



**BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

**Real Analysis (MAT221)**

# Cauchy Sequences and Completeness

Cauchy Criterion (CC), Completeness Revisited

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CONDUCTED BY

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## Convergence and Limit of a Sequence

### Convergence of a Sequence

A sequence  $\{a_n\}$  is said to **converge** to a real number  $a$  if for every real number  $\varepsilon > 0$ , there exists a natural number  $N \in \mathbb{N}$  such that for all  $n \geq N$ , the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number  $a$  exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \rightarrow a \text{ as } n \rightarrow \infty.$$



## Cauchy Sequence

### Cauchy Sequence

A sequence  $\{a_n\}$  is called a **Cauchy sequence** if for every real number  $\varepsilon > 0$ , there exists a natural number  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , the terms of the sequence satisfy the inequality

$$|a_n - a_m| < \varepsilon.$$



**② Problem**

Show that the sequence defined by  $a_n = \frac{1}{n}$  is a Cauchy sequence.





### **Theorem**

Every Cauchy sequence is bounded sequence.





### Theorem

Every Convergent sequence is a Cauchy sequence.





### Theorem

Every Cauchy Sequence in  $\mathbb{R}$  converges to a real number.





## Cauchy Criterion (CC) for Sequences in $\mathbb{R}$

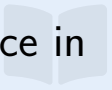
### Theorem

A sequence  $\{a_n\}$  of real numbers converges if and only if it is a Cauchy sequence.



## Completeness of a Metric Space

### Completeness of a Metric Space

A metric space is said to be **complete** if every Cauchy sequence in the space converges to a limit that is also within that space. 

### The incompleteness of $\mathbb{Q}$

Not every Cauchy Sequence in  $\mathbb{Q}$  converges to a rational number. 

Show that the sequence defined by

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}, \quad a_1 = 1$$

is a Cauchy sequence in  $\mathbb{Q}$  but does not converge to any rational number.



## Completeness Revisited

### Equivalence of Completeness Axioms for $\mathbb{R}$

The following are equivalent statements of the Completeness Axiom for the real numbers:

1. Axiom of Completeness (AoC)
2. Nested Interval Property (NIP) + Archimedean Property (AP)
3. Monotone Convergence Theorem (MCT)
4. Bolzano-Weierstrass Theorem (BWT)
5. Cauchy Criterion (CC) + Archimedean Property (AP)



## Assignment

### Prove the following statements

1.  $\text{AoC} \implies \text{NIP}$

2.  $\text{AoC} \implies \text{AP}$

3.  $\text{NIP} + \text{AP} \implies \text{AoC}$

4.  $\text{AoC} \implies \text{MCT}$

5.  $\text{MCT} \implies \text{AoC}$

6.  $\text{AoC} \implies \text{BWT}$

7.  $\text{BWT} \implies \text{AoC}$

8.  $\text{AoC} \implies \text{CC}$

9.  $\text{CC} + \text{AP} \implies \text{AoC}$



# Thank You!

We'd love your questions and feedback.

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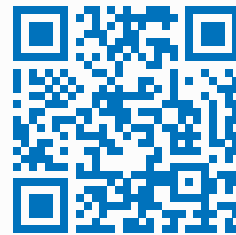
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(Lectures, walkthroughs, and course updates)



Scan for the channel

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## References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.