



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Cardinality of Sets

Countable and Uncountable Sets, Continuum Hypothesis


OCTOBER 21, 2025

CONDUCTED BY

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Injective / One-to-One Functions

Injective / One-to-One Function

Let A and B be two sets. A function $f : A \rightarrow B$ is called *injective* or *one-to-one* if for every $a_1, a_2 \in A$,

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

In other words, different elements in the domain map to different elements in the codomain.

Surjective / Onto Functions

Surjective / Onto Function

Let A and B be two sets. A function $f : A \rightarrow B$ is called *surjective* or *onto* if for every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$. In other words, every element in the codomain is the image of at least one element from the domain.

Bijjective Functions

Bijjective Function

Let A and B be two sets. A function $f : A \rightarrow B$ is called *bijjective* if it is both injective and surjective.

Idea of Cardinality and Historical Context

Equivalent Sizes of Sets

Comparing Sizes of Sets by Bijective Functions

If there exists an Bijective function from set A to set B , we say that the cardinality of A is equal to the cardinality of B , denoted as

$$\text{card}(A) = \text{card}(B)$$

Often denoted as $A \sim B$ to indicate that sets A and B have the same cardinality.

Problem

Show that the set of natural numbers \mathbb{N} and the set of even natural numbers $\mathbb{E} = \{2, 4, 6, 8, \dots\}$ have the same cardinality.

Problem

Show that the set of natural numbers \mathbb{N} and the set of perfect squares $\mathbb{S} = \{1, 4, 9, 16, 25, \dots\}$ have the same cardinality.

Finite Sets

Finite Set

A set A is called *finite* if either A is the empty set or there exists a bijective function between the set $N_m = \{1, 2, \dots, m\}$ for some natural number m and the set A . In this case, we say that the cardinality of A is m and write $\text{card}(A) = m$.

Infinite Sets

Infinite Set

A set A is called *infinite* if it is not finite. Thus, A is not the empty set and for every natural number m , there is no bijective function between the set $N_m = \{1, 2, \dots, m\}$ and the set A .

Countable Sets

Countable Set

A set A is called *countable* if it is either finite or has the same cardinality as the set of natural numbers \mathbb{N} .

To prove a set A is countable, we can either:

- Show that A is finite, or
- Construct a bijective function between \mathbb{N} and A .

Note: An infinite set which is countable is often referred to as *countably infinite* or *denumerable*. For this type of set we can list its elements in a sequence (like the natural numbers).

② Problem

Show that the set of all even integers is countable.



② Problem

Show that the set of all odd integers is countable.



② Problem

Show that the set of integers \mathbb{Z} is countable.



Uncountable Sets

Uncountable Set

A set A is called *uncountable* if it is not countable. This means that A is infinite and there is no bijective function between \mathbb{N} and A .

⌘ Problem

Show that the interval $(0, 1)$ is uncountable.



Problem

The set of real numbers \mathbb{R} is uncountable.



Problem

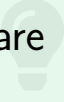
For any real numbers a and b with $a < b$, the open interval (a, b) and the closed interval $[a, b]$ are uncountable.

Theorems on Countability


Theorem

Every subset of a countable set is countable. That is, if A is a countable set and $B \subseteq A$, then B is countable.

Theorem

The union of two countable sets is countable. That is, if A and B are countable sets, then $A \cup B$ is countable. 

Theorem

The Cartesian product of two countable sets is countable. That is, if A and B are countable sets, then $A \times B$ is countable. 

② Problem

The set of rational numbers \mathbb{Q} is countable.



Theorem

The union of countably many countable sets is countable. That is, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Problem

The set of rational numbers \mathbb{Q} is countable.



The Schröder–Bernstein Theorem

💡 Schröder–Bernstein Theorem

Let A and B be two sets. If there exist injective functions $f : A \rightarrow B$ and $g : B \rightarrow A$, then there exists a bijective function $h : A \rightarrow B$. In other words, if $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A)$, then $\text{card}(A) = \text{card}(B)$.

Cardinal Numbers

Cardinal Number

For every set X , there exists a *cardinal number*, denoted $\text{card } X$, which represents the size of X . Two sets X and Y have the same cardinal number if and only if $X \sim Y$:

$$\text{card } X = \text{card } Y \iff \exists f : X \rightarrow Y \text{ that is bijective.}$$

Ordering of Cardinal Numbers

Ordering of Cardinal Numbers

For any sets A and B :



1. $\text{card}(A) \leq \text{card}(B)$ if and only if there exists an injective function $f : A \rightarrow B$.
2. $\text{card}(A) = \text{card}(B)$ if and only if there exists a bijective function $f : A \rightarrow B$.
3. $\text{card}(A) < \text{card}(B)$ if and only if $\text{card}(A) \leq \text{card}(B)$ but $A \not\sim B$.

Cantor's Theorem

💡 Cantor's Theorem

For every set A , the power set $\mathcal{P}(A)$ has strictly greater cardinality than A :

$$\text{card } A < \text{card } \mathcal{P}(A).$$

Cantor's theorem implies that there is no “largest” set, since from any set A we can always construct a larger one $\mathcal{P}(A)$. Hence, a “set of all things” cannot exist without contradiction.

Cardinality of Finite Sets

Cardinality of Finite Sets

If A is a finite set with n elements, then

$$\text{card } A = n.$$



Cardinality of Infinite Sets

Countable Infinity

The smallest infinite cardinal number is denoted by \aleph_0 (aleph-null):

$$\aleph_0 = \text{card}(\mathbb{N}).$$

It represents the size of any countably infinite set, such as \mathbb{Z} or \mathbb{Q} .

Continuum

The cardinality of the real numbers (or the power set of \mathbb{N}) is denoted by

$$\mathfrak{c} = \text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}.$$

By Cantor's theorem, we have $\aleph_0 < \mathfrak{c}$.

Cantor's Power Set Theorem

For every set A ,

$$\text{card}(\mathcal{P}(A)) = 2^{\text{card}(A)} > \text{card}(A).$$



Hierarchy of Infinite Cardinalities

Applying Cantor's theorem repeatedly yields an infinite sequence of strictly increasing cardinalities:

$$\begin{aligned}\aleph_0 &= \text{card}(\mathbb{N}), \\ 2^{\aleph_0} &= \mathfrak{c} = \text{card}(\mathbb{R}), \\ 2^{\mathfrak{c}} &= \text{card}(\mathcal{P}(\mathbb{R})), \\ 2^{2^{\mathfrak{c}}} &= \text{card}(\mathcal{P}(\mathcal{P}(\mathbb{R}))), \\ &\vdots\end{aligned}$$

Thus,

$$\aleph_0 < \mathfrak{c} < 2^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}} < \dots$$

Gaps Between Cardinalities

② Problem

Does there exist a set $A \subseteq \mathbb{R}$ such that $\aleph_0 < \text{card } A < \mathfrak{c}$? This question leads to the *Continuum Hypothesis*.

The Continuum Hypothesis (CH)

Continuum Hypothesis

The *Continuum Hypothesis* asserts that there is no set $A \subseteq \mathbb{R}$ such that

$$\aleph_0 < \text{card } A < \mathfrak{c}.$$

In other words, the cardinality of the continuum \mathfrak{c} is the immediate successor of \aleph_0 .

Thank You!

We'd love your questions and feedback.

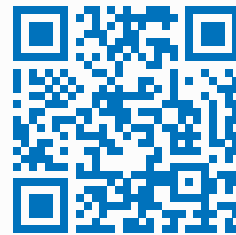
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(Lectures, walkthroughs, and course updates)



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References

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- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.