BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Constructing the Real Numbers

Dedekind Cuts

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Conducted by

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$\mathbb{Q}$ is not complete

Completeness

A field F is said to be **complete** if every nonempty subset $A \subseteq F$ that is bounded above has a least upper bound (supremum) in F.

Problem

Show that the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above in $\mathbb{Q}$ but does not have a supremum in $\mathbb{Q}$.

The ways to complete $\mathbb{Q}$

Dedekind Cuts

Dedekind Cut

A **cut** Σ is a subset of $\mathbb{Q}$ such that:

1. $\Sigma \neq \varnothing$ and $\Sigma \neq \mathbb{Q}$ *(non-trivial)*
2. If $p \in \Sigma$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \Sigma$ *(closed downward)*
3. If $p \in \Sigma$, then $\exists r \in \Sigma$ with $p < r$ *(no maximum element)*

Examples of Cuts

Example

Show that

$$\{x \in \mathbb{Q} : x < 2\}$$

is Dedekind cut.

Non-example

Show that

$$\{x \in \mathbb{Q} : x <= 2\}$$

is not a Dedekind cut.

Example

Show that

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

is not Dedekind cut.

Non-example

Show that

$$\{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$$

is Dedekind cut.

What is the set of real number?

Definition of $\mathbb{R}$

The set of real numbers $\mathbb{R}$ is defined as the set of all Dedekind cuts in $\mathbb{Q}$. Mathematically,

$$\mathbb{R} = \{\Sigma : \Sigma \text{ is a Dedekind cut in } \mathbb{Q}\}.$$

Addition of two Cuts

Addition of Cuts

If Σ and Γ are two cuts, then we define their sum as

$$\Sigma + \Gamma = \{a + b : a \in \Sigma, b \in \Gamma\}.$$

Well-definedness of Addition

If Σ and Γ are two cuts, then $\Sigma + \Gamma$ is also a cut.

Negative of a Cuts

Negative of a Cuts

If Σ is a cut, then we define its negative as

$$-\Sigma = \{x \in \mathbb{Q} : -x \notin \Sigma \text{ and } -x \text{ is not the least element of } \mathbb{Q}\setminus\Sigma\}.$$

or

$$-\Sigma = \{x \in \mathbb{Q} : \exists r > 0 \text{ s.t. } -p - r \notin \Sigma\}$$

Well-definedness of Negative

If Σ is a cut, then $-\Sigma$ is also a cut.

Multiplication of Cuts

If Σ and Γ are two cuts, then we define their product as

$$\Sigma \cdot \Gamma = \{a \cdot b : a \in \Sigma, b \in \Gamma, a > 0, b > 0\}.$$

Well-definedness of Multiplication

If Σ and Γ are two cuts, then $\Sigma \cdot \Gamma$ is also a cut.

Field Axioms of $\mathbb{R}$

Ordering on $\mathbb{R}$

Ordering on Cuts

If Σ and Γ are two cuts, then we define

$$\Sigma < \Gamma \iff \Sigma \subsetneq \Gamma.$$

Facts about real numbers

Least Upper Bound Property

Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$.

Gratest Lower Bound Property

Every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound in $\mathbb{R}$.

Uniqueness of $\mathbb{R}$

$\mathbb{R}$ is the only complete ordered field (up to isomorphism) containing $\mathbb{Q}$.

Thank You!

We'd love your questions and feedback.

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References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.