



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Real Analysis (MAT221)

Theorems on Limit of Sequences

Algebraic Properties, Order Properties and Squeeze Theorem

OCTOBER 28, 2025

CONDUCTED BY

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Convergence and Limit of a Sequence

Convergence of a Sequence

A sequence $\{a_n\}$ is said to **converge** to a real number a if for every real number $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n \geq N$, the terms of the sequence satisfy the inequality

$$|a_n - a| < \varepsilon.$$

If such a number a exists, it is called the **limit** of the sequence, and we write

$$\lim a_n = a \quad \text{or} \quad a_n \rightarrow a \text{ as } n \rightarrow \infty.$$

Uniqueness of Limit

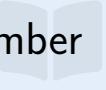
💡 Limit is Unique

If a sequence $\{a_n\}$ converges, then its limit is unique.



Bounded and Unbounded Sequences

Bounded Sequence

A sequence $\{a_n\}$ is said to be **bounded** if there exists a real number  $M > 0$ such that for all natural numbers $n \in \mathbb{N}$, the terms of the sequence satisfy the inequality

$$|a_n| \leq M.$$

If no such M exists, the sequence is said to be **unbounded**.

Every Convergent Sequence is Bounded

💡 Theorem

Every convergent sequence is bounded.



Theorem

Not every bounded sequence is convergent.



Bounded $\not\Rightarrow$ Convergent

Consider the sequence $a_n = (-1)^n$. Show that this sequence is bounded but does not converge.



Algebraic Properties of Limits

💡 Algebraic Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b , respectively. Then,

1. $\lim c = c$ for any constant sequence c .
2. $\lim |a_n| = |a|$.
3. $\lim(a_n + b_n) = a + b$.
4. $\lim(a_n - b_n) = a - b$.
5. $\lim(ca_n) = ca$, for any constant $c \in \mathbb{R}$.
6. $\lim(a_n b_n) = ab$.
7. $\lim\left(\frac{1}{a_n}\right) = \frac{1}{a}$, provided $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$.
8. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$.

Theorem

Prove that

$$\lim c = c$$

for any constant sequence c .

Theorem

If $\lim a_n = a$, then

$$\lim |a_n| = |a|.$$



Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n + b_n) = a + b.$$



Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n - b_n) = a - b.$$



Theorem

If $\lim a_n = a$, then for any constant $c \in \mathbb{R}$,

$$\lim(ca_n) = ca.$$



Theorem

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n b_n) = ab.$$



Theorem

If $\lim a_n = a$ with $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim \left(\frac{1}{a_n} \right) = \frac{1}{a}.$$



Theorem

If $\lim a_n = a$ and $\lim b_n = b$ with $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim \left(\frac{a_n}{b_n} \right) = \frac{a}{b}.$$

Order Properties of Limits

💡 Order Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b , respectively. Then,

1. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \geq 0 \text{ for all } n \geq N \implies a \geq 0.$$

2. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \leq b_n \text{ for all } n \geq N \implies a \leq b.$$

💡 Squeeze Theorem

Let $\{a_n\}$, $\{x_n\}$, and $\{b_n\}$ be three sequences such that $a_n \leq x_n \leq b_n$ for all n greater than or equal to some natural number $N \in \mathbb{N}$. If $\lim a_n = \lim b_n = L$, then $\lim x_n = L$.

 **Theorem:** $a_n \geq 0$ implies $a \geq 0$

Let $\{a_n\}$ be a convergent sequence with limit a . If there exists a natural number $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \geq 0 \text{ for all } n \geq N \implies a \geq 0.$$

② **Theorem:** $a_n > 0$ implies $a \geq 0$

Let $\{a_n\}$ be a convergent sequence with limit a . If there exists a natural number $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$, then $a \geq 0$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n > 0 \text{ for all } n \geq N \implies a \geq 0.$$

 **Theorem:** $a_n \leq b_n$ implies $a \leq b$

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b , respectively. If there exists a natural number $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n \leq b_n \text{ for all } n \geq N \implies a \leq b.$$

② **Theorem:** $a_n < b_n$ implies $a \leq b$

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with limits a and b , respectively. If there exists a natural number $N \in \mathbb{N}$ such that $a_n < b_n$ for all $n \geq N$, then $a \leq b$. In other words,

$$\exists N \in \mathbb{N} \text{ such that } a_n < b_n \text{ for all } n \geq N \implies a \leq b.$$

Squeeze Theorem

Let $\{a_n\}$, $\{x_n\}$, and $\{b_n\}$ be three sequences such that $a_n \leq x_n \leq b_n$ for all n greater than or equal to some natural number $N \in \mathbb{N}$. If $\lim a_n = \lim b_n = L$, then $\lim x_n = L$.

② Prove using Squeeze Theorem

Prove that

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0.$$



Ratio of Sequences

💡 Theorem

Let $\{a_n\}$ be a sequence of positive terms. If

$$\lim \frac{a_{n+1}}{a_n} = L,$$



then

1. if $L < 1$, the sequence $\{a_n\}$ converges to 0,
2. if $L > 1$, the sequence $\{a_n\}$ diverges,
3. if $L = 1$, the test is inconclusive.

② Problem

Prove that

$$\lim_{n \rightarrow \infty} \frac{n^n}{n!} = e$$



Root of Sequences

💡 Theorem

If $\{a_n\}$ is a sequence of positive terms and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists, then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

② Problem

Prove that

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$



② Problem

Prove that

$$\lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$$



Cesaro Means

💡 Theorem

If a sequence $\{a_n\}$ converges to a limit L , then the sequence $\{x_n\}$ defined by

$$x_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

converges to the same limit L .

② Problem

Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n} \right) = 1$$



Thank You!

We'd love your questions and feedback.

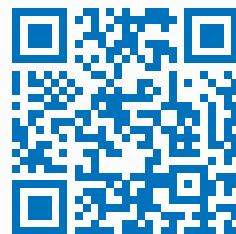
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(Lectures, walkthroughs, and course updates)



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References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.