



## **BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

### **Real Analysis (MAT221)**

# **Constructing the Real Numbers**

## **Dedekind Cuts**

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CONDUCTED BY

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## $\mathbb{Q}$ is not complete

### Completeness

An ordered field  $F$  is said to be **complete** if every nonempty subset of  $F$  that is bounded above has a least upper bound (supremum) in  $F$ .

### ② Why $\mathbb{Q}$ is not complete?

The set  $\{x \in \mathbb{Q} : x^2 < 2\}$  is bounded above in  $\mathbb{Q}$  but does not have a supremum in  $\mathbb{Q}$ .

## The ways to complete $\mathbb{Q}$

## Dedekind Cuts

### Dedekind Cut

A **cut**  $\Sigma$  is a subset of  $\mathbb{Q}$  such that:



1.  $\Sigma \neq \emptyset$  and  $\Sigma \neq \mathbb{Q}$  *(non-trivial)*
2. If  $p \in \Sigma$ ,  $q \in \mathbb{Q}$ , and  $q < p$ , then  $q \in \Sigma$  *(closed downward)*
3. If  $p \in \Sigma$ , then  $\exists r \in \Sigma$  with  $p < r$  *(no maximum element)*

## Examples of Cuts

### ① Example

Show that

$$\{x \in \mathbb{Q} : x < 2\}$$

is Dedekind cut.



### ② Non-example

Show that

$$\{x \in \mathbb{Q} : x \leq 2\}$$

is not a Dedekind cut.



## ② Non-example

Show that

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

is not Dedekind cut.



## ② Example

Show that

$$\{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$$

is Dedekind cut.



## What is the set of real numbers?

### Definition of $\mathbb{R}$

The set of real numbers  $\mathbb{R}$  is defined as the set of all Dedekind cuts in  $\mathbb{Q}$ . Mathematically,

$$\mathbb{R} = \{\Sigma : \Sigma \text{ is a Dedekind cut in } \mathbb{Q}\}.$$

## Addition of two Cuts

### ■ Addition of Cuts

If  $\Sigma$  and  $\Gamma$  are two cuts, then we define their sum as



$$\Sigma + \Gamma = \{a + b : a \in \Sigma, b \in \Gamma\}.$$

### ?

### Well-definedness of Addition

If  $\Sigma$  and  $\Gamma$  are two cuts, then  $\Sigma + \Gamma$  is also a cut.



## Negative of a Cuts

### ❑ Negative of a Cuts

If  $\Sigma$  is a cut, then we define its negative as



$-\Sigma = \{x \in \mathbb{Q} : -x \notin \Sigma \text{ and } -x \text{ is not the least element of } \mathbb{Q} \setminus \Sigma\}$ .

or

$$-\Sigma = \{x \in \mathbb{Q} : \exists r > 0 \text{ s.t. } -p - r \notin \Sigma\}$$

### 💡 Well-definedness of Negative

If  $\Sigma$  is a cut, then  $-\Sigma$  is also a cut.



## Multiplication of Cuts

If  $\Sigma$  and  $\Gamma$  are two cuts, then we define their product as



$$\Sigma \cdot \Gamma = \{a \cdot b : a \in \Sigma, b \in \Gamma, a > 0, b > 0\}.$$

## Well-definedness of Multiplication

If  $\Sigma$  and  $\Gamma$  are two cuts, then  $\Sigma \cdot \Gamma$  is also a cut.



## Field Axioms of $\mathbb{R}$

### Ordering on Cuts

If  $\Sigma$  and  $\Gamma$  are two cuts, then we define

$$\Sigma < \Gamma \iff \Sigma \subsetneq \Gamma.$$



## Facts about real numbers

### 💡 Least Upper Bound Property

Every nonempty subset of  $\mathbb{R}$  that is bounded above has a least upper bound in  $\mathbb{R}$ .

### 💡 Greatest Lower Bound Property

Every nonempty subset of  $\mathbb{R}$  that is bounded below has a greatest lower bound in  $\mathbb{R}$ .

### 💡 Uniqueness of $\mathbb{R}$

$\mathbb{R}$  is the only complete ordered field (up to isomorphism) containing  $\mathbb{Q}$ .

# Thank You!

We'd love your questions and feedback.

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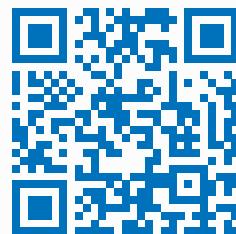
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(Lectures, walkthroughs, and course updates)



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## References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.