



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Properties of the Real Numbers

Consequences of the Axiom of Completeness

OCTOBER 18, 2025

CONDUCTED BY

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com
For updates subscribe on [@ParthoSutraDhor](#)

The Absolute Value Function

Absolute Value

For any real number x , the **absolute value** of x , denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem

Show that for any real numbers x , the following holds:

$$\sqrt{x^2} = |x|$$

The Triangle Inequality

💡 The Triangle Inequality for Real Numbers

For any real numbers x and y , the following inequality holds:

$$|x + y| \leq |x| + |y|$$



💡 Other forms of the Triangle Inequality

For any real numbers x and y , the following inequalities hold:

$$|x - y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

Equality of two Real Numbers

💡 Theorem

Two real numbers a and b are equal if and only if for every $\varepsilon > 0$,

$$|a - b| < \varepsilon$$



Axiom of Completeness (AoC)

Least Upper Bound (Supremum)

A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

- (i) s is an upper bound for A ;
- (ii) if u is any upper bound for A , then $s \leq u$.

Axiom of Completeness (AoC)

Every non-empty set of real numbers that is bounded above has a least upper bound (supremum) in \mathbb{R} .

Greatest Lower Bound (Infimum)

A real number t is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

- (i) t is a lower bound for A ;
- (ii) if l is any lower bound for A , then $t \geq l$.

Theorem: Existence of Infimum

Show that every non-empty set of real numbers that is bounded below has a greatest lower bound (infimum) in \mathbb{R} . 

Theorem: Characterization of Supremum

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$s - \epsilon < a$$

Theorem: Characterization of Infimum

Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying

$$a < t + \epsilon$$

Consequences of Completeness (AoC)

Nested Interval Property (NIP)

≡ Nested Interval Property (NIP)

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1} . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

💡 Theorem: AoC \implies NIP

The Axiom of Completeness implies the Nested Interval Property.

Archimedean Property (AP)

■ Archimedean Property (AP)

Given any number $x \in \mathbb{R}$ and any real number $y > 0$, the following two statements hold:

- (i) *There exists an $n \in \mathbb{N}$ satisfying $n > x$.*
- (ii) *There exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

💡 Theorem: AoC \implies AP

The Axiom of Completeness implies the Archimedean Property.

NIP + AP \implies AoC

💡 Theorem: NIP + AP \implies AoC

The Nested Interval Property and the Archimedean Property together imply the Axiom of Completeness.

Density of Rationals in Reals

💡 Density of \mathbb{Q} in \mathbb{R}

For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

The Existence of Square Roots

💡 Theorem

There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.



Thank You!

We'd love your questions and feedback.

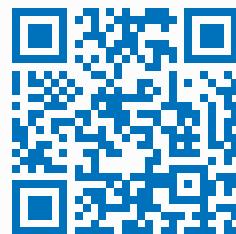
Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

 partho.dhor@bracu.ac.bd |  parthosutradhor@gmail.com



(Lectures, walkthroughs, and course updates)



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References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.