BRAC University
Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Constructing the Rationals

Equivalence Relation and Constructing the Rational Numbers

October 12, 2025

Conducted by

Partho Sutra Dhor
Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com
For updates subscribe on @ParthoSutraDhor

Cartesian Product and Relation

Relation Between Two Sets

A *relation* R from a set A to a set B is any subset of the Cartesian product $A \times B$, i.e.

$$R \subseteq A \times B.$$

If $(a, b) \in R$, we say that "a is related to b by R," written as $a\,R\,b$ or sometimes $a \sim b$.

Equivalence Relation and Examples

Equivalence Relation

A relation $\sim$ on a set A is called an **equivalence relation** if

1. **Reflexive:** $a \sim a$, for all $a \in A$.
2. **Symmetric:** If $a \sim b$, then $b \sim a$, for all $a, b \in A$.
3. **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$, for all $a, b, c \in A$.

Important examples of equivalence relations

1. **Congruence modulo n on $\mathbb{Z}$:** For a fixed integer $n > 1$, define

$$a \sim b \iff a \equiv b \pmod{n}.$$

2. **Rational Difference on $\mathbb{R}$:** On the set of real numbers, define

$$a \sim b \iff a - b \in \mathbb{Q}.$$

3. **Congruent Triangles:** For triangles $\triangle ABC$ and $\triangle DEF$, define

$$\triangle ABC \sim \triangle DEF \iff \triangle ABC \cong \triangle DEF$$

4. **Parallel Lines in the Plane:** For lines l_1 and l_2 in $\mathbb{R}^2$, define

$$l_1 \sim l_2 \iff l_1 \parallel l_2.$$

5. **Same Slope of Nonzero Vectors:** In $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Problem

Congruence modulo n on $\mathbb{Z}$: For a fixed integer $n > 1$, define

$$a \sim b \iff a \equiv b \pmod{n}.$$

Prove that $\sim$ is an equivalence relation.

Problem

Rational Difference on $\mathbb{R}$: On the set of real numbers, define

$$a \sim b \iff a - b \in \mathbb{Q}.$$

Prove that $\sim$ is an equivalence relation.

Problem

Same Slope of Nonzero Vectors: In $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Prove that $\sim$ is an equivalence relation.

Equivalence Classes

Equivalence Class

If $\sim$ is an equivalence relation on a set A and $a \in A$, the **equivalence class** of a is

$$[a] = \{ x \in A \mid x \sim a \}.$$

Problem

Define the equivalence relation $\sim$ on $\mathbb{Z}$ by

$$a \sim b \iff a \equiv b \pmod 5 \iff a - b \text{ is divisible by } 5.$$

Find the equivalence class of 3.

Problem

In $\mathbb{R}^2 \setminus \{(0,0)\}$, define the equivalence relation $\sim$ by

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Find the equivalence class of $(1, 2)$.

Theorem

Let $\sim$ be an equivalence relation on A. For any $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \varnothing$.

Partition from Equivalence Relations

Partition Theorem

Every equivalence relation on A partitions A into disjoint equivalence classes. Conversely, every partition of A defines an equivalence relation.

Problem

Define the equivalence relation $\sim$ on $\mathbb{Z}$ by

$$a \sim b \iff a \equiv b \pmod 5 \iff a - b \text{ is divisible by } 5.$$

Find the partition of $\mathbb{Z}$ by $\sim$.

Problem

In $\mathbb{R}^2 \setminus \{(0,0)\}$, define the equivalence relation $\sim$ by

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Give a physical interpretation of the partition of $\mathbb{R}^2 \setminus \{(0,0)\}$ by $\sim$.

Integers to Rational Numbers

"God made the integers; all else is the work of man."
— **Leopold Kronecker (1823–1891)**

Fractions as Ordered Pairs

Let

$$S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

be the set of all ordered pairs (a, b), where a is any integer and b is a nonzero integer. We think of (a, b) as the "fraction" $\dfrac{a}{b}$.

Theorem

Define a relation $\sim$ on $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

Show that the relation $\sim$ is an equivalence relation on S.

The Rational Numbers

The set of **rational numbers** is the quotient set / partition set

$$\mathbb{Q} = S/\sim = \{\, [(a,b)] \mid (a,b) \in S \,\}.$$

The equivalence class $[(a,b)]$ is informally written as $\dfrac{a}{b}$.

Example

The class $[(1,2)]$ is

$$[(1,2)] = \{(1,2),(2,4),(3,6),\dots,(-1,-2),(-2,-4),\dots\}.$$

All of these pairs represent the same rational number: $\dfrac{1}{2}$.

Integers as Rationals

The class $[(a,1)]$ is

$$[(a,1)] = \{(a,1),(2a,2),(3a,3),\dots,(-a,-1),(-2a,-2),\dots\}.$$

All of these pairs represent the integer $a \in \mathbb{Z}$.

Lowest Terms Representation of Rationals

If $(a,b) \in S$ with $b > 0$, write $g = \gcd(a,b)$. Then the class $[(a,b)]$ can be represented uniquely by

$$\frac{a}{b} \sim \frac{a/g}{b/g}$$

with $\gcd(a/g,\, b/g) = 1$ and denominator positive.

Uniqueness of Lowest Terms

Every nonzero rational has a unique expression

$$\frac{m}{n} \quad \text{with } n > 0 \text{ and } \gcd(m,n) = 1.$$

Addition and Multiplication of Rational Numbers

Definition

For $[(a,b)], [(c,d)] \in \mathbb{Q}$, define:

$$[(a,b)] + [(c,d)] := [(ad+bc,\ bd)], \qquad [(a,b)] \cdot [(c,d)] := [(ac,\ bd)].$$

Well-definedness

These operations do not depend on the representatives chosen.

Rationals as a Field

Order on the Rationals

Definition

For $[(a,b)],[(c,d)] \in \mathbb{Q}$ with $b,d > 0$, define

$$[(a,b)] < [(c,d)] \iff ad < bc.$$

Density of Rationals

Density

Between any two rationals $x < y$, there exists another rational z with $x < z < y$.

Thank You!

We'd love your questions and feedback.

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com

@ParthoSutraDhor

(Lectures, walkthroughs, and course updates)

Scan for the channel

References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.