let T be a tree with $x, y \in V(T)$ but $xy \notin E(T)$. Show that $T + xy$ has a unique cycle.

33. Find the different perfect matching in K_{2n} and $K_{n,n}$.

34. Show that a tree can have at most one perfect matching.

35. Show that if any two odd cycles of G have a vertex in common then $\chi(G) \leq 5$.

    Definition: If G and H are two graphs with $V(G) \cap V(H) = \emptyset$, then $G + H$ is the graph defined by $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(G)\}$.

36. Show that $\chi(G + H) = \chi(G) + \chi(H)$.

37. Prove that the number of edges of a graph G is at least $\frac{\chi(G)(\chi(G)-1)}{2}$.

38. Let G be a graph on n vertices and m edges. Show that $\chi(G) \geq \frac{n^2}{n^2-2m}$.

39. Let G be a simple graph with $\chi(G) = 51$. Prove that G has at least 1275 edges.

40. Let G be an r-regular graph on n vertices. Prove that $\chi(G) \geq \frac{n}{n-r}$.

41. Prove that the difference $\Delta(G) - \chi(G)$ may be arbitrarily large.

42. Show that if G is a simple planar graph with $|V(G)| \geq 11$, then G^c is not planar.

43. Construct plane embedding of the graphs (a) $K_5 - e$, (b) $K_{3,3} - e$ such that the edges are presented by straight line segments.

44. For which r there exists a planar r-regular graph?

45. Prove that a planar connected cubic graph whose every face has at least 5 vertices is of order at least 20.

46. Prove that a planar graph with n vertices ($n \geq 4$) has at least four vertices with degree five or less.

47. Prove or disprove:

    (a) Every subgraph of a planar graph is planar.

    (b) Every subgraph of a non-planar graph is non-planar.

48. Every planar graph with fewer than 12 vertices has a vertex of degree at most 4.

49. Prove or disprove: There is no simple bipartite planar graph G with $\delta(G) \geq 4$.

50. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of $n \geq 3$ points in the plane such that the distance between any two points is at least one. Show that there are at most $3n - 6$ pairs of points at distance exactly one.

51. Prove that every planar graph has a vertex of degree at most 5.

52. Show that every planar graph is 6-colorable.

53. Determine all r, s such that $K_{r,s}$ is planar.

54. Construct plane imbedding of the graph $K_{2,n}$ such that the edges are presented by straight line segments.

55. Let G be a simple connected graph with exactly two odd degree vertices. Show that G has an Euler trail.

56. Let G be a simple connected graph with exactly $2k$ ($k > 0$) odd degree vertices. Show that $E(G)$ can be partitioned in k open trails.

57. Let G be a simple graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of non-adjacent vertices u, v, then show that G has a Hamilon path.

58. Show that if a cubic graph has a spanning closed trail, then the graph is Hamilonian.

59. Show that a simple k-regular graph on $2k - 1$ vertices is Hamilonian.

60. If G is a bi-partitte Hamiltonian graph with bipartition $V(G) = X \cup Y$, then show that $|X| = |Y|$.

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Optional:

1. In a group of people, each girl knows exactly k boys ($k > 0$) and each boy knows exactly k girls. Find the number of girls and the number of boys in the group.

2. Let G be a simple graph with n vertices and m edges such that $m > \frac{(n-1)(n-2)}{2}$. Prove that G is connected. For each $n > 1$, find a disconnected simple graph G with n vertices and m edges such that $m = \frac{(n-1)(n-2)}{2}$.

3. Show that a simple connected graphs with n vertices, all of degree 2, is isomorphic to C_n.

4. Let $n \in \mathbb{N}$ be such that $n \equiv 0, 1 (\mathrm{mod}\ 4)$. Draw a self-complementary graph on n vertices.

5. Show that every simple graph on n vertices is isomorphic to a subgraph of K_n.

6. In a class of nine student, each student sends new year greetings cards to three others. Determine whether it is possible that each student receives cards from the same students to whom he or she sent cards.

7. Show that every tree with exactly two vertices of degree one is a path.

8. Let T be a tree. Prove that the vertices of T all have odd degree iff for all $e \in E(T)$, both the components of $T - e$ have odd order.

9. Prove that every connected subgraph of a tree is the induced one.

10. Show that every k-cube has a perfect matching, where $k \geq 2$.

11. Show that K_{2n} and $K_{n,n}$ can be expressed as a disjoint union of perfect matchings.

12. Let the graph G have a perfect matching and $S \subseteq V(G)$ with $|S| = k$. Show that the number of odd component of $G - S$ is at most k.

13. Prove that in some minimal proper coloring of a graph G, the neighborhood $N(v)$ of every vertex v is colored in the same color iff G is bipartite.

14. Prove that for any two different colors i, j of any minimal proper coloring of the vertices of a graph G there exists an edge uv such that u has color i and v has color j.

15. Prove that in every proper coloring of the line graph $L(G)$ of a graph G, every vertex is adjacent to at most two vertices of the same color.

16. Prove that $\chi(G) \leq l + 1$, where l is the length of a longest path of G.

17. Exhibit a graph G with a vertex v so that $\chi(G - v) < \chi(G)$ and $\chi(\overline{G} - v) < \chi(\overline{G})$.

18. Is it true that if $\deg(v) < \chi(G) - 1$ for some vertex v of a graph G then $\chi(G) = \chi(G - v)$?

19. Show that $\chi(G) \leq 1 + \max\ \delta(H)$, where the maximum is taken over all induced subgraphs H of the graph G.

20. Prove or disprove: If G is a connected graph, then $\chi(G) \leq 1 + a(G)$, where $a(G)$ is the average of the vertex degrees in G.

21. Let G be a connected planar graph with n vertices and m edges. Let $k(\geq 3)$ be the length of the shortest cycle. Prove that $m \leq \frac{k(n-2)}{k-2}$.

22. Find a simple planar graph G with $|V(G)| = 8$ such that G^c is also planar.

23. Prove that $m \leq 2(p+q-2)$ for a planar bipartite m-edge graph with the sizes $p, q(\geq 2)$ of the partite sets.

24. Let G be a planar connected cubic graph. Prove that $\sum\limits_{i \geq 3}(6-i)f_i = 12$, where f_i is the number of faces of G each of which is bounded i edges. [Here an edge that belongs to a single face is counted twice in f_i.]

    Also prove that G has at least one face with at most 5 edges.

25. Show that the Petersen graph is non-planar.

26. To how many faces of a plane graph may a vertex of degree $d \geq 1$ belong?

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