Q3 $\|x\|_w = \sqrt{\sum_{i=0}^{n} w_i x_i^2}$

$\|x\|_w = \sqrt{w_1 x_1^2 + w_2 x_2^2 \ldots}$

$w_i \in R, x_i \in R$

$\Rightarrow \|x\|_w \in R$

$\rightarrow \|.\|_w : R^n \rightarrow R.$

for all $w_i \in R$ $i \in [1, 2 \ldots n]$

$w_i > 0$

$\Rightarrow$

$\|x\|_w = \sqrt{w_1 x_1^2 + w_2 x_2^2 \ldots}$

$w_i > 0, \quad x_i^2 \geq 0$

$\Rightarrow$ $\|x\|_w$ is allways non negative

$\|x\|_w \geq 0$

Defitiness can be Proved by $x = 0 \Leftrightarrow \|x\|_w = 0$

$\|x\|_w = 0 = \sqrt{\underbrace{w_1 x^2}_{\geq 0} + \underbrace{w_2 x_2^2}_{\geq 0} \ldots \underbrace{w_n x_n^2}_{\geq 0}}$

$w_1, w_2 \ldots > 0$

$\Rightarrow x_1^2 = 0, x_2^2 = 0 \ldots \Rightarrow x = 0.$

Triangle Inequality :

let $f: R^n \to R^n$ such that

$$x \to y$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} \sqrt{w_1}\, x_1 \\ \sqrt{w_2}\, x_2 \\ \vdots \\ \sqrt{w_n}\, x_n \end{bmatrix} \Rightarrow f := \begin{bmatrix} \sqrt{w_1} & 0 & 0 \\ 0 & \sqrt{w_2} & 0 \\ 0 & \vdots & 0 \\ 0 & & \sqrt{w_n} \end{bmatrix}$$

This implies

$$||x||_w = ||y||_2$$

since f is a liniar transformation

$\Rightarrow f(x_1 + x_2) = f(x_1) + f(x_2)$
$= y_1 + y_2$

let $x_1 \to y_1$
$x_2 \to y_2$ Now from L2 norm's inequality

$$||y_1||_2 + ||y_2||_2 \geq ||y_1 + y_2||_2$$

$\Rightarrow ||x_1||_w = ||y_1||_2$

$||x_2||_w = ||y_2||_2 \quad \Rightarrow \quad ||x_1||_w + ||x_2||_w \geq ||x_1 + x_2||_w$

$||x_1 + x_2||_w = ||y_1 + y_2||_2$

as $f(x_1 + x_2) = f(x_1) + f(x_2)$

$\Rightarrow \therefore ||x||_w$ is a norm called weighted norm