BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Cauchy Sequences and Completeness

Cauchy Criterion (CC), Completeness Revisited

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CONDUCTED BY

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Convergence and Limit of a Sequence

Convergence of a Sequence

A sequence $\{a_n\}$ is said to **converge** to a real number a if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number a exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \to a \text{ as } n \to \infty.$$

Cauchy Sequence

Cauchy Sequence (Def-1)

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that

$$\forall m, n \geq N \quad \implies \quad |a_n - a_m| < \varepsilon.$$

Cauchy Sequence (Def-2)

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for every real number $\varepsilon > 0$, there exists a natural number $m \in \mathbb{N}$ such that

$$\forall n \geq m \quad \implies \quad |a_n - a_m| < \varepsilon.$$

Problem

Show that the sequence defined by

$$a_n = \frac{1}{n}$$

is a Cauchy sequence.

Problem

Show that the sequence defined by

$$a_n = \frac{1}{2^n}$$

is a Cauchy sequence.

Problem

Show that the sequence defined by

$$a_n = (-1)^n$$

is not a Cauchy sequence.

Problem

Show that the sequence defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not a Cauchy sequence.

Theorem

Every Cauchy sequence is bounded sequence.

Theorem

Every Convergent sequence is a Cauchy sequence.

Theorem

Every Cauchy Sequence in $\mathbb{R}$ converges to a real number.

Cauchy Criterion (CC) for Sequences in $\mathbb{R}$

Theorem

A sequence $\{a_n\}$ of real numbers converges if and only if it is a Cauchy sequence.

Completeness of a Metric Space

Completeness of a Metric Space

A metric space is said to be **complete** if every Cauchy sequence in the space converges to a limit that is also within that space.

The incompleteness of $\mathbb{Q}$

Not every Cauchy Sequence in $\mathbb{Q}$ converges to a rational number.

Show that the sequence defined by

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}, \quad a_1 = 1$$

is a Cauchy sequence in $\mathbb{Q}$ but does not converge to any rational number.

Completeness Revisited

Equivalence of Completeness Axioms for $\mathbb{R}$

The following are equivalent statements of the Completeness Axiom for the real numbers:

1. Axiom of Completeness (AoC)
2. Nested Interval Property (NIP) + Archimedean Property (AP)
3. Monotone Convergence Theorem (MCT)
4. Bolzano-Weierstrass Theorem (BWT)
5. Cauchy Criterion (CC) + Archimedean Property (AP)

Assignment

Prove the following statements

1. AoC $\implies$ NIP
2. AoC $\implies$ AP
3. NIP $+$ AP $\implies$ AoC
4. AoC $\implies$ MCT
5. MCT $\implies$ AoC
6. AoC $\implies$ BWT
7. BWT $\implies$ AoC
8. AoC $\implies$ CC
9. CC $+$ AP $\implies$ AoC

Thank You!

We'd love your questions and feedback.

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References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.