BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Theorems on Limit of Sequences

Algebraic Properties, Order Properties and Squeeze Theorem

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Conducted by

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Convergence and Limit of a Sequence

Convergence of a Sequence

A sequence $\{a_n\}$ is said to **converge** to a real number a if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number a exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \to a \text{ as } n \to \infty.$$

Unique Limit of a Sequence

Limit is Unique

If a sequence $\{a_n\}$ converges, then its limit is unique.

Bounded and Unbounded Sequences

Bounded Sequence

A sequence $\{a_n\}$ is said to be **bounded** if there exists a real number $M > 0$ such that for all natural numbers $n \in \mathbb{N}$, the terms of the sequence satisfy the inequality

$$|a_n| \leq M.$$

If no such M exists, the sequence is said to be **unbounded**.

Every Convergent Sequence is Bounded

Theorem

Every convergent sequence is bounded.

Theorem

Not every bounded sequence is convergent.

Bounded $\not\Rightarrow$ Convergent

Consider the sequence $a_n = (-1)^n$. Show that this sequence is bounded but does not converge.

Algebraic Properties of Limits

Algebraic Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. Then,

1. $\lim c = c$ for any constant sequence c.
2. $\lim |a_n| = |a|$.
3. $\lim(a_n + b_n) = a + b$.
4. $\lim(a_n - b_n) = a - b$.
5. c, $\lim(ca_n) = ca$, for any constant $c \in \mathbb{R}$.
6. $\lim(a_n b_n) = ab$.
7. $\lim \left(\frac{1}{a_n}\right) = \frac{1}{a}$, provided $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$.
8. $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$.

Theorem

Prove that

$$\lim c = c$$

for any constant sequence c.

Theorem

If $\lim a_n = a$, then

$$\lim |a_n| = |a|.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n + b_n) = a + b.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n - b_n) = a - b.$$

Theorem

If $\lim a_n = a$, then for any constant $c \in \mathbb{R}$,

$$\lim(ca_n) = ca.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n b_n) = ab.$$

Theorem

If $\lim a_n = a$ with $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim \left(\frac{1}{a_n}\right) = \frac{1}{a}.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$ with $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim \left(\frac{a_n}{b_n} \right) = \frac{a}{b}.$$

Order Properties of Limits

Order Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. Then,

1. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \geq 0 \text{ for all } n \geq N \implies a \geq 0.$$

2. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \leq b_n \text{ for all } n \geq N \implies a \leq b.$$

Squeeze Theorem

Let $\{a_n\}$, $\{x_n\}$, and $\{b_n\}$ be three sequences such that $a_n \leq x_n \leq b_n$ for all n greater than or equal to some natural number $N \in \mathbb{N}$. If $\lim a_n = \lim b_n = L$, then $\lim x_n = L$.

Theorem: $a_n \geq 0$ implies $a \geq 0$

Let $\{a_n\}$ be a convergent sequence with limit a. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \geq 0 \text{ for all } n \geq N \implies a \geq 0.$$

Theorem: $a_n > 0$ implies $a \geq 0$

Let $\{a_n\}$ be a convergent sequence with limit a. If there exists a natural number $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n > 0 \text{ for all } n \geq N \implies a \geq 0.$$

Theorem: $a_n \leq b_n$ implies $a \leq b$

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \leq b_n \text{ for all } n \geq N \implies a \leq b.$$

Theorem: $a_n < b_n$ implies $a \leq b$

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. If there exists a natural number $N \in \mathbb{N}$ such that $a_n < b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n < b_n \text{ for all } n \geq N \implies a \leq b.$$

Squeeze Theorem

Let $\{a_n\}$, $\{x_n\}$, and $\{b_n\}$ be three sequences such that $a_n \leq x_n \leq b_n$ for all n greater than or equal to some natural number $N \in \mathbb{N}$. If $\lim a_n = \lim b_n = L$, then $\lim x_n = L$.

Prove using Squeeze Theorem

Prove that
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Ratio of Sequences

Theorem

Let $\{a_n\}$ be a sequence such that $a_n \neq 0$ for any n. If

$$\lim \frac{a_{n+1}}{a_n} = L,$$

then

1. if $|L| < 1$, the sequence $\{a_n\}$ converges to 0,
2. if $|L| > 1$, the sequence $\{a_n\}$ diverges,
3. if $|L| = 1$, the test is inconclusive.

Problem

Prove that

$$\lim_{n\to\infty} \frac{(n^n)}{n!} = e$$

Root of Sequences

Theorem

If $\{a_n\}$ is a sequence of posite terms and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists, then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Problem

Prove that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Problem

Prove that

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$$

Cesaro Means

Theorem

If a sequence $\{a_n\}$ converges to a limit L, then the sequence $\{x_n\}$ defined by

$$x_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

converges to the same limit L.

Problem

Prove that

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}\right) = 1$$

Thank You!

We'd love your questions and feedback.

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References

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