



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Constructing the Real Numbers

Dedekind Cuts

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CONDUCTED BY

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\mathbb{Q} is not complete

Completeness

A field F is said to be **complete** if every nonempty subset $A \subseteq F$ that is bounded above has a least upper bound (supremum) in F .

Problem

Show that the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above in \mathbb{Q} but does not have a supremum in \mathbb{Q} .

The ways to complete \mathbb{Q}

Dedekind Cuts

Dedekind Cut

A **cut** Σ is a subset of \mathbb{Q} such that:



1. $\Sigma \neq \emptyset$ and $\Sigma \neq \mathbb{Q}$ *(non-trivial)*
2. If $p \in \Sigma$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \Sigma$ *(closed downward)*
3. If $p \in \Sigma$, then $\exists r \in \Sigma$ with $p < r$ *(no maximum element)*

Examples of Cuts

① Example

Show that

$$\{x \in \mathbb{Q} : x < 2\}$$

is Dedekind cut.



② Non-example

Show that

$$\{x \in \mathbb{Q} : x \leq 2\}$$

is not a Dedekind cut.



② Example

Show that

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

is not Dedekind cut.



② Non-example

Show that

$$\{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$$

is Dedekind cut.



What is the set of real number?

Definition of \mathbb{R}

The set of real numbers \mathbb{R} is defined as the set of all Dedekind cuts in \mathbb{Q} . Mathematically,

$$\mathbb{R} = \{\Sigma : \Sigma \text{ is a Dedekind cut in } \mathbb{Q}\}.$$

Addition of two Cuts

■ Addition of Cuts

If Σ and Γ are two cuts, then we define their sum as



$$\Sigma + \Gamma = \{a + b : a \in \Sigma, b \in \Gamma\}.$$

?

Well-definedness of Addition

If Σ and Γ are two cuts, then $\Sigma + \Gamma$ is also a cut.



Negative of a Cuts

❑ Negative of a Cuts

If Σ is a cut, then we define its negative as



$-\Sigma = \{x \in \mathbb{Q} : -x \notin \Sigma \text{ and } -x \text{ is not the least element of } \mathbb{Q} \setminus \Sigma\}$.

or

$$-\Sigma = \{x \in \mathbb{Q} : \exists r > 0 \text{ s.t. } -p - r \notin \Sigma\}$$

💡 Well-definedness of Negative

If Σ is a cut, then $-\Sigma$ is also a cut.



Multiplication of Cuts

If Σ and Γ are two cuts, then we define their product as



$$\Sigma \cdot \Gamma = \{a \cdot b : a \in \Sigma, b \in \Gamma, a > 0, b > 0\}.$$

Well-definedness of Multiplication

If Σ and Γ are two cuts, then $\Sigma \cdot \Gamma$ is also a cut.



Field Axioms of \mathbb{R}

Ordering on Cuts

If Σ and Γ are two cuts, then we define

$$\Sigma < \Gamma \iff \Sigma \subsetneq \Gamma.$$



Facts about real numbers

💡 Least Upper Bound Property

Every nonempty subset of \mathbb{R} that is bounded above has a least upper bound in \mathbb{R} .

💡 Greatest Lower Bound Property

Every nonempty subset of \mathbb{R} that is bounded below has a greatest lower bound in \mathbb{R} .

💡 Uniqueness of \mathbb{R}

\mathbb{R} is the only complete ordered field (up to isomorphism) containing \mathbb{Q} .

Thank You!

We'd love your questions and feedback.

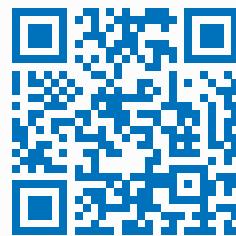
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(Lectures, walkthroughs, and course updates)



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References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.