



## **BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

### **Complex Variables & Laplace Transform (MAT221)**

## **Complex Functions**

**Exponential, Logarithmic, Circular and Hyperbolic Functions**

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CONDUCTED BY

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## Single valued and multi valued functions

### Single-valued Function

A function  $f : D \rightarrow \mathbb{C}$  is called single-valued if for every input  $z \in D$ , there is exactly one output  $w \in \mathbb{C}$  such that  $f(z) = w$ .

### Multi-valued Function

A function  $f : D \rightarrow \mathcal{P}(\mathbb{C})$  is called multi-valued if for some input  $z \in D$ , there are multiple outputs  $w_1, w_2, \dots, w_n \in \mathbb{C}$  such that  $f(z) = \{w_1, w_2, \dots, w_n\}$  with  $n > 1$ .

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a branch of the function.

## Why some functions are multi-valued?

# The Exponential Function

## Complex Exponential Function is Periodic

The complex exponential function  $e^z$  is periodic with period  $2\pi i$ .



 **Problem**

Find all values of  $z \in \mathbb{C}$  such that



$$e^z = -1 + i\sqrt{3}.$$

 **Problem**

Find all values of  $z \in \mathbb{C}$  such that

$$e^{4z} = -1$$



## The Logarithmic Function

## Complex Logarithm

The complex logarithm of a non-zero complex number  $z$  is defined as 

$$\log z = \ln |z| + i \arg z$$

where  $\arg z$  is the argument of  $z$ .

Note: The complex logarithm is a multi-valued function due to the multi-valued nature of the argument  $\arg z$ .

## ② Problem

Show that:



$$(i) \ln(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$$

$$(ii) \ln(1 - i) = \frac{1}{2}\ln 2 + \left(2n + \frac{7}{4}\right)\pi i$$

$$(iii) \ln(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i$$

### Complex Sine and Cosine Functions

The complex sine and cosine functions are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

for all  $z \in \mathbb{C}$ .



## Inverse Circular Functions

### Inverse Circular Functions

The inverse circular functions for complex numbers are defined as follows:

$$\sin^{-1} z = -i \ln \left( iz + \sqrt{1 - z^2} \right)$$

$$\cos^{-1} z = -i \ln \left( z + \sqrt{z^2 - 1} \right)$$

$$\tan^{-1} z = \frac{i}{2} \ln \left( \frac{i+z}{i-z} \right)$$

$$\cot^{-1} z = \frac{i}{2} \ln \left( \frac{z-i}{z+i} \right)$$

$$\sec^{-1} z = -i \ln \left( \frac{1 + \sqrt{1 - z^2}}{z} \right)$$

$$\operatorname{cosec}^{-1} z = -i \ln \left( \frac{i + \sqrt{z^2 - 1}}{z} \right)$$

for all  $z \in \mathbb{C}$ .

## ② Problem

Prove that for all  $z \in \mathbb{C}$ :

$$\sin^{-1} z = -i \ln \left( iz + \sqrt{1 - z^2} \right)$$

## ② Problem

Prove that for all  $z \in \mathbb{C}$ :

$$\sec^{-1} z = -i \ln \left( \frac{1 + \sqrt{1 - z^2}}{z} \right)$$



## ② Problem

Prove that for all  $z \in \mathbb{C}$ :

$$\cot^{-1} z = \frac{i}{2} \ln \left( \frac{z - i}{z + i} \right)$$



 **Problem**

Solve for  $z \in \mathbb{C}$ :

$$\sin z = 2$$



## Hyperbolic Functions

### Sine Hyperbolic and Cosine Hyperbolic Functions

The complex sine and cosine functions are defined as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

for all  $z \in \mathbb{C}$ .

# Inverse Hyperbolic Functions

## Inverse Hyperbolic Functions

The inverse hyperbolic functions for complex numbers are defined as follows:

$$\sinh^{-1} z = \ln \left( z + \sqrt{z^2 + 1} \right)$$

$$\cosh^{-1} z = \ln \left( z + \sqrt{z^2 - 1} \right)$$

$$\tanh^{-1} z = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right)$$

$$\coth^{-1} z = \frac{1}{2} \ln \left( \frac{z+1}{z-1} \right)$$

$$\operatorname{sech}^{-1} z = \ln \left( \frac{1 + \sqrt{1 - z^2}}{z} \right)$$

$$\operatorname{cosech}^{-1} z = \ln \left( \frac{1 + \sqrt{1 + z^2}}{z} \right)$$

for all  $z \in \mathbb{C}$ .

## ② Problem

Prove that for all  $z \in \mathbb{C}$ :

$$\cosh^{-1} z = \log \left( z + \sqrt{z^2 - 1} \right)$$



## ② Problem

Prove that for all  $z \in \mathbb{C}$ :

$$\operatorname{cosech}^{-1} z = \ln \left( \frac{1 + \sqrt{1 + z^2}}{z} \right)$$



## ② Problem

Prove that for all  $z \in \mathbb{C}$ :

$$\tanh^{-1} z = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right)$$



 **Problem**

Solve for  $z \in \mathbb{C}$ :

$$\cosh z = 3i$$



## Properties of Circular and Hyperbolic Functions

### 💡 Theorem

For all  $z \in \mathbb{C}$ :

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

where  $z = x + iy$ ,  $x, y \in \mathbb{R}$ .



### 💡 Theorem

For all  $z \in \mathbb{C}$ :

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y$$

where  $z = x + iy$ ,  $x, y \in \mathbb{R}$ .



### 💡 Theorem

For all  $z \in \mathbb{C}$ :

$$\sin(iz) = i \sinh z,$$

$$\cos(iz) = \cosh z,$$

$$\sinh(iz) = i \sin z,$$

$$\cosh(iz) = \cos z.$$



## Theorem

For all  $z \in \mathbb{C}$ :

$$\cos^2 z + \sin^2 z = 1$$

$$\cosh^2 z - \sinh^2 z = 1$$



## Theorem

For all  $z \in \mathbb{C}$ :

$$\frac{d}{dz}(\sin z) = \cos z, \quad \frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\sinh z) = \cosh z, \quad \frac{d}{dz}(\cosh z) = \sinh z$$



# Thank You!

We'd love your questions and feedback.

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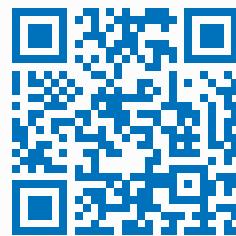
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(Lectures, walkthroughs, and course updates)



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## References

- [1] Murray R. Spiegel, *Schaum's Outline of Complex Variables*, 2nd Edition, McGraw–Hill, 2009.
- [2] Dennis G. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th Edition, Cengage, 2018.