BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Subsequences and Limit Points

Bolzano-Weierstrass Theorem (BWT)

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Conducted by

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Subsequence

Subsequence

Let (a_n) be a sequence of real numbers, and let

$$n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$$

be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \cdots$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Other Definition of Subsequence

Subsequence

Let (a_n) be a sequence, and let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Then the sequence $(a_{f(k)})$ is called a **subsequence** of (a_n). In other words, a subsequence is obtained by choosing elements of (a_n) in the same order, but possibly omitting some of them.

Examples of Subsequence

Theorem

Any subsequence of a convergent sequence converge to the same limit as the original sequence.

Theorem

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Bolzano–Weierstrass Theorem

Bolzano–Weierstrass Theorem

Every bounded sequence contains a convergent subsequence.

NIP $\implies$ BWT

Let (a_n) be a bounded sequence. Using the Nested Interval property, show that there exists a convergent subsequence.

Aoc $\implies$ BWT

Let (a_n) be a bounded sequence. Using the Axiom of Completeness, show that there exists a convergent subsequence.

Theorem

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Theorem

Every sequence contains a monotonic subsequence.

Limit Point or Cluster Point

Limit Point

A real number l is said to be a limit point or cluster point of a sequence (a_n) if for every $\varepsilon > 0$, there exists a term a_m such that

$$|a_m - a| < \varepsilon.$$

Examples of Limit Points

Other Definition of Limit Point or Cluster Point

Limit Point

A real number l is said to be a limit point or cluster point of a sequence (a_n) if for every $\varepsilon > 0$, the ε neighbouhood

$$(l - \varepsilon, l + \varepsilon)$$

contains at least one term of the sequence.

Theorem

If a real number l is a limit point or cluster point of a sequence (a_n), then for every $\varepsilon > 0$, the ε neighbouhood

$$(l - \varepsilon, l + \varepsilon)$$

contains infinitely many terms of the sequence.

Theorem

If a sequence (a_n) converges to l, then l is the only limit point of the sequence.

Bolzano–Weierstrass Theorem (Limit Point Vesion)

Bolzano–Weierstrass Theorem

Every hounded sequence has at least one limit point.

Limit Superior and Inferior

Limit Superior

The Limit Superior of a bounded sequence is the latgest limit point.

Limit Inferior

The Limit Inferior of a bounded sequence is the smallest limit point.

Thank You!

We'd love your questions and feedback.

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References

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[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.