



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Constructing the Rationals

Equivalence Relation and Constructing the Rational Numbers


OCTOBER 12, 2025

CONDUCTED BY

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Cartesian Product and Relation

Relation Between Two Sets


A *relation* R from a set A to a set B is any subset of the Cartesian product $A \times B$, i.e.

$$R \subseteq A \times B.$$

If $(a, b) \in R$, we say that “ a is related to b by R ,” written as $a R b$ or sometimes $a \sim b$.


Equivalence Relation and Examples

Equivalence Relation

A relation \sim on a set A is called an **equivalence relation** if 

1. **Reflexive:** $a \sim a$, for all $a \in A$.
2. **Symmetric:** If $a \sim b$, then $b \sim a$, for all $a, b \in A$.
3. **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$, for all $a, b, c \in A$.

Important examples of equivalence relations

1. **Congruence modulo n on \mathbb{Z} :** For a fixed integer $n > 1$, define 

$$a \sim b \iff a \equiv b \pmod{n}.$$

2. **Rational Difference on \mathbb{R} :** On the set of real numbers, define

$$a \sim b \iff a - b \in \mathbb{Q}.$$

3. **Congruent Triangles:** For triangles $\triangle ABC$ and $\triangle DEF$, define

$$\triangle ABC \sim \triangle DEF \iff \triangle ABC \cong \triangle DEF$$

4. **Parallel Lines in the Plane:** For lines l_1 and l_2 in \mathbb{R}^2 , define

$$l_1 \sim l_2 \iff l_1 \parallel l_2.$$

5. **Same Slope of Nonzero Vectors:** In $\mathbb{R}^2 \setminus \{(0, 0)\}$, define

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

❓ Problem

Congruence modulo n on \mathbb{Z} : For a fixed integer $n > 1$, define 

$$a \sim b \iff a \equiv b \pmod{n}.$$

Prove that \sim is an equivalence relation.

🔍 Problem

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Prove that \sim is an equivalence relation.

Equivalence Classes

Equivalence Class

If \sim is an equivalence relation on a set A and $a \in A$, the **equivalence class** of a is

$$[a] = \{ x \in A \mid x \sim a \}.$$

Problem

Define the equivalence relation \sim on \mathbb{Z} by

$$a \sim b \iff a \equiv b \pmod{5} \iff a - b \text{ is divisible by } 5.$$

Find the equivalence class of 3.

Problem

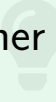
In $\mathbb{R}^2 \setminus \{(0, 0)\}$, define the equivalence relation \sim by



$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Find the equivalence class of $(1, 2)$.

Theorem

Let \sim be an equivalence relation on A . For any $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$. 

Partition from Equivalence Relations

Partition Theorem

Every equivalence relation on A partitions A into disjoint equivalence classes. Conversely, every partition of A defines an equivalence relation.

Problem

Define the equivalence relation \sim on \mathbb{Z} by



$$a \sim b \iff a \equiv b \pmod{5} \iff a - b \text{ is divisible by } 5.$$

Find the partition of \mathbb{Z} by \sim .

❓ Problem

In $\mathbb{R}^2 \setminus \{(0, 0)\}$, define the equivalence relation \sim by

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Give a physical interpretation of the partition of $\mathbb{R}^2 \setminus \{(0, 0)\}$ by \sim .

Integers to Rational Numbers

“God made the integers; all else is the work of man.”

— Leopold Kronecker (1823–1891)

Fractions as Ordered Pairs

Let

$$S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

be the set of all ordered pairs (a, b) , where a is any integer and b is a nonzero integer. We think of (a, b) as the “fraction” $\frac{a}{b}$.

Theorem


Define a relation \sim on $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

Show that the relation \sim is an equivalence relation on S .



The Rational Numbers

The set of **rational numbers** is the quotient set / partition set 

$$\mathbb{Q} = S / \sim = \{ [(a, b)] \mid (a, b) \in S \}.$$

The equivalence class $[(a, b)]$ is informally written as $\frac{a}{b}$.


Example

The class $[(1, 2)]$ is 

$$[(1, 2)] = \{(1, 2), (2, 4), (3, 6), \dots, (-1, -2), (-2, -4), \dots\}.$$

All of these pairs represent the same rational number: $\frac{1}{2}$.

Integers as Rationals

The class $[(a, 1)]$ is 

$$[(a, 1)] = \{(a, 1), (2a, 2), (3a, 3), \dots, (-a, -1), (-2a, -2), \dots\}.$$

All of these pairs represent the integer $a \in \mathbb{Z}$.

Lowest Terms Representation of Rationals

If $(a, b) \in S$ with $b > 0$, write $g = \gcd(a, b)$. Then the class $[(a, b)]$ can be represented uniquely by

$$\frac{a}{b} \sim \frac{a/g}{b/g}$$

with $\gcd(a/g, b/g) = 1$ and denominator positive.

Uniqueness of Lowest Terms

Every nonzero rational has a unique expression

$$\frac{m}{n} \quad \text{with } n > 0 \text{ and } \gcd(m, n) = 1.$$

Addition and Multiplication of Rational Numbers

Definition

For $[(a, b)], [(c, d)] \in \mathbb{Q}$, define:



$$[(a, b)] + [(c, d)] := [(ad + bc, bd)], \quad [(a, b)] \cdot [(c, d)] := [(ac, bd)].$$

Well-definedness

These operations do not depend on the representatives chosen.



Rationals as a Field

Order on the Rationals

Definition

For $[(a, b)], [(c, d)] \in \mathbb{Q}$ with $b, d > 0$, define

$$[(a, b)] < [(c, d)] \iff ad < bc.$$

Density of Rationals

Density

Between any two rationals $x < y$, there exists another rational z with $x < z < y$.

Thank You!

We'd love your questions and feedback.

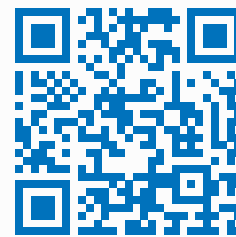
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(Lectures, walkthroughs, and course updates)



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References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.