BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Properties of the Real Numbers

Consequences of the Axiom of Completeness

October 17, 2025

Conducted by

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

✉ partho.dhor@bracu.ac.bd | ✉ parthosutradhor@gmail.com
For updates subscribe on   ▶ @ParthoSutraDhor

The Absolute Value Function

📖 Absolute Value

For any real number x, the **absolute value** of x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

💡 Theorem

Show that for any real numbers x, the following holds:

$$\sqrt{x^2} = |x|$$

The Triangle Inequality

> **💡 The Triangle Inequality for Real Numbers**
>
> For any real numbers x and y, the following inequality holds:
>
> $$|x + y| \leq |x| + |y|$$

💡 Other forms of the Triangle Inequality

For any real numbers x and y, the following inequalities hold:

$$|x - y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

Equality of two Real Numbers
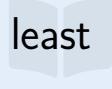
> **Theorem**
>
> Two real numbers a and b are equal if and only if for every $\varepsilon > 0$,
>
> $$|a - b| < \varepsilon$$

Axiom of Completeness

Axiom of Completeness

Every non-empty set of real numbers that is bounded above has a least upper bound (supremum) in $\mathbb{R}$.

Least Upper Bound (Supremum)

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if u is any upper bound for A, then $s \leq u$.

> **💡 Theorem**
>
> Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying
> $$s - \epsilon < a$$

📖 Greatest Lower Bound (Infimum)

A real number t is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) t is a lower bound for A;

(ii) if l is any lower bound for A, then $t \geq l$.

💡 Theorem

Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$a < t + \epsilon$$

Consequences of Completeness

Nested Interval Property (NIP)

💡 Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{ x \in \mathbb{R} : a_n \leq x \leq b_n \}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Archimedean Property (AP)

> **♀ Archimedean Property**
>
> Given any number $x \in \mathbb{R}$ and any real number $y > 0$, the following two statements hold:
>
> (i) *There exists an $n \in \mathbb{N}$ satisfying $n > x$.*
>
> (ii) *There exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

Density of Rationals in Reals

> 💡 **Density of $\mathbb{Q}$ in $\mathbb{R}$**
>
> *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

The Existence of Square Roots

> ### 💡 Theorem
> There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

Thank You!

We'd love your questions and feedback.
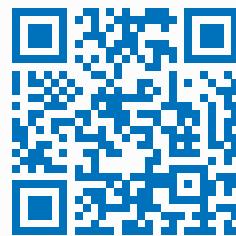
Partho Sutra Dhor

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▶ **@ParthoSutraDhor**

(Lectures, walkthroughs, and course updates)

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📖 References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.