BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Theorems on Limit of Sequences

Algebraic Properties, Order Properties and Squeeze Theorem

OCTOBER 28, 2025

CONDUCTED BY

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com

For updates subscribe on @ParthoSutraDhor

Convergence and Limit of a Sequence

Convergence of a Sequence

A sequence $\{a_n\}$ is said to **converge** to a real number a if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number a exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \to a \text{ as } n \to \infty.$$

Unique Limit of a Sequence

Limit is Unique

If a sequence $\{a_n\}$ converges, then its limit is unique.

Bounded and Unbounded Sequences

Bounded Sequence

A sequence $\{a_n\}$ is said to be **bounded** if there exists a real number $M > 0$ such that for all natural numbers $n \in \mathbb{N}$, the terms of the sequence satisfy the inequality

$$|a_n| \leq M.$$

If no such M exists, the sequence is said to be **unbounded**.

Every Convergent Sequence is Bounded

Theorem

Every convergent sequence is bounded.

Theorem

Not every bounded sequence is convergent.

Bounded $\not\Rightarrow$ Convergent

Consider the sequence $a_n = (-1)^n$. Show that this sequence is bounded but does not converge.

Algebraic Properties of Limits

Algebraic Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. Then,

1. $\lim c = c$ for any constant sequence c.
2. $\lim |a_n| = |a|$.
3. $\lim(a_n + b_n) = a + b$.
4. $\lim(a_n - b_n) = a - b$.
5. c, $\lim(ca_n) = ca$, for any constant $c \in \mathbb{R}$.
6. $\lim(a_n b_n) = ab$.
7. $\lim\left(\frac{1}{a_n}\right) = \frac{1}{a}$, provided $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$.
8. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$.

Theorem

Prove that

$$\lim c = c$$

for any constant sequence c.

Theorem

If $\lim a_n = a$, then

$$\lim |a_n| = |a|.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n + b_n) = a + b.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n - b_n) = a - b.$$

Theorem

If $\lim a_n = a$, then for any constant $c \in \mathbb{R}$,

$$\lim(ca_n) = ca.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n b_n) = ab.$$

Theorem

If $\lim a_n = a$ with $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim \left(\frac{1}{a_n} \right) = \frac{1}{a}.$$

Theorem

If $\lim a_n = a$ and $\lim b_n = b$ with $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then
$$\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}.$$

Order Properties of Limits

Order Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. Then,

1. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \geq 0 \text{ for all } n \geq N \implies a \geq 0.$$

2. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \leq b_n \text{ for all } n \geq N \implies a \leq b.$$

Squeeze Theorem

Let $\{a_n\}, \{x_n\}$, and $\{b_n\}$ be three sequences such that $a_n \leq x_n \leq b_n$ for all n greater than or equal to some natural number $N \in \mathbb{N}$. If $\lim a_n = \lim b_n = L$, then $\lim x_n = L$.

Theorem: $a_n \geq 0$ implies $a \geq 0$

Let $\{a_n\}$ be a convergent sequence with limit a. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \geq 0 \text{ for all } n \geq N \implies a \geq 0.$$

Theorem: $a_n > 0$ implies $a \geq 0$

Let $\{a_n\}$ be a convergent sequence with limit a. If there exists a natural number $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n > 0 \text{ for all } n \geq N \implies a \geq 0.$$

Theorem: $a_n \leq b_n$ implies $a \leq b$

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \leq b_n \text{ for all } n \geq N \implies a \leq b.$$

Theorem: $a_n < b_n$ implies $a \le b$

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b, respectively. If there exists a natural number $N \in \mathbb{N}$ such that $a_n < b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n < b_n \text{ for all } n \geq N \implies a \leq b.$$

Squeeze Theorem

Let $\{a_n\}, \{x_n\}$, and $\{b_n\}$ be three sequences such that $a_n \leq x_n \leq b_n$ for all n greater than or equal to some natural number $N \in \mathbb{N}$. If $\lim a_n = \lim b_n = L$, then $\lim x_n = L$.

Prove using Squeeze Theorem

Prove that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Ratio of Sequences

Theorem

Let $\{a_n\}$ be a sequence such that $a_n \neq 0$ for any n. If

$$\lim \frac{a_{n+1}}{a_n} = L,$$

then

1. if $|L| < 1$, the sequence $\{a_n\}$ converges to 0,
2. if $|L| > 1$, the sequence $\{a_n\}$ diverges,
3. if $|L| = 1$, the test is inconclusive.

Problem

Prove that
$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$$

Root of Sequences

Theorem

If $\{a_n\}$ is a sequence of posite terms and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists, then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Problem

Prove that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Cesaro Means

Theorem

If a sequence $\{a_n\}$ converges to a limit L, then the sequence $\{x_n\}$ defined by

$$x_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

converges to the same limit L.

Problem

Prove that

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}\right) = 1$$

Thank You!

We'd love your questions and feedback.

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com

@ParthoSutraDhor

(Lectures, walkthroughs, and course updates)

Scan for the channel

References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.