



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Cauchy Sequences and Completeness

Cauchy Criterion (CC), Completeness Revisited

NOVEMBER 09, 2025

CONDUCTED BY

Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

partho.dhor@bracu.ac.bd | parthosutradhor@gmail.com

For updates subscribe on [@ParthoSutraDhor](#)

Convergence and Limit of a Sequence

Convergence of a Sequence

A sequence $\{a_n\}$ is said to **converge** to a real number a if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number a exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \rightarrow a \text{ as } n \rightarrow \infty.$$

Cauchy Sequence

■ Cauchy Sequence (Def-1)

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that

$$\forall m, n \geq N \implies |a_n - a_m| < \varepsilon.$$



■ Cauchy Sequence (Def-2)

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for every real number $\varepsilon > 0$, there exists a natural number $m \in \mathbb{N}$ such that

$$\forall n \geq m \implies |a_n - a_m| < \varepsilon.$$



② Problem

Show that the sequence defined by

$$a_n = \frac{1}{n}$$

is a Cauchy sequence.

② Problem

Show that the sequence defined by

$$a_n = \frac{1}{2^n}$$

is a Cauchy sequence.

② Problem

Show that the sequence defined by

$$a_n = (-1)^n$$

is not a Cauchy sequence.

② Problem

Show that the sequence defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not a Cauchy sequence.



Theorem

Every Cauchy sequence is bounded sequence.



Theorem

Every Convergent sequence is a Cauchy sequence.



Theorem

Every Cauchy Sequence in \mathbb{R} converges to a real number.



Cauchy Criterion (CC) for Sequences in \mathbb{R}

💡 Theorem

A sequence $\{a_n\}$ of real numbers converges if and only if it is a Cauchy sequence.

Completeness of a Metric Space

Completeness of a Metric Space

A metric space is said to be **complete** if every Cauchy sequence in the space converges to a limit that is also within that space.



② The incompleteness of \mathbb{Q}

Not every Cauchy Sequence in \mathbb{Q} converges to a rational number. 

Show that the sequence defined by

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}, \quad a_1 = 1$$

is a Cauchy sequence in \mathbb{Q} but does not converge to any rational number.

Completeness Revisited



Equivalence of Completeness Axioms for \mathbb{R}

The following are equivalent statements of the Completeness Axiom for the real numbers:



1. Axiom of Completeness (AoC)
2. Nested Interval Property (NIP) + Archimedean Property (AP)
3. Monotone Convergence Theorem (MCT)
4. Bolzano-Weierstrass Theorem (BWT)
5. Cauchy Criterion (CC) + Archimedean Property (AP)

Assignment

█ Prove the following statements

1. AoC \implies NIP
2. AoC \implies AP
3. NIP + AP \implies AoC
4. AoC \implies MCT
5. MCT \implies AoC
6. AoC \implies BWT
7. BWT \implies AoC
8. AoC \implies CC
9. CC + AP \implies AoC



Thank You!

We'd love your questions and feedback.

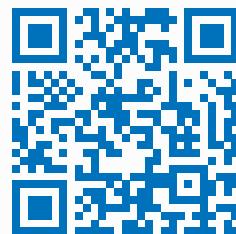
Partho Sutra Dhor

Lecturer, BRAC University, Dhaka-1212

 partho.dhor@bracu.ac.bd |  parthosutradhor@gmail.com



(Lectures, walkthroughs, and course updates)



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References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.