



## **BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

### **Real Analysis (MAT221)**

## **Convergence of Sequences in $\mathbb{R}$**

### **Limit of a Sequence**

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CONDUCTED BY

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## Definition of a Sequence

### Sequence

A **sequence** is a function whose domain is the set of natural numbers  $\mathbb{N}$  and whose codomain is a non-empty set  $S$ . 

### Real Sequence or Sequence in $\mathbb{R}$

A **real sequence** is a sequence whose codomain is the set of real numbers  $\mathbb{R}$ . 

## Common Notations for Sequences

### Sequence Notation

A sequence is usually denoted by  $\{a_n\}_{n=1}^{\infty}$  or simply  $\{a_n\}$ , where  $a_n$  represents the  $n$ -th term of the sequence. Here,  $n$  is a natural number indicating the position of the term in the sequence.



### Examples

Each of the following are common ways to describe a sequence.



- $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ ,
- $\left(\frac{1+n}{n}\right)_{n=1}^{\infty} = \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \dots\right)$ ,
- $\{a_n\}$ , where  $a_n = 2^n$  for each  $n \in \mathbb{N}$ ,
- $(x_n)$ , where  $x_1 = 2$  and  $x_{n+1} = \frac{x_n + 1}{2}$ .

## Convergence and Limit of a Sequence

### Convergence of a Sequence

A sequence  $\{a_n\}$  is said to **converge** to a real number  $a$  if for every real number  $\varepsilon > 0$ , there exists a natural number  $N \in \mathbb{N}$  such that for all  $n \geq N$ , the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number  $a$  exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \rightarrow a \text{ as } n \rightarrow \infty.$$

## ② Example

Consider the sequence

$$a_n = \left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty}.$$

We will show that this sequence converges to 0.

## Quantifier Breakdown of the Definition

## Template for a proof that $\lim a_n = a$

### ② Proof Template

To prove that  $\lim a_n = a$ , we need to show that for every  $\varepsilon > 0$ , there exists a natural number  $N \in \mathbb{N}$  such that for all  $n \geq N$ , the inequality  $|a_n - a| < \varepsilon$  holds.

**Step 1:** Let  $\varepsilon > 0$  be given.

**Step 2:** Find a suitable  $N \in \mathbb{N}$  (often in terms of  $\varepsilon$ ) such that for all  $n \geq N$ , the desired inequality holds.

**Step 3:** Show that for all  $n \geq N$ , we have

$$|a_n - a| < \varepsilon.$$

**Conclusion:** Since we have shown that for every  $\varepsilon > 0$ , there exists an  $N$  such that for all  $n \geq N$ , the inequality holds, we conclude that  $\lim a_n = a$ .

## ② Example Proof

Show that the sequence  $a_n = \frac{n+1}{n}$  converges to 1.



## ② Example Proof

Show that the sequence  $a_n = \frac{3n+1}{2n+5}$  converges to  $\frac{3}{2}$ .



## ② Example Proof

Show that

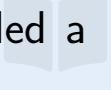
$$\lim_{n \rightarrow \infty} \frac{1}{3^n} = 0.$$



## Negation of Quantifiers

## Divergent Sequences

### Divergent Sequence

A sequence that does not converge to any real number is called a  **divergent sequence**.

### Divergence

A sequence  $\{a_n\}$  is said to **diverge** if there exists a real number  $\varepsilon > 0$  such that for every natural number  $N \in \mathbb{N}$ , there exists an index  $n \geq N$  such that the terms of the sequence satisfy the inequality

$$|a_n - a| \geq \varepsilon.$$

 **Problem**

Discuss why the sequence  $a_n = (-1)^n$  does not converge.



## ② Problem

Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right).$$

How can we argue that this sequence does not converge to zero?

## ② Problem

Argue that the sequence

1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, (5 zeros), 1, ...

does not converge to zero. For what values of  $\varepsilon > 0$  does there exist a response  $N$ ? For which values of  $\varepsilon > 0$  is there no suitable response?

## Diverge to Positive Infinity

### Divergence to positive Infinity

A sequence  $\{a_n\}$  is said to **diverge to positive infinity** if for every real number  $M > 0$ , there exists a natural number  $N \in \mathbb{N}$  such that for all  $n \geq N$ , the terms of the sequence satisfy the inequality

$$a_n > M.$$

If such a condition holds, we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

### Problem

Show that the sequence  $a_n = n^2$  diverges to infinity.



## Diverge to Negative Infinity

### Divergence to negative Infinity

A sequence  $\{a_n\}$  is said to **diverge to negative infinity** if for every real number  $M < 0$ , there exists a natural number  $N \in \mathbb{N}$  such that for all  $n \geq N$ , the terms of the sequence satisfy the inequality

$$a_n < M.$$

If such a condition holds, we write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty \text{ as } n \rightarrow \infty.$$

### Problem

Show that the sequence  $a_n = -n^3$  diverges to negative infinity.

## Oscillatory Sequences

### Oscillatory Sequence

A sequence that neither converges nor diverges to infinity or negative infinity is called an **oscillatory sequence**.

## Some More Examples on Convergence

### ① Problem

Show that the sequence  $a_n = \sqrt{n+1} - \sqrt{n}$  converges to 0.



## ② Problem

Show that

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 + 3} = \frac{1}{2}.$$



② Can you prove this?

Show that

$$\lim_{n \rightarrow \infty} \frac{3n + 4}{2n - 5} = \frac{3}{2}.$$



## ② Problem

Show that

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 - 3} = \frac{1}{2}.$$



## ② Problem

Show that

$$\lim_{n \rightarrow \infty} \frac{5n^2 + 3}{2n^3 + 1} = 0.$$



# Thank You!

We'd love your questions and feedback.

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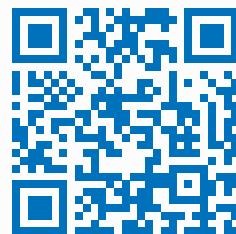
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(Lectures, walkthroughs, and course updates)



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## References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.