BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Gaps in the Rational Numbers

Supremum and Infimum

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CONDUCTED BY

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There are holes or gaps in $\mathbb{Q}$

Problem

Show that there is no rational number x such that $x^2 = 2$.

Upper Bound and Lower Bound

Upper Bound

Let $S \subseteq \mathbb{Q}$. An element $u \in \mathbb{Q}$ is called an **upper bound** of S if

$$x \leq u \quad \text{for all } x \in S.$$

If such an element exists, we say that S is *bounded above* in $\mathbb{Q}$.

Lower Bound

Let $S \subseteq \mathbb{Q}$. An element $l \in \mathbb{Q}$ is called a **lower bound** of S if

$$x \leq l \quad \text{for all } x \in S.$$

If such an element exists, we say that S is *bounded below* in $\mathbb{Q}$.

Supremum and Infimum

Least Upper Bound (Supremum)

Let $S \subseteq \mathbb{Q}$. An element $s \in \mathbb{Q}$ is called the **least upper bound** or the **supremum** of S in $\mathbb{Q}$ if

1. s is an upper bound of S, and
2. for every upper bound u of S, we have $s \leq u$.

Greatest Lower Bound (Infimum)

Let $S \subseteq \mathbb{Q}$. An element $i \in \mathbb{Q}$ is called the **greatest lower bound** or the **infimum** of S in $\mathbb{Q}$ if

1. i is a lower bound of S, and
2. for every lower bound l of S, we have $i \geq l$.

Problem

Find the **supremum** and **infimum** of each of the following set:

$$\{\, n \in \mathbb{N} : n^2 < 10 \,\}$$

Problem

Find the **supremum** and **infimum** of each of the following set:

$$\left\{ \frac{n}{m+n} : m, n \in \mathbb{N} \right\}$$

Problem

Find the **supremum** and **infimum** of each of the following set:

$$\left\{ \frac{n}{2n+1} : n \in \mathbb{N} \right\}$$

Problem

Find the **supremum** and **infimum** of each of the following set:

$$\left\{ \frac{n}{m} : m, n \in \mathbb{N} \text{ with } m + n \leq 10 \right\}$$

Problem

Show that the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above in $\mathbb{Q}$ but does not have a supremum in $\mathbb{Q}$.

Properties of Supremum

Properties of Supremum

Let $A \subseteq \mathbb{Q}$ be a nonempty set that is bounded above. Denote $\sup A$ by the least upper bound of A. Then the following properties hold:

(a) u is an upper bound for $A \iff \sup A \leq u$.

(b) If $u < \sup A$, then $\exists\, a \in A$ such that $u < a \leq \sup A$.

(c) If $A \subseteq B$, then $\sup A \leq \sup B$.

(d) Let $A + B = \{\, a + b : a \in A,\ b \in B \,\}$. Then

$$\sup(A + B) = \sup A + \sup B.$$

(e) Let $A \cdot B = \{\, ab : a \in A,\ b \in B \,\}$ and suppose $A, B \subseteq \mathbb{R}_+$. Then

$$\sup(A \cdot B) = (\sup A)(\sup B).$$

Theorem

Assume $s \in \mathbb{Q}$ is an upper bound for a set $A \subseteq \mathbb{Q}$. Then $s = \sup A$ if and only if, for every $\varepsilon > 0$, there exists an element $a \in A$ such that

$$s - \varepsilon < a.$$

Properties of Infimum

Properties of Infimum

Let $A \subseteq \mathbb{Q}$ be a nonempty set that is bounded below. Denote $\inf A$ by the greatest lower bound of A. Then the following properties hold:

(a) l is a lower bound for $A \iff \inf A \geq l$.

(b) If $l > \inf A$, then

(c) If $A \subseteq B$, then $\inf A \geq \inf B$.

(d) Let $A + B = \{ a + b : a \in A, b \in B \}$. Then

$$\inf(A + B) = \inf A + \inf B.$$

(e) Let $A \cdot B = \{ ab : a \in A, b \in B \}$ and suppose $A, B \subseteq \mathbb{R}_+$. Then

$$\inf(A \cdot B) = (\inf A)(\inf B).$$

Theorem

Assume $s \in \mathbb{Q}$ is an least lower bound for a set $A \subseteq \mathbb{Q}$. Then $s = \inf A$ if and only if, for every $\varepsilon > 0$, there exists an element $a \in A$ such that

$$s + \varepsilon > a.$$

Not every bounded above subset of $\mathbb{Q}$ has a supremum in $\mathbb{Q}$

Problem

Show that the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above in $\mathbb{Q}$ but does not have a supremum in $\mathbb{Q}$.

Thank You!

We'd love your questions and feedback.

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References

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[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.