BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Complex Variables & Laplace Transform (MAT215)

Complex Functions

Exponential, Logarithmic, Circular and Hyperbolic Functions

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Conducted by

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Single valued and multi valued functions

Single-valued Function

A function $f : D \to \mathbb{C}$ is called single-valued if for every input $z \in D$, there is exactly one output $w \in \mathbb{C}$ such that $f(z) = w$.

Multi-valued Function

A function $f : D \to \mathcal{P}(\mathbb{C})$ is called multi-valued if for some input $z \in D$, there are multiple outputs $w_1, w_2, \ldots, w_n \in \mathbb{C}$ such that $f(z) = \{w_1, w_2, \ldots, w_n\}$ with $n > 1$.

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a branch of the function.

Why some functions are multi-valued?

The Exponential Function

Complex Exponential Function is Periodic

The complex exponential function e^z is periodic with period $2\pi i$.

Problem

Find all values of $z \in \mathbb{C}$ such that

$$e^z = -1 + i\sqrt{3}.$$

Problem

Find all values of $z \in \mathbb{C}$ such that

$$e^{4z} = -1$$

The Logarithmic Function

Complex Logarithm

The complex logarithm of a non-zero complex number z is defined as

$$\log z = \ln |z| + i \arg z$$

where $\arg z$ is the argument of z.

Note: The complex logarithm is a multi-valued function due to the multi-valued nature of the argument $\arg z$.

Problem

Show that:

(i) $\ln(-1+\sqrt{3}i) = \ln 2 + 2\left(n+\frac{1}{3}\right)\pi i$

(ii) $\ln(1-i) = \frac{1}{2}\ln 2 + \left(2n+\frac{7}{4}\right)\pi i$

(iii) $\ln(i^{1/2}) = \left(n+\frac{1}{4}\right)\pi i$

Circular Functions

Complex Sine and Cosine Functions

The complex sine and cosine functions are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

for all $z \in \mathbb{C}$.

Inverse Circular Functions

Inverse Circular Functions

The inverse circular functions for complex numbers are defined as follows:

$$\sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$$

$$\cos^{-1} z = -i \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right)$$

$$\cot^{-1} z = \frac{i}{2} \ln\left(\frac{z - i}{z + i}\right)$$

$$\sec^{-1} z = -i \ln\left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$

$$\operatorname{cosec}^{-1} z = -i \ln\left(\frac{i + \sqrt{z^2 - 1}}{z}\right)$$

for all $z \in \mathbb{C}$.

Problem

Prove that for all $z \in \mathbb{C}$:

$$\sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$$

Problem

Prove that for all $z \in \mathbb{C}$:

$$\sec^{-1} z = -i \ln \left(\frac{1 + \sqrt{1 - z^2}}{z} \right)$$

Problem

Prove that for all $z \in \mathbb{C}$:

$$\cot^{-1} z = \frac{i}{2} \ln\left(\frac{z-i}{z+i}\right)$$

Problem

Solve for $z \in \mathbb{C}$:

$$\sin z = 2$$

Hyperbolic Functions

Sine Hyperbolic and Cosine Hyperbolic Functions

The complex sine and cosine functions are defined as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

for all $z \in \mathbb{C}$.

Inverse Hyperbolic Functions

Inverse Hyperbolic Functions

The inverse hyperbolic functions for complex numbers are defined as follows:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

$$\operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\operatorname{cosech}^{-1} z = \ln\left(\frac{1+\sqrt{1+z^2}}{z}\right)$$

for all $z \in \mathbb{C}$.

Problem

Prove that for all $z \in \mathbb{C}$:

$$\cosh^{-1} z = \log\left(z + \sqrt{z^2 - 1}\right)$$

Problem

Prove that for all $z \in \mathbb{C}$:

$$\operatorname{cosech}^{-1} z = \ln\left(\frac{1+\sqrt{1+z^2}}{z}\right)$$

Problem

Prove that for all $z \in \mathbb{C}$:

$$\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$$

Problem

Solve for $z \in \mathbb{C}$:

$$\cosh z = 3i$$

Properties of Circular and Hyperbolic Functions

Theorem

For all $z \in \mathbb{C}$:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

where $z = x + iy$, $x, y \in \mathbb{R}$.

Theorem

For all $z \in \mathbb{C}$:

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y$$

where $z = x + iy$, $x, y \in \mathbb{R}$.

Theorem

For all $z \in \mathbb{C}$:

$$\sin(iz) = i \sinh z,$$

$$\cos(iz) = \cosh z,$$

$$\sinh(iz) = i \sin z,$$

$$\cosh(iz) = \cos z.$$

Theorem

For all $z \in \mathbb{C}$:

$$cos^2 z + \sin^2 z = 1$$

$$cosh^2 z - sinh^2 z = 1$$

Theorem

For all $z \in \mathbb{C}$:

$$\frac{d}{dz}(\sin z) = \cos z, \quad \frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\sinh z) = \cosh z, \quad \frac{d}{dz}(\cosh z) = \sinh z$$

Thank You!

We'd love your questions and feedback.

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References

[1] Murray R. Spiegel, *Schaum's Outline of Complex Variables*, 2nd Edition, McGraw–Hill, 2009.

[2] Dennis G. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th Edition, Cengage, 2018.