



## **BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

### **Real Analysis (MAT221)**

## **Properties of the Real Numbers**

### **Consequences of the Axiom of Completeness**

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CONDUCTED BY

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# The Absolute Value Function

## Absolute Value

For any real number  $x$ , the **absolute value** of  $x$ , denoted by  $|x|$ , is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

## Theorem

Show that for any real numbers  $x$ , the following holds:

$$\sqrt{x^2} = |x|$$

## The Triangle Inequality

### 💡 The Triangle Inequality for Real Numbers

For any real numbers  $x$  and  $y$ , the following inequality holds:

$$|x + y| \leq |x| + |y|$$



## 💡 Other forms of the Triangle Inequality

For any real numbers  $x$  and  $y$ , the following inequalities hold:

$$|x - y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

## Equality of two Real Numbers

### 💡 Theorem

Two real numbers  $a$  and  $b$  are equal if and only if for every  $\varepsilon > 0$ ,

$$|a - b| < \varepsilon$$



## Axiom of Completeness (AoC)

### Least Upper Bound (Supremum)

A real number  $s$  is the *least upper bound* for a set  $A \subseteq \mathbb{R}$  if it meets the following two criteria:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $u$  is any upper bound for  $A$ , then  $s \leq u$ .

### Axiom of Completeness (AoC)

Every non-empty set of real numbers that is bounded above has a least upper bound (supremum) in  $\mathbb{R}$ .

## Greatest Lower Bound (Infimum)

A real number  $t$  is the *greatest lower bound* for a set  $A \subseteq \mathbb{R}$  if it meets the following two criteria:

- (i)  $t$  is a lower bound for  $A$ ;
- (ii) if  $l$  is any lower bound for  $A$ , then  $t \geq l$ .

### Theorem: Existence of Infimum

Show that every non-empty set of real numbers that is bounded below has a greatest lower bound (infimum) in  $\mathbb{R}$ . 

## Theorem: Characterization of Supremum

Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then,  $s = \sup A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying

$$s - \epsilon < a$$

## Theorem: Characterization of Infimum

Assume  $t \in \mathbb{R}$  is a lower bound for a set  $A \subseteq \mathbb{R}$ . Then,  $t = \inf A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying

$$a < t + \epsilon$$

## Consequences of Completeness (AoC)

## Nested Interval Property (NIP)

### ≡ Nested Interval Property (NIP)

For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ . Assume also that each  $I_n$  contains  $I_{n+1}$ . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

### 💡 Theorem: AoC $\implies$ NIP

The Axiom of Completeness implies the Nested Interval Property.

## Archimedean Property (AP)

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Given any number  $x \in \mathbb{R}$  and any real number  $y > 0$ , the following two statements hold:

- (i) *There exists an  $n \in \mathbb{N}$  satisfying  $n > x$ .*
- (ii) *There exists an  $n \in \mathbb{N}$  satisfying  $1/n < y$ .*

### 💡 Theorem: AoC $\implies$ AP

The Axiom of Completeness implies the Archimedean Property.

## NIP + AP $\implies$ AoC

### 💡 Theorem: NIP + AP $\implies$ AoC

The Nested Interval Property and the Archimedean Property together imply the Axiom of Completeness.

## Density of Rationals in Reals

### 💡 Density of $\mathbb{Q}$ in $\mathbb{R}$

For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

## The Existence of Square Roots

### 💡 Theorem

*There exists a real number  $\alpha \in \mathbb{R}$  satisfying  $\alpha^2 = 2$ .*



# Thank You!

We'd love your questions and feedback.

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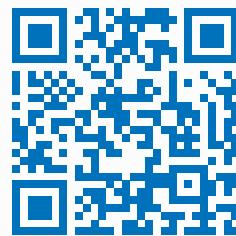
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(Lectures, walkthroughs, and course updates)



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## References

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- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.