BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Properties of the Real Numbers

Consequences of the Axiom of Completeness

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Conducted by

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The Absolute Value Function

Absolute Value

For any real number x, the **absolute value** of x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem

Show that for any real numbers x, the following holds:

$$\sqrt{x^2} = |x|$$

The Triangle Inequality

The Triangle Inequality for Real Numbers

For any real numbers x and y, the following inequality holds:

$$|x + y| \leq |x| + |y|$$

Other forms of the Triangle Inequality

For any real numbers x and y, the following inequalities hold:

$$|x - y| \le |x| + |y|$$

$$||x| - |y|| \le |x - y|$$

Equality of two Real Numbers

Theorem

Two real numbers a and b are equal if and only if for every $\varepsilon > 0$,

$$|a - b| < \varepsilon$$

Axiom of Completeness (AoC)

Least Upper Bound (Supremum)

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if u is any upper bound for A, then $s \leq u$.

Axiom of Completeness (AoC)

Every non-empty set of real numbers that is bounded above has a least upper bound (supremum) in $\mathbb{R}$.

Greatest Lower Bound (Infimum)

A real number t is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) t is a lower bound for A;

(ii) if l is any lower bound for A, then $t \geq l$.

Theorem: Existence of Infimum

Show that every non-empty set of real numbers that is bounded below has a greatest lower bound (infimum) in $\mathbb{R}$.

Theorem: Characterization of Supremum

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$s - \epsilon < a$$

Theorem: Characterization of Infimum

Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$a < t + \epsilon$$

Consequences of Completeness (AoC)

Nested Interval Property (NIP)

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For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{\, x \in \mathbb{R} : a_n \leq x \leq b_n \,\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Theorem: AoC $\implies$ NIP

The Axiom of Completeness implies the Nested Interval Property.

Archimedean Property (AP)

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Given any number $x \in \mathbb{R}$ and any real number $y > 0$, the following two statements hold:

(i) *There exists an* $n \in \mathbb{N}$ *satisfying* $n > x$.

(ii) *There exists an* $n \in \mathbb{N}$ *satisfying* $1/n < y$.

Theorem: AoC $\implies$ AP

The Axiom of Completeness implies the Archimedean Property.

NIP + AP $\implies$ AoC

Theorem: NIP + AP $\implies$ AoC

The Nested Interval Property and the Archimedean Property together imply the Axiom of Completeness.

Density of Rationals in Reals

Density of $\mathbb{Q}$ in $\mathbb{R}$

For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

The Existence of Square Roots

Theorem

There exists a real number $\alpha \in \mathbb{R}$ *satisfying* $\alpha^2 = 2$.

Thank You!

We'd love your questions and feedback.

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References

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[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.