24. Confidence Intervals - Practicals

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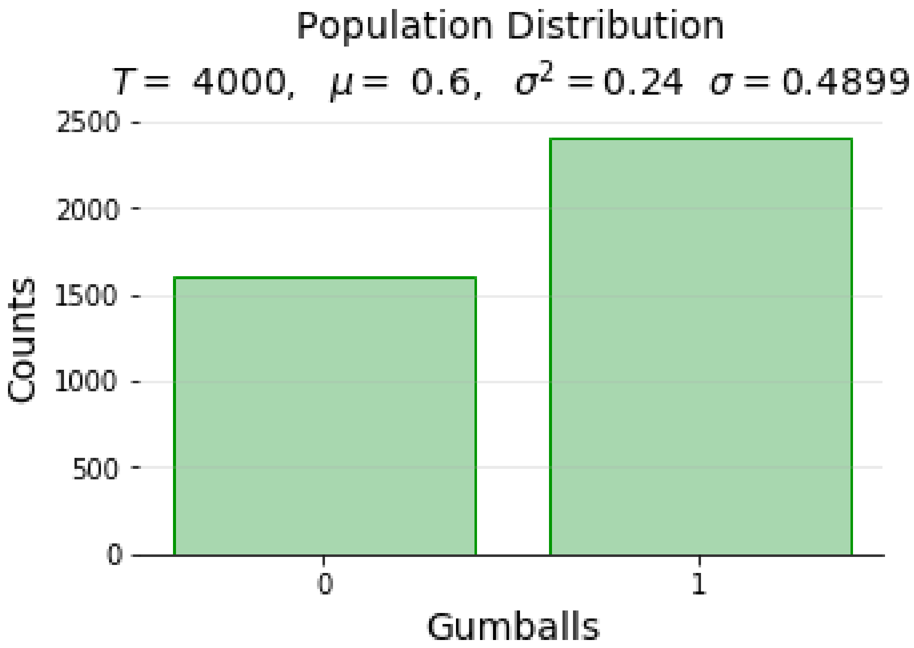
1 Deep Examples

1.1 Confidence Intervals for Sampling Proportions

1.1.1 Create Population

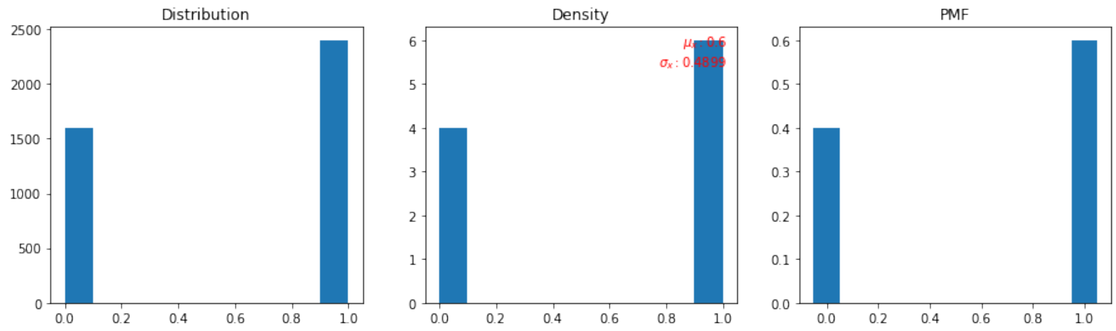
Let us create a population of 10000 balls, with 60% yellow balls. Programmatically, our population contains 1s and 0s, 1 indicating yellow.

```
In[1]: %matplotlib inline
       import matplotlib.pyplot as plt
       from SDSPSM import get_metrics, drawBarGraph
       from ci_helpers import create_bernoulli_population

       T = 4000   # total size of population
       p = 0.6    # 60% has yellow balls

       # create population
       population, population_freq = create_bernoulli_population(T,p)

       # population metrics
       mu, var, sigma = get_metrics(population)

       # visualize
       fig, (ax1) = plt.subplots(1,1, figsize=(5,3))
       drawBarGraph(population_freq, ax1, [T, mu, var, sigma], 'Population
       Distribution','Gumballs', 'Counts',xmin=0)
       plt.show()
```



Deriving and visualizing the probability Mass function (the intermediate density function, where total area of bars will be 1, is just for fitting normal continous approximation later)

```
In[2]: from ci_helpers import mini_plot_SDSP

       fig, (ax1,ax2,ax3) = plt.subplots(1,3,figsize=(15,4))
       mini_plot_SDSP(population, ax1,ax2,ax3, norm_off=True)
       plt.show()
```



1.1.2 Sampling from the Population

Let us sample from population, N no of times, each time with sample set of size n. If $np \geq 30$ and $nq \geq 30$, the resulting sampling distribution should be approximately normal. Remember, for Population described by random variable Y, we describe the sampling distribution by

$$\begin{aligned} \text{for any sample set k, sample mean is } \ \overline{\widehat{Y_k}} &= \frac{1}{n}\sum_{i=1}^{n} Y_{ki} \\ \text{Random Variable } \ \widehat{p} = \overline{\widehat{Y}} &= \overline{\widehat{Y_1}}, \overline{\widehat{Y_2}}, \cdots \overline{\widehat{Y_k}} \cdots \overline{\widehat{Y_N}} \\ \mu_{\widehat{p}} &= \mu(\overline{\widehat{Y}}) \\ \sigma_{\widehat{p}} &= \sigma(\overline{\widehat{Y}}) \end{aligned} \tag{1}$$

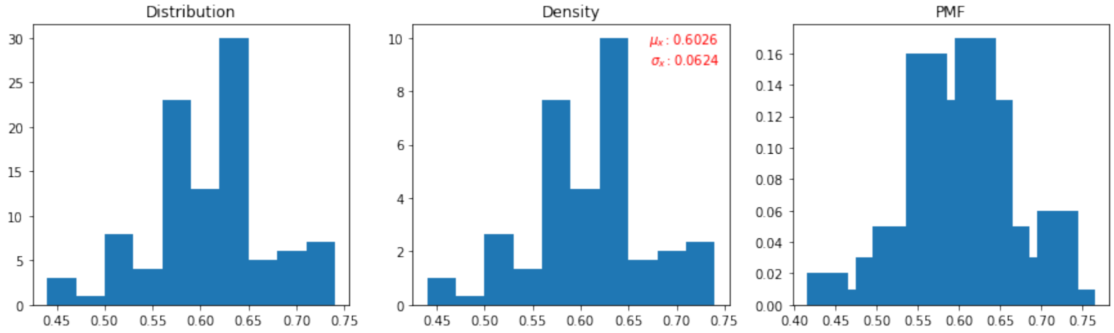
where the hat$\widehat{}$indicates the statistical outcome. And statistically by CLT,

$$\begin{aligned} \mu_{\widehat{p}} \approx 0.6 = \mu = p \\ \sigma_{\widehat{p}} \approx 0.0693 \approx \frac{0.4898}{\sqrt{50}} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}} \end{aligned} \tag{2}$$

Note we have sampled WITH REPLACEMENT, so the samples are independent. If you sample without replacement, you need to factor in FPC (finite population correction) for each sample set's SD.

```
In[3]: from ci_helpers import sample_with_CI
       from random import seed

       N = 100
       n = 50

       #seed(0)

       # sample from population
       Y_mean_list, CI_list = sample_with_CI(N, n, population, sigma=sigma, mode='z')

       # sample metrics
       mu, var, sigma = get_metrics(Y_mean_list)

       # visualize
       fig, (ax1,ax2,ax3) = plt.subplots(1,3,figsize=(15,4))
       mini_plot_SDSP(Y_mean_list,ax1,ax2,ax3,width=0.05,  norm_off=True)

       from IPython.display import display, Math
       display(Math(r'\mu_{{\hat{{p}}}}:{} \ \ \ \ \sigma_{{\hat{{p}}}}:{}'.format(mu, sigma)))
```

$$\mu_{\hat{p}} : 0.6026 \quad \sigma_{\hat{p}} : 0.0624$$



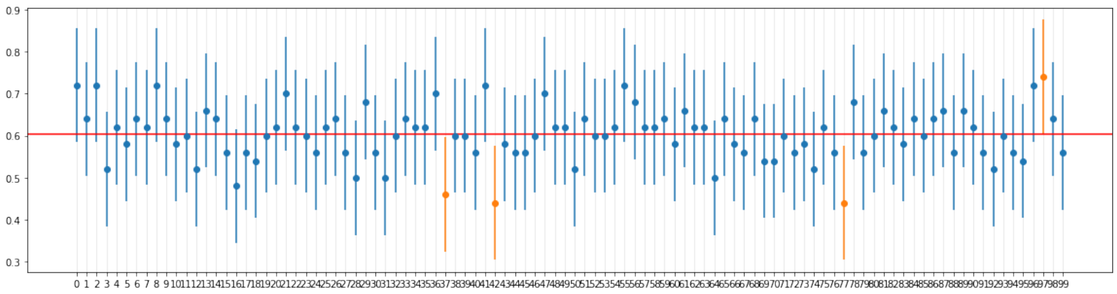
1.1.3 When σ is known

For each of above sample set of size 'n', let us calculate confidence interval using population SD σ as below. 1.96 is from Z tranformation for 95% confidence interval, like we saw earlier in our theoretical section.

$$CI = Y \pm 1.96\frac{\sigma}{\sqrt{n}} \tag{3}$$

```
In[4]: from ci_helpers import plot_ci_accuracy_1

       fig, ax = plt.subplots(1,1, figsize=(20,5))

       plot_ci_accuracy_1(ax, CI_list, mu)
       plt.show()
```

```
CI containing pop.mean:96.0%
```



As expected we observe that out of all CIs above, 95% of them or above contain population mean.

1.1.4 When σ is not known

For each sample mean $\overline{X_k}$ calculated, the confidence interval is calculated as below. Note, the constant value t_{n-1} depends on degrees of freedom (n-1).

$$CI = Y \pm t_{n-1}\frac{S_k}{\sqrt{n}} \tag{4}$$

Hope you noted. This time, for each sample mean, we also calculate unbiased sample variable of that set (that is, divided by n-1), and use that for calculating M_k. We sample again, because, for each sample, this time, we calculate CI using t distribution.

t value for 95% CI:

Degrees of Freedom $df = n-1$. For 95% confidence level, the confidence coefficient, $1-\alpha = 1-0.05 = 0.95$.

To calculate t in python, we simply need to pass, $(1-\alpha, df)$. A sample calculation shown below for sample size $n = 10$

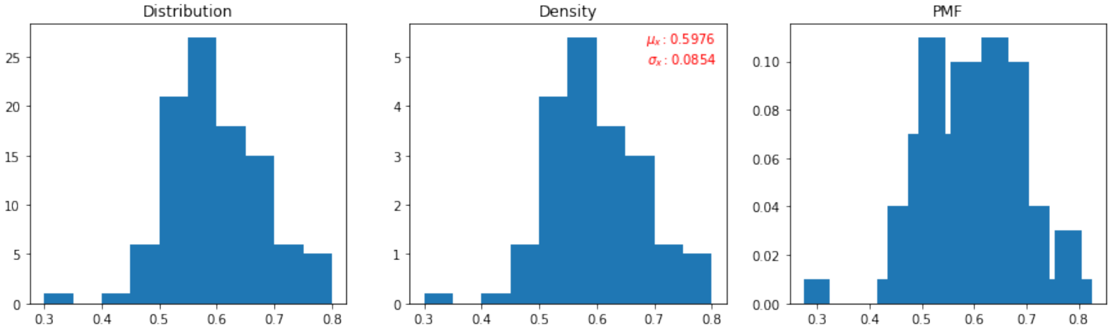
```
In[5]: from scipy import stats
       print(stats.t.ppf(1-0.025, 10-1))
```

```
2.2621571627409915
```

Now to our sampling distribution. Note, we are getting an approximate normal distribution.

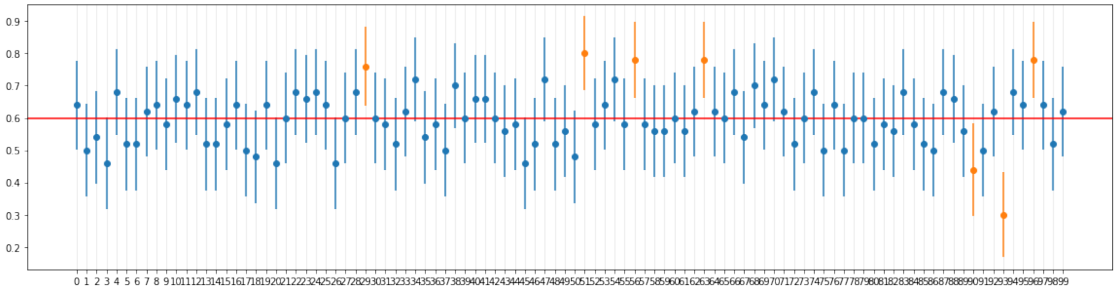
```
In[6]: from ci_helpers import sample_with_CI

       N = 100
       n = 50

       #seed(0)

       # sample from population, this time in t mode,
       # so CI intervals are calculated with t value 2.093
       Y_mean_list, CI_list = sample_with_CI(N, n, population, sigma=sigma, mode='t')

       # sample metrics
       mu, var, sigma = get_metrics(Y_mean_list)

       # visualize
       fig, (ax1,ax2,ax3) = plt.subplots(1,3,figsize=(15,4))
       mini_plot_SDSP(Y_mean_list,ax1,ax2,ax3,width=0.05,  norm_off=True)

       from IPython.display import display, Math
       display(Math(r'\mu_{{\hat{{p}}}}:{} \ \ \ \ \sigma_{{\hat{{p}}}}:{}'.format(mu, sigma)))

       plt.show()
```

$$\mu_{\hat{p}} : 0.5976 \quad \sigma_{\hat{p}} : 0.0854$$



```
In[7]: from ci_helpers import plot_ci_accuracy_1

       fig, ax = plt.subplots(1,1, figsize=(20,5))

       plot_ci_accuracy_1(ax, CI_list, mu)
       plt.show()
```

```
CI containing pop.mean:93.0%
```



Generally we should get more than 95% as above. Above result just means, if we take a sampling size, and calculate CI, and do that 100 times, about 95 times our CI would contain population mean, and our result gave 97 times. We could expect at least 95% most of the time. But can we get any idea, how that ”success” of getting population mean in our CI, 95% of time, depends on sample size? We get it, greater the sample size, better, but how it would be? Let us take our simulation to next scale as below, trying with various experiment and sample sizes.

1.1.5 Digging deeper 1

What if I use Z distribution and unbiased sample SD even for CI? What happens when I use t distribution but population SD for CI? We will find out what happens in such cases below.

Environment:

1. Population size T, fixed
2. Sample size n, varied
3. Experiment size N, varied
4. Sampling with or without replacement, varied.

Applied methods:

1. Z distribution and population SD
2. Z distribution and unbiased sample SD
3. T distribution and population SD
4. T distribution and unbiased sample SD

Note, in case of sampling without replacement, each sample SD is corrected with FPC (Finite Population Correction)

```
In[8]: from ci_helpers import plot_summary


       max_sample_size = int(T/4)  # 25% of total population
       N_list = range(5,500,20)
       n_list = range(5,max_sample_size,50)  # different sample sizes


       plot_summary(population, N_list, n_list)
```

Sampling Without Replacement

Fig 01: Using Z distribution and population SD

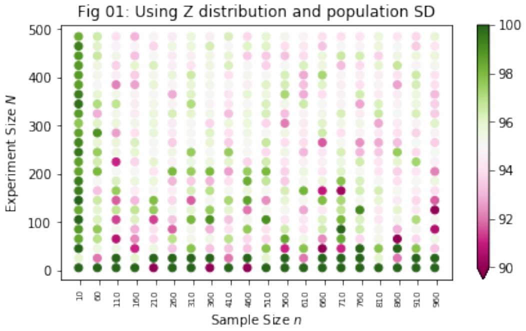


Fig 02: Using Z distribution and unbiased sample SD

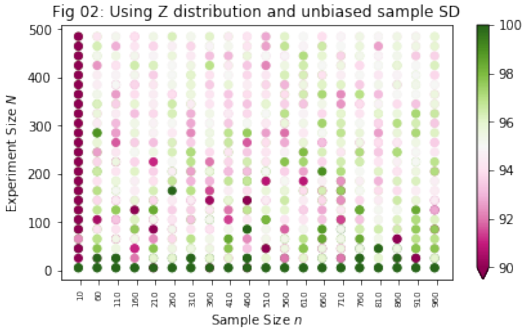


Fig 11: Using T distribution and population SD

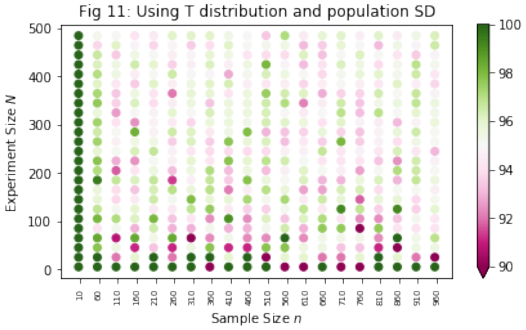
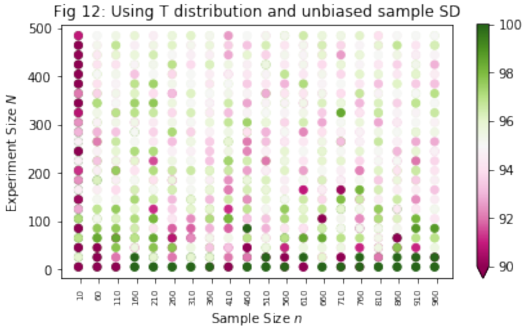


Fig 12: Using T distribution and unbiased sample SD



Sampling With Replacement

Fig 21: Using Z distribution and population SD

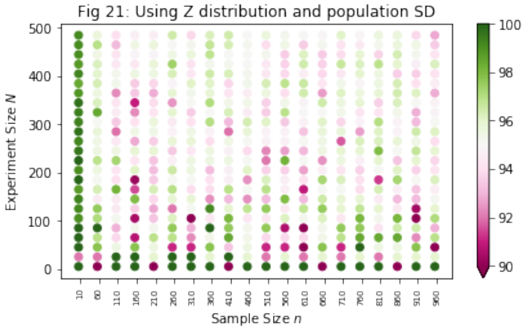


Fig 22: Using Z distribution and unbiased sample SD

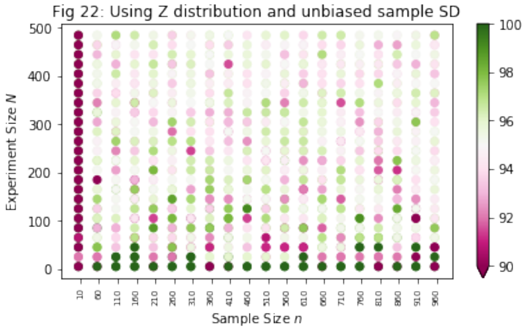


Fig 31: Using T distribution and population SD

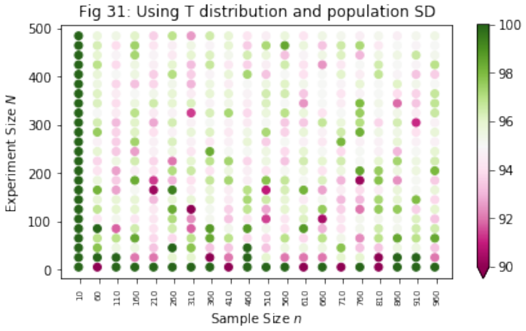
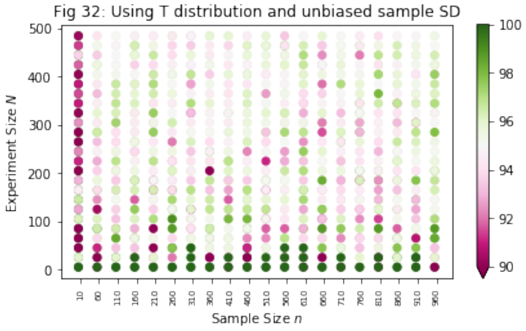


Fig 32: Using T distribution and unbiased sample SD



Note that, as per color gradient used, lighter the dots, nearer they are to 95%. And if green they are above 95%. And if pink, they are below 95%. So more the green dots or lighter dots, the better, the CI performance.

1. Compared to graphs using sample SDs on right hand side, the graphs using population SDs on left hand side, has more dots that are green and lighter indicating better CI performance on LHS. This is especially very pronounced, when sample sizes are small (observe dark dots at $n = 10$). LHS almost always have green dots at $n = 10$ while RHS has mostly pinky dots.
2. For a common SD usage, there is not much a difference between using Z or t distribution when $n \geq 30$. For eg, compare figures 01 and 11 both using population SD. Or compare 02 and 12 both using sample SD.
3. Comparing figures 01 and 11 at $n = 10$ we observe, figure 11 performs better (more darker green dots). So when you know σ, and if $n < 30$ using Z distribution is better.
4. Comparing figures 02 and 12 at $n = 10$ we observe, figure 12 performs better (lighter pink dots). So when you do not know σ and if $n < 30$, using T distribution with unbiased sample SD is better.
5. Similar observation also applies to sampling with replacement.

Though the limit 30 is not obvious from above graphs, this number has been arrived at by statisticians after extensive research

Warning

The CI for proportions have been always blotchy. Though above formula are straight forward, they have been proven ineffective, effectively by dasgupta2001. When you use CI for proportions problem in a practical scenario do use the alternatives provided there. In a nutshell, for smaller sizes, $n < 40$, Wilson or the equal-tailedJeffreys prior interval are recommended. For larger n, the Wilson, the Jeffreys andthe Agresti–Coull intervals are all comparable, and the Agresti–Coull interval is the simplest to present.

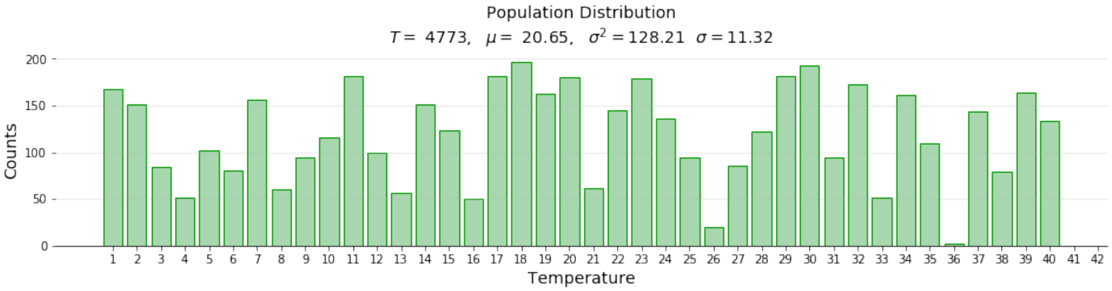
1.2 Confidence Intervals for Sample Means

1.2.1 Create Population

Let Y be the random variable indicating temperature over a distribution of certain values. If limiting values are say, 0 deg C to 40 deg C, our population would thus look like this: $[23, 13, 35, 50, 10, 2, 5, 0, 33, \cdots, 21]$ Unlike Sample proportions,we do not know or designate any proportion of temperatures in this example, but we know the mean and variance by simply calculating all values in the distribution. These would be our population parameters.

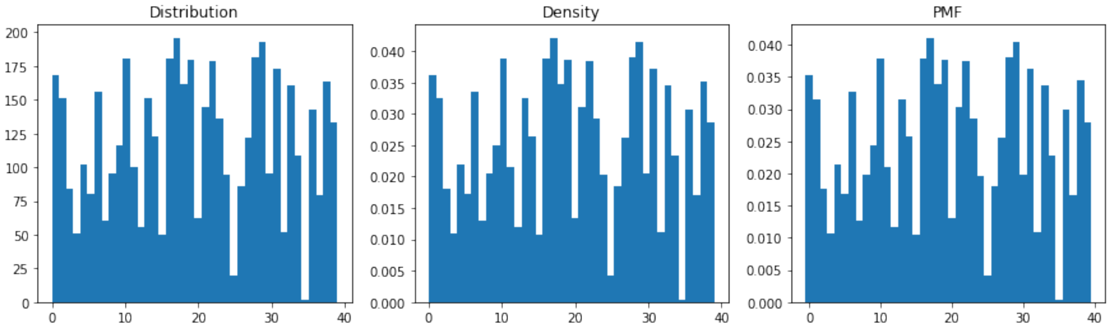
Population mean $\mu = \mu_y$
Population variance $\sigma^2 = \sigma_y^2$

```
In[9]: %matplotlib inline
       from math import floor
       import matplotlib.pyplot as plt
       from random import random, seed, shuffle
       from SDSPSM import get_metrics, drawBarGraph, getPopulationStatistics
       from ci_helpers import createRandomPopulation

       seed(0)

       popMin = 1    # Min population
       popMax = 40   # Max population
       freqMax = 200 # freq of any set of population (for eg, no of occurances of temperatures
       at 25 deg C)

       population, population_freq = createRandomPopulation(popMax - popMin + 1, freqMax)

       N, mu, var, sigma = getPopulationStatistics(population_freq, popMin)

       #visualize
       fig, (ax1) = plt.subplots(1,1, figsize=(16,3))
       drawBarGraph(population_freq, ax1, [N, mu, var, sigma], 'Population
       Distribution','Temperature', 'Counts')
       plt.show()
```



Let us visualize the density function and PMF as usual.

```
In[10]: from ci_helpers import mini_plot_SDSM

        fig, (ax1,ax2,ax3) = plt.subplots(1,3,figsize=(15,4))
        mini_plot_SDSM(population, ax1, ax2, ax3, popMax, width=1)
        plt.show()
```



1.2.2 Sampling from the Population

Let us sample from above population, N no of times, each time with sample set of size n. If $n > 30$, the resulting sampling distribution should be approximately normal (always if population itself was normally distributed)

Remember, for Population described by random variable Y, we describe the sampling distribution of sample means by

$$\begin{gathered}\mu_{\overline{Y}} = \mu(\widehat{\overline{Y}}) \\ \\ \sigma_{\overline{Y}} = \sigma(\widehat{\overline{Y}})\end{gathered} \tag{5}$$

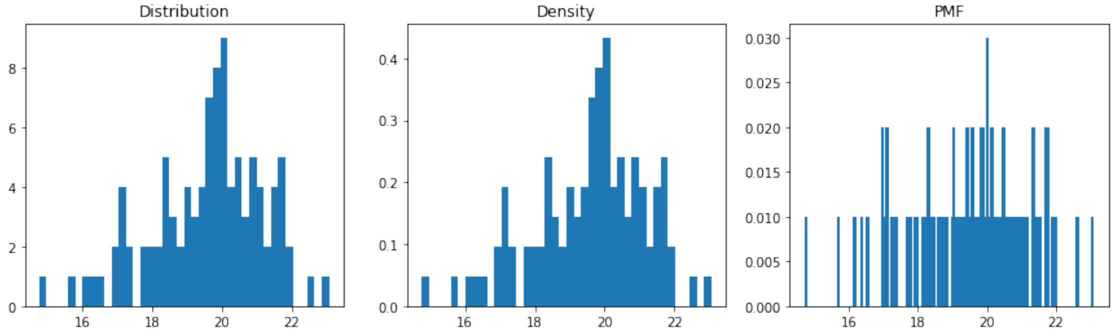
where the $\widehat{}$ indicates the statistical outcome. And statistically by CLT,

$$\begin{gathered}\mu_{\overline{Y}} = 19.4 \approx 20 = \mu \\ \\ \sigma_{\overline{Y}} \approx 1.52 \approx \frac{11.32}{\sqrt{50}} = \frac{\sigma}{\sqrt{n}}\end{gathered} \tag{6}$$

$\overline{Y}$ is called the sample means which is a random variable.

```
In[11]: from ci_helpers import sample_with_CI
        from random import seed

        N = 100
        n = 50

        #seed(0)

        # sample from population
        Y_mean_list, CI_list = sample_with_CI(N, n, population, sigma=sigma, mode='z')

        # sample metrics
        mu, var, sigma = get_metrics(Y_mean_list)

        # visualize
        fig, (ax1,ax2,ax3) = plt.subplots(1,3,figsize=(15,4))
        mini_plot_SDSM(Y_mean_list, ax1, ax2, ax3, popMax, width=0.1)

        from IPython.display import display, Math
        display(Math(r'\mu_{{\hat{{p}}}}:{} \ \ \ \ \sigma_{{\hat{{p}}}}:{}'.format(mu, sigma)))
```

$$\mu_{\hat{p}} : 19.5912 \quad \sigma_{\hat{p}} : 1.5865$$



Ok I get it, the resulting distribution and density functions look abnormal (ugly, slightly normal). Try increasing experiment size N, and you will see much better approximation of normal distribution. We had to stick with N=100 because we have to see how CI from each sample mean performs, so bear with me here.

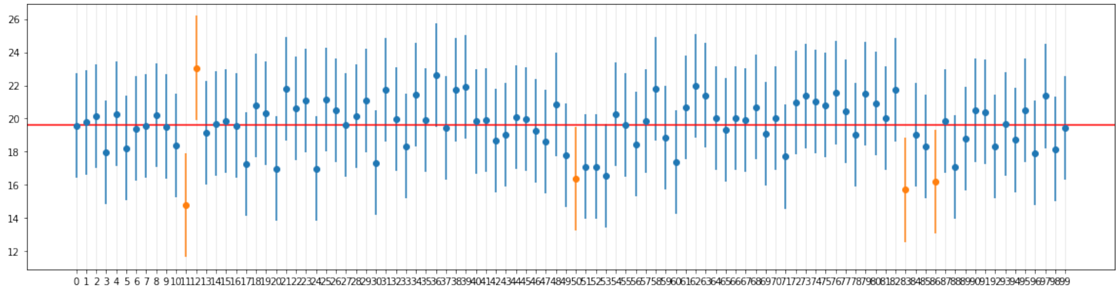
1.2.3 When σ is known

For each of above sample set of size 'n', let us calculate confidence interval using population SD σ as below. 1.96 is from Z tranformation for 95% confidence interval, like we saw earlier in our theoretical section.

$$CI = Y \pm 1.96 \frac{\sigma}{\sqrt{n}} \tag{7}$$

```
In[12]: from ci_helpers import plot_ci_accuracy_1

        fig, ax = plt.subplots(1,1, figsize=(20,5))

        plot_ci_accuracy_1(ax, CI_list, mu)
        plt.show()
```

```
CI containing pop.mean:95.0%
```



1.2.4 When σ is not known

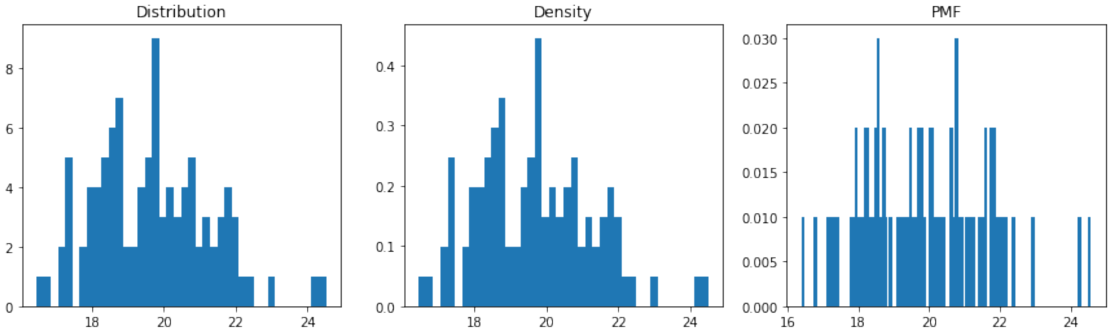
When we do not know population SD

Just like earlier, for each sample mean $\overline{X_k}$ calculated, the confidence interval is calculated as below. Note, the constant value t_{n-1} depends on degrees of freedom (n-1).

$$CI = Y \pm t_{n-1}\frac{S_k}{\sqrt{n}} \tag{8}$$

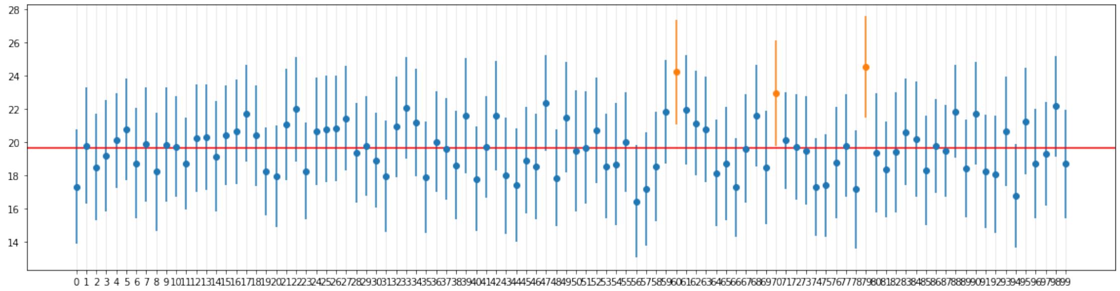
```
In[13]: from ci_helpers import sample_with_CI

        N = 100
        n = 50

        #seed(0)

        # sample from population, this time in t mode,
        # so CI intervals are calculated with t value 2.093
        Y_mean_list, CI_list = sample_with_CI(N, n, population, sigma=sigma, mode='t')

        # sample metrics
        mu, var, sigma = get_metrics(Y_mean_list)

        # visualize
        fig, (ax1,ax2,ax3) = plt.subplots(1,3,figsize=(15,4))
        mini_plot_SDSM(Y_mean_list, ax1, ax2, ax3, popMax, width=0.1)

        from IPython.display import display, Math
        display(Math(r'\mu_{{\hat{{p}}}}:{} \ \ \ \ \sigma_{{\hat{{p}}}}:{}'.format(mu, sigma)))

        plt.show()
```

$$\mu_{\hat{p}} : 19.6824 \quad \sigma_{\hat{p}} : 1.5962$$



```
In[14]: from ci_helpers import plot_ci_accuracy_1

         fig, ax = plt.subplots(1,1, figsize=(20,5))

         plot_ci_accuracy_1(ax, CI_list, mu)
         plt.show()
```

```
CI containing pop.mean:97.0%
```



1.2.5 Digging deeper 2

What if I use Z distribution and unbiased sample SD even for CI? What happens when I use t distribution but population SD for CI? We will find out what happens in such cases below.

Environment:

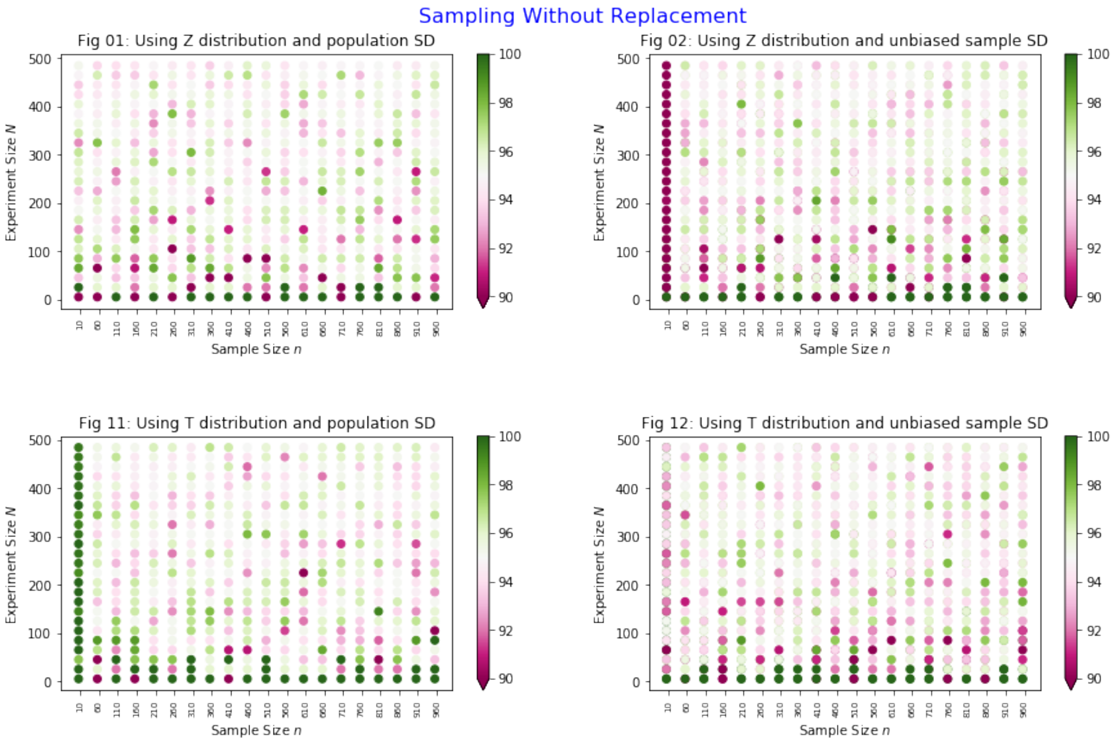
1. Population size T, fixed
2. Sample size n, varied
3. Experiment size N, varied
4. Sampling with or without replacement, varied.

Applied methods:

1. Z distribution and population SD
2. Z distribution and unbiased sample SD
3. T distribution and population SD
4. T distribution and unbiased sample SD

Note, in case of sampling without replacement, each sample SD is corrected with FPC (Finite Population Correction)

```
In[15]: max_sample_size = int(T/4)  # 25% of total population
         N_list = range(5,500,20)
         n_list = range(5,max_sample_size,50)  # different sample sizes


         plot_summary(population, N_list, n_list)
```



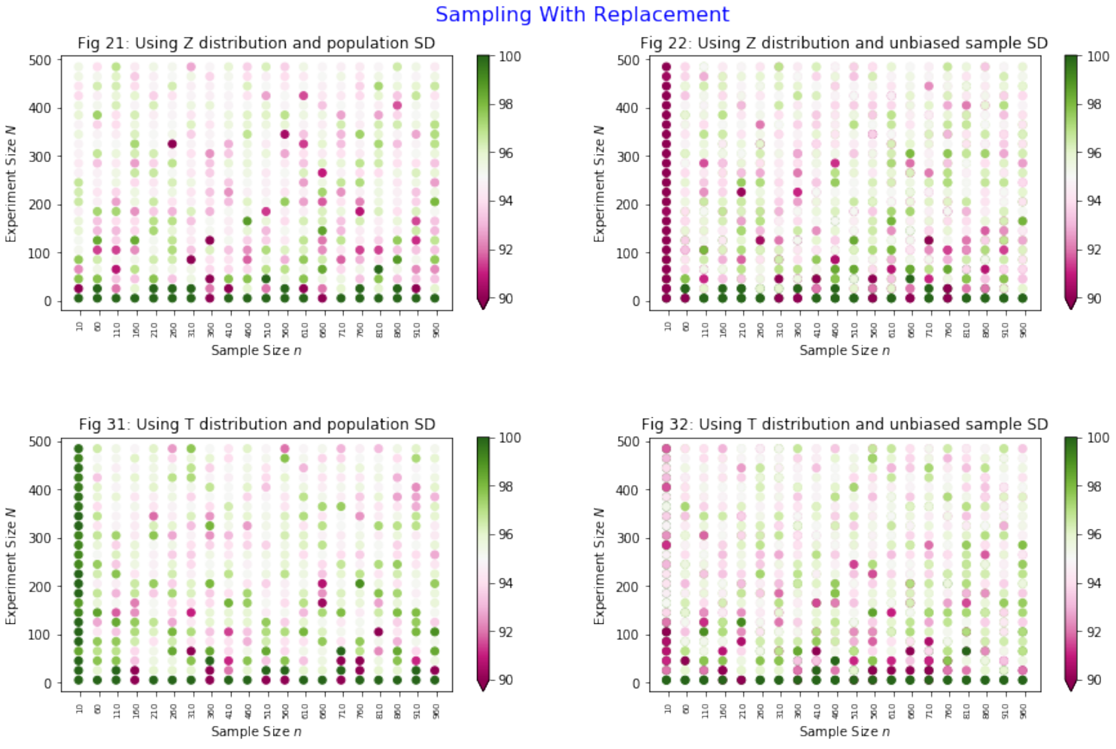
Sampling Without Replacement

Fig 01: Using Z distribution and population SD

Fig 02: Using Z distribution and unbiased sample SD

Fig 11: Using T distribution and population SD

Fig 12: Using T distribution and unbiased sample SD



Sampling With Replacement

Fig 21: Using Z distribution and population SD

Fig 22: Using Z distribution and unbiased sample SD

Fig 31: Using T distribution and population SD

Fig 32: Using T distribution and unbiased sample SD

Note that, as per color gradient used, lighter the dots, nearer they are to 95%. And if green they are above 95%. And if pink, they are below 95%. So more the green dots or lighter dots, the better, the CI performance.

1. Compared to graphs using sample SDs on right hand side, the graphs using population SDs on left hand side, has more dots that are green and lighter indicating better CI performance on LHS. This is especially very pronounced, when sample sizes are small (observe dark dots at $n = 10$). LHS almost always have green dots at $n = 10$ while RHS has mostly pinky dots.
2. For a common SD usage, there is not much a difference between using Z or t distribution when $n \geq 30$. For eg, compare figures 01 and 11 both using population SD. Or compare 02 and 12 both using sample SD.
3. Comparing figures 01 and 11 at $n = 10$ we observe, figure 11 performs better (more darker green dots). So when you know σ, and if $n < 30$ using Z distribution is better.
4. Comparing figures 02 and 12 at $n = 10$ we observe, figure 12 performs better (lighter pink dots). So when you do not know σ and if $n < 30$, using T distribution with unbiased sample SD is better.
5. Similar observation also applies to sampling with replacement.

Though the limit 30 is not obvious from above graphs, this number has been arrived at by statisticians after extensive research.

Yes, the inferences are same as Section 1.1.5 except that the differences are much more clearer in this case. For eg, compare figures 02 and 12 at $n = 10$. It is very clear now, why figure 12 (using t distribution) is far better at lower sample sizes.