**BRAC University**

Department of Mathematics and Natural Sciences

**Real Analysis (MAT221)**

# Monotone Sequences and Subsequences

## Monotone Convergence, Bolzano-Weierstrass Theorem

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Conducted by

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# Monotone Sequences

### Monotonically Increasing Sequence

A sequence $\{a_n\}$ is said to be **monotonically increasing** if for all natural numbers $n$, the terms of the sequence satisfy the inequality

$$a_n \leq a_{n+1}.$$

### Monotonically Decreasing Sequence

A sequence $\{a_n\}$ is said to be **monotonically decreasing** if for all natural numbers $n$, the terms of the sequence satisfy the inequality

$$a_n \geq a_{n+1}.$$

## Examples of Monotone Sequences

**② Example**

The sequence defined by
$$x_n = \frac{1}{n}$$
is a monotone sequence.

**② Example**

The sequence defined by $x_1 = 3$ and
$$x_{n+1} = \frac{1}{4 - x_n}$$
is a monotone sequence.

The Sequence

$$\sqrt{2}, \ \sqrt{2\sqrt{2}}, \ \sqrt{2\sqrt{2\sqrt{2}}}, \ \ldots$$

is a monotone sequence.

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

is a monotone sequence.

# Monotone Convergence Theorem (MCT)

> **♀ Monotone Convergence Theorem**
>
> If a sequence is monotone and bounded, then it converges.

## 💡 Aoc $\implies$ MCT

Let $(a_n)$ be a monotone and bounded sequence. Using the Axiom of Completeness, show that the sequence $(a_n)$ converges.

Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges and find the limit.

Prove that the sequence defined by $x_1 = 1$ and

$$x_{n+1} = 4 - \frac{1}{x_n}$$

converges and find the limit.

Show that

$$\sqrt{2}, \ \sqrt{2\sqrt{2}}, \ \sqrt{2\sqrt{2\sqrt{2}}}, \ \ldots$$

converges and find the limit.

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

1. Show that $x_n^2$ is always greater than $2$, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

2. Modify the sequence $(x_n)$ so that it converges to $\sqrt{c}$.

# Subsequence

## Subsequence

Let $(a_n)$ be a sequence of real numbers, and let

$$n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$$

be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \cdots$$

is called a *subsequence* of $(a_n)$ and is denoted by $(a_{n_j})$, where $j \in \mathbb{N}$ indexes the subsequence.

# Examples of Subsequence

> **💡 Theorem**
>
> Subsequences of a convergent sequence converge to the same limit as the original sequence.

# Bolzano–Weierstrass Theorem

> 📖 **Bolzano–Weierstrass Theorem**
>
> Every bounded sequence contains a convergent subsequence.

## 💡 NIP ⟹ BWT

Let $(a_n)$ be a bounded sequence. Using the Nested Interval property, show that there exists a convergent subsequence.

## 💡 Aoc ⟹ BWT

Let $(a_n)$ be a bounded sequence. Using the Axiom of Completeness, show that there exists a convergent subsequence.

> **💡 Theorem**
>
> Assume $(a_n)$ is a bounded sequence with the property that every convergent subsequence of $(a_n)$ converges to the same limit $a \in \mathbb{R}$. Show that $(a_n)$ must converge to $a$.

# Thank You!

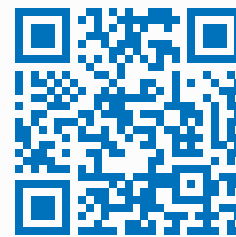We'd love your questions and feedback.

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(Lectures, walkthroughs, and course updates)

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### 📖 References

[1] Stephen Abbott, *Understanding Analysis*, 2nd Edition, Springer, 2015.

[2] Terence Tao, *Analysis I*, 3rd Edition, Texts and Readings in Mathematics, Hindustan Book Agency, 2016.