BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Monotone Sequences

Monotone Convergence Theorem (MCT)

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CONDUCTED BY

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Monotone Sequences

Monotonically Increasing Sequence

A sequence $\{a_n\}$ is said to be **monotonically increasing** if for all natural numbers n, the terms of the sequence satisfy the inequality

$$a_n \leq a_{n+1}.$$

Monotonically Decreasing Sequence

A sequence $\{a_n\}$ is said to be **monotonically decreasing** if for all natural numbers n, the terms of the sequence satisfy the inequality

$$a_n \geq a_{n+1}.$$

Examples of Monotone Sequences

Example

The sequence defined by

$$x_n = \frac{1}{n}$$

is a monotone sequence.

Example

The sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

is a monotone sequence.

Example

The Sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

is a monotone sequence.

Example

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

is a monotone sequence.

Monotone Convergence Theorem (MCT)

Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Aoc $\implies$ MCT

Let (a_n) be a monotone and bounded sequence. Using the Axiom of Completeness, show that the sequence (a_n) converges.

Problem

Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges and find the limit.

Problem

Prove that the sequence defined by $x_1 = 1$ and

$$x_{n+1} = 4 - \frac{1}{x_n}$$

converges and find the limit.

Problem

Show that

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

converges and find the limit.

Problem

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

1. Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.
2. Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Limit Superior

Definition

The **limit superior** of a bounded sequence (a_n) is defined as

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right).$$

Explain

Why limit superior exists?

Limit Inferior

Definition

The **limit inferior** of a bounded sequence (a_n) is defined as

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right).$$

Explain

Why limit inferior exists?

Problem

Find the Limit Superior and Limit Inferior of the sequence

$$x_n = (-1)^n + \frac{1}{n}$$

Theorem

For every bounded sequence a_n

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

Theorem

Any sequence a_n converges if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

Thank You!

We'd love your questions and feedback.

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References

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[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.