BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Property of Real Numbers

Consequences of the Axiom of Completeness

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CONDUCTED BY

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The Absolute Value Function

Absolute Value

For any real number x, the **absolute value** of x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem

Show that for any real numbers x, the following holds:

$$\sqrt{x^2} = |x|$$

The Triangle Inequality

> ### 💡 The Triangle Inequality for Real Numbers
>
> For any real numbers x and y, the following inequality holds:
>
> $$|x + y| \leq |x| + |y|$$

💡 Other forms of the Triangle Inequality

For any real numbers x and y, the following inequalities hold:

$$|x - y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

Equality of two Real Numbers
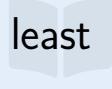
> **Theorem**
>
> Two real numbers a and b are equal if and only if for every $\varepsilon > 0$,
>
> $$|a - b| < \varepsilon$$

Axiom of Completeness

📖 Axiom of Completeness

Every non-empty set of real numbers that is bounded above has a least upper bound (supremum) in $\mathbb{R}$.

📖 Least Upper Bound (Supremum)

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if u is any upper bound for A, then $s \leq u$.

> **💡 Theorem**
>
> Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying
> $$s - \epsilon < a$$

📘 Greatest Lower Bound (Infimum)

A real number t is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) t is a lower bound for A;

(ii) if l is any lower bound for A, then $t \geq l$.

💡 Theorem

Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$a < t + \epsilon$$

Consequences of Completeness

Nested Interval Property (NIP)

💡 Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{ x \in \mathbb{R} : a_n \leq x \leq b_n \}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Archimedean Property (AP)

> **💡 Archimedean Property**
>
> Given any number $x \in \mathbb{R}$ and any real number $y > 0$, the following two statements hold:
>
> (i) *There exists an $n \in \mathbb{N}$ satisfying $n > x$.*
>
> (ii) *There exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

Density of Rationals in Reals

> **💡 Density of $\mathbb{Q}$ in $\mathbb{R}$**
>
> *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

The Existence of Square Roots

> **Theorem**
>
> *There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*

Thank You!

We'd love your questions and feedback.
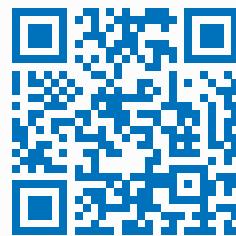
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(Lectures, walkthroughs, and course updates)

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📖 References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.