A set is an object [illegible]

$a \in A$ $\quad a \notin A$

Principle of Extensionality :- for any 2 sets A & B if for every object x, $x \in A$ iff $x \in B$ then $A = B$.

Empty set :- A set with no members is the empty set $\emptyset$

$\boxed{\emptyset \neq \{\emptyset\}}$

If every member of A is also a member of B then $A \subseteq B$ subset and $B \supseteq A$ superset

If $A \subseteq B$ and $A \neq B$ then $A \subset B$, $A \subsetneq B$ (A is a proper subset of B)

$A \cup B$: the set of all of objects in A or in B

$A \cap B$: ______ and in B

$U - A$: ______ not in A, but in U

* A stands for a predicate $a(x)$ (a first order predicate with one variable)
  $x \in A$

$A = \{x \mid a(x)\}$

$A \cup B = \{x \mid a(x) \vee b(x)\}$

$\cup \leftrightarrow \vee$ (union corresponds to [illegible])
$\cap \leftrightarrow \wedge$ (intersection corresponds to [illegible])
$U^c \leftrightarrow \neg$ (complement corresponds to [illegible])

$\cup, \cap, U^c$] define Boolean algebra

Set Theory

→ A set is a collection of objects called its members

→ A set is an object.

$a \in A$ $\quad a \notin A$.

Principle of Extensionality :- for any 2 sets A & B if for every object x, $x \in A$ iff $x \in B$

then $A = B$.

Empty set :- A set with no members exists

— empty set $\emptyset$

$\boxed{\emptyset \neq \{\emptyset\}}$

★ If every member of A is also a member of B then

$A \subseteq B$ subset.

and $B \supseteq A$ superset.

if $A \neq B$ and $A \subseteq B$ then $A \subset B$, $A \subsetneq B$

(A is a proper subset of B).

$A \cup B$: the set of all of objects in A or in B

$A \cap B$: ———————————————— and in B

$U - A$: ———————————————— not in A, but in U

★ A stands for a predicate $\alpha(x)$

$x \in A$

↳ First order predicate with one variable.

$A \equiv \{x \mid \alpha(x)\}$

$A \cup B$: $\{x \mid \alpha_A(x) \vee \alpha_B(x)\}$

\# $\cup : \vee$ (union corresponds to ~~and~~ OR)

$\cap : \wedge$ (Intersection corresponds to AND)

$U^- : \neg$ (complement corresponds to NOT)

$\cup, \cap, U^-$] Define Boolean algebra

Properties:

1. $A \cap U = A$ $A \cup \phi = A$ (Identity)
   $x \wedge 1 = x$ $x \vee 0 = x$

2. $A \cup U = U$ $A \cap \phi = \phi$ (Domination)

3. $A \cup A = A$ $A \cap A = A$ (Idempotent)

4. $U - (U - A) = A$ (Double negation law)

5. $A \cup B = B \cup A$ (commutative)
   $A \cap B = B \cap A$

6. $A \cup (B \cup C) = (A \cup B) \cup C$ (associative)
   $A \cap (B \cap C) = (A \cap B) \cap C$

7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive)
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8. $U - (A \cap B) = (U - A) \cup (U - B)$ (Demorgan's law)
   $U - (A \cup B) = (U - A) \cap (U - B)$

9. $A \cup (A \cap B) = A$ (Absorption)
   $A \cap (A \cup B) = A$

10. $A \cup (U - A) = U$ (Negation)
    $A \cap (U - A) = \phi$

Power set:
The set of all subset of A

2^A $P(A)$

is the power set of A.

$|A| = n$
then $|2^A| = 2^n$

$\cup A = \{x \mid \exists b (b \in A \wedge x \in b)\}$
↳ union of all members of members of A

e.g. – $\cup \{\underbrace{\{2, 3\}}_{\text{set}}, \underbrace{4}_{\text{not a set}}, \{5\}\} = \{2, 3, 5\}$

$\cap A = \{x \mid \forall b \in A (x \in b)\}$
↳ every member of A has to be set. x should belong to every set of A.

Scanned by CamScanner

Embedding of Natural no.s in Set Theory

→ successor operator

for set a $\quad a^+ = a \cup \{a\}$

$a = \{2,3,4\}$

$a^+ = \{\{2,3,4\}, 2,3,4\}$

→ Inductive:

A set A is inductive

if $\phi \in A$

and $\forall a \, (a \in A \longrightarrow a^+ \in A)$

| A is closed under the successor operator

$\omega = \bigcap \{A \mid A \text{ is inductive}\}$

→ $x \in \omega$ iff x belongs to every inductive set.

∴ ω is a subset of every inductive set.

★ ω is inductive

$\phi \in \omega$

$a \in \omega \longrightarrow a$ belongs to all Is,

$a^+ \in \omega \longleftarrow a^+$ belongs to all Is,

★ $x \subseteq \omega \ \wedge \ x$ is inductive $\longrightarrow x = \omega$

OR

No proper subset of ω is inductive.

★ ω is the smallest Inductive set.

How ω looks like?

$0 = \phi \in \omega$

$1 = 0^+ = \phi^+ \in \omega \qquad \phi^+ = \phi \cup \{\phi\} = \{\phi\}$

$2 = 1^+ = 0^{++} = \phi^{++} \in \omega \qquad \phi^{++} = \phi^+ \cup \{\phi^+\} = \{\phi\} \cup \{\{\phi\}\}$

$= \{\phi, \{\phi\}\}$

⋮

$0 = \phi$

$1 = \{\phi\} = \{0\}$

$2 = \{\phi, \{\phi\}\} = \{\phi, 1\}$.

$3 = \{0, 1, \{0,1\}\} = \{0,1,2\}$

$\vdots$

$n = \{0, 1, 2, \ldots, n-1\}$

$0 \in 1$
$1 \in 2$
$2 \in 3$
$3 \in 4$
$\vdots$

$0 \subset 1$
$1 \subset 2$
$2 \subset 3$
$3 \subset 4$
$\vdots$

For any two distinct objects x, y

$\{x, y\}$ — unordered pair

Ordered pair.

$(x, y) = \{x, \{x,y\}\}$

A Relation is a set of OPs

dom $R = \{x \mid \exists y \; (x,y) \in R\}$

ran $R = \{y \mid \exists x \; (x,y) \in R\}$

field of R

fld $R = (\text{dom} R) \cup (\text{ran } R)$

$A \times B$: The set of all OPs s.t. the first is from A, the 2^{nd} is from B.

★ R from A to B is a subset of $A \times B$

★ A binary relation R on A is a subset of $A \times A$.

Extending to n elements:

$(a, b, c) = ((a,b), c)$

$(a_1, \ldots, a_n) = ((a_1, \ldots a_{n-1}), a_n)$

n-ary relation:

$(A_1 \times \ldots \times A_n) = (A_1 \times \ldots A_{n-1}) \times A_n$

If $A_1, A_2 \ldots A_n = A$ then the n-ary relations can be written as:- $A^2, A^3, \ldots$

A ~~function~~ f ~~is a relation~~ s.t.
$(\forall x \cancel{\in dom\, f}) \cancel{(\exists! y\ (x,y) \in f)}$

1) For

A Function F

is a relation
s.t. $\forall x \in dom F\ (\exists! y\ xFy)$

2) For

$F(x)$: the unique y st. xFy.

3) F

F maps A into B, if $dom F = A$ and $ran F \subseteq B$

4)

F maps A onto B, if $dom F = A$ and $ran F = B$

★ A set R is single Rooted
if for each $y \in ran R$.
$\exists! x$ s.t. $(x,y) \in R$.

5)

$2 \to 0$, $2 \to 1$

$\{(2,0), (2,1), e\}$ $\cancel{(3,0)}$
is single Rooted.

★ A function that is 1-rooted is 1-1

6)

Bijection = one-one, onto
Injection = one-one, into
Surjection = onto

bijection iff injection and surjection

• A
• A

$F^{-1} = \{(u,v) \mid (v,u) \in F\}$

• A

$F \circ G = \{(u,v) \mid (u,t) \in G$ and $(t,v) \in F$ for some t $\}$

the Restriction of set F to set A

$F \restriction A = \{(u,v) \mid (u,v) \in F \wedge u \in A\}$

$F[\![A]\!]$ image of A under F

$= ran\, F \restriction A = \{v \mid \exists u \in A\ (u,v) \in F\}$

1) For a set F;

$$\operatorname{dom} F^{-1} = \operatorname{ran} F$$

$$\operatorname{ran} F^{-1} = \operatorname{dom} F$$

2) For a relation F,

$$(F^{-1})^{-1} = F$$

3) For a set F, F^{-1} is a function iff F is 1-rooted.

4) if F is a 1-1 function,
   if $x \in \operatorname{dom} F$, then $F^{-1}(F(x)) = x$
   if $y \in \operatorname{ran} F$, then $F(F^{-1}(y)) = y$

5) if F and G are functions, then $F \circ G$ is a function

[There $\circ$ - composition is an operator on set of functions. (it is closed)]

its domain :-

$$\{x \in \operatorname{dom} G \mid G(x) \in \operatorname{dom} F\}$$

for $x \in \operatorname{dom} F \circ G$

$$(F \circ G)(x) = F(G(x))$$

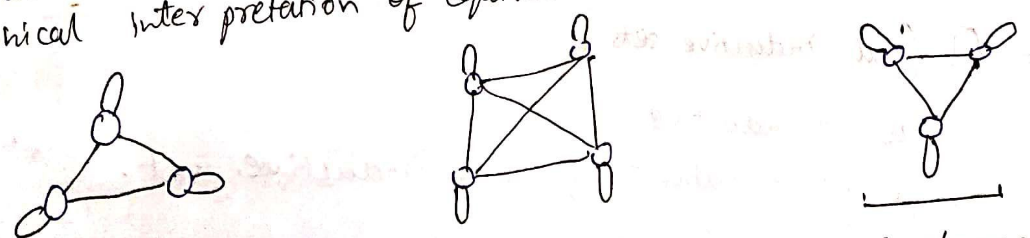
6) For any two sets F and G

$$(F \circ G)^{-1} = G^{-1} \circ F^{-1}$$

- A binary Rel^n ~~R~~ R on A is <u>reflexive</u> if $\forall x \in A\ (xRx)$
- A binary Rel^n R on A is <u>symmetric</u> if
  $\forall x, y \in A\ (xRy \rightarrow yRx)$
- A binary Rel^n R on A is <u>transitive</u> if
  $\forall x, y, z \in A\ (xRy \wedge yRz \longrightarrow xRz)$

R is an equivalence relation if R satisfies all the above three properties.

Graphical interpretation of equivalence rel^n :-



equivalence class

clique are being formed.

If there is a path b/w x and y then there is a direct edge b/w them also.

equivalence classes are being formed.

e.g. $xRy \equiv x$ is $y \pmod 5$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19

$\{0, 1, 2, 3, 4\}$

$[0]$ $\equiv \bmod 5$

$x \equiv x \pmod 5$ ~~(symmetric)~~ (reflexive)

if $x \equiv y$ then $y \equiv x \pmod 5$ (symmetric)

if $x \equiv y$, $y \equiv z \rightarrow x \equiv z \pmod 5$ (transitive)

Trichotomy is a binary Relation R

s.t. xRy , $x = y$, yRx

exactly one holds $\forall\ x, y \in A$

e.g. $<$ ✓ $\leq$ ✗

If a Relation is transitive and trichotomy : strict linear order

Antisymmetry property :

$\forall xy\ (xRy \wedge yRx \rightarrow x = y) \rightarrow$ e.g. $\leq$

reflexive , antisymmetric , transitive : linear order
total order

Successor
$a^+ = a \cup \{a\}$

inductive set : contains ϕ , closed under $^+$

$w = \bigcap \{$all inductive sets$\}$.

- w is inductive
- w is a subset of every inductive set.

ω is the "smallest" inductive set.

$\omega = \mathbb{N}$

$\phi \quad 0 \qquad n = \{0, 1, 2, \cdots, n-1\}$

$\phi^{+} \quad 1 \qquad \Rightarrow \quad 0 \in 1 \in 2 \in 3 \ldots$

$1^{+} = \phi^{++} \quad , 2 \qquad 0 \subset 1 \subset 2 \subset 3 \ldots$

successor : $+1$

$m + n = m^{+(n)}$

$m \times n = +$ operator applied on m itself $(n-1)$ times.

$m \times 1 = m \qquad m \times 2 = m^{+(m)}$

Integers in Set Theory

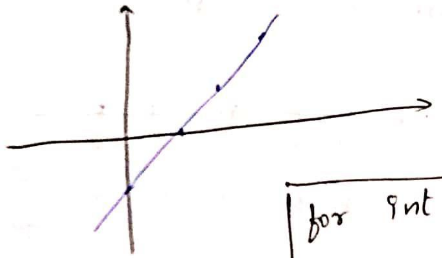
$\sim$ on $\mathbb{N} \times \mathbb{N}$

$(m, n) \sim (p, q)$ iff $m + q = n + p$

equivalence rel^n

$(4,3)\ (3,2)\ (2,1) \quad : 1$

$(1,2) \quad : -1$



$x - y = 1$

$y = x - 1$: represent integer 1

$$\boxed{\text{for int } i :- \quad y = x - i}$$

Integer addition :-

$+_{\mathbb{Z}} \quad : \quad \underset{m-n}{[(m,n)]_{\sim}} \; +_{\mathbb{Z}} \; \underset{p-q}{[(p,q)]_{\sim}} \longrightarrow \underset{m+p-n-q}{[(m+p, n+q)]}$

$\times_{\mathbb{Z}} \quad : \quad [(m,n)]_{\sim} \; \times_{\mathbb{Z}} \; [(p,q)]_{\sim} \longrightarrow [(mp +_{\mathbb{N}} nq, pn +_{\mathbb{N}} mq)]_{\sim}$

$\star \; \mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$

$+_{\mathbb{Z}}$

$[(m,n)]_{\sim} +_{\mathbb{Z}} [(p,q)]_{\sim} = [(m+p, n+q)]_{\sim}$

$[(m,n)]_{\sim} \times_{\mathbb{Z}} [(p,q)]_{\sim} = [(mp +_{\mathbb{N}} nq, pn +_{\mathbb{N}} mq)]_{\sim}$

Natural Number multiplication

$\times_{\mathbb{Z}}$ is commutative, associative

$\times_{\mathbb{Z}}$ distributive ove $+_{\mathbb{Z}}$

$[(2,1)]_{\sim} = 1_{\mathbb{Z}}$ is id of $\times_{\mathbb{Z}}$

$[(1,1)]_{\sim} = 0_{\mathbb{Z}} \neq 1_{\mathbb{Z}}$

$(\mathbb{Z}, +_{\mathbb{Z}}, \times_{\mathbb{Z}}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}})$ Form an integral domain.

★ $[(m,n)]_{\sim} <_{\mathbb{Z}} [(p,q)]_{\sim} \longrightarrow$ Trichotomy and Transitive $\Rightarrow$ strict linear order

when $m +_{N} q \in n +_{N} q'$

$m - n < p - q$

$m +_{N} q <_{N} p +_{N} n$

$\Rightarrow m +_{N} q \in p +_{N} n$ (w.r.t natural numbers)

★ $\mathbb{Z}$ is countable

(S is countable iff
$\exists$ a 1-1 mapping from S into $\mathbb{N}$)

One-One and onto mapping

<u>Rational Number</u>

$q = \frac{p}{q}$ where $p, r \in \mathbb{Z}$ $r \neq 0$

$\mathbb{Z}' = \mathbb{Z} - \{0_{\mathbb{Z}}\}$

$(a,b) \bowtie (c,d)$ $a, c \in \mathbb{Z}$

bow tie $b, d \in \mathbb{Z}'$

stands for stands for

$\frac{a}{b} = \frac{c}{d}$

iff $a \times_{\mathbb{Z}} d = b \times_{\mathbb{Z}} c$

$(a,b) \bowtie (a,b)$

$(a,b) \bowtie (c,d) \rightarrow (c,d) \bowtie (a,b)$

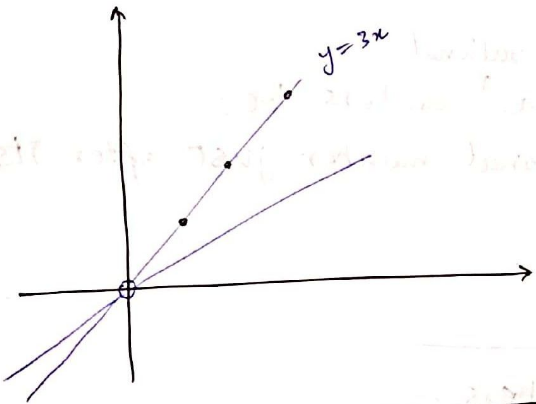
If $(a,b) \bowtie (c,d)$ and $(c,d) \bowtie (e,f)$ then $(a,b) \bowtie (e,f)$

$$(\mathbb{Z} \times \mathbb{Z}')/_{\bowtie} = \mathbb{Q}$$

$$\frac{1}{3} \equiv \left\{ \begin{array}{l} (1,3),(2,6),(3,9),\ldots \\ (-1,-3),(-2,-6),\ldots \end{array} \right\}$$

$$3 \equiv \left\{ \begin{array}{l} (3,1)(6,2)(9,3),\ldots \\ (-3,-1),(-6,-2),(-9,-3),\ldots \end{array} \right\}$$

In general $\quad p/q \equiv \left\{ \begin{array}{l} (p,q),(2p,2q),(3p,3q),\ldots \\ (-p,-q),(-2p,-2q),\ldots \end{array} \right\}$



$\frac{1}{3} \equiv \{(x,y)$ that satisfies $y = 3x\}$

$+_{\mathbb{Q}} : [(a,b)]_{\bowtie} +_{\mathbb{Q}} [(c,d)]_{\bowtie} = [(ad+bc, bd)]_{\bowtie}$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Integer addition and multiplication

$\times_{\mathbb{Q}} : [(a,b)]_{\bowtie} \times_{\mathbb{Q}} [(c,d)]_{\bowtie} = [(ac,bd)]_{\bowtie}$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

$(\mathbb{Q}, +_{\mathbb{Q}}, \times_{\mathbb{Q}}, 0_{\mathbb{Q}}, 1_{\mathbb{Q}})$ field

---

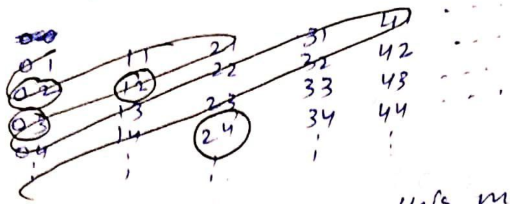
$[(a,b)]_{\bowtie} <_{\mathbb{Q}} [(c,d)]_{\bowtie}$ iff $ad <_{\mathbb{Z}} cb$

$\left(\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc\right)$ Trichotomy and Transitive

linear Order

$\mathbb{Q}$ is countable

Consider $\mathbb{N} \times \mathbb{N}^+$



(when we count in this manner, we are going through every rational number in 1st quadrant.

but we need a 1-1 mapping, i.e., a number should be counted only once.

→ First assign numbers ~~as~~ in the pattern as above then remove the numbers/mapping for equal numbers except when encountered first.

Now we need to count −ve rational numbers too;
introduce the −ve rational number just after its positive equivalent.

Hence $\mathbb{Q}$ is countable

There are irrational numbers.

$\sqrt{2}$ is irrational

$\sqrt{2} = \dfrac{p}{q}$ $\quad$ $\gcd(p, q) = 1$

$p = \sqrt{2}q$

$p^2 = 2q^2$ $\quad \Rightarrow$ p^2 is an even square

$\Rightarrow p^2 = (2k)^2 = 4k^2$

thus $4k^2 = 2q^2$

$2k^2 = q^2 \Rightarrow q^2$ is an even number

(q is also even)

thus p and q both are even

$\Rightarrow$ they have 2 as a common factor

contradiction to the fact that

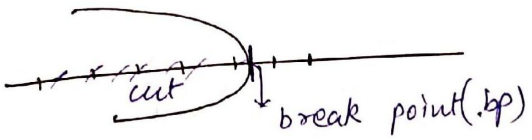
Real Numbers Done using Dedekind cuts

Dedekind Cut :- is a subset of $\mathbb{Q}$ such that

① $\phi \neq x \neq \mathbb{Q}$

② x is closed downwards
$\forall\, q, r \in \mathbb{Q}\,\big((q \in x) \wedge (r < q) \rightarrow (r \in x)\big)$.

③ x has no largest member belonging to $\mathbb{Q}$



$\mathfrak{z} \equiv$ the set of all rational number less than $\mathfrak{z}$

Every dedekind cut is a Real number.
set of all dedekind cut is equal to set of all ~~rationa~~ Real Numbers

$\mathbb{R}$ is uncountable.

$(0,1)$

.1 $0.\, a_1 \; a_2 \; a_3 \; a_4 \cdots$

3 $0.\, b_1 \; b_2 \; b_3 \; b_4 \cdots$

4 $0.\, c_1 \; c_2 \; c_3 \; c_4 \cdots$

construct a number as :-

$0.\, \underbrace{(a_1+1) \bmod 10}_{\text{1st digit is not equal to the 1st rational}} \; \overbrace{0}^{2^{nd}} \; \underbrace{(b_3+1) \bmod 10}_{\text{3rd digit is not equal to the 3rd number}}$

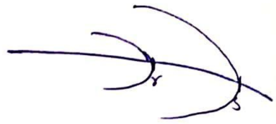
In this way, we can find that the number that we constructed has never been counted.
thus $(0,1)$ is uncountable

$\Rightarrow \mathbb{R}$ is uncountable

Dedekind cut (continued)

if the break point (bp) is rational ;
then the complement of the cut includes the b.p

★ the set of all Dedekind cuts $\equiv \mathbb{R}$

$r <_R s$ iff. $r \subset s$ 

$\Rightarrow$ strict linear order

$x +_R y = \{ q+r \mid q \in x$ and $r \in y \}$.

$x \times_R y = \{ qr \mid 0 \le q \in x \wedge 0 \le r \in y \} \cup 0_R$

- if x and y are non negative real

If both x and y are negative real, then

$x \times_R y = |x| \times_R |y|$

$x \times_R y = -|x| \times_R |y|$

★ $(\mathbb{R}, +_R, \times_R, 0_R, 1_R)$ is a field.

Naive set Theory

- By Georg Cantor

$\{x \mid \alpha(x)\}$ → this is the base for defining sets.

→ this has many problems and paradoxes:-
One of them is — (Berry's Paradox)

$A = \{ x \mid x \in \mathbb{N}$ and x can not be defined in at most 100 characters $\}$

$n =$ the smallest member/natural no. not defined ≤ 100 cha b.p.

Russell's Paradox

$S = \{ x \mid x \notin x \}$

Abnormal: self membership. (Paradox)

$S \in S$ iff $S \notin S$

* Russels Type theory

Zermelo Fraenkel Axioms of set Theory :-

① Extensionality Axiom

$\forall x \forall y \, (\forall z \, (z \in x \leftrightarrow z \in y) \rightarrow x = y \}$

② empty set Axiom

$\exists x \, \forall y (y \notin x)$

③ Pairing Axiom

$\forall x \forall y \, \exists z \, \forall u \, (u \in z \leftrightarrow u = x \vee u = y)$

given any two elements, we can pair them into a set (an unordered pair)

④ Power set Axiom

$\forall x \, \exists y \, \forall z \, (z \in y \longleftrightarrow z \subseteq x)$

⑤ Subset Axiom Schema

for any formula α with free variable $t_1 \dots t_k$ u and x the following is an axiom

$\forall t_1 \dots \forall t_k \; \forall u \; \exists x \; \forall y \left(y \in x \longleftrightarrow y \in u \wedge \alpha(t_1, \dots t_k, u, x) \right)$

= for every set u, there is a subset x s.t. $y \in x$ iff $y \in u \wedge \alpha(t_1, \dots t_k, u, x)$

* e.g. $\{ x \mid x \notin x \}$ ⇒ the above axiom says that you cannot talk about self membership in the complement in unrestricted manner.

char

$\forall t \, \forall u \, \exists x \, \exists y \, (y \in x \leftrightarrow y \in t \wedge y \in u)$

there exist intersection t and u

⑥ Union Axiom:- there exist y that is made of members of member of x

⑦ Infinity Axiom :- there exist an inductive set

⑧ Regularity Axiom :- Every non-empty set x has a member y s.t. $x \cap y = \phi$

\# For any formula $\psi(x,y)$ in which z is not free

if every member of x has a nominee then there is a set made up of these nominees.

⑨ Axiom of choice : For any relation R, there is a function $F \subseteq R$ s.t. dom F = dom R

Relation R on A :-

is reflexive if $\forall x \in A (xRx)$

antisymmetric if $\forall x,y \in A \; (xRy \wedge yRx \rightarrow x=y)$

transitive if $\forall \; x,y,z \in A(xRy \wedge yRz \rightarrow xRz)$

then R is partial order on A

A is a partially ordered set (poset) w.r.t. R

e.g. $(N, \leq)$

For any

$x, y \in A$

$\forall t \forall u \exists x \exists y (y \in x \leftrightarrow y \in t \wedge y \in u)$

there exist intersection t and u

(6) Union Axiom:- there exist y that is made of members of member of x

(7) Infinity Axiom :- there exist an inductive set

(8) Regularity Axiom :- Every non-empty set x has a member y s.t. $x \cap y = \phi$

For any formula $\mathcal{V}(x,y)$ in which z is not free

if every member of x has a nominee then there is a set made up of these nominees.

(9) Axiom of choice : For any relation R, there is a function $F \subseteq R$ s.t. dom F = dom R

Relation R on A :-

is reflexive if $\forall x \in A (xRx)$

antisymmetric if $\forall x,y \in A (xRy \wedge yRx \rightarrow x=y)$

transitive if $\forall x,y,z \in A (xRy \wedge yRz \rightarrow xRz)$

then R is partial order on A

A is a partially ordered set (poset) w.r.t. R

e.g. $(\mathbb{N}, \leq)$

For any $x,y \in A$

$xRy \vee yRx$

Not necessarily

* $a \precsim b$ a precedes b

or b succeeds a.

$b \succsim a$ $\succsim \; = \; \precsim^{-1}$] dual

Trichotomy : $\underline{aRb}$ xor $\underline{bRa}$ xor $\underline{a=b}$

* irreflexive, transitive :- quasi order
* quasi order with trichotomy :- strict partial order

$a \prec b \equiv a \precsim b$ and $a \neq b$

* e.g. $\subseteq$ on 2^A $\longrightarrow$ partial order

$A = \{a, b\}$

$2^A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$\{a\} \nsubseteq \{b\}$
$\nsupseteq$

e.g. Divides on $\mathbb{N}^+$ $\longrightarrow$ partial order

$x \mid x$

$x \mid y \wedge y \mid x \rightarrow x = y$

$x \mid y \wedge y \mid z \rightarrow x \mid z$

Hasse diagram

S is a poset
$a, b \in S$.

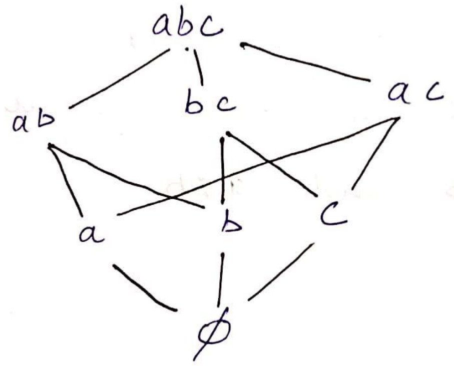
a is an immediate predecessor of b $(a \ll b)$

if $a < b$ and $\neg \exists c \, (a < c < b)$

b is an immediate successor of a $(b \gg a)$

Directed Graph

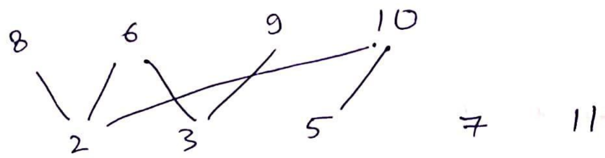
Take a vertex for each $a \in S$
if $a \ll b$ then construct $(a,b) \in E$
$(a \rightarrow b)$



we can draw all order pairs using transitivity

we do not draw a direct edge from a to abc ; but there is an intermediate path via ab or ac.

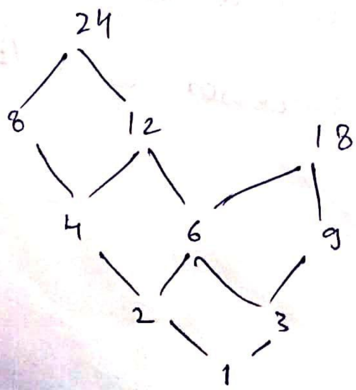
e.g. Divide Relation :-



since $\mathbb{N}$ has infinite ~~diagram~~ elements, Hasse diagram seems useless.

But when there are finite elements, Hasse diagram seems good to implement.

e.g. Restriction of $|$ onto $\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$



a and b are comparable in R if

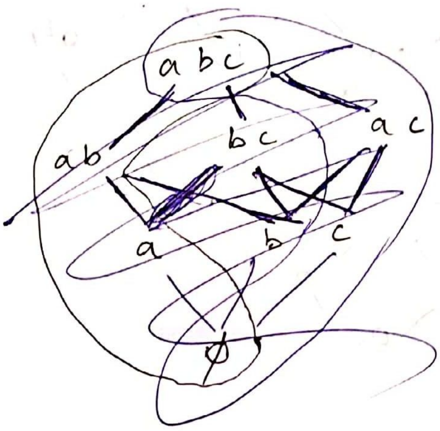
aRb or bRa. (and can have a path)

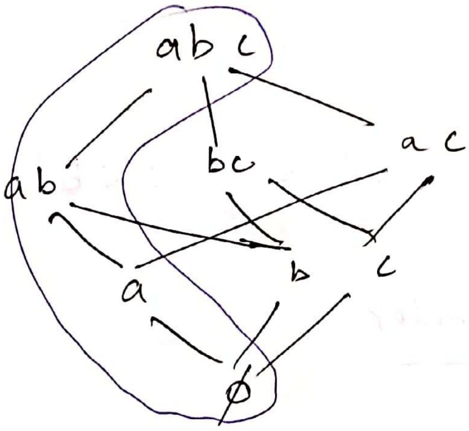
12 and 18 are incomparable $12 \parallel_R 18$

★ Total order / Linear order has every pair comparable while this need not be a case in proper partial order.

★ every subset of linear order is a linear order.

But —
A linear order taken from a partial order through restriction is a chain of the POSET.





Anti- Chain: of a POSET is a subset of which no two elements are comparable

Ordering of ordered tuples

1. Product order

S, T are both linearly ordered sets.

$S \times T$ $\quad (a, b) \leq (a', b')$ iff

$a \leq a'$

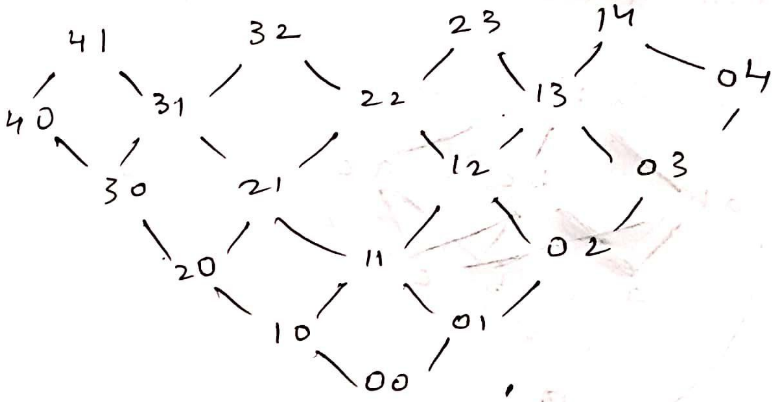
and $b \leq b'$

41 32 23 14 → Not comparable

$4 \geq 3$ ×

but $1 \leq 2$

But



2. Lexicographic order

$(a, b) < (a', b')$

If $a < a'$ or $a = a'$ and $b < b'$

20
⋮
1n
⋮
12
11
10
⋮
0n
⋮
03
02
01
00

Linear order

String : A finite sequence of symbols

alphabet $A = \{a, b, c\}$

$$A^* = \left\{ \begin{array}{lll} \epsilon \longrightarrow & \text{String of zero length} & \\ a, b, c \longrightarrow & \text{unit length.} & (3 \times 1) \\ \begin{array}{l} aa, ab, ac \\ ba, bb, bc \\ ca, cb, cc \end{array} \longrightarrow & \text{2 unit length} & (3 \times 3) \\ \longrightarrow & \text{3 unit length} & (3 \times 3 \times 3) \end{array} \right\}$$

$A^* \longrightarrow$ kleene closure of A

Lexicographic order :

\# $u = a u'$
$v = b v'$ $\quad a, b \in A$

$u, v, u', v' \in A^*$

$u < v$ iff $a < b$ or $a = b$ and $u' < v'$

$\epsilon < w$ for any non empty w

e.g.
cat < catch
at < atch
t < tch
ϵ < ch

Short lex. order

$u < v$ if $|u| < v$
or $|u| = |v|$
and u precedes v lexicographically

Finite poset S

Draw the Hasse diagram.

It is a Directed Acyclic Graph.

Topographically Sorted

... then $f(a) < f(b)$

Strings :- A finite sequence of symbols

alphabet $A = \{a, b, c\}$

$$A^* = \left\{ \begin{array}{lll} \epsilon \longrightarrow \text{String of zero length} & & \\ a, b, c \longrightarrow \text{unit length.} & & (3 \times 1) \\ \begin{matrix} aa, ab, ac \\ ba, bb, bc \\ ca, cb, cc \end{matrix} \longrightarrow \text{2 unit length} & & (3 \times 3) \\ \quad - \text{3 unit length} & & (3 \times 3 \times 3) \end{array} \right\}$$

$A^* \longrightarrow$ kleene closure of A

Lexicographic order :

\# $u = a u'$
$v = b v'$ $\quad a, b \in A$

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$u < v$ iff $a < b$ or $a = b$ and $u' < v'$

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Finite poset S

Draw the Hasse diagram

It is a Directed Acyclic Graph.

Topographically sorted

$a < b$ then $f(a) < f(b)$

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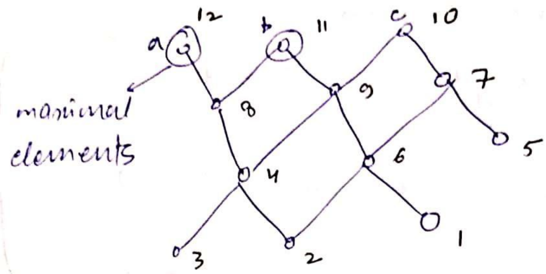
Maximal Element

$a \in S$ is maximal if ~~...~~ $\neg\exists x(x \geq a)$



a b c
↓
all 3 are maximal, we can pick any of these three

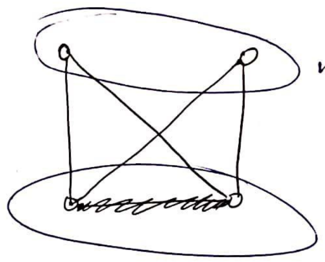
Pick a maximal → name is n
then rest can be numbered.

Basis :- $n = 1$

→ Consistent enumeration of partially ordered set.

Minimal element :-

if $\neg\exists x \in A \ (x \leq A)$



Chain :- A subset of the poset that is in Linear Order

Antichain :- A subset of the poset s.t. no 2 members of it are comparable.

if a poset has a chain of length n
then it has n disjoint anti-chain

→ Remove all the maximal length(then they will form an antichain)
Now the path with n nodes will loose 1 node
⇒ By induction they will have $n-1$ antichains
⇒ we have a total of $n-1+1$ antichain

If a finite poset has $nm+1$ elements atleast then either it has a chain of length $\geq n$ or it has an antichain of ~~length~~ size $\geq m$.

Proof ⇒ Assume that it is wrong → suppose that it has the maximum chain length $= n$

then we can have a total of $n-1$ antichains.

If each antichain has elements $\leq m$

then total elements $\leq m(n)$

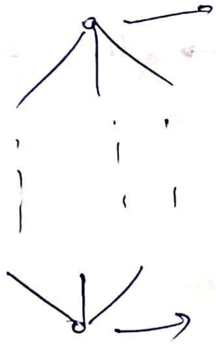
$\leq nm$

But total number of elements $= nm+1$

thus we have a contradiction.

The First element of a poset is an element a s.t. $\forall x \in S \ (a \leq s)$

If the number of minimal elements $= 1$, then that elements is the first element, similarly of Last element.



Last element

First element

e.g. integer → $m \in \mathbb{N}^+$ $\quad m = m_1 + m_2 + \dots + m_k \quad m_1, \dots, m_k \in \mathbb{N}^+$

suppose $m = m_1 + m_2 + \dots + m_k$

assume $m_1 \geq m_2 \geq \dots \geq m_k$

then we say the sequence $m_1, m_2, \dots m_k$ is a partition of m

$P_1 \leq P_2$ → If P_1 is a refinement of P_2

(Take P_2, split some of ~~integers~~ integers)

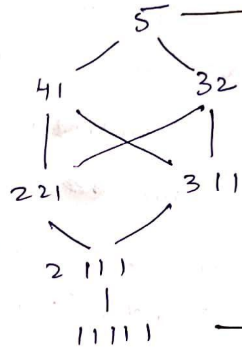
e.g. $m = 5$

$\Rightarrow 5 = 4 + 1$ 41

$5 = 2 + 2 + 1$ 221

Then say :- $221 \leq 41$ ★

Partial order:



5 → ~~Last~~ element

11111 — First element

M is an upper bound of $A \subseteq S$,

if $\forall x \in A \ (M \geq x)$

$\{41, 32\} \longrightarrow 5$ ↓ upper bound

m is a lower bound of $A \subseteq S$

if $\forall x \in A \ (m \leq x)$

$\underbrace{\{221, 311, 2111, 11111\}}_{LB} \longrightarrow \{41, 32\} \xrightarrow[UB]{} 5$

$\{221, 311\} \xrightarrow{UB} 41, 32, 5$

$\xrightarrow{LB} 2111, 11111$

Supremum (A) :-

LUB / Least Upper Bound (A)

is an UB of A that precedes every UB of A

$\{41, 32\} \longrightarrow 5$

$\{221, 311\} \xrightarrow{LB} 41, 32, 5$

$\xrightarrow{LUB}$ Doesn't exist (41 and 32 are not comparable.)

Infimum (A) :-

GLB / Greatest Lower Bound (A)

is a LB of A that succeeds every LB of A

$\{221, 311\} \xrightarrow{LB} 2111, 11111$

$\xrightarrow{GLB} 2111$

$\{41, 32\} \xrightarrow{LB} 221, 311, 2111, 11111$

$\xrightarrow{GLB}$ Doesn't exist (221 and 311 are not comparable)

e.g. $(N^+, |)$

the set of UB (a,b) = the set of all common multiples

LUB (a,b) = LCM(a,b)

GLB(a,b) = GCD(a,b)

the set of LB(a,b) = the set of all common divisors of a and b.

Isomorphic Posets

$f: X \longrightarrow Y$ 1-1

$a \leq a' \longrightarrow f(a) \leq f(a')$

if $a \| a'$ then $f(a) \| f(a')$

An ordered set is well ordered if every subset has a first element.

A well ordered set is a LO

Every subset of a WOS is a WOS.

If X is WO they $Y \simeq X$ (isomorphic)

then Y is also WO.

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★ All WOS, of size n are isomorphic to $\{1, 2, \dots n\}$.

★ If S is WO, then every $a \in S$ other than the last of S has an imm. successor

Proof :- take any element; say a
then take the set of all elements which succeeds a;
then this will be a subset of S,
and since S is WO, this is also WO
Hence it will have a minimal
that will be the immediate successor
of a.

Lattices

A lattice is poset in which every pair a, b has a GLB and a LUB.

→ Set of Natural numbers under division.

L is a non-empty set, closed under meet ($\wedge$) and join ($\vee$)

L is a ~~is~~ lattice if

(L1) $a \wedge b = b \wedge a$ } commutative
$a \vee b = b \vee a$

(L2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ } associative.
$a \vee (b \vee c) = (a \vee b) \vee c$

(L3) $a \wedge (a \vee b) = a$ } Absorptive
$a \vee (a \wedge b) = a$

Swapping of $\wedge$ with $\vee$ in any theorem will also give us a theorem.

Dual Theorems.

$a \wedge a = a \wedge (a \vee (a \wedge b))$ L3 (b)

$= a$ L3(a)

$a \vee a = a$ (Idempotent Law)

* $(L, \wedge, \vee)$

$a \leq b$ iff $a \wedge b = a$

Theorem :-

If L is a lattice $a \wedge b = a$ iff $a \vee b = b$

and $\leq$ is a partial order (PO)

(reflexive, antisymmetric and transitive)

Proof :-

$\Rightarrow$ $b = b \vee (b \wedge a)$ L 3b

$= b \vee (a \wedge b)$

$= b \vee a$

$= a \vee b$

$\boxed{b = a \vee b}$ H.P.

$\Leftarrow$ If $a \vee b = b$

to show - $a \wedge b = a$

$a = a \wedge (a \vee b)$ L3a

$\boxed{a = a \wedge b}$ H.P.

thus $a \leq b$ iff $a \wedge b = a$

$a \vee b = b$

To Prove now :- $(L, \leq)$ is a partial order

① $\forall a \in L (a \wedge a = a) \Rightarrow \forall a \in L (a \preceq a)$

thus $\preceq$ is reflexive on L

② $a \preceq b$ and $b \preceq a$

$\Rightarrow a \wedge b = a$ and $b \wedge a = b = a \wedge b$

$\Rightarrow a \wedge b = a$ as well as $a \wedge b = b$

$\Rightarrow a = b$

thus $\preceq$ is antisymmetric on L

③ $a \preceq b$ and $b \preceq c$

$\Rightarrow a \wedge b = a$ and $b \wedge c = b$

Now: $a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$

$\Rightarrow a \preceq c$

thus $\preceq$ is transitive on L

Hence $(L, \preceq)$ is a partial order

<u>Claim</u>: LUB and GLB exist for any $a, b \in L$

Consider :- (a, b)

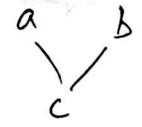
if $a \preceq b$ then $GLB(a,b) = a$

and $LUB(a,b) = b$

if $a \succeq b$ then $GLB(a,b) = b$

and $LUB(a,b) = a$

If $a \parallel b$ then let $c = a \wedge b \neq a, b$



$a \wedge c = a \wedge (a \wedge b)$

$= (a \wedge a) \wedge b$

$= a \wedge b$

$= c \quad \Rightarrow \quad c \preceq a$

the defⁿ :- $a \preceq b$ iff $a \wedge b = a$ provides the required ordering such that meet comes down in Hasse diagram

By similar argument :-

$b \wedge c = c \Rightarrow c \preceq b$

New c is a common lower bound of both a and b,
Now we want to show that c is infact the greatest lower bound.

→ say some y is also a LB of a & b
$y \leq a$ and $y \leq b$

$c \wedge y = (a \wedge b) \wedge y = a \wedge (b \wedge y) = a \wedge y = y.$

$\Rightarrow \quad y \leq c$

thus every other LB should precede c;
Hence $c \rightarrow GLB$.

Hence :- If $a \parallel b$ then $GLB = a \wedge b$

Hence we can say that for any (a, b) in L

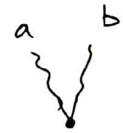
$GLB : a \wedge b$
$LUB : a \vee b$

Let P be a poset with GLB and LUB defined for all $a, b \in P$

$a \wedge b = GLB(a, b)$ set is closed under $\wedge$ and $\vee$.
$a \vee b = LUB(a, b)$

Now we need to show that the 3 properties of lattice are defined.

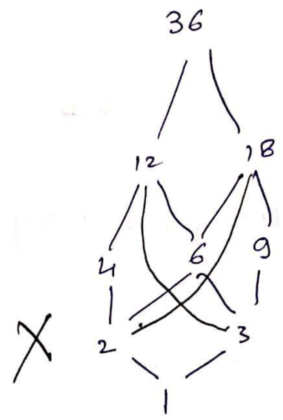
Commutativity :- $a \wedge b = b \wedge a$
$GLB(a, b) = GLB(b, a)$



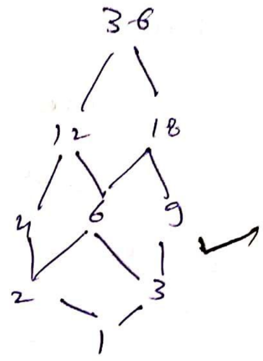
associativity :-

Try This !! absorptive :-

D_{36} :- 1, 2, 3, 4, 6, 9, 12, 18, 36



join $\longrightarrow$ only immediate members



★ $(D_{36}, |)$ is a sublattice of $(N^+, |)$

<u>Isomorphic</u> : $f(a \wedge b) = f(a) \wedge f(b)$

<u>Bounded lattice</u>:-

L has a LB 0, $\forall\, x \in L\ (0 \leq x)$

L has an UB $1, I$; $\forall\, x \in L\ (x \leq 1)$

If L has a LB and an UB, it is bounded.

What if L has finite elements?

Can we say that it has a LB and UB.

Consider :-

$$a_i \wedge (\underbrace{a_1 \wedge a_2 \ldots \wedge a_n}_{b}) = b$$

$\Rightarrow b \leq a_i$ for every i

thus b is a LB

Similar argument for UB.

<u>Distributive Lattice</u>

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

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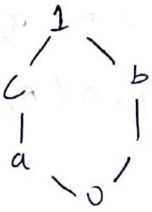
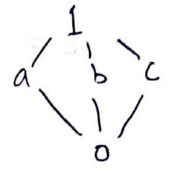
Non distributive lattice :-





If a Lattice has a part isomorphic to any of these two, then the lattice is Non - distributive lattice

⇓

$a \vee (b \wedge c) = a \vee 0$
$= a$

$(a \vee b) \wedge (a \vee c)$
$= 1 \wedge c = c$

both are not equal

Similarly for the 2nd :-

$a \vee (b \wedge c) = a \vee 0 = a$

But $(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$

Complements

L is bounded LB - 0
UB - 1

for $a \in L$, the complement $\bar{a}$ of a

$x = \bar{a}$ iff $a \vee x = 1$ and $x \wedge a = 0$

Theorem :- If L is a bounded distributive lattice, then Complements are unique (if they exist)

and L is a complemented lattice, if L is bounded and every element has a complement

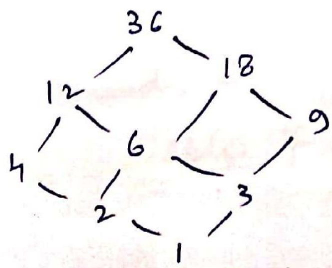
* $(L, \wedge, \vee)$ is distributive.

if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$\wedge$ GCD
$\vee$ LCM.
(Divides)



Q→ $(a,b,c) \to GCD(a,b,c) = g$

$a = Ag$ $\quad b = \hat{B}g = BGg$ $\quad c = \hat{C}g = CGg$

$GCD(\hat{B},\hat{C}) = G$

$GCD(B,C) = 1$

$GCD(A,B,C) = 1$

• $a \underline{LCM} (b \underline{GCD} c) = Ag \underline{LCM} (BGg \underline{GCD} CGg)$
$= Ag \underline{LCM} Gg$
$= g \times LCM(A,G) = gAG$

• $a \underline{LCM} b = Ag \underline{LCM} BGg = ABGg$

• $a \underline{LCM} c = ACGg = AGg$

• $a \underline{GCD} (b \underline{LCM} c) = a \underline{GCD} (BGg \underline{LCM} CGg)$
$= Ag \underline{GCD} (BCGg)$
$= g \times GCD(A, BC)$

$a \underline{GCD} b = Ag \ GCD \ BGg$
$= g \times GCD(A, BG)$
$= g \times GCD(A,B)$
$= g \times GCD(A,C)$

For any lattice L, 0 is LB, 1 is UB

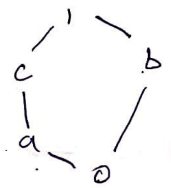
for $a \in L$ $\underbrace{\bar{a} = x}_{\text{complement of } x}$ iff $a \vee x = 1$ and $a \wedge x = 0$

complements may not exist.

e.g.

without complements.

complements may not be unique,



Theorem :- Let L be a distributive lattice, complements are unique if they exist.

Proof :- x and y are two complements of a same

$a \vee x = a \vee y = 1$

$a \wedge x = a \wedge y = 0$

Now $x = x \vee 0 = x \vee (a \wedge y)$

$= (x \vee a) \wedge (x \vee y)$

$= 1 \wedge (x \vee y)$

$= x \vee y$

similarly $y = y \vee x$; we have $x \vee y = y \vee x$

By commutativity of $\vee$

$\Rightarrow \boxed{x = y}$ H.P.

---

lattice L, Lower Bound 0

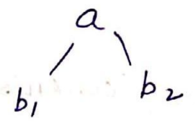
$a \in L$ is join irreducible, if $a = x \vee y \rightarrow a = x \vee a = y$

(if p is a prime
if $p = x \times y \longrightarrow p = x \vee p = y$)

0 is Join irreducible

$\Longrightarrow 0 = x \vee y \longrightarrow x = 0 \vee y = 0$

if a has 2 immediate predecessors -

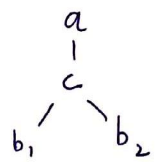


$a = b_1 \vee b_2$

$a \neq b_1, \ a \neq b_2$

} a is not join irreducible

if a has only 1 immediate predecessor



if $a = x \vee y$

$\rightarrow x = a \vee y = a$

} a is join irreducible

★ a has ≥ 2 immediate predecessors $\Longrightarrow$ Not join irreducible

a has < 2 Immediate predecessors $\Rightarrow$ join irreducible

---

★ Immediate successors of 0 are called atoms
(In any divisibility lattice, the atoms are prime numbers)

$\longrightarrow$ ★ All prime powers are also join irreducible.

Finite lattice L

$a \in L$ is not J.I.

$a = b_1 \vee b_2$ where $b_1 \neq a$ and $b_2 \neq a$

Further if both b_1 and b_2 are not J.I.

then $a = c_1 \vee c_2 \vee c_3 \vee c_4$

we can continue like this until we find an exp. of the form $\rightarrow$

$$a = d_1 \vee d_2 \vee \ldots \vee d_n$$

where d_i is J.I.

$\longrightarrow$ every non JI element can be expressed as a join of JI elements.

If $\left.\begin{array}{l} d_i \leq d_j \\ d_i \vee d_j = d_j \end{array}\right\}$ d_i is reduntant.

Let us get rid of all such d_i's

then we will have

$a =$ join of irreduntant J.I. elements

(In fact there is a unique such factorisation)

Let L be a distributive lattice, Then every $a \in L$ can be written uniquely (except for order) as the join of irreduntant JI elements.

Proof :-

$a = b_1 \vee \dots \vee b_r = c_1 \vee \dots \vee c_s$

b's and c's are irredundant and JI

$b_i \leq (b_1 \vee \dots \vee b_r) = a = c_1 \vee \dots \vee c_s$

$b_i = b_i \wedge a = b_i \wedge (c_1 \vee \dots \vee c_s)$

$= \underbrace{(b_i \wedge c_1) \vee (b_i \wedge c_2) \dots (b_i \wedge c_s)}$

$b_i = b_i \wedge c_j$ — b_i is J.I. and it is written as join of different elements.

$\Rightarrow b_i \leq c_j$

Similarly we can argue that $c_j \leq b_k$

Then by transitivity we have $b_i \leq c_j \leq b_k$

$\Rightarrow b_i \leq b_k$

but we removed all reduncencies

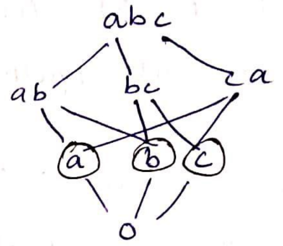
$\Rightarrow$ $i = k$

thus $\begin{array}{l} b_i \leq c_j \\ c_j \leq b_i \end{array} \Rightarrow b_i = c_j$

Hence b_i is identical to some c_j

Hence for every b_i there is a unique c_j (which is equal to b_i itself).

Theorem:- Let L be a complemented lattice with unique complements.
Then the J.I. elements of L other than 0 are its atoms.



Proof:- Say a is JI $a \neq 0$ and a is not an atom.
Then a has a unique predecessor b $b \neq 0$
then $\bar{b}$ exists. (unique ; property of L)
$b \neq 0 \longrightarrow \bar{b} \neq 1$

① If $a \leq \bar{b}$ then $b \leq a \leq \bar{b}$

$\longrightarrow b \leq \bar{b}$

$\longrightarrow b \vee \bar{b} = \bar{b} = 1$ (contradiction)

thus $a \not\leq \bar{b}$

$\Rightarrow$ Now $a \wedge \bar{b}$ must strictly preced a.

If b is the imm. pred. of $a \longrightarrow a \wedge \bar{b}$ precedes b

$a \wedge \bar{b} \leq \inf(b, \bar{b}) = b \wedge \bar{b} = 0$

$\Rightarrow a \wedge \bar{b} = 0$ $\quad$ $a \vee b = a$

$a \vee \bar{b} = (a \vee b) \vee \bar{b} = a \vee (b \vee \bar{b}) = a \vee 1 = 1$

$\Rightarrow \bar{a} = \bar{b}$

$\Rightarrow a = b$

(but b is immediate predecessor of a thus contradiction)

Discrete Numeric Functions

$f : N \to Z$

$\langle a_0, a_1, \ldots \rangle$

e.g. $1, 1, 1, 1, \ldots$

$1, 2, 3, 4, \ldots$

$1, -2, 3, -4, 5, -6, \ldots$

e.g. $a_r = 7r^3 + 1, \quad r \geq 0$

$$b_r = \begin{cases} 2r & 0 \leq r \leq 11 \\ 3^r - 1 & r > 11 \end{cases}$$

$\bar{a} = \langle a_0, a_1, \ldots \rangle$
$\bar{b} = \langle b_0, b_1, \ldots \rangle$ notations.

① $c = a + b = \langle a_0 + b_0, a_1 + b_1, \ldots \rangle$

② $a \times b = \langle a_0 b_0, a_1 b_1, \ldots \rangle$

③ $|a| = \langle |a_0|, |a_1|, \ldots \rangle$

*④ $S^i a = \langle \underbrace{0, 0, \ldots 0}_{i}, a_0, a_1, \ldots \rangle$ → i shifted (Right)

⑤ $S^{-i} a = \langle a_i, a_{i+1}, \ldots a_{i+2}, \ldots \rangle$ → (Left shifted)

⑥ forward difference

$\Delta a = \langle a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots \rangle$

backward diff.

$\nabla a = \langle 0, a_1 - a_0, a_2 - a_1, \ldots \rangle$

$\boxed{\nabla a = S^1(\Delta a)}$

⑦

$c = a * b$ convolution

$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \ldots a_{r-2} b_2 + a_{r-1} b_1 + a_r b_0$

$$= \sum_{i=0}^{r} a_i b_{r-i}$$

* DNF $a = \langle a_0, a_1, a_2, \ldots \rangle$

$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$

$A(z) \longrightarrow$ the generating f^n of a

$a = \langle 1, 1, 1, 1, \ldots \rangle$

then $A(z) = 1 + z + z^2 + \ldots z^j \ldots$

$= \frac{1}{1-z}$

$a = \langle 1, 3, 3^2, 3^3, \ldots \rangle$ $\quad 1 + 3z + 3^2 z^2 \ldots = \frac{1}{1-3z}$

$a = \langle 7.1, 7.3, 7.3^2, \ldots \rangle$ $\quad \frac{7}{1-3z}$

$a_r = 3^{r+2}, r \geq 0 \quad \Rightarrow \quad \frac{9}{1-3z}$

if $c = a + b$

$C(z) = A(z) + B(z)$

e.g. $a_r = 3^r + 2^r \; ; r \geq 0$

$A(z) = \frac{1}{1-3z} + \frac{1}{1-2z}$

$A(z) = \frac{2 + 3z - 6z^2}{1-2z}$

$= 3z + \frac{2}{1-2z}$

$\downarrow$

$+ 3z \quad 2(1 + 2z + 2^2 z^2 + \ldots + 2^r z^r + \ldots)$

$\Rightarrow$

$2 + 7z + 2^3 z^2 + 2^4 z^3 + \ldots$

$$a_r = \begin{cases} 2 & r = 0 \\ 7 & r = 1 \\ 2^{r+1} & r > 1 \end{cases}$$