Image Processing and Visual Communications

Statistical Image Modeling

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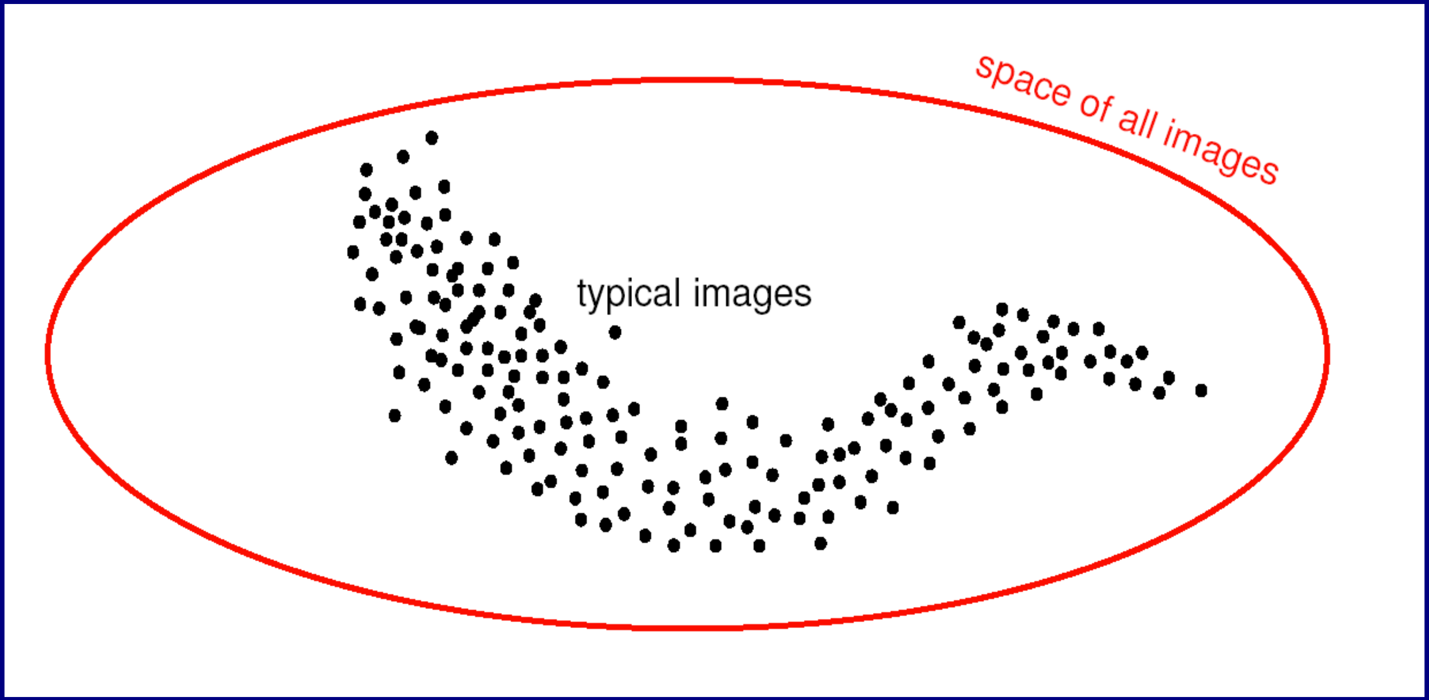
Outline

- **Statistical Image Modeling**
  - Why statistical image modeling?
  - Pixel intensity models
  - Markov random field models
  - Fourier models
  - Wavelet marginal models
  - Wavelet joint models
  - Advanced statistics: phase and orientation
- **Applications of Statistical Image Modeling**
  - Image denoising
  - Other applications

Why Statistical Image Modeling?

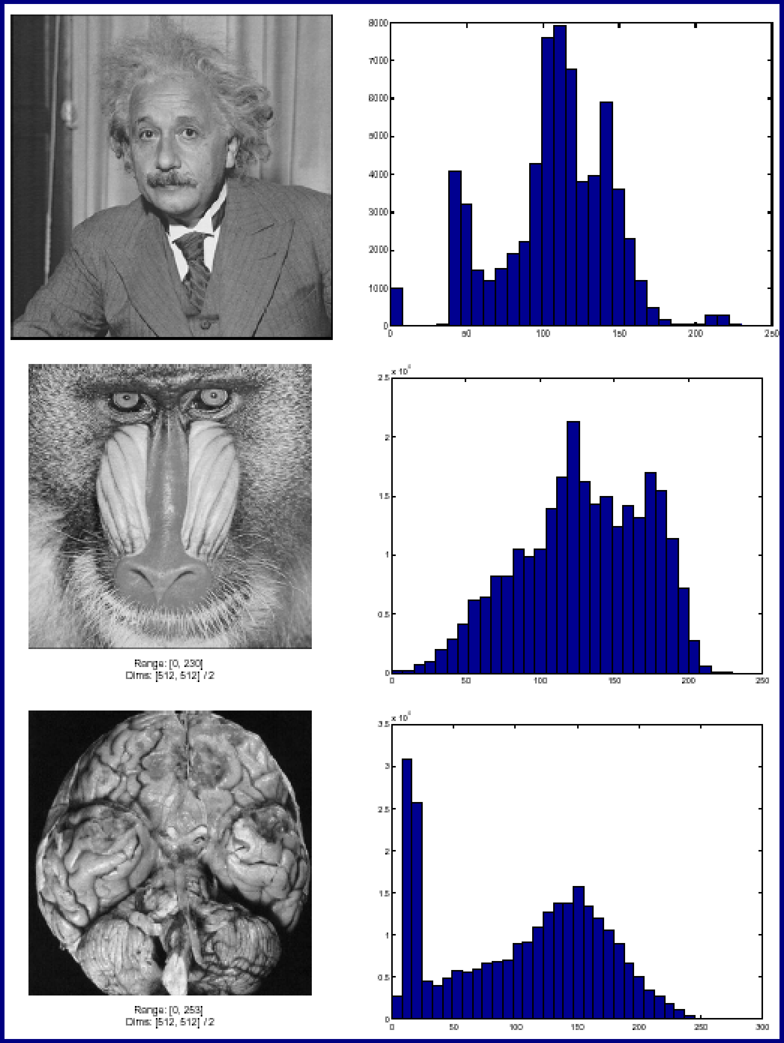
- **Prior Image Probability Distribution**
  - Typical (natural) images occupy an extremely tiny (and unknown-shape) space in the space of all images



- **What's the Benefit of Knowing Image Prior?**

Ref: *Representation and Analysis of Visual Images* lecture notes by Prof. Eero P. Simoncelli

Pixel Intensity Statistics: Marginal



- Does not tell much about what is special of natural images

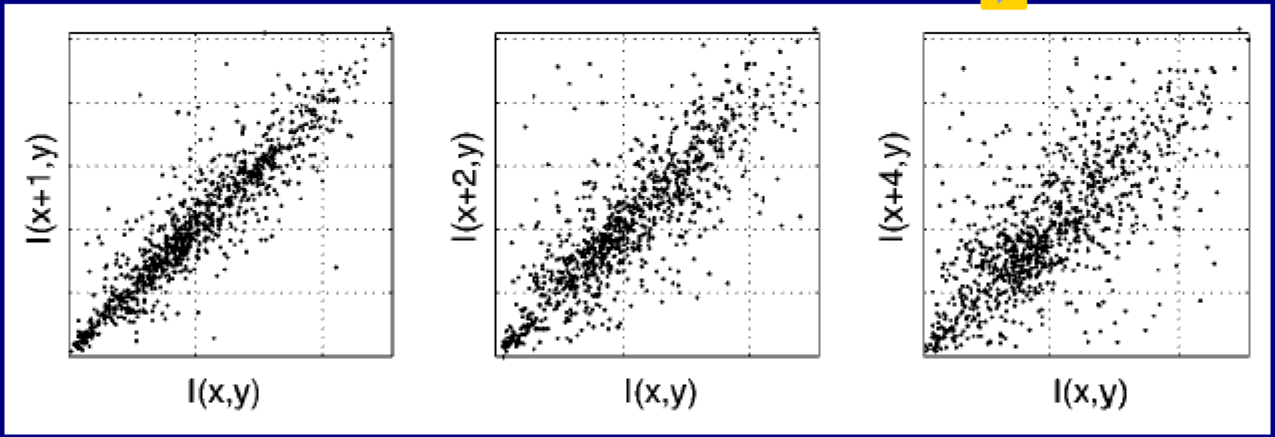
- What if I scramble the order of image pixels?

- What's the problem?

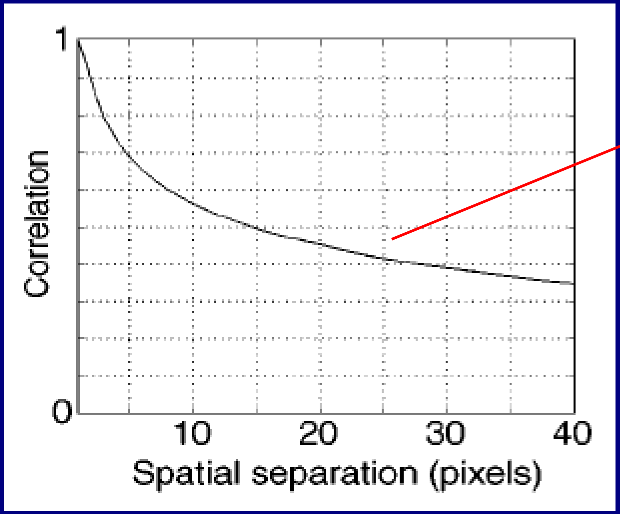
  images are highly "structured", or image pixel intensities are highly dependent (correlated), which marginal statistics completely ignore

Ref: *Representation and Analysis of Visual Images* lecture notes by Prof. Eero P. Simoncelli

Pixel Intensity Statistics: Joint



joint distributions of image pixel intensities separated by 1, 2, and 4 pixels



autocorrelation function

autocorrelation function

Fourier transform

power spectrum

Q: What's the underlying assumption here?

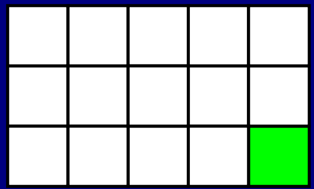
A: Stationary (good or bad?)

Simoncelli & Olshausen '01

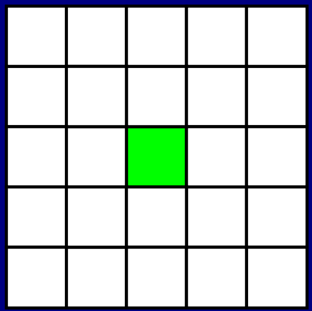
Pixel Intensity Model: Markov Random Field

- **Random Field x**
  - A collection of RVs on a lattice
  - Difficult to define $p(\mathbf{x})$ for images
  - "Implicit" but "easier" models
- **Markovianity:**
  - 1-D: decouple "past" and "future"
  - From 1-D to 2-D
  - Causal and non-causal neighbors
- **Variations**
  - Gaussian MRF: $p(\mathbf{x})$ Gaussian
  - Gibbs MRF: $p(\mathbf{x}) \propto \exp[-\beta E(\mathbf{x})]$
  - Extensions:

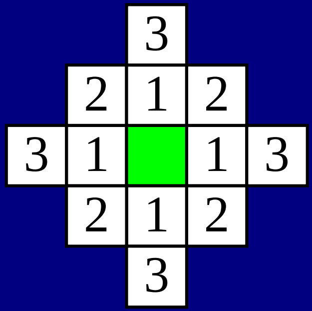
    Gaussian mixture; multi-scale …



causal neighbors

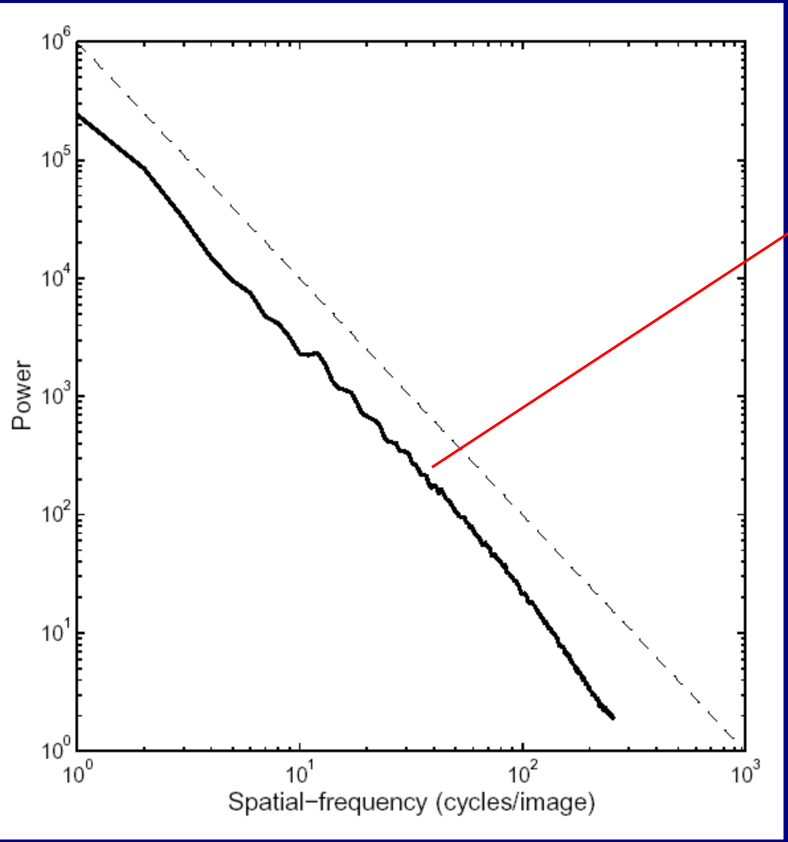


non-causal neighbors



1st, 2nd, 3rd order neighbors

Fourier Magnitude Statistics



power spectrum of images
(in log-log scale)

straight line!!

$$P(f) = 1/f^{\beta}$$

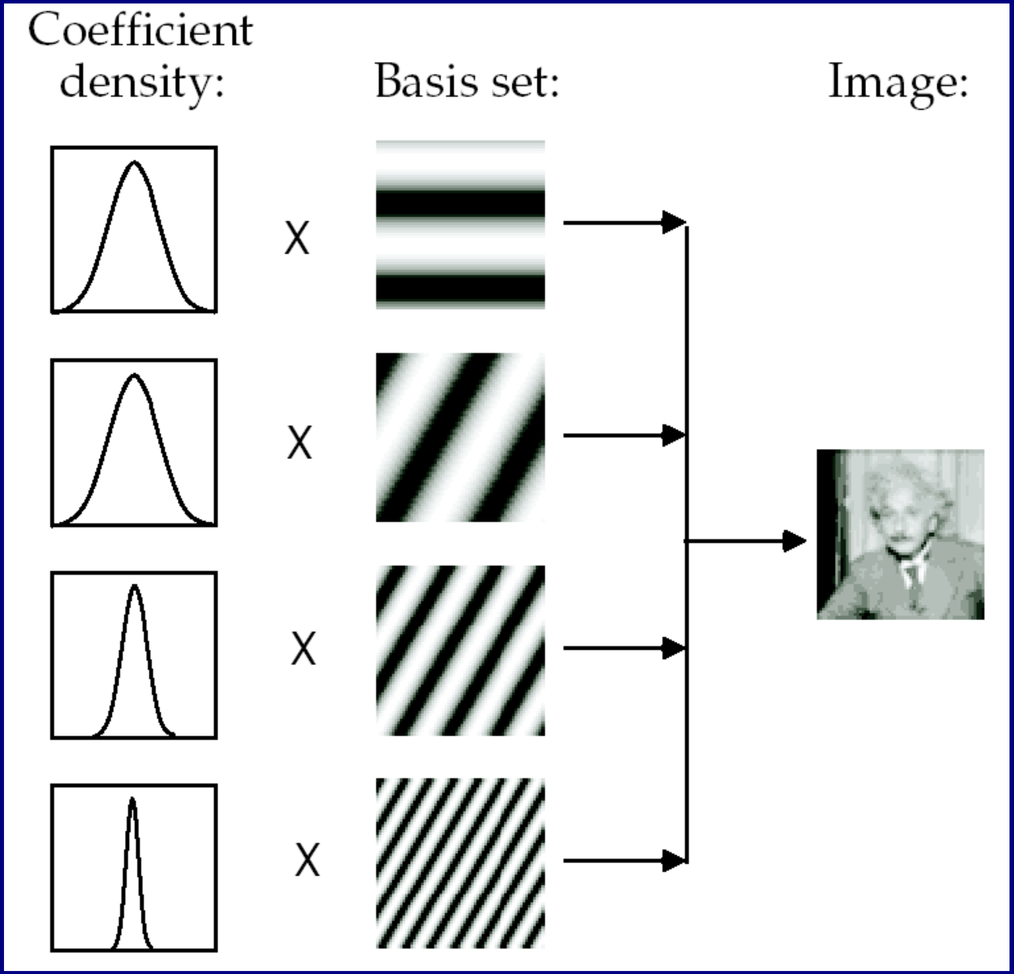
β: typically around 2

Explanations:

1. Scale-invariance
2. Edges

Simoncelli & Olshausen '01

Fourier Magnitude + Gaussian

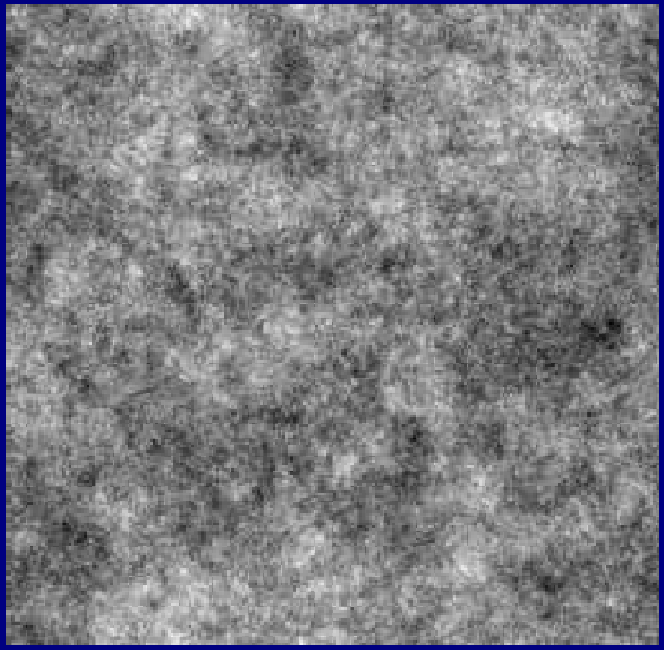


Note: projected Gaussian is still Gaussian

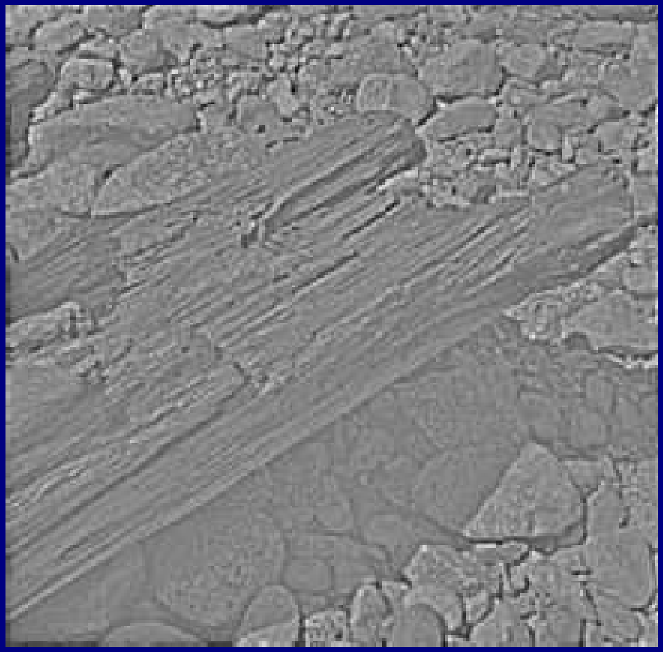
Ref: *Representation and Analysis of Visual Images* lecture notes by Prof. Eero P. Simoncelli

Fourier Magnitude + Gaussian

- **Failures of [Fourier Magnitude + Gaussian] Model**



A sample drawn from a
1/f Fourier + Gaussian model
Doesn’t look natural! ☹



A natural image after whitening
in Fourier domain
Doesn’t look uncorrelated! ☹

Simoncelli & Olshausen ’01

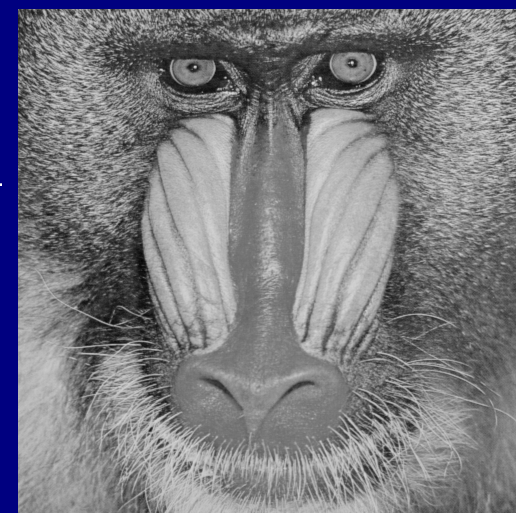
Fourier Magnitude + Gaussian

- **Explanations of the Failure**
  - Probability distribution of natural images is not Gaussian
  - Image signals are not stationary
  - Fourier phase is more important than Fourier magnitude

Fourier phase

Fourier magnitude

combine



Modeling on the wrong aspect of the Fourier transform?!

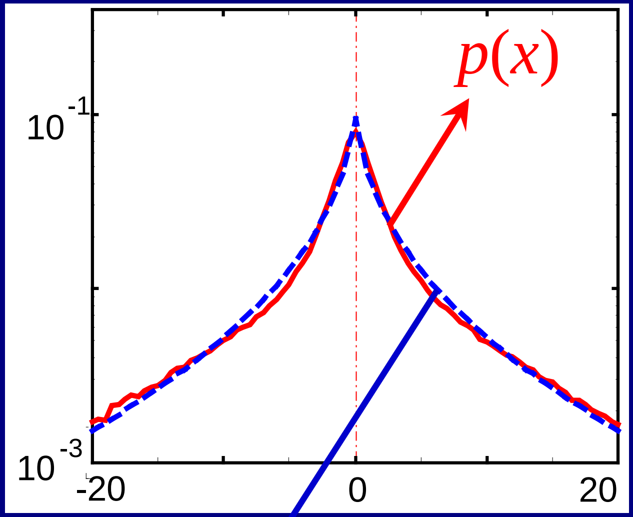
After [Oppenheim & Lim ’81]

Wavelet Marginal Model

DWT

+

subband statistics



coefficient histogram in log-scale

Generalized Gaussian Density [Mallat '89]:

$$p_m(x) = \frac{\beta}{2\alpha\Gamma(1/\beta)} e^{-(|x|/\alpha)^{\beta}}$$

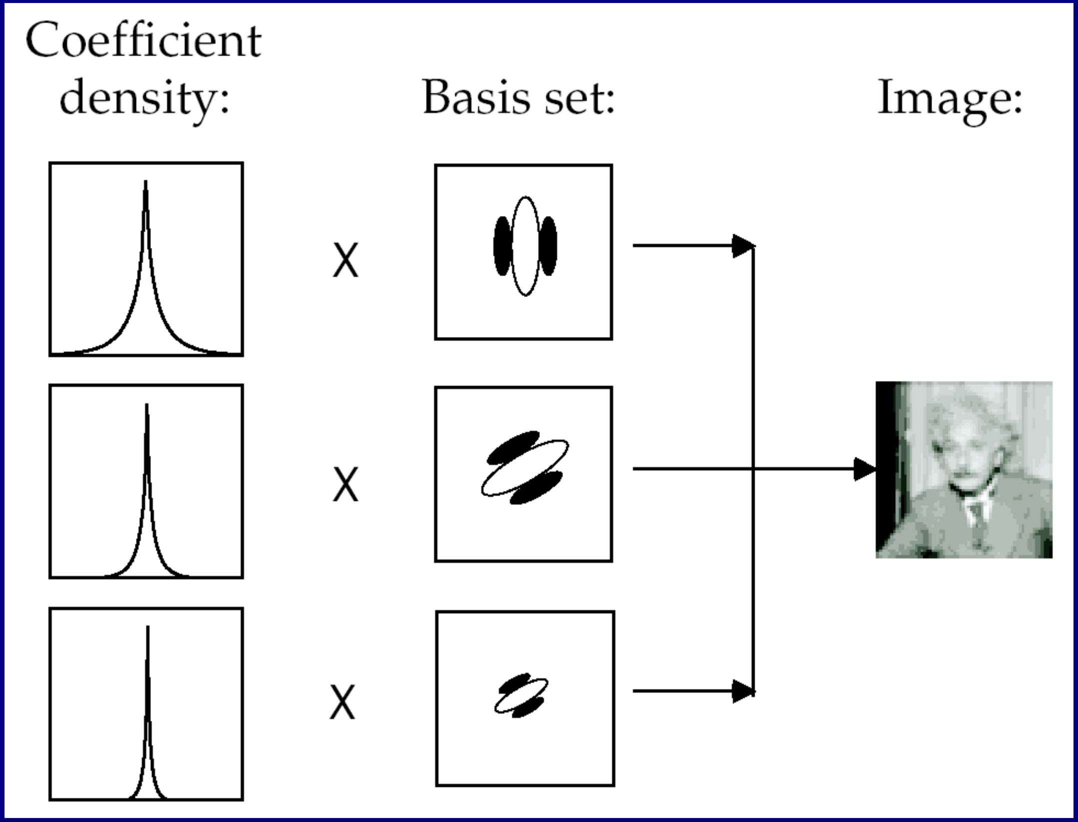
α: scale (variance) parameter

β: peakedness parameter, typically between 0.3 ~ 1.2

Gaussian: $\beta = 2$ → higher entropy, lower kurtosis

Laplacian: $\beta = 1$

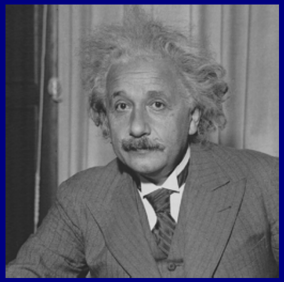
Wavelet + Non-Gaussian Marginal



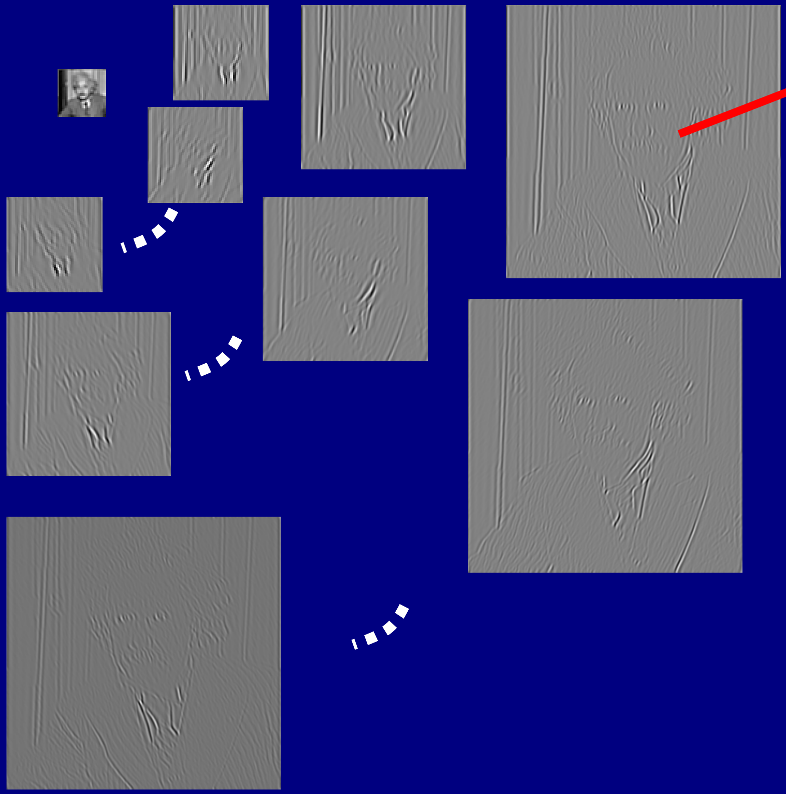
- **Limitation**
  - Still cannot capture the dependency between neighboring coefficients (both intra- and inter-channel dependencies)

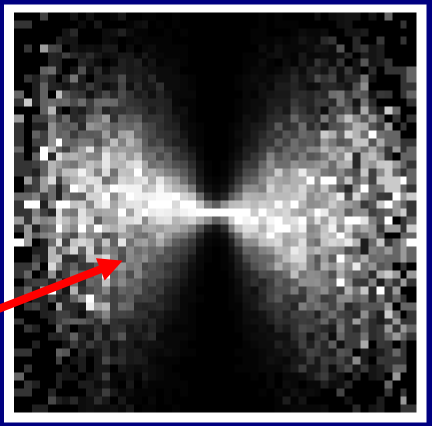
Ref: *Representation and Analysis of Visual Images* lecture notes by Prof. Eero P. Simoncelli

Wavelet Joint Statistics



steerable pyramid decomposition
[Simoncelli *et al.* ’92]



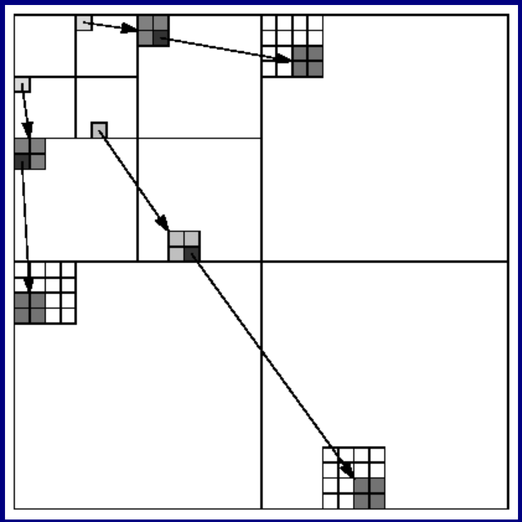


conditional histogram of neighboring coefficients
A “bow-tie” structure [Simoncelli ’97]

- Significant (large magnitude) coefficients tend to appear in clusters (neighbors in intra-channel spatial location, and across scales and orientations)

Wavelet Joint Models

- **Hidden Markov Tree** [Crouse, Nowak & Baraniuk ’98]



  - Hidden states: S → small coefficient; L → large coefficient
  - Hidden state transition matrix

$$A = \begin{bmatrix} p^{\mathsf{S}\to\mathsf{S}} & p^{\mathsf{S}\to\mathsf{L}} \\ p^{\mathsf{L}\to\mathsf{S}} & p^{\mathsf{L}\to\mathsf{L}} \end{bmatrix}$$ where $$p^{\mathsf{S}\to\mathsf{S}} > \frac{1}{2}, \quad p^{\mathsf{L}\to\mathsf{L}} > \frac{1}{2}$$

Why?

Wavelet Joint Models

- **Gaussian Scale Mixture**

$$\mathbf{x} \overset{d}{=} \sqrt{z}\,\mathbf{u}$$

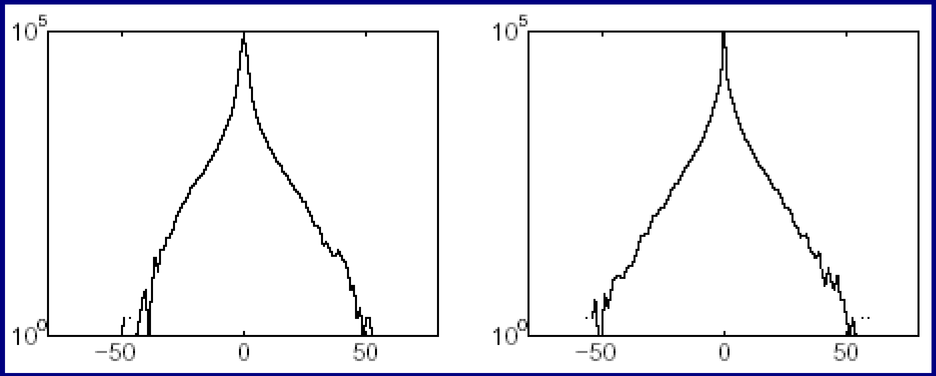
x: A vector of neighboring coefficients

z: multiplier

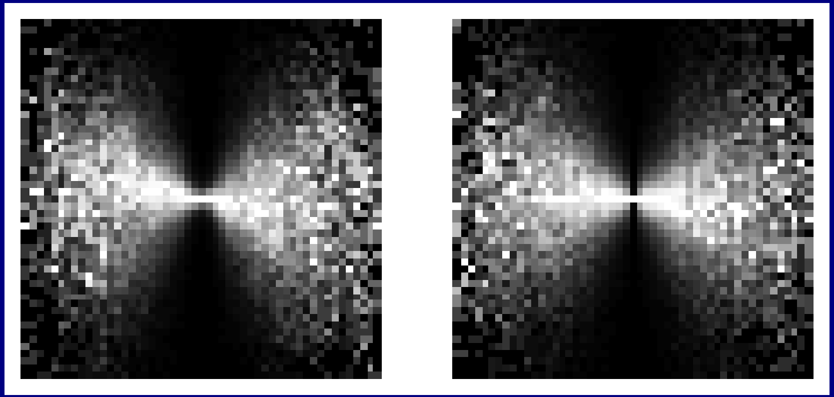
u: zero-mean Gaussian vector with covariance $\mathbf{C}_\mathbf{u}$

PDF:

$$\begin{aligned} p_\mathbf{x}(\mathbf{x}) &= \int p(\mathbf{x}|z)\, p_z(z) dz \\ &= \int \frac{\exp\left(-\mathbf{x}^T (z\mathbf{C}_\mathbf{u})^{-1}\mathbf{x}/2\right)}{(2\pi)^{N/2}|z\mathbf{C}_\mathbf{u}|^{1/2}}\, p_z(z) dz \end{aligned}$$



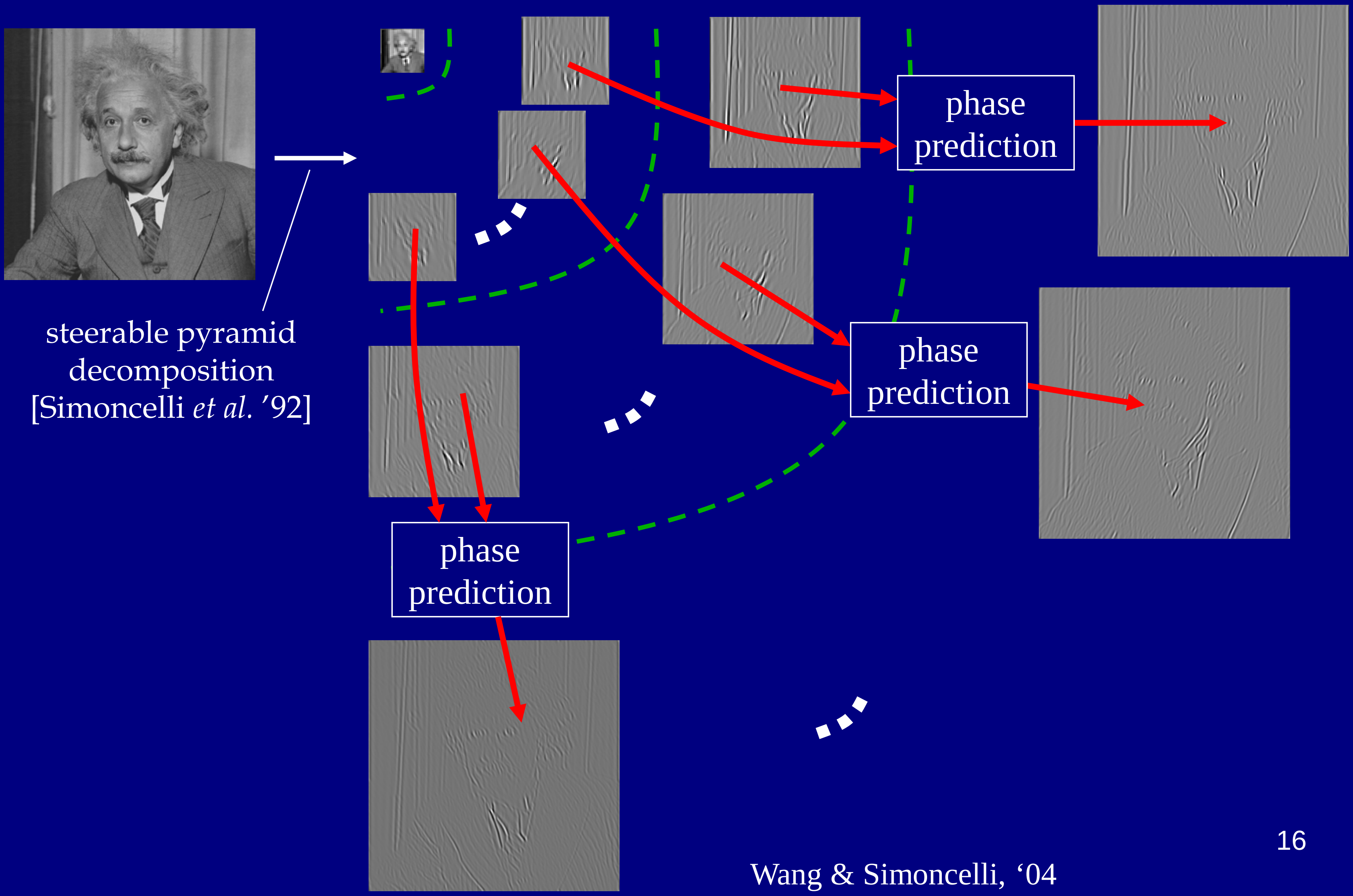
real simulated



real simulated

[Wainwright, Schwartz & Simoncelli ’00], [Portilla, Strela, Wainwright & Simoncelli ’03]

Wavelet Phase Statistics

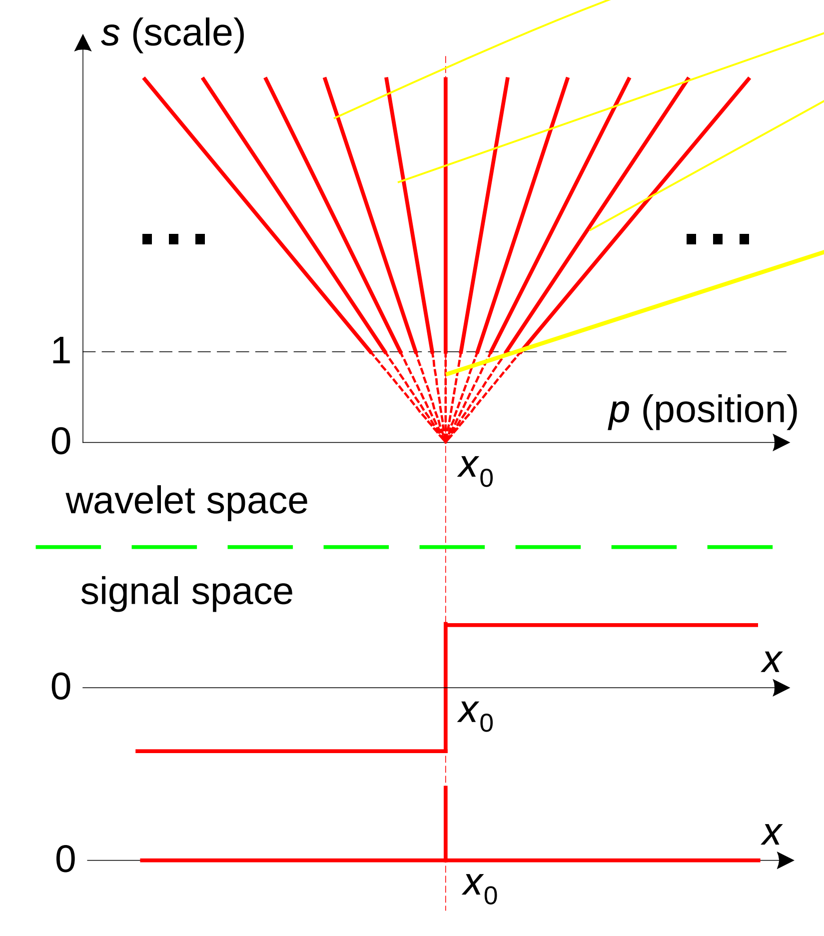


Wang & Simoncelli, ‘04

Wavelet Phase Statistics

$$\Phi\big(F(s,p)\big) = \Phi\left(F\left(1,\ x_0 + \frac{p - x_0}{s}\right)\right)$$

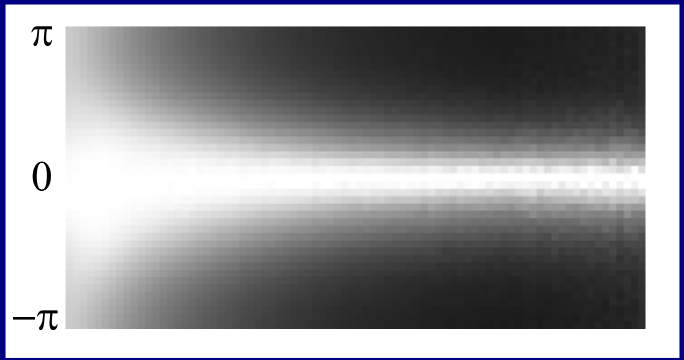
$$x_0 + \frac{p - x_0}{s} = Const$$



convergent straight line

1000 natural images

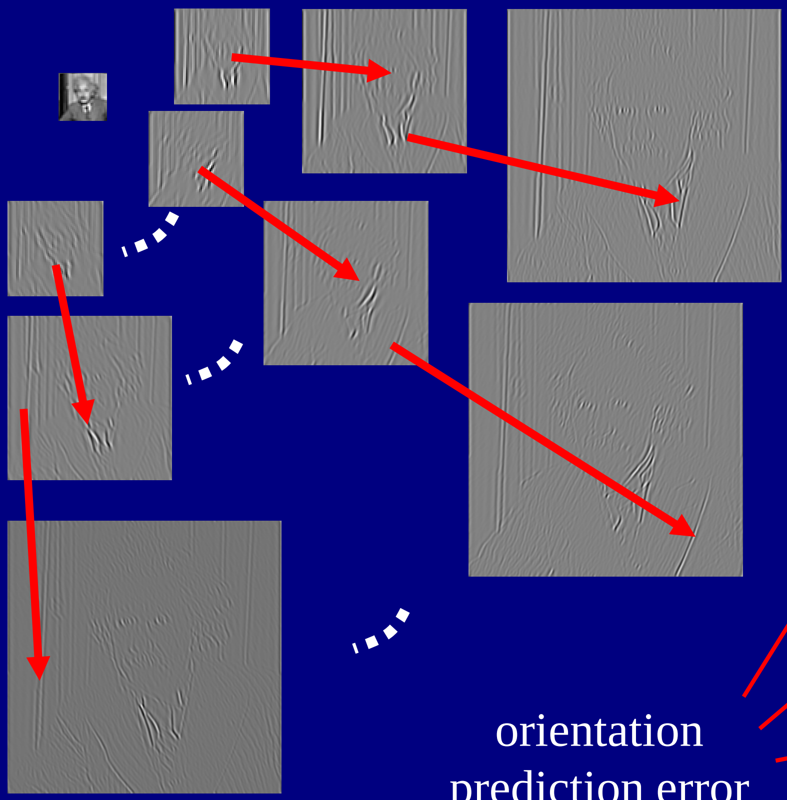
phase prediction error



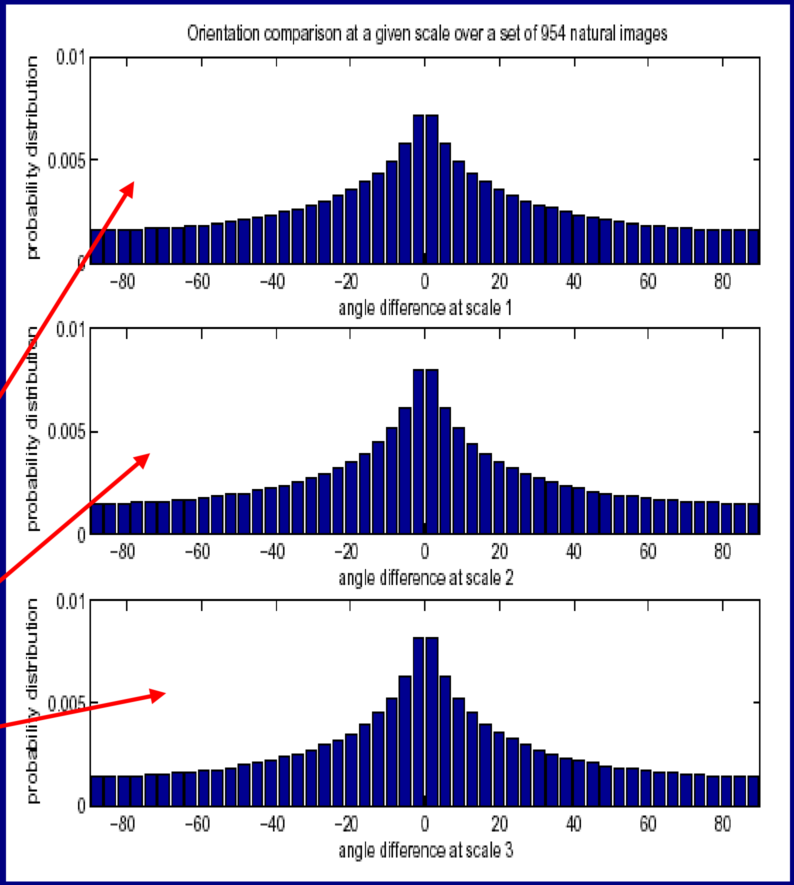
coefficient magnitude

Wang & Simoncelli, ‘04

Wavelet Orientation Statistics



orientation
prediction error

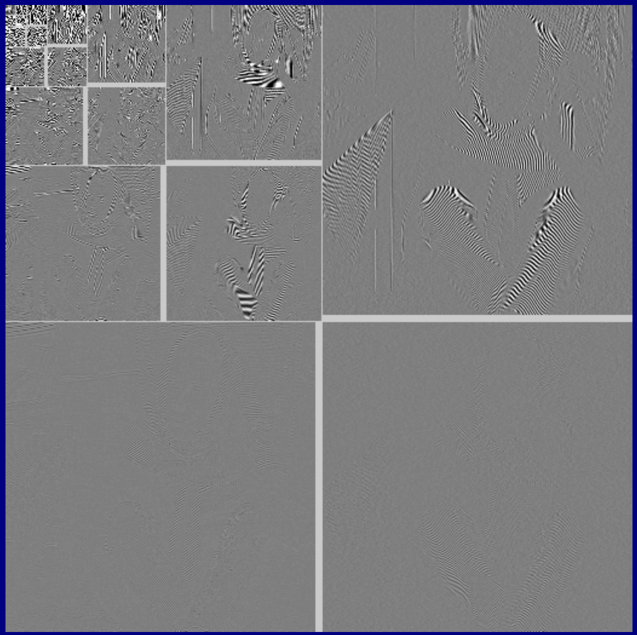


– Orientation at the same spatial location but different scale are correlated.

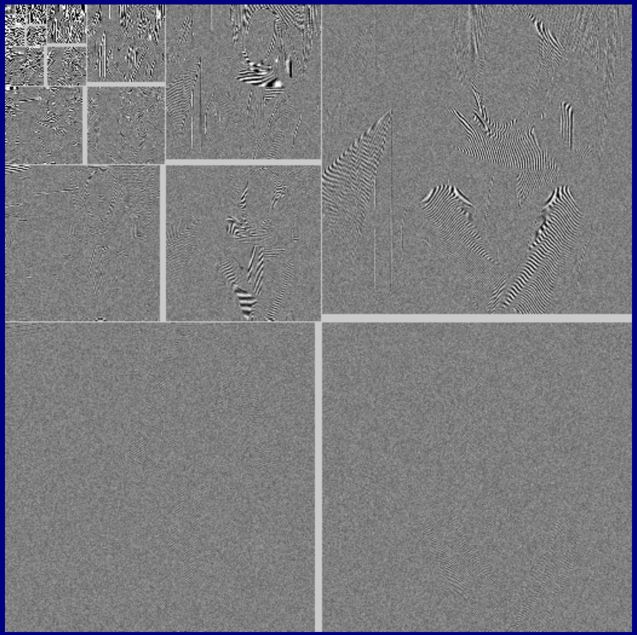
Marcovici & Simoncelli '03

Application: Image Denoising

- **Noisy Image: Observations**



DWT of clean “Barbara” image



DWT of noisy “Barbara” image

  - More and more noisy from coarser to finer scales (low-pass filtering?)
  - More interference with smaller than larger coefficients (thresholding?)
  - Signal-to-noise ratio varies over space (locally adaptive filtering?)
  - Large coefficients appear in clusters (……?)
  - Large coefficients have correlated orientations and phases (……?)

Application: Image Denoising

- **Denoising Problem**

$$y = x + w$$

x: original
w: i.i.d. noise, independent of x (zero-mean Gaussian)
y: observed

  - **Goal:** given y, estimate x

- **Estimators**
  - Maximum likelihood: $\hat{x}_{ML} = \arg\max_{\hat{x}} \; p(y \mid \hat{x})$
  - (Bayes) maximum a posterior: $\hat{x}_{MAP} = \arg\max_{\hat{x}} \; p(\hat{x} \mid y)$
  - Bayes least square: $\hat{x}_{BLS} = \arg\min_{\hat{x}} \; \mathrm{E}[(x-\hat{x})^2 \mid y]$

$$\hat{x}_{BLS} = \mathrm{E}(x \mid y) = \int x \, p(x \mid y) dx = \int x \frac{p(y \mid x) p(x)}{p(y)} dx = \frac{\int x \, p(y \mid x) p(x) dx}{\int p(y \mid x) p(x) dx}$$

Why?

Application: Image Denoising

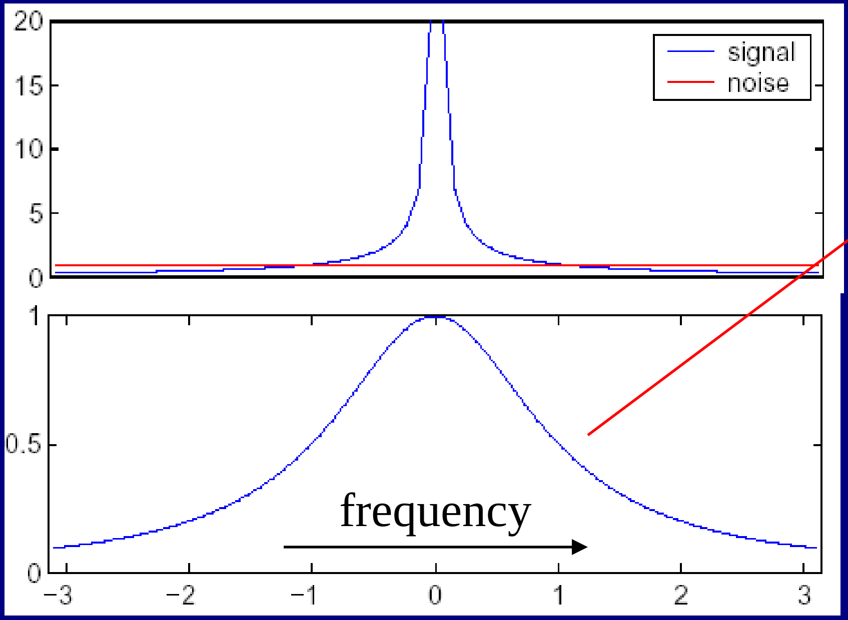
- **Signal Model: Gaussian**

  x: zero-mean Gaussian with variance σ_x^2

  w: zero-mean Gaussian with variance σ_w^2

  – ML estimator: $\hat{x}_{ML} = y$

  – MAP and BLS estimator: $\hat{x}_{MAP} = \hat{x}_{BLS} = \dfrac{\sigma_x^2}{\sigma_x^2 + \sigma_w^2} y$



equivalent to Weiner filtering when applied in the Fourier domain

- This explains why Weiner filtered images are always blurred
- Note: the low-pass feature comes from the signal and noise models (not assumed)

Ref: *Representation and Analysis of Visual Images* lecture notes by Prof. Eero P. Simoncelli

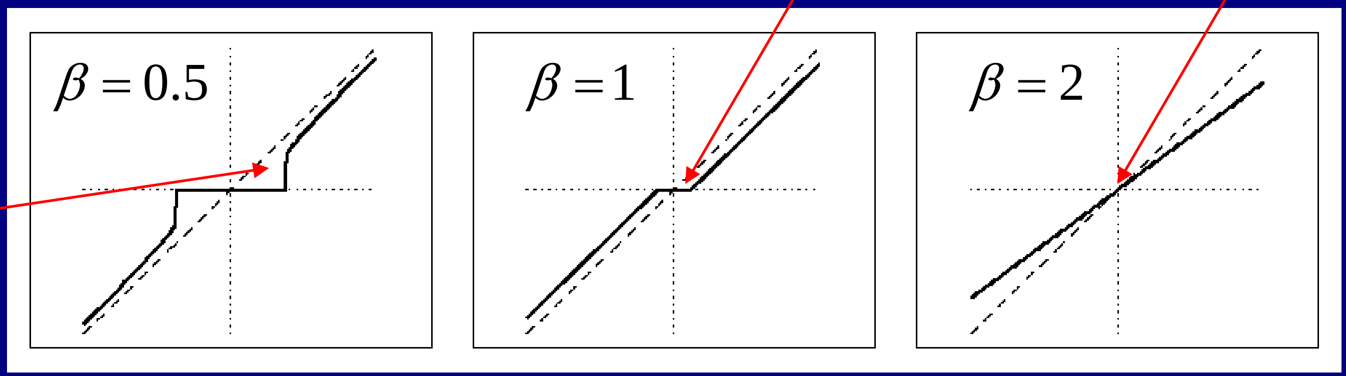
Application: Image Denoising

- **Signal Model: Generalized Gaussian**
  - ML estimator: $\hat{x}_{ML} = y$
  - MAP estimators:

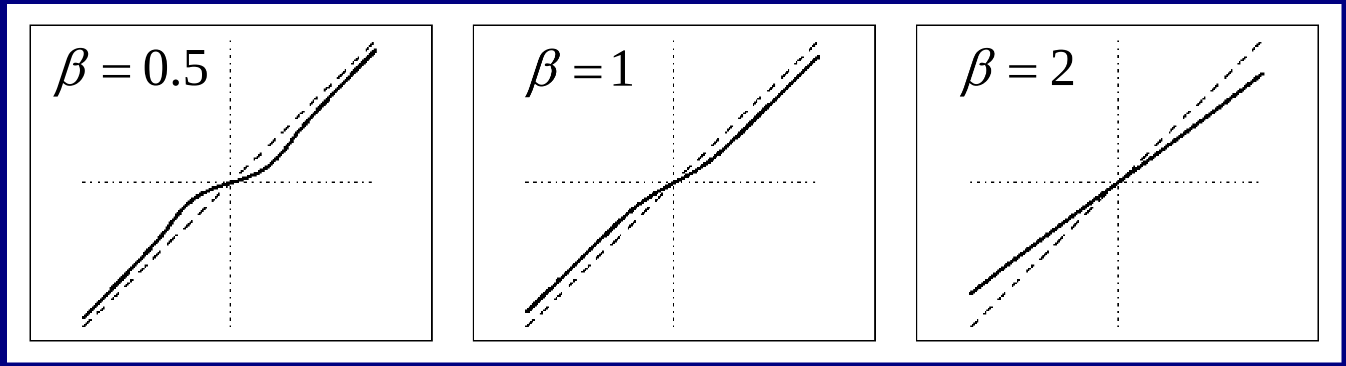
soft thresholding

linear solution

hard thresholding



  - BLS estimators:



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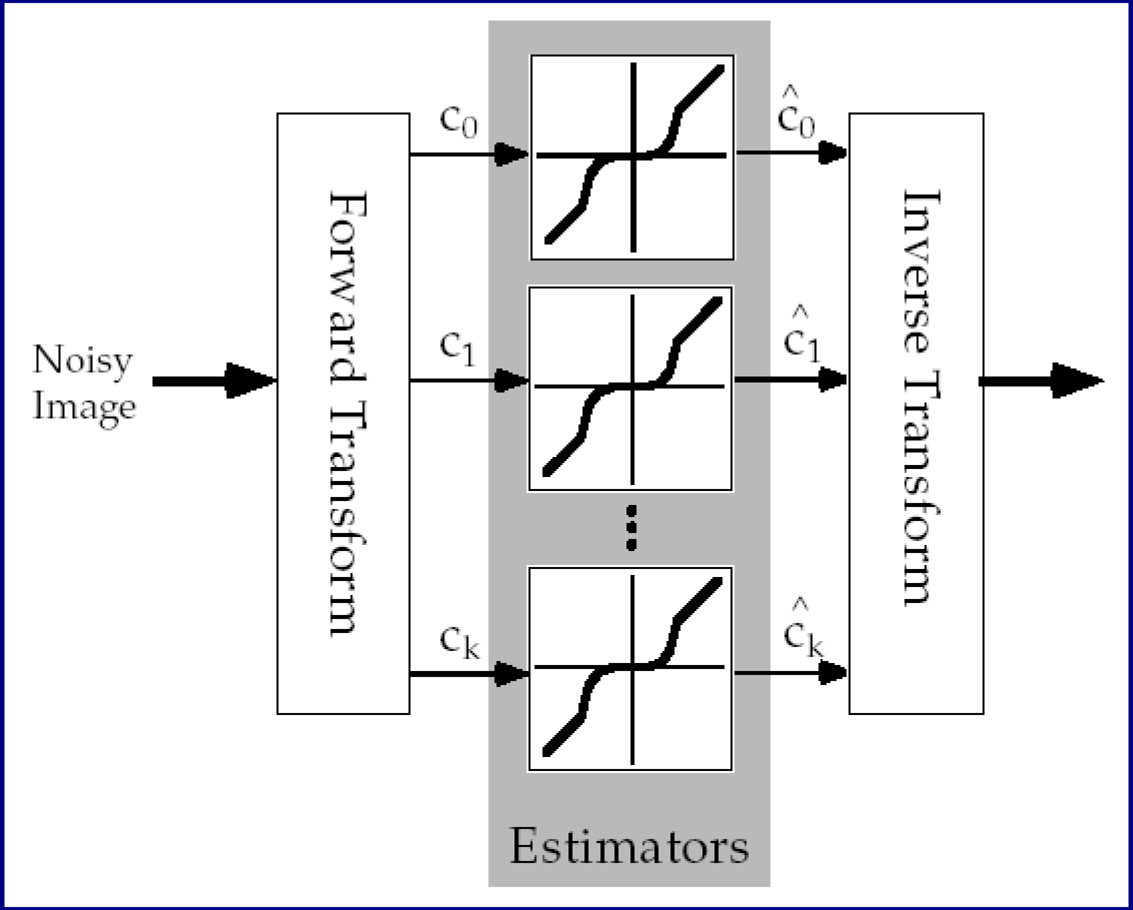
Application: Image Denoising

- **Application to Images**



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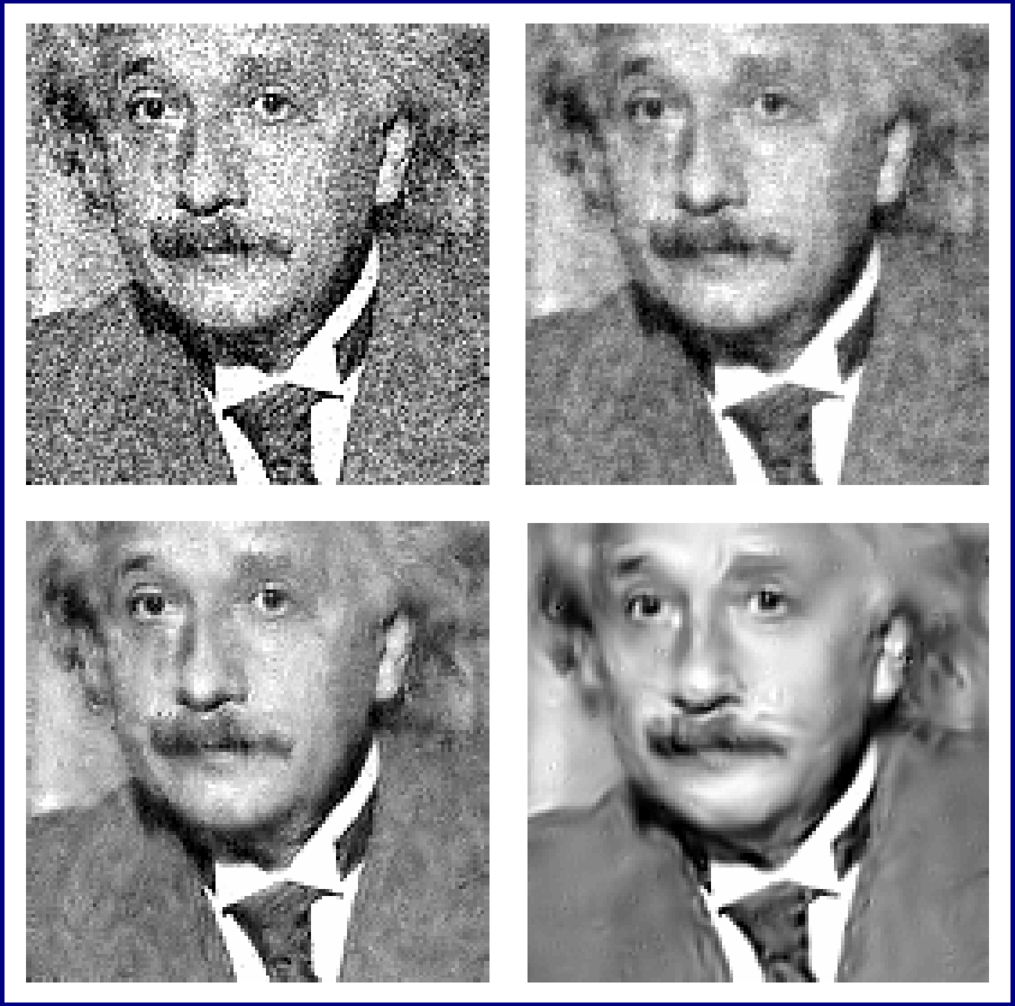
Application: Image Denoising

- **Application to Images**

noisy image
SNR = 4.8dB

linear filtering
SNR = 10.61dB



nonlinear:
marginal model
+ BLS filtering
SNR = 11.98dB

nonlinear:
joint (GSM) model
+ BLS filtering
SNR = 13.60dB

[Portilla *et al.* ’03]

Ref: *Representation and Analysis of Visual Images* lecture notes by Prof. Eero P. Simoncelli

Other Applications of Statistical Image Models

- **Image Compression**
  - Explicit: [Buccigrossi & Simoncelli ’99]
  - Implicit: EZW [Shapiro ’93], SPIHT [Said & Pearlman ’96], JPEG2000 …
- **Image Restoration**
  - Image model + blur model + noise model
- **Image Enhancement**
  - Moving toward more probable direction in image space
- **Image Quality Assessment**
  - How far an image departs from “natural image clusters”
- **Image (Texture) Segmentation and Classification**
- **Image (Texture) Synthesis**

……