BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Convergence of Sequences in $\mathbb{R}$

Limit of a Sequence

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CONDUCTED BY

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Definition of a Sequence

Sequence

A **sequence** is a function whose domain is the set of natural numbers $\mathbb{N}$ and whose codomain is a non-empty set S.

Real Sequence or Sequence in $\mathbb{R}$

A **real sequence** is a sequence whose codomain is the set of real numbers $\mathbb{R}$.

Common Notations for Sequences

Sequence Notation

A sequence is usually denoted by $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$, where a_n represents the n-th term of the sequence. Here, n is a natural number indicating the position of the term in the sequence.

Examples

Each of the following are common ways to describe a sequence.

- $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$,
- $\left(\frac{1+n}{n}\right)_{n=1}^{\infty} = \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\right)$,
- $\{a_n\}$, where $a_n = 2^n$ for each $n \in \mathbb{N}$,
- (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n + 1}{2}$.

Convergence and Limit of a Sequence

Convergence of a Sequence

A sequence $\{a_n\}$ is said to **converge** to a real number a if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number a exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \to a \text{ as } n \to \infty.$$

Example

Consider the sequence

$$a_n = \left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty}.$$

We will show that this sequence converges to 0.

Quantifier Breakdown of the Definition

Template for a proof that $\lim a_n = a$

Proof Template

To prove that $\lim a_n = a$, we need to show that for every $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the inequality $|a_n - a| < \varepsilon$ holds.

Step 1: Let $\varepsilon > 0$ be given.

Step 2: Find a suitable $N \in \mathbb{N}$ (often in terms of ε) such that for all $n \geq N$, the desired inequality holds.

Step 3: Show that for all $n \geq N$, we have

$$|a_n - a| < \varepsilon.$$

Conclusion: Since we have shown that for every $\varepsilon > 0$, there exists an N such that for all $n \geq N$, the inequality holds, we conclude that $\lim a_n = a$.

Example Proof

Show that the sequence $a_n = \dfrac{n+1}{n}$ converges to 1.

Example Proof

Show that the sequence $a_n = \dfrac{3n+1}{2n+5}$ converges to $\dfrac{3}{2}$.

Example Proof

Show that

$$\lim_{n\to\infty} \frac{1}{3^n} = 0.$$

Negation of Quantifiers

Divergent Sequences

Divergent Sequence

A sequence that does not converge to any real number is called a **divergent sequence**.

Digergence

A sequence $\{a_n\}$ is said to **diverge** if there exists a real number $\varepsilon > 0$ such that for every natural number $N \in \mathbb{N}$, there exists an index $n \geq N$ such that the terms of the sequence satisfy the inequality

$$|a_n - a| \geq \varepsilon.$$

Problem

Discuss why the sequence $a_n = (-1)^n$ does not converge.

Problem

Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \ldots\right).$$

How can we argue that this sequence does not converge to zero?

Problem

Argue that the sequence

$$1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ \text{(5 zeros)},\ 1,\ \ldots$$

does not converge to zero. For what values of $\varepsilon > 0$ does there exist a response N? For which values of $\varepsilon > 0$ is there no suitable response?

Digerge to Positive Infinity

Divergence to positive Infinity

A sequence $\{a_n\}$ is said to **diverge to positive infinity** if for every real number $M > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$a_n > M.$$

If such a condition holds, we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty \text{ as } n \to \infty.$$

Problem

Show that the sequence $a_n = n^2$ diverges to infinity.

Digerge to Negative Infinity

Divergence to negative Infinity

A sequence $\{a_n\}$ is said to **diverge to negative infinity** if for every real number $M < 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$a_n < M.$$

If such a condition holds, we write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty \text{ as } n \to \infty.$$

Problem

Show that the sequence $a_n = -n^3$ diverges to negative infinity.

Oscillatory Sequences

Oscillatory Sequence

A sequence that neither converges nor diverges to infinity or negative infinity is called an **oscillatory sequence**.

Some More Examples on Convergence

Problem

Show that the sequence $a_n = \sqrt{n+1} - \sqrt{n}$ converges to 0.

Problem

Show that

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}.$$

Can you prove this?

Show that

$$\lim_{n\to\infty} \frac{3n+4}{2n-5} = \frac{3}{2}.$$

Problem

Show that

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2-3} = \frac{1}{2}.$$

Problem

Show that

$$\lim_{n\to\infty} \frac{5n^2+3}{2n^3+1} = 0.$$

Thank You!

We'd love your questions and feedback.

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References

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[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.