BRAC University

Department of Mathematics and Natural Sciences

Lecture on

Real Analysis (MAT221)

Constructing the Rationals

Equivalence Relation and Constructing the Rational Numbers

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Conducted by

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Cartesian Product and Relation

Relation Between Two Sets

A *relation* R from a set A to a set B is any subset of the Cartesian product $A \times B$, i.e.

$$R \subseteq A \times B.$$

If $(a, b) \in R$, we say that "a is related to b by R," written as $a\,R\,b$ or sometimes $a \sim b$.

Equivalence Relation and Examples

Equivalence Relation

A relation $\sim$ on a set A is called an **equivalence relation** if

1. **Reflexive:** $a \sim a$, for all $a \in A$.
2. **Symmetric:** If $a \sim b$, then $b \sim a$, for all $a, b \in A$.
3. **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$, for all $a, b, c \in A$.

Important examples of equivalence relations

1. **Congruence modulo n on $\mathbb{Z}$:** For a fixed integer $n > 1$, define

$$a \sim b \iff a \equiv b \pmod{n}.$$

2. **Rational Difference on $\mathbb{R}$:** On the set of real numbers, define

$$a \sim b \iff a - b \in \mathbb{Q}.$$

3. **Congruent Triangles:** For triangles $\triangle ABC$ and $\triangle DEF$, define

$$\triangle ABC \sim \triangle DEF \iff \triangle ABC \cong \triangle DEF$$

4. **Parallel Lines in the Plane:** For lines l_1 and l_2 in $\mathbb{R}^2$, define

$$l_1 \sim l_2 \iff l_1 \parallel l_2.$$

5. **Same Slope of Nonzero Vectors:** In $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Problem

Congruence modulo n on $\mathbb{Z}$: For a fixed integer $n > 1$, define

$$a \sim b \iff a \equiv b \pmod{n}.$$

Prove that $\sim$ is an equivalence relation.

Problem

Rational Difference on $\mathbb{R}$: On the set of real numbers, define

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Same Slope of Nonzero Vectors: In $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Prove that $\sim$ is an equivalence relation.

Equivalence Classes

Equivalence Class

If $\sim$ is an equivalence relation on a set A and $a \in A$, the **equivalence class** of a is

$$[a] = \{\, x \in A \mid x \sim a \,\}.$$

Problem

Define the equivalence relation $\sim$ on $\mathbb{Z}$ by

$$a \sim b \iff a \equiv b \pmod 5 \iff a - b \text{ is divisible by } 5.$$

Find the equivalence class of 3.

Problem

In $\mathbb{R}^2 \setminus \{(0,0)\}$, define the equivalence relation $\sim$ by

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists k \neq 0 \text{ such that } (x_1, y_1) = k(x_2, y_2).$$

Find the equivalence class of $(1, 2)$.

Partitions from Equivalence Relations

Integers to Rational Numbers

"God made the integers; all else is the work of man."

— **Leopold Kronecker (1823–1891)**

Rational Numbers to Reals (complete)

Reals to Complex (algebraically closed)

Basic Topology of $\mathbb{R}$

Sequence and Limit

Functional Limits and Continuity

The Derivative

Sequences and Series of Functions

The Riemann Integral

Thank You!

We'd love your questions and feedback.

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References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.