BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Complex Variables & Laplace Transform (MAT215)

Limit and Continuity of Complex Functions

L'Hospital's Rule and Existence of Limits

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Conducted by

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Undefined vs Indeterminate Forms

An Intuitive Approach to Limits

Problem

Find the limit, if it exists:

$$\lim_{x \to 2} x^2$$

Problem

Find the limit, if it exists:

$$\lim_{x \to 2} \frac{x^2 - 2}{x - 2}$$

Problem

Find the limit, if it exists:

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

Left and Right Hand Limits

Problem

Find the limit, if it exists:

$$\lim_{x \to 5} \frac{x}{5 - x}$$

Limit of a Real-valued Function

Limit of a Real-valued Function

Let $f : D \to \mathbb{R}$ be a function defined on a domain $D \subseteq \mathbb{R}$. We say that the limit of $f(x)$ as x approaches c is L, denoted by

$$\lim_{x \to c} f(x) = L,$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$ with $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \epsilon$.

Limit of a Complex-valued Function

Limit of a Complex-valued Function

Let $f : D \to \mathbb{C}$ be a function defined on a domain $D \subseteq \mathbb{C}$. We say that the limit of $f(z)$ as z approaches z_0 is L, denoted by

$$\lim_{z \to z_0} f(z) = L,$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z \in D$ with $0 < |z - z_0| < \delta$, it follows that $|f(z) - L| < \epsilon$.

How many ways can z approach a point z_0?

Limit (path approach)

If in any path that z approaches z_0, the limit of $f(z)$ is the same value L, then we say that the limit of $f(z)$ as z approaches z_0 exists and is equal to L. Otherwise, we say that the limit does not exist.

Problem

Show that the limit

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

does not exist.

Problem

Show that the limit

$$\lim_{z\to a} \frac{\bar{z}}{z}$$

does not exist for any $a \in \mathbb{C}$.

Problem

Show that the limit

$$\lim_{z \to 0} \frac{xy}{|z|^2}$$

does not exist where $z = x + iy$.

Properties of Limits

Theorem

Let $f(z)$ and $g(z)$ be functions defined on a domain $D \subseteq \mathbb{C}$, and let z_0 be a limit point of D. If

$$\lim_{z \to z_0} f(z) = L_1 \quad \text{and} \quad \lim_{z \to z_0} g(z) = L_2,$$

then the following properties hold:

1. $\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2$
2. $\lim_{z \to z_0} [f(z) - g(z)] = L_1 - L_2$
3. $\lim_{z \to z_0} [f(z) g(z)] = L_1 L_2$
4. If $L_2 \neq 0$, then $\lim_{z \to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{L_1}{L_2}$

Some Important Techniques for Finding the Limit of $\frac{P(z)}{Q(z)}$

Limit of Polynomials

If $f(z)$ is a polynomials in z, then the limit $\lim_{z\to z_0} f(z) = f(z_0)$ for any $z_0 \in \mathbb{C}$.

Something / Non-zero

If $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials, and if $Q(z_0) \neq 0$, then the limit $\lim_{z\to z_0} f(z) = \frac{P(z_0)}{Q(z_0)}$.

Non-zero / Zero

If $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials, and if $Q(z_0) = 0$ while $P(z_0) \neq 0$, then the limit $\lim_{z\to z_0} f(z)$ does not exist.

Zero / Zero

If $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials, and if $Q(z_0) = 0$ also $P(z_0) = 0$, then the limit $\lim_{z\to z_0} f(z)$ may exist. In this case, we can try to simplify $f(z)$ by factoring both $P(z)$ and $Q(z)$ and then canceling out the common factors. After simplification, we can re-evaluate the limit.

Problem

Find the limit, if it exists:

$$\lim_{z \to 1+i} z^2 - 5z + 10$$

Problem

Find the limit, if it exists:

$$\lim_{z \to \frac{i}{2}} \frac{(2z-3)(4z+i)}{(iz-1)^2}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to i} \frac{z^2 - 1}{z^6 + 1}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to 1+i} \left\{ \frac{z - 1 - i}{z^2 - 2z + 2} \right\}^2$$

Problem

Let $f(z) = \frac{z^2+1}{z-i}$. Prove that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{7}{(3z_0 + 2)^2}, \quad z_0 \neq -\tfrac{2}{3}.$$

L'Hospital's Rule

L'Hospital's Rule

Let $f(z)$ and $g(z)$ be functions that are differentiable on an open interval containing z_0, except possibly at z_0 itself. If

$$\lim_{z \to z_0} f(z) = 0 \quad \text{and} \quad \lim_{z \to z_0} g(z) = 0,$$

and if $g'(z) \neq 0$ for all z in the interval except possibly at z_0, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)},$$

provided the limit on the right side exists or is infinite.

Problem

Find the limit, if it exists:

$$\lim_{z \to 0} \frac{z - \sin z}{z^3}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to 0} \frac{z - \tan z z}{z^3}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to e^{i\pi/3}} \left(z - e^{i\pi/3}\right) \frac{z}{z^3 + 1}$$

Exponential Forms

Problem

Find the limit, if it exists:

$$\lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{\frac{1}{z^2}}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{\frac{\sin z}{z - \sin z}}$$

Problem

Find the limit, if it exists:

$$\lim_{z\to 0}\left(\frac{\tan z}{z}\right)^{\frac{1}{z^2}}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to 0} (\sec z)^{\frac{1}{z^2}}$$

Problem

Find the limit, if it exists:

$$\lim_{z \to 0} (\cos z)^{\frac{1}{z^3}}$$

Continuity of Complex Functions

Continuity of Complex Functions

A function $f : D \to \mathbb{C}$ is said to be continuous at a point $z_0 \in D$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If f is continuous at every point in its domain D, then we say that f is continuous on D.

Note

3 conditions must be satisfied for a function f to be continuous at a point z_0:

1. $f(z_0)$ is defined.
2. $\lim_{z \to z_0} f(z)$ exists.
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

If any of these conditions fail, then f is not continuous at z_0.

Problem

Let

$$f(z) = \begin{cases} \dfrac{z^2 - 4}{z^2 - 3z + 2}, & z \neq 2, \\ kz^2 + 6, & z = 2 \end{cases}$$

Find the value of k such that $f(z)$ is continuous at $z = 2$.

Problem

Let

$$f(z) = \begin{cases} \dfrac{1 - \cos(az)}{z^2}, & z \neq 0, \\ 1, & z = 0 \end{cases}$$

Find the value of a such that $f(z)$ is continuous at $z = 0$.

Problem

Let

$$f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$$

1. Is f continuous at $z = i$?
2. If f is not continuous at $z = i$, redefine $f(z)$ such that it becomes continuous at $z = i$.

Discontinuity of Complex Functions

Discontinuity of Complex Functions

A function $f : D \to \mathbb{C}$ is said to be discontinuous at a point $z_0 \in D$ if it is not continuous at z_0. In other words, f is discontinuous at z_0 if

$$\lim_{z \to z_0} f(z) \neq f(z_0),$$

or if the limit does not exist, or if $f(z_0)$ is not defined.

Problem

Find all points of discontinuity for the following function

$$f(z) = \frac{2z-3}{z^2+2z+2}$$

Problem

Find all points of discontinuity for the following function

$$f(z) = \frac{3z^2 + 4}{z^4 - 16}$$

Problem

Find all points of discontinuity for the following function

$$f(z) = \cot z$$

Problem

Find all points of discontinuity for the following function

$$f(z) = \frac{1}{z} - \sec z$$

Problem

Find all points of discontinuity for the following function

$$f(z) = \frac{\tanh z}{z^2 + 1}$$

Thank You!

We'd love your questions and feedback.

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(Lectures, walkthroughs, and course updates)

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References

[1] Murray R. Spiegel, *Schaum's Outline of Complex Variables*, 2nd Edition, McGraw–Hill, 2009.

[2] Dennis G. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th Edition, Cengage, 2018.