BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Cardinality of Sets

Countable and Uncountable Sets, Continuum Hypothesis

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Conducted by

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com

For updates subscribe on @ParthoSutraDhor

Injective / One-to-One Functions

Injective / One-to-One Function

Let A and B be two sets. A function $f : A \to B$ is called *injective* or *one-to-one* if for every $a_1, a_2 \in A$,

$$f(a_1) = f(a_2) \quad \Longrightarrow \quad a_1 = a_2$$

In other words, different elements in the domain map to different elements in the codomain.

Surjective / Onto Functions

Surjective / Onto Function

Let A and B be two sets. A function $f : A \to B$ is called *surjective* or *onto* if for every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$. In other words, every element in the codomain is the image of at least one element from the domain.

Bijective Functions

Bijective Function

Let A and B be two sets. A function $f : A \to B$ is called *bijective* if it is both injective and surjective.

Idea of Cardinality and Historical Context

Equivalent Sizes of Sets

Comparing Sizes of Sets by Bijective Functions

If there exists an Bijective function from set A to set B, we say that the cardinality of A is equal to the cardinality of B, denoted as

$$\operatorname{card}(A) = \operatorname{card}(B)$$

Often denoted as $A \sim B$ to indicate that sets A and B have the same cardinality.

Problem

Show that the set of natural numbers $\mathbb{N}$ and the set of even natural numbers $\mathbb{E} = \{2, 4, 6, 8, \ldots\}$ have the same cardinality.

Problem

Show that the set of perfect squares $\{1, 4, 9, 16, 25, \ldots\}$ is countable.

Finite Sets

Finite Set

A set A is called *finite* if either A is the empty set or there exists a bijective function between the set $N_m = \{1, 2, \ldots, m\}$ for some natural number m and the set A. In this case, we say that the cardinality of A is m and write $\mathrm{card}(A) = m$.

Infinite Sets

Infinite Set

A set A is called *infinite* if it is not finite. Thus, A is not the empty set and for every natural number m, there is no bijective function between the set $N_m = \{1, 2, \ldots, m\}$ and the set A.

Countable Sets

Countable Set

A set A is called *countable* if it is either finite or has the same cardinality as the set of natural numbers $\mathbb{N}$.

To prove a set A is countable, we can either:

- Show that A is finite, or
- Construct a bijective function between $\mathbb{N}$ and A.

Problem

Show that the set of all even integers is countable.

Problem

Show that the set of all odd integers is countable.

Problem

Show that the set of integers $\mathbb{Z}$ is countable.

Uncountable Sets

Uncountable Set

A set A is called *uncountable* if it is not countable. This means that A is infinite and there is no bijective function between $\mathbb{N}$ and A.

Problem

Show that the interval $(0, 1)$ is uncountable.

Problem

The set of real numbers $\mathbb{R}$ is uncountable.

Problem

For any real numbers a and b with $a < b$, the open interval $(a, b)$and the closed interval $[a, b]$ are uncountable.

Theorems on Countability

Theorem

Every subset of a countable set is countable. That is, if A is a countable set and $B \subseteq A$, then B is countable.

Theorem

The union of two countable sets is countable. That is, if A and B are countable sets, then $A \cup B$ is countable.

Theorem

The Cartesian product of two countable sets is countable. That is, if A and B are countable sets, then $A \times B$ is countable.

The set of rational numbers $\mathbb{Q}$ is countable.

Theorem

The union of countably many countable sets is countable. That is, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of countable sets, then $\cup_{n=1}^{\infty} A_n$ is countable.

Problem

The set of rational numbers $\mathbb{Q}$ is countable.

Cardinal Numbers

Cardinal Number

For every set X, there exists a *cardinal number*, denoted $\operatorname{card} X$, which represents the size of X. Two sets X and Y have the same cardinal number if and only if $X \sim Y$:

$$\operatorname{card} X = \operatorname{card} Y \iff \exists f : X \to Y \text{ that is bijective.}$$

Cantor's Theorem

Cantor's Theorem

For every set A, the power set $\mathcal{P}(A)$ has strictly greater cardinality than A:

$$\operatorname{card} A < \operatorname{card} \mathcal{P}(A).$$

Cantor's theorem implies that there is no "largest" set, since from any set A we can always construct a larger one $\mathcal{P}(A)$. Hence, a "set of all things" cannot exist without contradiction.

Ordering of Cardinal Numbers

Ordering of Cardinal Numbers

For any sets A and B:

1. $\mathrm{card}(A) \leq \mathrm{card}(B)$ if and only if there exists an injective function $f : A \to B$.
2. $\mathrm{card}(A) = \mathrm{card}(B)$ if and only if there exists a bijective function $f : A \to B$.
3. $\mathrm{card}(A) < \mathrm{card}(B)$ if and only if $\mathrm{card}(A) \leq \mathrm{card}(B)$ but $A \not\sim B$.

Cardinality of Finite Sets

Cardinality of Finite Sets

If A is a finite set with n elements, then

$$\operatorname{card} A = n.$$

Cardinality of Infinite Sets

Countable Infinity

The smallest infinite cardinal number is denoted by $\aleph_0$ (aleph-null):

$$\aleph_0 = \mathrm{card}(\mathbb{N}).$$

It represents the size of any countably infinite set, such as $\mathbb{Z}$ or $\mathbb{Q}$.

Continuum

The cardinality of the real numbers (or the power set of $\mathbb{N}$) is denoted by

$$\mathfrak{c} = \mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}.$$

By Cantor's theorem, we have $\aleph_0 < \mathfrak{c}$.

Cantor's Power Set Theorem

For every set A,

$$\mathrm{card}(\mathcal{P}(A)) = 2^{\mathrm{card}(A)} > \mathrm{card}(A).$$

Hierarchy of Infinite Cardinalities

Applying Cantor's theorem repeatedly yields an infinite sequence of strictly increasing cardinalities:

$$\begin{aligned}
\aleph_0 &= \mathrm{card}(\mathbb{N}), \\
2^{\aleph_0} &= \mathfrak{c} = \mathrm{card}(\mathbb{R}), \\
2^{\mathfrak{c}} &= \mathrm{card}(\mathcal{P}(\mathbb{R})), \\
2^{2^{\mathfrak{c}}} &= \mathrm{card}(\mathcal{P}(\mathcal{P}(\mathbb{R}))), \\
&\vdots
\end{aligned}$$

Thus,

$$\aleph_0 < \mathfrak{c} < 2^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}} < \cdots$$

Gaps Between Cardinalities

Problem

Does there exist a set $A \subseteq \mathbb{R}$ such that $\aleph_0 < \operatorname{card} A < \mathfrak{c}$? This question leads to the *Continuum Hypothesis*.

The Continuum Hypothesis (CH)

Continuum Hypothesis

The *Continuum Hypothesis* asserts that there is no set $A \subseteq \mathbb{R}$ such that

$$\aleph_0 < \operatorname{card} A < \mathfrak{c}.$$

In other words, the cardinality of the continuum $\mathfrak{c}$ is the immediate successor of $\aleph_0$.

Thank You!

We'd love your questions and feedback.

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com

@ParthoSutraDhor

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References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.