



## **BRAC University**

Department of Mathematics and Natural Sciences

LECTURE ON

### **Real Analysis (MAT221)**

## **Subsequences and Limit Points**

### **Bolzano-Weierstrass Theorem (BWT)**

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NOVEMBER 02, 2025

CONDUCTED BY

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## Subsequence

### Subsequence

Let  $(a_n)$  be a sequence of real numbers, and let



$$n_1 < n_2 < n_3 < n_4 < n_5 < \dots$$

be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots$$

is called a *subsequence* of  $(a_n)$  and is denoted by  $(a_{n_k})$ , where  $k \in \mathbb{N}$  indexes the subsequence.

## Other Definition of Subsequence

### Subsequence

Let  $(a_n)$  be a sequence, and let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a strictly increasing function. Then the sequence  $(a_{f(k)})$  is called a **subsequence** of  $(a_n)$ . In other words, a subsequence is obtained by choosing elements of  $(a_n)$  in the same order, but possibly omitting some of them.

## Examples of Subsequence

## Theorem

Any subsequence of a convergent sequence converge to the same limit as the original sequence.

## Theorem

Assume  $(a_n)$  is a bounded sequence with the property that every convergent subsequence of  $(a_n)$  converges to the same limit  $a \in \mathbb{R}$ .

Show that  $(a_n)$  must converge to  $a$ .

## Bolzano–Weierstrass Theorem

### Bolzano–Weierstrass Theorem

Every bounded sequence contains a convergent subsequence.

## **NIP $\implies$ BWT**

Let  $(a_n)$  be a bounded sequence. Using the Nested Interval property, show that there exists a convergent subsequence.

## Aoc $\implies$ BWT

Let  $(a_n)$  be a bounded sequence. Using the Axiom of Completeness, show that there exists a convergent subsequence.

## Theorem

Assume  $(a_n)$  is a bounded sequence with the property that every convergent subsequence of  $(a_n)$  converges to the same limit  $a \in \mathbb{R}$ .

Show that  $(a_n)$  must converge to  $a$ .

## Theorem

Every sequence contains a monotonic subsequence.



## Limit Point or Cluster Point

### Limit Point

A real number  $l$  is said to be a limit point or cluster point of a sequence  $(a_n)$  if for every  $\varepsilon > 0$ , there exists a term  $a_m$  such that

$$|a_m - l| < \varepsilon.$$

## Examples of Limit Points

## Other Definition of Limit Point or Cluster Point

### Limit Point

A real number  $l$  is said to be a limit point or cluster point of a sequence  $(a_n)$  if for every  $\varepsilon > 0$ , the  $\varepsilon$  neighbourhood

$$(l - \varepsilon, l + \varepsilon)$$

contains at least one term of the sequence.

### Theorem

If a real number  $l$  is a limit point or cluster point of a sequence  $(a_n)$ ,  
then for every  $\varepsilon > 0$ , the  $\varepsilon$  neighbourhood

$$(l - \varepsilon, l + \varepsilon)$$

contains infinitely many terms of the sequence.

## **Theorem**

If a sequence  $(a_n)$  converges to  $l$ , then  $l$  is the only limit point of the sequence.

## Bolzano–Weierstrass Theorem (Limit Point Version)

### 💡 Bolzano–Weierstrass Theorem

Every bounded sequence has at least one limit point.



## Limit Superior and Inferior

### Limit Superior

The Limit Superior of a bounded sequence is the largest limit point.



### Limit Inferior

The Limit Inferior of a bounded sequence is the smallest limit point.



# Thank You!

We'd love your questions and feedback.

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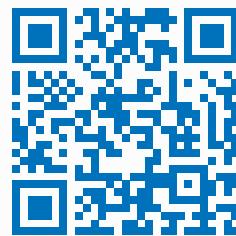
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(Lectures, walkthroughs, and course updates)



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## References

- [1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.
- [2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.