Modes of convergence

- Pointwise
- Convergence in rth mean,
- Convergence in probab.
- Convergence in distribution & Useful

Almost sure convergence :

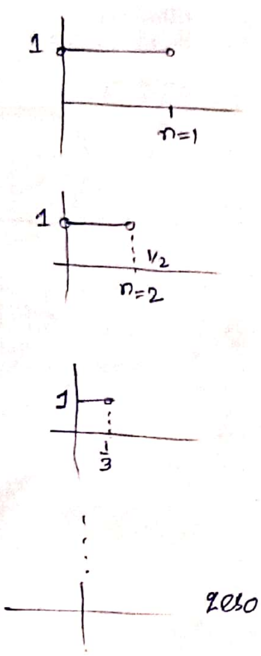
$$P(\omega : X_n(\omega) \to X(\omega)) = 1$$

eg

P → uniform

Probab. of any interval = length of interval

$$X_n = 1_{[0, \frac{1}{n}]}$$



$\boxed{\text{w.p. } 1 = \text{with probability } 1}$

or

$\boxed{\text{a.s.} = \text{almost sure}}$

zero random var.

$x \in [0,1]$

$X_n(0) = 1 \not\to X(0) = 0$

$\omega > 0$

$X_n(\omega) = 0$ for all n sufficiently large

$\to X(\omega) = 0$

Point-wise convergence

Convergence in rth mean

$$E|X_n - X|^r = E|X_n|^r = EX_n^r = \frac{1}{n} \to 0$$

$r=1$ $\quad E|X_n - X| = \int_0^1$

$$E|X+Y|^r \leq 2^r(E|X|^r + |Y|^r)$$

Convergence $\boxed{\text{in}}$ probability $\qquad$ converges with probab 1

$$P(|X_n - X| > \epsilon) \to 0$$

As you take n larger & larger, probab. of problematic set is smaller & smaller

Eg 34 : $\quad X_n(\omega) = n1_{[0,\frac{1}{n}]}$

$$X_n = n \quad \text{w.p. } \frac{1}{n}$$
$$= 0 \quad \text{w.p. } 1-\frac{1}{n}$$

$\epsilon > 0$

$$P(|X_n| > \epsilon) = P(X_n > \epsilon) = \frac{1}{n} \to 0$$

$X_n \to 0$ in 1^{st} mean

$E|X_n| = 1 \not\to 0$

Counter eg.

* If converges in rth mean then converges in probab.

Not the other way round. (above is a counter eg of converse.)

Convergence in distribution

Ex 35 : $P(X_n = \frac{1}{n}) = 1$

by intuition, it should go to zero rv. ie.
rv that takes 0 with probab 1.

$$F_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

$X = c$ w.p. 1

$$F_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

$x < 0$

$F_n(x) \to F(0)$

$x > 0$

$0 = F_n(0) \not\to F(0) = 1$

converge at all x but might not converge where $F(x)$ is not continuous.

Intuition:

25th Sept

Def: Convergence in distribution:

$$F_n(x) \to F(x)$$

$P(X_n = \frac{1}{n}) = 1$

$P(X_1 = 1) = 1$

$P(X_2 = \frac{1}{2}) = 1$

$P(X_3 = \frac{1}{3}) = 1$

$\vdots$

$P(\lim_{i \to \infty} X_i = \frac{1}{i}) = P(X_n \approx 0) = 1$ ← zero r.v.

~~$F(x)$~~

Theorem: $\{X_n\}$, $\{Y_n\}$ be 2 sequences of rv

$X_n \to X$ in distribution

i.i.d = independent & identically distributed

Example of $X_n + Y_n \not\to X + Y$.

take X & Y are i.i.d. & X

let $X_n = X$

$Y_n = X$ in distribution.

& ~~X~~.

$X_n \to X$ in distr.

$Y_n \to Y$ in distribution

$X_n + Y_n = 2X$

$\downarrow$

$X + Y \stackrel{?}{=} 2X$

eg: X, Y be $N(0,1)$

$2N(0,1) = N(0,4)$ var 4.

$X + Y \to var(1+1) \Rightarrow var = 2.$

both independent of each other

$$\begin{array}{l} a.s \searrow \\ \qquad\qquad i.p. \Rightarrow D \\ \qquad \nearrow \\ r^{th}\text{mean} \end{array}$$

* mean $\Rightarrow$ i.p. proof in tutorial.
* No relation betn a.s. & r^{th} mean.
* We won't go into a.s. $\Rightarrow$ ip. & ip. $\Rightarrow$ D

Counterexamples of ulta arrows

Ex 36 :

a.s. $\not\Rightarrow$ r^{th} mean

i.p. $\not\Rightarrow$ r^{th} mean

$$X_n = n \mathbb{1}_{[0,\frac{1}{n}]}$$

$$X_n \xrightarrow{i.p.} 0 \qquad X_n \xrightarrow{a.s.} 0 \qquad X_n \not\xrightarrow{r^{th}} 0$$

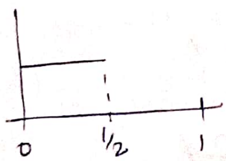
$$P(|X_n| > \epsilon) = \frac{1}{n} \to 0$$

$$E|X_n|^r = \frac{n^r}{n} = n^{r-1} \not\to 0$$

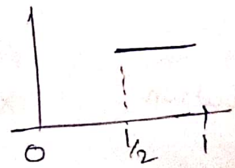
Ex 37

$$X_n \xrightarrow{r^{th}\text{mean}} X, \quad X_n \xrightarrow{ip} X, \quad X_n \not\xrightarrow{a.s.} X.$$

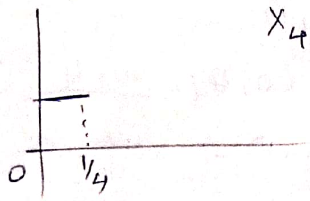
X_1 :



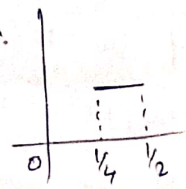
X_2 :



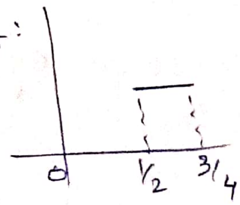
X_3 :



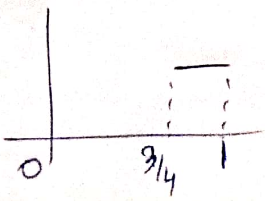
X_4 :



X_5 :



X_6 :



expectation of the rv is decreasing

$$\frac{1}{2} \to \frac{1}{4} \to \frac{1}{8} \to \cdots$$

$$\lim_{n \to \infty} \frac{1}{2^n}$$

when the stick is above 0, its 1 else 0.

1, 0, 1, 0, 0, 0, 1, ...

~~So 1~~ a.s. convergence D the convergence of a real sequence.

1 will keep popping up after every little while.

So this sequence doesnt converge.

$X_n \xrightarrow{D} -X$, $X_n \not\xrightarrow{P} -X$

Ex 38

$-X$ also has Normal distr.

X & $-X$ have the same distr.

if $X_n = X$

& $X_n \xrightarrow{D} -X$

$\Rightarrow$

$$P(|X_n - (-X)| > \epsilon)$$
$$= P(|2X| > \epsilon) = P\left(X > \frac{\epsilon}{2} \cup X < -\frac{\epsilon}{2}\right)$$
$$= \alpha \not\to 0$$

(smthn +ve)

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The Special Case where $D \Rightarrow$ i.p.

Limit = constant

Thm: Suppose $\{X_n\}$ seq. of RVs defined on a single probab. space & X_n converges in distribution to some constant c, then
$X_n \xrightarrow{ip} c$

Proof:

$F_n(x) \longrightarrow F(x)$ where F is distribution funcn.

$$\begin{cases} F_n(x) \longrightarrow 0 & x < c \\ \quad\quad\;\; \longrightarrow 1 & x > c \end{cases}$$

$X_n \xrightarrow{distri} c$

$\epsilon > 0$

$$P(|X_n - c| > \epsilon) \longrightarrow 0$$

$$\Leftrightarrow \quad P(|X_n - c| \leq \epsilon) \longrightarrow 1$$

$$\Rightarrow \quad P(c-\epsilon \leq X_n \leq c+\epsilon)$$

$$\geq P(c-\epsilon < X \leq c+\epsilon)$$
$$= \underset{\downarrow \atop 1.}{F_n(c+\epsilon)} - \underset{\downarrow \atop 0}{F_n(c-\epsilon)}$$

Thm: $\{X_n\}$, $M_n(t)$

X, $M(t)$

If $M_n(t) \longrightarrow M(t)$ for all $t \in$ open int. containing 0.

then $X_n \xrightarrow{D} X$

* Proof is tough. Don't do.

Ex 39 : $X_n \sim Bin(n, p_n)$

$p_n \longrightarrow 0$

$np_n = \lambda \ (>0)$

$X \sim Poi(\lambda)$

Then $X_n \xrightarrow{D} X$

Proof :

$M_n(t) = (1 - p_n + p_n e^t)^n$ $\qquad n \to \infty$

$M(t) = e^{\lambda(e^t - 1)}$ $\qquad p_n \to 0$

$np_n = \lambda$

$(1 + p_n(e^t - 1))^n$

$= \left(1 + \frac{\lambda(e^t-1)}{n}\right)^n \longrightarrow e^{\lambda(e^t-1)}$

hence by the previous theorem on MGFs,

$X_n \xrightarrow{D} X.$

$\therefore$ Poisson rv is an approximation of Binomial.

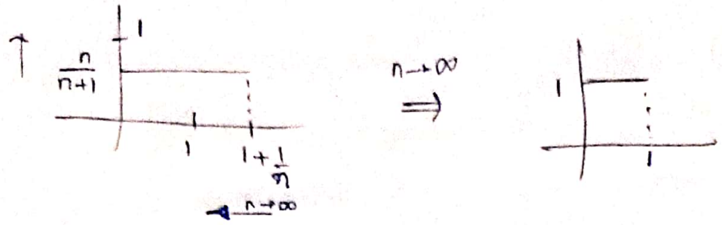
Theorem : DRV/CRV both PMF/PDF $f(\cdot)$

if $f_n(x) \longrightarrow f(x)$ for all x then

$X_n \xrightarrow{D} X$

26th Sept

$$X_n \sim U\left(0, 1+\frac{1}{n}\right) \longrightarrow \frac{1}{1+\frac{1}{n}} \mathbb{1}_{\left[0, \frac{n+1}{n}\right]}$$



Theorem : (Strong Law of Large Nos.)

Let $\{X_n\}$ be a seq. of i.i.d RVs with finite mean μ.

Define $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Then $\{\overline{X_n}\}$ converges to μ almost surely.

$X_i = 1$ if ith toss is head.

$\sum_{i=1}^{n} X_i \rightarrow$ no. of heads in n tosses

$\frac{1}{n}\sum_{i=1}^{n} X_i \rightarrow$ proportion.

$$X_i = 1 \quad \text{w.p. } p$$
$$= 0 \quad \text{w.p. } 1-p$$

$EX_i = p$.

Monte Carlo integration :

$$\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b (b-a) f(x)\,dx$$

$$= E((b-a) f(X)) \quad \text{where } X \sim U(a,b)$$

$\{X_i\}$ are iid $U(a,b)$

$$\frac{1}{n}\sum_{i=1}^{n}(b-a) f(x_i) \longrightarrow E[(b-a) f(X)]$$

A probabilistic way to do this integration

Suppose g is pdf supported on (a,b)

$$\int_a^b \frac{f(x)}{g(x)} g(x)\,dx$$

$$E\left[\frac{f(x)}{g(x)}\right] \qquad \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{g(x_i)}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

$$p.\ \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\int_{-\infty}^{+\infty} \frac{f(x)}{\phi(x)}\phi(x)\,dx = E\left(\frac{f(X)}{\phi(X)}\right) \quad \text{where } X \sim N(0,1)$$

Central Limit Thm :

$$P\left(\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \le a\right)$$

$$\rightarrow \Phi = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \xrightarrow{dist} N(0,1)$$

limit does not depend on ~~priviti~~ state.

Ex 44 : $X_n \sim Bin(n,p)$

Then

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \le a\right) \rightarrow \Phi(a)$$

$Bin(n,p)$ is sum of iid $Ber(p)$

$$\underset{X_n}{\overset{\|}{}} \qquad X_1 + \cdots + X_n$$

$$X_i \sim Ber(p)$$

CLT proof / Heuristic proof :

$X_i \rightarrow$ life of i^{th} battery

$$P(X_1 + \cdots + X_{25} > 1100)$$

$$P(S_{25} - 1000 > 100)$$

$$= 1 - \Phi(1)$$

Exercise :

If $X_i \sim N(\mu, \sigma^2)$

$$\frac{\sqrt{n}\,(\overline{X}_n - \mu)}{\sigma} = \frac{S_n - n\mu}{\quad}$$

$$\cancel{\Phi_X\left(\frac{t}{X_n}\right)}$$

$$\Phi_X\left(\frac{t}{\sqrt{n}}\right) = E\left(e^{\frac{t X_1}{\sqrt{n}}}\right)$$

$$\sim 1 + \frac{t\, E X_1}{\sqrt{n}} + \frac{t^2}{n} E X_1^2$$

$\cancel{\Phi}$ X_n has mean 0 , var 1

$$\frac{S_n}{\sqrt{n}} \xrightarrow{dnt} N(0,1)$$

$$\Phi_{\frac{S_n}{\sqrt{n}}}(t) = E\left[e^{\frac{t}{\sqrt{n}}(X_1 + X_n)}\right]$$

$$= \left(\Phi_X\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$= \left(1 + \frac{t^2}{2n}\right)^n \xrightarrow{n \to \infty} e^{t^2/2}$$

$$\downarrow$$

MGF of $N(0,1)$

$=$

$\cancel{\Phi_X}$ proved

27th Sept

Bivariate Normal :

$X \sim N(\mu_1, \sigma_1^2)$
$Y \sim N(\mu_2, \sigma_2^2)$ > independent

$(X, Y) \rightarrow$ multivariate normal.

$aX + bY \sim$ Normal.

Thm 1: $\mu = E(X)$

Σ = var-covar matrix of X

then for any fixed $\vec{u}^T = (a, b) \in \mathbb{R}^2 \setminus (0,0)$.

$u'X \sim N(u'\mu, u'\Sigma u)$.

$$\Sigma = E(X-\mu)(X-\mu)^T = \begin{pmatrix} \text{Var } X & \text{Cov}(X,Y) \\ \text{Cov}(X,Y) & \text{Var } Y \end{pmatrix}$$

prime = transpose

Proof of thm 1 :

$$E(aX + bY) = aE(X) + bE(Y)$$
$$= a\mu_x + b\mu_y$$
$$= u^T\mu. \quad \text{ie. } u'\mu.$$

$\text{Var}(aX + bY)$

$$= E(aX+bY)^2 - (a\mu_x + b\mu_y)^2$$
$$= a^2 EX^2 + b^2 EY^2 + 2ab\, EXY - (a\mu_x + b\mu_y)^2$$

$\vdots$

$$= [a \;\; b] \begin{bmatrix} \text{Var}(X) & \text{Cov}(X,Y) \\ \text{Cov}(X,Y) & \text{var}(Y) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Thm 2 : let X be a bivariate normal rv then

$$M_X(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t} \quad \text{for all } t \in \mathbb{R}^2$$

Proof : $X = (x, y)$

$$M_X(t_1, t_2) = E\left[e^{t_1 x + t_2 y}\right] \quad (\mathbb{R}^2)$$

$$= e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

$X = (X, Y) \sim N_2(\mu, \Sigma)$

$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$X = 1 \cdot X + 0 \cdot Y$

$(1 \ 0)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$(1 \ 0)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sigma_{11}$$

Thm 3 : If $X \sim N_2(\mu, \Sigma)$, then $X \sim N(\mu_1, \sigma_{11})$ & $Y \sim N(\mu_2, \sigma_{22})$

Counter example of converse of above thm.

$X \sim N(0,1)$

Z ind of X, and $Z = 1$ w.p. $1/2$
$= -1$ w.p. $1/2$

Define $Y = ZX$

$\therefore Y \sim N(0,1)$

(X, Y)

$X + Y = 0$ w.p. $1/2$

One way to show X & Y are independent is showing that joint MGF is the product of marginal MGFs.

$$\Sigma = \begin{pmatrix} \sigma_{11} & \cancel{\sigma_{12}} \\ \cancel{\sigma_{21}} & \sigma_{22} \end{pmatrix}$$

0 0

$$e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

$$= e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}t_1^2\sigma_{11} + \frac{1}{2}t_2^2\sigma_{22}}$$

$$= e^{t_1\mu_1 + \frac{1}{2}t_1^2\sigma_{11}} \, e^{t_2\mu_2 + \frac{1}{2}t_2^2\sigma_{22}}$$

$$N_2(\mu\,\Sigma) \sim \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \overset{d}{\equiv} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + A\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

i.i.d. $N(0,1)$

$f: \mathbb{R}^n \to \mathbb{R}^n$

$x \to A_{nm} x$

$|J| = |A|$

$AA^T = \Sigma$

$\det \Sigma \neq 0$

$\Rightarrow \det A = \det A^P \neq 0$

$\Rightarrow A$ is inv.

$$f_Y(y_1, y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}y^t y} \qquad y^t y = y_1^2 + y_2^2$$

$(\det A)^2 = \det \Sigma$

$|\det A| = |\det \Sigma|^{1/2}$

$Y = A^{-1}(X - \mu)$

$|J| = |A^{-1}| = |\Sigma|^{-1/2}$

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20th sept (lec 21 contd)

1) $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ $\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$

ρ – corr, coefficient

2)

$X \sim N(0,1)$

$X = \begin{pmatrix} X \\ X \end{pmatrix}$ $aX + bX = (a+b)X$

$\Downarrow$

bivariate normal.

$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ where $Y \sim N(0,1)$ indepe. of X.

$\Sigma = AA^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Non-invertible matrix det $=0$

ρ doesnt exist (?)

Hence we see that need not always have a Multivariate normal density.

3)

dependence $\Rightarrow$ uncorelated.

Uncorelated $\Rightarrow$ independence.

4)

4) $\dfrac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$

Conditional

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \rho^2\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right) - \rho\left(\frac{y-\mu_y}{\sigma_y}\right)\right]^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma_x^2}\left[x-\left(\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)\right]^2}$$

5) Chi square distribution with n - degrees of freedom

6) $X \sim N(0,1)$

$$E[e^{tX^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx^2} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2t)x^2}\,dx \qquad 1-2t>0, \quad t<\frac{1}{2}$$

$x\sqrt{(1-2t)} = y$

$dx = \dfrac{dy}{\sqrt{1-2t}}$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-2t}}\int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \frac{1}{\sqrt{1-2t}}$$

$\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$

~~1 2 5 4 1~~

4 1 5 3 2

13 2 4 1

$$E\left[e^{t(x_1^2+\dots+x_n^2)}\right] = E\left[\prod_{i=1}^{n} e^{t x_i^2}\right]$$

$$= \frac{1}{(1-2t)^{n/2}} \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

7) Orthogonal matrix.

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ & \vdots & \end{pmatrix}_{n \times n} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

$A \rightarrow$ orthogonal.

$$AA^T = A^T A = I$$

$$A^{-1} = A^T$$

$$\det A = \det A^{-1} = 1$$

$$Y = AX$$

$$X = A^{-1} Y$$

$$|J| = |\det A^{-1}| = 1$$

$$f_{X_1 \dots X_n}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \sigma^n} e^{-\frac{1}{2\sigma^2}(x-\mu)(x-\mu)}$$

$$f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = \frac{1}{(2\pi)^{n/2} \sigma^n} e^{-\frac{1}{2\sigma^2}(A^{-1}y-\mu)(A^{-1}y-\mu)}$$

$$= \text{"} \quad e^{-\frac{1}{2\sigma^2}(y-\mu)^T} \qquad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

?

$$= \frac{1}{(2\pi)^{1/2} \sigma^n} e^{-\frac{1}{2\sigma}(y-\beta)^T(y-\beta)(AA^T)^{-1}}$$

$Y_i \sim N(\beta_i, \sigma^2)$ So we have found distribution of Y_1 to Y_n

Let us find β_i

$$? \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ 0 & 0 & & 0 \\ & \vdots & & \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\beta_1 = \sqrt{n}\,\mu$$

$$\sum_{i=1}^{n} \beta_i^2 = \beta^t \beta = (A\underline{\mu})^T (A\underline{\mu})$$

$$= \underline{\mu}^T A^T A \underline{\mu}.$$

$$= \underline{\mu}^T \underline{\mu}$$

$$= \sum_{i=1}^{n} \mu^2 = n\mu^2$$

$\Rightarrow \beta_2 = \beta_3 = \beta_n = 0$

all $\beta_i = 0$ except for β_1.

$$Y_1 \sim N(\sqrt{n}\mu, \sigma^2)$$

$$\frac{\sqrt{n}}{n} \sum_{i=1}^{n} x_i$$

$$\bar{X} \sim \frac{1}{\sqrt{n}} N(\sqrt{n}\mu, \sigma^2)$$

$$= N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\sum_{i=1}^{n} Y_i^2 = Y^T Y = (AX)^T (AX) = X^T A^T A X = X^T X$$

$$= \sum_{i=1}^{n} X_i^2$$

$$n\bar{X}^2 + \sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2$$

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$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=2}^{n}\left(\frac{Y_i}{\sigma}\right)^2 - \chi^2_{n-1}$$

$$\cancel{n\bar{x}^2 + \sum Y_i^2 = \sum x_i^2}$$

$$\frac{(n-1)}{\sigma^2} s^2$$

$Y_i \sim N(0, \sigma^2)$

$i = 2, \cdots, n$

$\frac{Y_i}{\sigma} \sim N(0, 1)$