BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Constructing the Rationals

Equivalence Relation and Constructing the Rational Numbers

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Conducted by

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Cartesian Product and Relation

Relation Between Two Sets

A *relation* R from a set A to a set B is any subset of the Cartesian product $A \times B$, i.e.

$$R \subseteq A \times B.$$

If $(a, b) \in R$, we say that "a is related to b by R," written as $a\,R\,b$ or sometimes $a \sim b$.

Equivalence Relation and Examples

Equivalence Relation

A relation $\sim$ on a set A is called an **equivalence relation** if

1. **Reflexive:** $a \sim a$, for all $a \in A$.
2. **Symmetric:** If $a \sim b$, then $b \sim a$, for all $a, b \in A$.
3. **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$, for all $a, b, c \in A$.

Important examples of equivalence relations

1. **Congruence modulo n on $\mathbb{Z}$:** For a fixed integer $n > 1$, define
$$a \sim b \iff a \equiv b \pmod{n}.$$
2. **Rational Difference on $\mathbb{R}$:** On the set of real numbers, define
$$a \sim b \iff a - b \in \mathbb{Q}.$$
3. **Congruent Triangles:** For triangles $\triangle ABC$ and $\triangle DEF$, define
$$\triangle ABC \sim \triangle DEF \iff \triangle ABC \cong \triangle DEF$$
4. **Parallel Lines in the Plane:** For lines l_1 and l_2 in $\mathbb{R}^2$, define
$$l_1 \sim l_2 \iff l_1 \parallel l_2.$$
5. **Same Slope of Nonzero Vectors:** In $\mathbb{R}^2 \setminus \{(0,0)\}$, define
$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Problem

Congruence modulo n on $\mathbb{Z}$: For a fixed integer $n > 1$, define

$$a \sim b \iff a \equiv b \pmod{n}.$$

Prove that $\sim$ is an equivalence relation.

Problem

Rational Difference on $\mathbb{R}$: On the set of real numbers, define

$$a \sim b \iff a - b \in \mathbb{Q}.$$

Prove that $\sim$ is an equivalence relation.

Problem

Same Slope of Nonzero Vectors: In $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Prove that $\sim$ is an equivalence relation.

Equivalence Classes

Equivalence Class

If $\sim$ is an equivalence relation on a set A and $a \in A$, the **equivalence class** of a is

$$[a] = \{\, x \in A \mid x \sim a \,\}.$$

Problem

Define the equivalence relation $\sim$ on $\mathbb{Z}$ by

$$a \sim b \iff a \equiv b \pmod 5 \iff a - b \text{ is divisible by } 5.$$

Find the equivalence class of 3.

Problem

In $\mathbb{R}^2 \setminus \{(0,0)\}$, define the equivalence relation $\sim$ by

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Find the equivalence class of $(1, 2)$.

Theorem

Let $\sim$ be an equivalence relation on A. For any $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \varnothing$.

Partition from Equivalence Relations

Partition Theorem

Every equivalence relation on A partitions A into disjoint equivalence classes. Conversely, every partition of A defines an equivalence relation.

Problem

Define the equivalence relation $\sim$ on $\mathbb{Z}$ by

$$a \sim b \iff a \equiv b \pmod 5 \iff a - b \text{ is divisible by } 5.$$

Find the partition of $\mathbb{Z}$ by $\sim$.

Problem

In $\mathbb{R}^2 \setminus \{(0,0)\}$, define the equivalence relation $\sim$ by

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Give a physical interpretation of the partition of $\mathbb{R}^2 \setminus \{(0,0)\}$ by $\sim$.

Integers to Rational Numbers

"God made the integers; all else is the work of man."
— **Leopold Kronecker (1823–1891)**

Fractions as Ordered Pairs

Let

$$S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

be the set of all ordered pairs (a, b), where a is any integer and b is a nonzero integer. We think of (a, b) as the "fraction" $\dfrac{a}{b}$.

Theorem

Define a relation $\sim$ on $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

Show that the relation $\sim$ is an equivalence relation on S.

The Rational Numbers

The set of **rational numbers** is the quotient set / partition set

$$\mathbb{Q} = S/\sim = \{\, [(a,b)] \mid (a,b) \in S \,\}.$$

The equivalence class $[(a,b)]$ is informally written as $\dfrac{a}{b}$.

Example

The class $[(1,2)]$ is

$$[(1,2)] = \{(1,2),(2,4),(3,6),\dots,(-1,-2),(-2,-4),\dots\}.$$

All of these pairs represent the same rational number: $\dfrac{1}{2}$.

Integers as Rationals

The class $[(a,1)]$ is

$$[(a,1)] = \{(a,1),(2a,2),(3a,3),\dots,(-a,-1),(-2a,-2),\dots\}.$$

All of these pairs represent the integer $a \in \mathbb{Z}$.

Lowest Terms Representation of Rationals

If $(a, b) \in S$ with $b > 0$, write $g = \gcd(a, b)$. Then the class $[(a, b)]$ can be represented uniquely by

$$\frac{a}{b} \sim \frac{a/g}{b/g}$$

with $\gcd(a/g,\, b/g) = 1$ and denominator positive.

Uniqueness of Lowest Terms

Every nonzero rational has a unique expression

$$\frac{m}{n} \quad \text{with } n > 0 \text{ and } \gcd(m, n) = 1.$$

Addition and Multiplication of Rational Numbers

Definition

For $[(a,b)],[(c,d)] \in \mathbb{Q}$, define:

$$[(a,b)]+[(c,d)] := [(ad+bc,\ bd)], \qquad [(a,b)]\cdot[(c,d)] := [(ac,\ bd)].$$

Well-definedness

These operations do not depend on the representatives chosen.

Rationals as a Field

Order on the Rationals

Definition

For $[(a,b)], [(c,d)] \in \mathbb{Q}$ with $b, d > 0$, define

$$[(a,b)] < [(c,d)] \iff ad < bc.$$

Density of Rationals

Density

Between any two rationals $x < y$, there exists another rational z with $x < z < y$.

Thank You!

We'd love your questions and feedback.

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References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.