



## **BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

### **Real Analysis (MAT221)**

## **Cardinality of Sets**

**Countable and Uncountable Sets, Continuum Hypothesis**

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CONDUCTED BY

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## Injective / One-to-One Functions

### Injective / One-to-One Function

Let  $A$  and  $B$  be two sets. A function  $f : A \rightarrow B$  is called *injective* or *one-to-one* if for every  $a_1, a_2 \in A$ ,

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

In other words, different elements in the domain map to different elements in the codomain.

## Surjective / Onto Functions

### ■ Surjective / Onto Function

Let  $A$  and  $B$  be two sets. A function  $f : A \rightarrow B$  is called *surjective* or *onto* if for every  $b \in B$ , there exists at least one  $a \in A$  such that  $f(a) = b$ . In other words, every element in the codomain is the image of at least one element from the domain.

## Bijective Functions

### Bijective Function

Let  $A$  and  $B$  be two sets. A function  $f : A \rightarrow B$  is called *bijective* if it is both injective and surjective.

## Idea of Cardinality and Historical Context

## Equivalent Sizes of Sets

### Comparing Sizes of Sets by Bijective Functions

If there exists an Bijective function from set  $A$  to set  $B$ , we say  that the cardinality of  $A$  is equal to the cardinality of  $B$ , denoted as

$$\text{card}(A) = \text{card}(B)$$

Often denoted as  $A \sim B$  to indicate that sets  $A$  and  $B$  have the same cardinality.

### ① Problem

Show that the set of natural numbers  $\mathbb{N}$  and the set of even natural numbers  $\mathbb{E} = \{2, 4, 6, 8, \dots\}$  have the same cardinality.

## ② Problem

Show that the set of natural numbers  $\mathbb{N}$  and the set of perfect squares  $\mathbb{S} = \{1, 4, 9, 16, 25, \dots\}$  have the same cardinality.

### ■ Finite Set

A set  $A$  is called *finite* if either  $A$  is the empty set or there exists a bijective function between the set  $N_m = \{1, 2, \dots, m\}$  for some natural number  $m$  and the set  $A$ . In this case, we say that the cardinality of  $A$  is  $m$  and write  $\text{card}(A) = m$ .

### Infinite Set

A set  $A$  is called *infinite* if it is not finite. Thus,  $A$  is not the empty set and for every natural number  $m$ , there is no bijective function between the set  $N_m = \{1, 2, \dots, m\}$  and the set  $A$ .

## Countable Sets

### ■ Countable Set

A set  $A$  is called *countable* if it is either finite or has the same cardinality as the set of natural numbers  $\mathbb{N}$ .



To prove a set  $A$  is countable, we can either:

- Show that  $A$  is finite, or
- Construct a bijective function between  $\mathbb{N}$  and  $A$ .

Note: An infinite set which is countable is often referred to as *countably infinite* or *denumerable*. For this type of set we can list its elements in a sequence (like the natural numbers).

 **Problem**

Show that the set of all even integers is countable.



 **Problem**

Show that the set of all odd integers is countable.



## ?(?) Problem

Show that the set of integers  $\mathbb{Z}$  is countable.



## Uncountable Sets

### ≡ Uncountable Set

A set  $A$  is called *uncountable* if it is not countable. This means that  $A$  is infinite and there is no bijective function between  $\mathbb{N}$  and  $A$ .

 **Problem**

Show that the interval  $(0, 1)$  is uncountable.



## ① Problem

The set of real numbers  $\mathbb{R}$  is uncountable.



### ② Problem

For any real numbers  $a$  and  $b$  with  $a < b$ , the open interval  $(a, b)$  and the closed interval  $[a, b]$  are uncountable.

## Theorems on Countability

### Theorem

Every subset of a countable set is countable. That is, if  $A$  is a countable set and  $B \subseteq A$ , then  $B$  is countable.

## Theorem

The union of two countable sets is countable. That is, if  $A$  and  $B$  are countable sets, then  $A \cup B$  is countable.

## Theorem

The Cartesian product of two countable sets is countable. That is, if  $A$  and  $B$  are countable sets, then  $A \times B$  is countable.

## ② Problem

The set of rational numbers  $\mathbb{Q}$  is countable.



## Theorem

The union of countably many countable sets is countable. That is, if  $\{A_n\}_{n=1}^{\infty}$  is a sequence of countable sets, then  $\cup_{n=1}^{\infty} A_n$  is countable.

## ② Problem

The set of rational numbers  $\mathbb{Q}$  is countable.



## The Schröder–Bernstein Theorem

### 💡 Schröder–Bernstein Theorem

Let  $A$  and  $B$  be two sets. If there exist injective functions  $f : A \rightarrow B$  and  $g : B \rightarrow A$ , then there exists a bijective function  $h : A \rightarrow B$ . In other words, if  $\text{card}(A) \leq \text{card}(B)$  and  $\text{card}(B) \leq \text{card}(A)$ , then  $\text{card}(A) = \text{card}(B)$ .

### ■ Cardinal Number

For every set  $X$ , there exists a *cardinal number*, denoted  $\text{card } X$ , which represents the size of  $X$ . Two sets  $X$  and  $Y$  have the same cardinal number if and only if  $X \sim Y$ :

$$\text{card } X = \text{card } Y \iff \exists f : X \rightarrow Y \text{ that is bijective.}$$

## Ordering of Cardinal Numbers

### ■ Ordering of Cardinal Numbers

For any sets  $A$  and  $B$ :



1.  $\text{card}(A) \leq \text{card}(B)$  if and only if there exists an injective function  $f : A \rightarrow B$ .
2.  $\text{card}(A) = \text{card}(B)$  if and only if there exists a bijective function  $f : A \rightarrow B$ .
3.  $\text{card}(A) < \text{card}(B)$  if and only if  $\text{card}(A) \leq \text{card}(B)$  but  $A \not\sim B$ .

## Cantor's Theorem

### 💡 Cantor's Theorem

For every set  $A$ , the power set  $\mathcal{P}(A)$  has strictly greater cardinality than  $A$ :

$$\text{card } A < \text{card } \mathcal{P}(A).$$

Cantor's theorem implies that there is no “largest” set, since from any set  $A$  we can always construct a larger one  $\mathcal{P}(A)$ . Hence, a “set of all things” cannot exist without contradiction.

## Cardinality of Finite Sets

### Cardinality of Finite Sets

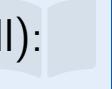
If  $A$  is a finite set with  $n$  elements, then

$$\text{card } A = n.$$



# Cardinality of Infinite Sets

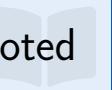
## Countable Infinity

The smallest infinite cardinal number is denoted by  $\aleph_0$  (aleph-null): 

$$\aleph_0 = \text{card}(\mathbb{N}).$$

It represents the size of any countably infinite set, such as  $\mathbb{Z}$  or  $\mathbb{Q}$ .

## Continuum

The cardinality of the real numbers (or the power set of  $\mathbb{N}$ ) is denoted by 

$$\mathfrak{c} = \text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}.$$

By Cantor's theorem, we have  $\aleph_0 < \mathfrak{c}$ .

## Cantor's Power Set Theorem

For every set  $A$ ,

$$\text{card}(\mathcal{P}(A)) = 2^{\text{card}(A)} > \text{card}(A).$$



## Hierarchy of Infinite Cardinalities

Applying Cantor's theorem repeatedly yields an infinite sequence of strictly increasing cardinalities:

$$\aleph_0 = \text{card}(\mathbb{N}),$$

$$2^{\aleph_0} = \mathfrak{c} = \text{card}(\mathbb{R}),$$

$$2^{\mathfrak{c}} = \text{card}(\mathcal{P}(\mathbb{R})),$$

$$2^{2^{\mathfrak{c}}} = \text{card}(\mathcal{P}(\mathcal{P}(\mathbb{R}))),$$

⋮

Thus,

$$\aleph_0 < \mathfrak{c} < 2^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}} < \dots$$



## Gaps Between Cardinalities

### ② Problem

Does there exist a set  $A \subseteq \mathbb{R}$  such that  $\aleph_0 < \text{card } A < \mathfrak{c}$ ? This question leads to the *Continuum Hypothesis*.

## The Continuum Hypothesis (CH)

### Continuum Hypothesis

The *Continuum Hypothesis* asserts that there is no set  $A \subseteq \mathbb{R}$  such that

$$\aleph_0 < \text{card } A < \mathfrak{c}.$$

In other words, the cardinality of the continuum  $\mathfrak{c}$  is the immediate successor of  $\aleph_0$ .

# Thank You!

We'd love your questions and feedback.

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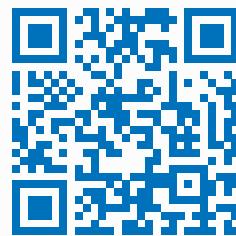
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(Lectures, walkthroughs, and course updates)



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## References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.