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Q1 $P_n(R)$ = set of all polynomials with real coefficients)

$P_n(R) = a_0 + a_1x + a_2x^2 + \dots a_nx^n$

; $a_1, a_2 \dots a_n \in R$

(a) $P_n(R)$ is a real vector space.

Let $p, q \in P_n(R)$

$p: a_0 + a_1x \dots a_nx^n$

$q: b_0 + b_1x \dots b_nx^n$

$p+q: (a_0+b_0) + (a_1+b_1)x \dots (a_n+b_n)x^n$

$(a_0+b_0) = c_0 \in R$, $(a_1+b_1) = c_1 \in R$, $(a_n+b_n) = c_n \in R$

$\Rightarrow p+q \in P_n(R)$

additive inverse

Let $p = a_0 + a_1x \dots a_nx^n$

$q = (-a_0) + (-a_1x) \dots (-a_n)x^n$

since $p \in P_n(R)$ & $q \in P_n(R)$

and $p+q = 0$

$\Rightarrow$ additive inverse exists

$0 \in P_n(R)$

$\Rightarrow \quad 0 + 0.x \cdots 0.x^n$

$\Rightarrow 0 \in P_n(R)$

let $P = a_0 + a_1x + \cdots a_nx^n$

$P + 0 = (a_0+0) + (a_1+0)x \cdots (a_n+0)x^n$

$= P$

scalar multiplication

$\Rightarrow$ ~~$\alpha \in \alpha$~~

let $P = a_0 + a_1x \cdots a_nx^n$

$\alpha P = \alpha(a_0 + a_1x \cdots a_nx^n)$

$= (\alpha a_0) + (\alpha a_1)x \cdots (\alpha a_n)x^n$

$= \underset{b_0 \in R}{\shortparallel} \quad \underset{b_1 \in R}{\shortparallel} \quad \underset{b_n \in R}{\shortparallel}$

$\Rightarrow \alpha P \in P_n(R)$ for $\alpha \in R, P \in P_n(R)$

$(\alpha\beta)(a_0 + a_1x \cdots) = (\alpha\beta a_0) + (\alpha\beta a_1)x \cdots (\alpha\beta a_n)x^n$

$= \alpha(\beta a_0 + (\beta a_1)x + (\beta a_n)x^n)$

$1 * (a_0 + a_1x \cdots a_nx^n) = a_0 + a_1x \cdots a_nx^n$

$\Rightarrow P_n(R)$ is a vector space.

(b) $\mathcal{F} : P_n(\mathbb{R}) \longrightarrow \mathbb{R}$

$$\mathcal{F}(p(x)) = \frac{d}{dx}(p(x))_{x=0}$$

for $\mathcal{F}$ to be a linear functional.

$$\mathcal{F}(\alpha p(x) + \beta(q(x)) = \frac{d}{dx}(\alpha p(x) + \beta q(x))_{x=0}$$

$$= \alpha \frac{d}{dx}(p(x))_{x=0} + \beta \frac{d}{dx} \mathcal{F}(q(x))_{x=0}$$

$$= \alpha \mathcal{F}(p(x)) + \beta(\mathcal{F}(q(x))$$

$\Rightarrow$ For $\alpha, \beta \in \mathbb{R}$.

$$F(\alpha p(x) + \beta(q(x)) = \alpha F(p(x)) + \beta F(q(x))$$

$\Rightarrow$ It is a linear functional

(c) $p(x) = a_0 + a_1 x \ldots a_n x^n$

$$p = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$F[p(x)] = a_1 + 2 a_2 x + n a_n x^{n-1} \Big|_{x=0}$$

$$= a_1$$

$\Rightarrow$ $F[p(x)] = e_1^T p$