**BRAC University**

Department of Mathematics and Natural Sciences

Lecture on

**Real Analysis (MAT221)**

# Properties of the Real Numbers

## Consequences of the Axiom of Completeness

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Conducted by

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# The Absolute Value Function

## Absolute Value

For any real number $x$, the **absolute value** of $x$, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

## Theorem

Show that for any real numbers $x$, the following holds:

$$\sqrt{x^2} = |x|$$

## The Triangle Inequality

### The Triangle Inequality for Real Numbers

For any real numbers $x$ and $y$, the following inequality holds:

$$|x + y| \leq |x| + |y|$$

### Other forms of the Triangle Inequality

For any real numbers $x$ and $y$, the following inequalities hold:

$$|x - y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

## Equality of two Real Numbers

### Theorem

Two real numbers $a$ and $b$ are equal if and only if for every $\varepsilon > 0$,

$$|a - b| < \varepsilon$$

# Axiom of Completeness (AoC)

## Least Upper Bound (Supremum)

A real number $s$ is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) $s$ is an upper bound for $A$;

(ii) if $u$ is any upper bound for $A$, then $s \leq u$.

## Axiom of Completeness (AoC)

Every non-empty set of real numbers that is bounded above has a least upper bound (supremum) in $\mathbb{R}$.

## Greatest Lower Bound (Infimum)

A real number $t$ is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) $t$ is a lower bound for $A$;

(ii) if $l$ is any lower bound for $A$, then $t \geq l$.

## Theorem: Existence of Infimum

Show that every non-empty set of real numbers that is bounded below has a greatest lower bound (infimum) in $\mathbb{R}$.

### Theorem: Characterization of Supremum

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$s - \epsilon < a$$

### Theorem: Characterization of Infimum

Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$a < t + \epsilon$$

# Consequences of Completeness (AoC)

# Nested Interval Property (NIP)

## Nested Interval Property (NIP)

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{ x \in \mathbb{R} : a_n \leq x \leq b_n \}$. Assume also that each $I_n$ contains $I_{n+1}$. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

## Theorem: AoC $\implies$ NIP

The Axiom of Completeness implies the Nested Interval Property.

# Archimedean Property (AP)

## Archimedean Property (AP)

Given any number $x \in \mathbb{R}$ and any real number $y > 0$, the following two statements hold:

(i) *There exists an* $n \in \mathbb{N}$ *satisfying* $n > x$.

(ii) *There exists an* $n \in \mathbb{N}$ *satisfying* $1/n < y$.

## Theorem: AoC $\implies$ AP

The Axiom of Completeness implies the Archimedean Property.

## NIP + AP $\implies$ AoC

**Theorem: NIP + AP $\implies$ AoC**

The Nested Interval Property and the Archimedean Property together imply the Axiom of Completeness.

## Density of Rationals and Irrationals in $\mathbb{R}$

### Density of $\mathbb{Q}$ in $\mathbb{R}$

*For every two real numbers $a$ and $b$ with $a < b$, there exists a rational number $r$ satisfying $a < r < b$.*

### Density of Irrationals in $\mathbb{R}$

*For every two real numbers $a$ and $b$ with $a < b$, there exists an irrational number $s$ satisfying $a < s < b$.*

## The Existence of Roots

### Existence of Square Roots

*There exists a real number* $\alpha \in \mathbb{R}$ *satisfying* $\alpha^2 = 2$.

### Existence of n-th Roots

*For any given real number $x \geq 0$ and any integer $n \in \mathbb{N}$, there exists a unique real number $y \geq 0$ satisfying $y^n = x$.*

# Thank You!

We'd love your questions and feedback.


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## References


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[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.